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Compact quantum ergodic systems

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Abstract

We develop theory of multiplicity maps for compact quantum groups. As an application, we obtain a complete classification of right coideal C^*-algebras of $C(SU_q(2))$ for $q \in [-1, 1) \setminus \{0\}$. They are labeled with Dynkin diagrams, but classification results for positive and negative cases of q are different. Many of the coideals are quantum spheres or quotient spaces by quantum subgroups, but we do have other ones in our classification list.

Keywords: Compact quantum groups; Ergodic actions; Right coideals

1. Introduction

The core of this paper consists of studying general compact quantum ergodic systems and classifying right coideals of $C(SU_q(2))$ for $q \in [-1, 1) \setminus \{0\}$. Our motivation relies on the works [20] and [21] by A. Wassermann, where he has established theory of multiplicity maps and classified ergodic systems of the compact group $SU(2)$. It is natural to ask whether his theory can be adapted to the classification of ergodic systems of a compact quantum group $SU_q(2)$, which is an example of a compact quantum group and we regard it as a deformed $SU(2)$ group with a parameter q. First of all, we look back upon ergodic actions of compact groups.

Ergodicity means that the fixed point algebra of given action becomes trivial. This strong condition derives several special properties. For example, the invariant state must be tracial and the multiplicities of irreducible representations are bounded by dimensions of their representation spaces [10]. In quantum setting, the multiplicities of them also become finite [4], however an

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invariant state has no longer the tracial property and the multiplicities are bounded not by usual dimensions but by quantum dimensions which are larger than or equal to usual dimensions. Actually in [3], a great deal of examples of compact quantum ergodic systems which have the multiplicities strictly larger than their dimensions are constructed. Keeping in mind these phenomena, one shall notice there are rather differences between the ergodic actions of classical and quantum groups. In studying compact quantum ergodic systems, the main machineries treated in this paper are multiplicity maps and their diagrams developed by A. Wassermann [20]. Since multiplicity maps are defined via equivariant K-theory, one easily obtains its quantum version by using the work due to, for example, [1,4,19]. The most non-trivial important problem is whether there exists a common eigenvector of multiplicity maps or not. One shall notice that its existence is guaranteed by a tracial invariant state. For that problem, we show its existence in two cases, (1) A has a (not necessarily faithful) tracial state and $G = SU_q(2)$, (2) A is a right coideal.

With these results, we can proceed to classify all right coideals of $C(SU_q(2))$, which is the aim of the latter half of this paper. In the classical case, we know the continuous function algebra on the homogeneous space $H \setminus G$ by a closed subgroup $H \subset G$ gives a right coideal and its converse holds via the Gelfand–Naimark theorem on abelian C^*-algebras. However in the case of a quantum group, such a correspondence breaks down in general. For example, in [16], a one-parameter family of quantum spheres $C(S^2_{q,\lambda})$ is constructed, which consists of right coideals and most of them are not obtained by taking quotient by subgroups. Therefore, it is interesting to investigate the other non-quotient type right coideals. The most important information of a right coideal is its spectral pattern, that is, multiplicities of irreducible representations. We will see that multiplicity diagrams are labeled by closed subgroups of $SU(2)$ or $SU_{-1}(2)$ as McKay diagrams with respect to the fundamental two-dimensional representation and from those diagrams, one can compute the possible spectral patterns. These data enable us to classify right coideals into the several types by connected graphs of norm 2 as is used in the classification of ergodic systems of $SU(2)$. Then one can find absence in some spectral patterns. Their gaps often become a good obstruction for existence of ergodic systems. Then we carry out case-by-case study of them. In that procedure, one has to be careful of the essential effect of $|q| \neq 1$ and its sign. In fact, when we work on the negative q case, it is also needed to treat the graphs with a single loop at a vertex. As a result, right coideals of some types such as the regular polyhedrons do not appear in those cases. We state the main result on this classification.

(1) The case $0 < q < 1$. A right coideal must be one of type 1, $SU(2)$, $\mathbb{T}_n$, $\mathbb{T}$ and D^*_∞. When it is of type $\mathbb{T}$, then it is one of series of the quantum spheres. Otherwise it is uniquely determined by the type.
(2) The case $-1 < q < 0$. A right coideal must be one of type 1, $SU(2)$, $\mathbb{T}_n$, $\mathbb{T}$, D^*_∞ and D_1. When it is of type $\mathbb{T}$, then it is one of series of the quantum spheres. Otherwise it is uniquely determined by the type.
(3) The case $q = -1$. If a right coideal A is not of type $\mathbb{T}_n$ (odd $n \geqslant 3$) or D_n (odd $n \geqslant 1$), there exists a closed subgroup H in $SO_{-1}(3)$ such that A is $C(H \setminus SO_{-1}(3))$. If a right coideal A is of type $\mathbb{T}_n$ (odd $n \geqslant 3$), A is conjugated to $C(\mathbb{T}_n \setminus SU_{-1}(2))$ or $C^*(\eta^{\frac{n}{2}}, \hat{\eta}^{\frac{n}{2}})$. If a right coideal A is of type D_1, then A is conjugated to $C(D_1 \setminus SU_{-1}(2))$. If a right coideal A is of type D_n (odd $n \geqslant 3$), A is conjugated to $C(D_n \setminus SU_{-1}(2))$ or $C^*(\eta^{\frac{n}{2}})$. Here conjugation is given by the left action β^L_z of the maximal torus for some $z \in \mathbb{T}$.

In each of the above cases, uniqueness is up to conjugation by the left action of the maximal torus $\mathbb{T}$. On right coideals which are of type D^*_∞ in the case $0 < q < 1$, of types D^*_∞ and D_1

in the case $-1 < q < 0$ and $C^*(\eta^{\frac{n}{2}}, \hat{\eta}^{\frac{n}{2}})$, $C^*(\eta^{\frac{n}{2}})$ in the case $q = -1$, one can see that they are not the quantum spheres nor the quotient spaces. Moreover, in the case $q = -1$, a right coideal which is not of type $\mathbb{T}_n$ (odd $n \geqslant 3$) is $SU_{-1}(2)$-isomorphic to each other in the same type, however, $C(\mathbb{T}_n \setminus SU_{-1}(2))$ is not $SU_{-1}(2)$-isomorphic to $C^*(\eta^{\frac{n}{2}}, \hat{\eta}^{\frac{n}{2}})$. Hence there exist at least two non-conjugate ergodic systems of type $\mathbb{T}_n$ (odd $n \geqslant 3$).

We briefly explain the content of each section. In Section 2, we collect well-known facts on compact quantum groups and their actions and also prepare their notations. In Section 3, general compact quantum ergodic systems are studied. In Section 4, we extend the theory of multiplicity maps for compact groups to the case of compact quantum groups. Section 5 is a summary of representation theory of $SU_q(2)$. In Sections 6–8, we carry out classification of right coideals of $C(SU_q(2))$ for $0 < q < 1$, $-1 < q < 0$ and $q = -1$, respectively. In Appendix A, all the connected graphs of norm 2 are listed. If one has only interest in classification of right coideals, he or she can skip results in Sections 1–4 except for Corollary 4.21.

2. Preliminaries

We collect basic notions on compact quantum groups and their actions. Our standard references are [24] for theory of compact quantum groups and [2] for their actions. In this paper we only treat minimal tensor products for C^*-algebras and use the notation simply $\otimes$. Although it is not essential, separability of compact quantum groups is always assumed.

2.1. Compact quantum groups and theory of their representations

Definition 2.1. (See [24, p. 853].) Let A be a unital C^*-algebra and $\delta : A \to A \otimes A$ be a faithful unital $*$-homomorphism. If they satisfy the following conditions, the pair (A, δ) is called a compact quantum group.

(1) The map δ satisfies coassociativity condition

$$(\delta \otimes \mathrm{id}_A) \circ \delta = (\mathrm{id}_A \otimes \delta) \circ \delta.$$

(2) The vector spaces $\delta(A)(\mathbb{C} \otimes A)$ and $\delta(A)(A \otimes \mathbb{C})$ are dense in $A \otimes A$.

Let (A, δ) be a compact quantum group. Then there exists the unique state called the *Haar state* h with the invariance condition,

$$(h \otimes \mathrm{id}_A) \circ \delta(a) = (\mathrm{id}_A \otimes h) \circ \delta(a) = h(a)1_A, \quad \text{for all } a \in A.$$

We always consider a compact quantum group whose Haar state is faithful. A compact quantum group (A, δ) is often regarded as "the continuous function algebra" on a non-commutative space G. From this concept, we write $G = (A, \delta)$ and $A = C(G)$.

Let H be a Hilbert space and v be a unitary in $M(\mathbb{K}(H) \otimes A)$. A unitary v is called a *unitary representation* of G if it satisfies the following equality:

$$(\mathrm{id} \otimes \delta)(v) = v_{12} v_{13}.$$

Let $\{v_i\}_{i \in I}$ be a family of unitary representations on Hilbert spaces $\{H_{v_i}\}_{i \in I}$, respectively. The direct sum representation $\prod_{i \in I} v_i$ is defined as the direct sum of unitary operators via the

natural inclusion $\prod_{i\in I} M(\mathbb{K}(H_{v_i})\otimes C(G))\subset M(\mathbb{K}(\bigoplus_{i\in I} H_v)\otimes C(G))$. Let v, w be unitary representations on Hilbert spaces H_v, H_w, respectively. The tensor product representation $v \dot{\otimes} w$ is defined as a unitary operator $v_{13}w_{23}$ which is an element of $M(\mathbb{K}(H_v\otimes H_w)\otimes C(G))$. An operator T in $\mathbb{B}(H_v, H_w)$ is called an *intertwiner* of v and w if it satisfies $(T\otimes 1)v = w(T\otimes 1)$. The set of intertwiners between v and w is written as $\mathrm{Hom}(v, w)$ and it is called a *hom-space*. For one unitary representation v, its hom-space $\mathrm{Hom}(v, v)$ becomes a C^*-subalgebra of $\mathbb{B}(H_v)$. If it is trivial (that is, $\mathrm{Hom}(v, v) = \mathbb{C}1_{H_v}$), the unitary representation v is called *irreducible*. Let v be a unitary representation of a compact quantum group G and v is a direct sum of finite-dimensional irreducible representations [24, p. 864]. In particular, any irreducible unitary representation is finite-dimensional.

In the set of all irreducible unitary representations of G, we define unitary equivalence relation naturally and denote its quotient set by $\widehat{G}$. We can select one unitary representation $w(\pi)\in \mathbb{B}(H_\pi)\otimes C(G)$ for each equivalence class $\pi\in\widehat{G}$. Let $\{\xi_i\}_{i\in I_\pi}$ be an orthonormal basis for H_π. We can identify $w(\pi)$ with the matrix $\{w(\pi)_{i,j}\}_{i,j\in I_\pi}$ where $w(\pi)_{i,j}$ are elements of $C(G)$. From the definition of unitary representation, we obtain the following formula:

$$\delta\big(w(\pi)_{i,j}\big) = \sum_{k\in I_\pi} w(\pi)_{i,k}\otimes w(\pi)_{k,j}, \quad \text{for all } i, j\in I_\pi.$$

Let us define the *smooth function algebra* on G, $A(G) = \mathrm{span}\{w(\pi)_{i,j} \mid i, j\in I_\pi,\ \pi\in\widehat{G}\}$. It is in fact a unital $*$-subalgebra dense in $C(G)$ and the set $\{w(\pi)_{i,j} \mid i, j\in I_\pi,\ \pi\in\widehat{G}\}$ is a linear basis for $A(G)$. Two maps $\varepsilon : A(G)\to\mathbb{C}$ and $\kappa : A(G)\to A(G)$ defined by

$$\varepsilon\big(w(\pi)_{i,j}\big) = \delta_{i,j}, \qquad \kappa\big(w(\pi)_{i,j}\big) = w(\pi)_{j,i}^*$$

for each $\pi\in\widehat{G}$ and $i, j\in I_\pi$ give the algebra $A(G)$ a *Hopf $*$-algebra* structure:

(1) The map $\delta : A(G)\to A(G)\otimes A(G)$ satisfies coassociativity

$$(\delta\otimes\mathrm{id})\circ\delta = (\mathrm{id}\otimes\delta)\circ\delta.$$

(2) The map (called the *counit*) $\varepsilon : A(G)\to\mathbb{C}$ is a unital $*$-homomorphism and satisfies

$$(\varepsilon\otimes\mathrm{id})\circ\delta = (\mathrm{id}\otimes\varepsilon)\circ\delta = \mathrm{id}.$$

(3) The map (called the *antipode*) $\kappa : A(G)\to A(G)$ is a linear anti-multiplicative map and satisfies

$$m(\kappa\otimes\mathrm{id})\circ\delta = m(\mathrm{id}\otimes\kappa)\circ\delta = \varepsilon$$

and

$$\kappa\big(\kappa(a^*)^*\big) = a \quad \text{for all } a\in A(G).$$

Let $A(G)^*$ be an algebraic dual space of $A(G)$. For $a\in A(G)$ and $\theta, \omega_1, \omega_2\in A(G)^*$, we define their convolution products and the involution,

$$\theta * a = (\mathrm{id} \otimes \theta)\big(\delta(a)\big),$$
$$a * \theta = (\theta \otimes \mathrm{id})\big(\delta(a)\big),$$
$$\omega_1 * \omega_2 = (\omega_1 \otimes \omega_2) \circ \delta,$$
$$\theta^*(a) = \overline{\theta\big(\kappa(a)^*\big)}.$$

With these operations the dual space $A(G)^*$ becomes a unital $*$-algebra (its unit is the counit) and acts on $A(G)$ from the left and right. On the smooth function algebra $A(G)$, there exists a unique family of characters $\{f_z\}_{z\in\mathbb{C}}$ which are called *Woronowicz characters* with the following properties:

(1) $f_z(1) = 1$ for all $z \in \mathbb{C}$.
(2) For any element $a \in A(G)$, the mapping $z \in \mathbb{C} \mapsto f_z(a) \in \mathbb{C}$ is an entire holomorphic function.
(3) $f_{z_1} * f_{z_2} = f_{z_1+z_2}$ for all $z_1, z_2 \in \mathbb{C}$.
(4) $f_0 = \varepsilon$.
(5) $f_z(\kappa(a)) = f_{-z}(a)$ for all $z \in \mathbb{C}$, $a \in A(G)$.
(6) $f_z(a^*) = \overline{f_{-\bar{z}}(a)}$ for all $z \in \mathbb{C}$, $a \in A(G)$.

We can define one-parameter automorphism groups $\{\sigma_t^h\}_{t\in\mathbb{R}}$ and $\{\tau_t\}_{t\in\mathbb{R}}$ and $*$-anti-multiplicative linear map R by

$$\sigma_t^h(a) = f_{it} * a * f_{it} \quad \text{for all } a \in A(G),$$
$$\tau_t(a) = f_{-it} * a * f_{it} \quad \text{for all } a \in A(G),$$
$$R(a) = f_{\frac{1}{2}} * \kappa(x) * f_{-\frac{1}{2}} \quad \text{for all } a \in A(G).$$

They are called the *modular automorphism group*, the *scaling automorphism group* and the *unitary antipode*, respectively. They are norm continuously extendable to the continuous function algebra $C(G)$. The Haar state h is a σ^h-KMS state. For any $a, b \in A(G)$, we have

$$h(ab) = h\big(\sigma_i^h(b)a\big).$$

Relations of these maps are as follows:

$$\sigma_s^h \circ \tau_t = \tau_t \circ \sigma_s^h \quad \text{for all } s, t \in \mathbb{R},$$
$$R \circ \sigma_s^h = \sigma_{-s}^h \circ R, \qquad R \circ \tau_t = \tau_t \circ R,$$
$$\kappa = R \circ \tau_{-\frac{i}{2}} = \tau_{-\frac{i}{2}} \circ R.$$

Let v be a unitary representation on a finite-dimensional Hilbert space H_v and $j_v : H_v \to \overline{H_v}$ be a conjugate unitary map where $\overline{H_v}$ is the conjugate Hilbert space. The transpose map $t_v : \mathbb{B}(H_v) \to \mathbb{B}(\overline{H_v})$ is defined by $t_v(x) = j_v x^* j_v^{-1}$. We define the *contragradient representation* (which is not necessarily a unitary operator) v^{c} and the conjugate unitary representation $\bar{v}$ of a finite-dimensional unitary representation v by

$$v^{\mathrm{c}} = (t_v \otimes \kappa)(v), \qquad \bar{v} = (t_v \otimes R)(v).$$

The *F-matrix* of a finite-dimensional unitary representation $v \in \mathbb{B}(H_v) \otimes C(G)$ is defined by

$$F_v = (\mathrm{id} \otimes f_1)(v) \in \mathbb{B}(H_v).$$

This matrix is strictly positive definite. It is known that $F_v^z = (\mathrm{id} \otimes f_z)(v)$ for any $z \in \mathbb{C}$ and $\mathrm{Tr}(F_v) = \mathrm{Tr}(F_v^{-1})$. For a unitary irreducible representation v, its second contragradient (not necessarily unitary) representation v^{cc} is also irreducible and equivalent to v. In fact we have $\mathrm{Hom}(v, v^{\mathrm{cc}}) = \mathbb{C}F_v$. The value $\mathrm{Tr}(F_v)$ is called the *quantum dimension* and denoted by D_v. Since we have $F_{\bar{v}} = t_v(F_v^{-1})$, the quantum dimension of v is equal to that of $\bar{v}$. We write d_v for the usual dimension of the vector space H_v. The quantum dimension is larger than or equal to the usual dimension. When we fix a selection of $\{w(\pi)\}_{\pi \in \widehat{G}}$, we write H_π, F_π, D_π and d_π for $H_{w(\pi)}$, $F_{w(\pi)}$, $D_{w(\pi)}$ and $d_{w(\pi)}$, respectively.

Using F-matrices, we have the following equalities about the Haar state:

$$h\big(w(\pi)_{i,j} w(\rho)_{r,s}^*\big) = D_\pi^{-1}(F_\pi)_{s,j}\delta_{\pi,\rho}\delta_{i,r},$$
$$h\big(w(\pi)_{i,j}^* w(\rho)_{r,s}\big) = D_\pi^{-1}\big(F_\pi^{-1}\big)_{r,i}\delta_{\pi,\rho}\delta_{j,s}$$

for any $\pi, \rho \in \widehat{G}$, $i, j \in I_\pi$ and $r, s \in I_\rho$. For the tensor product $H \otimes K$ of two Hilbert spaces H and K, we define its conjugate unitary map by $j_{H\otimes K} : H \otimes K \to \overline{K} \otimes \overline{H}$, $j_{H\otimes K}(\overline{\xi \otimes \eta}) = j_K\eta \otimes j_H\xi$. Then for two finite-dimensional unitary representation v and w, we obtain $\overline{v \mathbin{\dot{\otimes}} w} = \overline{w} \mathbin{\dot{\otimes}} \bar{v}$.

2.2. *Multiplicative unitaries, group C^*-algebras*

Let $L^2(G)$ be the GNS-representation space of $C(G)$ associated to the Haar state h. Its cyclic vector is denoted by $\hat{1}_h$. We define the *modular conjugation* J and $\hat{J}$ by

$$Jx\hat{1}_h = \sigma_{\frac{i}{2}}^h(x)^*\hat{1}_h,$$
$$\hat{J}x\hat{1}_h = R(x)^*\hat{1}_h$$

where x is an element of $A(G)$. We also define a unitary $U = J\hat{J} = \hat{J}J$. When we consider $L^\infty(G)$ which is the σ-weak closure of $C(G)$ in $\mathbb{B}(L^2(G))$, it becomes a *von Neumann algebraic compact quantum group*. Precisely speaking, the coproduct δ and the Haar state h extend to $L^\infty(G)$ as a normal $*$-homomorphism and the faithful normal invariant state, respectively. By the invariance of h, we define the following two unitaries on $L^2(G) \otimes L^2(G)$, for all $x, y \in C(G)$:

$$V_\ell^*(x\hat{1}_h \otimes y\hat{1}_h) = \delta(y)(x\hat{1}_h \otimes \hat{1}_h),$$
$$V(x\hat{1}_h \otimes y\hat{1}_h) = \delta(x)(\hat{1}_h \otimes y\hat{1}_h).$$

They satisfy the following *pentagonal equality* and therefore are called *multiplicative unitaries*,

$$V_{12}V_{13}V_{23} = V_{23}V_{12}.$$

We also use other unitaries:

$$\widetilde{V} = \Sigma(1 \otimes U)V(1 \otimes U)\Sigma,$$
$$W = (U \otimes 1)V(U \otimes 1),$$

where $\Sigma : L^2(G) \otimes L^2(G) \to L^2(G) \otimes L^2(G)$ is a flipping unitary $\xi \otimes \eta \mapsto \eta \otimes \xi$. The unitary $\widetilde{V}$ satisfies the above pentagonal equality and W satisfies $W_{23}W_{13}W_{12} = W_{12}W_{23}$. It is easy to see $V_\ell = \Sigma(\hat{J} \otimes \hat{J})V^*(\hat{J} \otimes \hat{J})\Sigma = (1 \otimes U)\Sigma V \Sigma(1 \otimes U)$. We define a left G-action on H_π by $\xi_i^\pi \mapsto \sum_{j \in I_\pi} w(\pi)_{i,j} \otimes \xi_j^\pi$ and denote it by H_π^ℓ. Let Θ_π be a unitary map $H_\pi^\ell \otimes H_\pi \to L^2(G)$ defined by $\Theta_\pi(\xi_i \otimes \xi_j) = \sqrt{D_\pi (F_\pi)_{i,i}}\, w(\pi)_{i,j}$. It intertwines the left and right G-actions. Then we have the *Peter–Weyl decomposition* $\Theta : \bigoplus_{\pi \in \widehat{G}} H_\pi^\ell \otimes H_\pi \to L^2(G)$ with $\Theta = \bigoplus_{\pi \in \widehat{G}} \Theta_\pi$. The *left, right reduced group C^*-algebras* are defined by

$$C_\ell^*(G) = \overline{\operatorname{span}}\bigl\{(\omega \otimes \mathrm{id})(V_\ell) \bigm| \omega \in \mathbb{B}\bigl(L^2(G)\bigr)_*\bigr\},$$
$$C_r^*(G) = \overline{\operatorname{span}}\bigl\{(\mathrm{id} \otimes \omega)(V) \bigm| \omega \in \mathbb{B}\bigl(L^2(G)\bigr)_*\bigr\},$$

where the closure is taken with respect to the operator norm of $\mathbb{B}(L^2(G))$. Note that $C_r^*(G)$ is contained in the commutant algebra of $C_\ell^*(G) = \hat{J}C_r^*(G)\hat{J}$. There is a distinguished projection $p_0 = (\mathrm{id} \otimes h)(V)$ in $C_r^*(G)$. It is also included in $C_\ell^*(G)$ and hence it is a central projection of $C_r^*(G)$. Moreover it is a minimal projection of $\mathbb{K}(L^2(G))$, in fact, we have $p_0 x \hat{1}_h = h(x)\hat{1}_h$ for $x \in C(G)$. We now have a map $\rho : \mathbb{B}(L^2(G))_* \to C_r^*(G)$ by $\rho(\omega) = (\mathrm{id} \otimes \omega)(V)$. Let us define a normal functional $\theta(\pi)_{i,j}(x) = D_\pi (F_\pi)_{j,j}^{-1} h(x w(\pi)_{i,j}^*)$ for $x \in \mathbb{B}(L^2(G))$. Then the linear space $\operatorname{span}\{\rho(\theta(\pi)_{i,j}) \mid i, j \in I_\pi,\ \pi \in \widehat{G}\}$ is dense in $C_r^*(G)$ and moreover we have $\rho(\theta(\pi)_{i,j})\rho(\theta(\sigma)_{p,q}) = \delta_{\pi,\sigma}\delta_{j,p}\rho(\theta(\pi)_{i,q})$. Hence we obtain a $*$-isomorphism

$$C_r^*(G) \to \bigoplus_{\pi \in \widehat{G}} \operatorname{End}(H_\pi)$$

defined by $\rho(\theta(\pi)_{i,j}) \mapsto E_{i,j}^\pi$ which is a matrix unit. With this identification, the action of $E_{i,j}^\pi$ on $L^2(G)$ is given by

$$E_{i,j}^\pi w(\sigma)_{p,q}\hat{1}_h = \delta_{\pi,\sigma}\delta_{j,q} w(\pi)_{p,i}\hat{1}_h.$$

Using this fact and the Peter–Weyl decomposition, we see that the C^*-algebras $C_\ell^*(G)$ and $C_r^*(G)$ act on the first and the second tensor components of the Hilbert space $\bigoplus_{\pi \in \widehat{G}} H_\pi^\ell \otimes H_\pi$, respectively. This is just the differential representation of $\mathbb{B}(L^2(G))_* \subset A(G)^*$. Notice that $E_{i,j}^\pi$ is an operator of rank n and hence $C_\ell^*(G)$ and $C_r^*(G)$ are contained in $\mathbb{K}(L^2(G))$. The projection $*$-homomorphism $C_r^*(G) \to \operatorname{End}(H_\pi)$ is denoted by pr_π.

The *dual coproduct* is defined by $\hat{\delta} : C_r^*(G) \to M(C_r^*(G) \otimes C_r^*(G))$ by

$$\hat{\delta}(x) = V^*(1 \otimes x)V \quad \text{for all } x \in C_r^*(G)$$

and it follows that linear subspaces $\hat{\delta}(C_r^*(G))(\mathbb{C} \otimes C_r^*(G))$ and $\hat{\delta}(C_r^*(G))(C_r^*(G) \otimes \mathbb{C})$ are dense in $C_r^*(G) \otimes C_r^*(G)$. On $C_\ell^*(G)$, we define $\widehat{\delta_\ell}(x) = \Sigma V_\ell^*(x \otimes 1)V_\ell\Sigma$.

2.3. Representation rings

Let $R(G)$ be a $\mathbb{Z}$-module $\bigoplus_{\pi\in\widehat{G}}\mathbb{Z}\pi$. For two $\pi,\sigma\in\widehat{G}$, we define their product via irreducible decomposition of the tensor product representation $w(\pi)\dot{\otimes}w(\sigma)$. More precisely, let $N_\tau^{\pi\sigma}$ be the dimension of the *hom-space* $\mathrm{Hom}(w(\tau), w(\pi)\dot{\otimes}w(\sigma))$, which does not depend on the choice of representatives of $\{w(\pi)\}_{\pi\in\widehat{G}}$. Then we have

$$\pi\cdot\sigma=\sum_{\tau\in\widehat{G}}N_\tau^{\pi\sigma}\tau.$$

The module $R(G)$ has an involution, that is, $\bar{\pi}$ is an equivalence class of $\overline{w(\pi)}$, which also does not depend on the choice of representatives of $\{w(\pi)\}_{\pi\in\widehat{G}}$. By these operations, $R(G)$ has an involutive $\mathbb{Z}$-ring structure and it is called the *representation ring* of G. We define the positive cone $R(G)_+=\bigoplus_{\pi\in\widehat{G}}\mathbb{Z}_{\geqslant 0}\pi$.

Since the hom-space $\mathrm{Hom}(w(\tau), w(\pi)\dot{\otimes}w(\sigma))$ is naturally isomorphic to the hom-space $\mathrm{Hom}(w(\pi), w(\tau)\dot{\otimes}\overline{w(\sigma)})$ or $\mathrm{Hom}(w(\sigma), \overline{w(\pi)}\dot{\otimes}w(\tau))$, we have symmetry of $N_\tau^{\pi\sigma}=N_\pi^{\tau\bar{\sigma}}=N_\sigma^{\bar{\pi}\tau}$.

From now on, we use the letter A not for a continuous function algebra on a quantum group but for an arbitrary C^*-algebra.

2.4. Actions, spectral patterns, crossed products

The following definition of an action is standard.

Definition 2.2. Let A be a (not necessarily unital) C^*-algebra and $(C(G),\delta)$ be a compact quantum group and $\alpha: A\to A\otimes C(G)$ be a $*$-homomorphism. The triple $\{A,G,\alpha\}$ is called a compact quantum covariant system (or simply covariant system) if the following statements hold:

(1) The map α is injective and satisfies $(\alpha\otimes\mathrm{id})\circ\alpha=(\mathrm{id}\otimes\delta)\circ\alpha$.
(2) The vector space $\alpha(A)(\mathbb{C}\otimes C(G))$ is dense in $A\otimes C(G)$.

If we consider a compact quantum covariant system $\{A,G,\alpha\}$ where A is unital, we always assume that α is a unital $*$-homomorphism. Let $\{A,G,\alpha\}$ be a compact quantum covariant system. We can select representatives $\{w(\pi)\}_{\pi\in\widehat{G}}$ of $\widehat{G}$ whose F-matrices are diagonal and fix them in this section. We give comodule structure $\Gamma_\pi: H_\pi\to H_\pi\otimes A(G)$ to a representation space H_π by fixing an orthonormal basis $\{\xi_j^\pi\}_{j\in I_\pi}$: $\Gamma_\pi(\xi_j^\pi)=\sum_{k\in I_\pi}\xi_k^\pi\otimes w(\pi)_{k,j}$ for any $j\in I_\pi$. For $\pi\in\widehat{G}$, define the functional θ_π on $C(G)$ by $\theta_\pi(x)=D_\pi h(x(\sum_{i\in I_\pi}(F_\pi^{-1})_{i,i}w(\pi)_{i,i}))$ for any $x\in C(G)$. As in the case of a compact group [18, Theorem 2], A is completely decomposable. Define the linear map $P_\pi=(\mathrm{id}\otimes\theta_\pi)\circ\alpha$ on A. From simple calculations, we have $P_\pi P_\rho=\delta_{\pi,\rho}P_\pi$. Put A_π for the range space of the projection P_π. Then the linear subspace $\mathcal{A}=\bigoplus_{\pi\in\widehat{G}}A_\pi$ is norm dense in A. Let $\mathrm{Hom}_G(H_\pi,A)$ be a set of intertwiners, that is, $\mathrm{Hom}_G(H_\pi,A)\ni S$ if and only if $\alpha\circ S=(S\otimes\mathrm{id})\circ\Gamma_\pi$. Then we have $A_\pi=\overline{\mathrm{span}}\{S\xi\mid\xi\in H_\pi, S\in\mathrm{Hom}_G(H_\pi,A)\}$. The dimension of $\mathrm{Hom}_G(H_\pi,A)$ is called the *multiplicity* of π. Especially for the trivial representation π_0, the closed linear subspace A_{π_0} coincides with the fixed point algebra $A^\alpha=\{x\in A\mid\alpha(x)=x\otimes 1\}$. Let $m(\pi)$ be the multiplicity of π. We often use a formal symbol $\bigoplus_{\pi\in\widehat{G}}m(\pi)\pi$ which is called the *spectral pattern* of $\{A,G,\alpha\}$ (or simply A).

We recall the notion of the *crossed product*. Let $\{A, G, \alpha\}$ be a covariant system. Consider the Hilbert A-module $A \otimes L^2(G)$. In a C^*-algebra $\mathbb{K}(A \otimes L^2(G)) = A \otimes \mathbb{K}(L^2(G))$ define the crossed product C^*-algebra

$$A \rtimes_\alpha G = \overline{\alpha(A)\big(\mathbb{C} \otimes C_r^*(G)\big)},$$

where the closure of linear space is taken with respect to the operator norm. It is also characterized as a fixed point algebra of $\{A \otimes \mathbb{K}(L^2(G)), G, \tilde{\alpha}\}$: $A \rtimes_\alpha G = (A \otimes \mathbb{K}(L^2(G)))^{\tilde{\alpha}}$, where $\tilde{\alpha}(a \otimes k) = W_{23}\alpha(a)_{1,3}(1 \otimes k \otimes 1)W_{23}^*$ for all $a \in A$ and $k \in \mathbb{K}(L^2(G))$. Note that $\alpha(A)$, $C_r{}^*(G)$ are C^*-subalgebras of $M(A \rtimes_\alpha G)$ naturally. When we consider $A = C(G)$ and $\alpha = \delta$, we have a natural isomorphism

$$C(G) \rtimes_\delta G \to \mathbb{K}\big(L^2(G)\big)$$

defined by $\delta(x)(1 \otimes y) \mapsto xy$. The *dual coaction* $\hat{\alpha} : A \rtimes_\alpha G \to M(A \rtimes_\alpha G \otimes C_r^*(G))$ is defined by

$$\hat{\alpha}(x) = \widetilde{V}_{23}(x \otimes 1)\widetilde{V}_{23}^*$$

which satisfies the density condition that a linear space $\hat{\alpha}(A \rtimes_\alpha G)(\mathbb{C} \otimes C_r^*(G))$ is dense in $A \rtimes_\alpha G \otimes C_r^*(G)$. We define a $*$-homomorphism $\hat{\alpha}_\pi : A \rtimes_\alpha G \to A \rtimes_\alpha G \otimes \mathrm{End}(H_\pi)$ by the composition $(\mathrm{id} \otimes \mathrm{pr}_\pi) \circ \hat{\alpha}$. Now we introduce the famous duality theorem. Let $\{A, G, \alpha\}$ be a covariant system. We consider the $*$-homomorphism $\hat{\alpha}^o = (\mathrm{id} \otimes \mathrm{Ad}\, U) \circ \hat{\alpha} : A \rtimes_\alpha G \to M(A \rtimes_\alpha G \otimes C_\ell^*(G))$. It satisfies $(\mathrm{id} \otimes \widehat{\delta_\ell}^{\mathrm{op}}) \circ \hat{\alpha}^o = (\hat{\alpha}^o \otimes \mathrm{id}) \circ \hat{\alpha}^o$, that is, $\hat{\alpha}^o$ is a coaction of the opposite discrete quantum group $C_\ell^*(G)$. Define the crossed product $A \rtimes_\alpha G \rtimes_{\hat{\alpha}^o} (\widehat{G})'^{\mathrm{op}}$ by the norm closure of the linear space $\hat{\alpha}^o(A \rtimes_\alpha G)(\mathbb{C} \otimes \mathbb{C} \otimes C(G))$. Its dual action $\widehat{\hat{\alpha}^o}$ is given by $\widehat{\hat{\alpha}^o}(x) = V_{34}(x \otimes 1)V_{34}^*$. The duality theorem says there exists an isomorphism between $A \rtimes_\alpha G \rtimes_{\alpha^o} (\widehat{G})'^{\mathrm{op}}$ and $A \otimes \mathbb{K}(L^2(G))$ such that $\hat{\alpha}^o(\alpha(a)(1 \otimes x))(1 \otimes 1 \otimes y)$ is mapped to $\alpha(a)(1 \otimes x)(1 \otimes UyU)$ for all $a \in A$, $x \in C_r^*(G)$ and $y \in C(G)$. This isomorphism conjugates $\widehat{\hat{\alpha}^o}$ and $\tilde{\alpha}$.

2.5. *Quantum subgroups and their quotient spaces*

Let $G = (C(G), \delta_G)$ and $H = (C(H), \delta_H)$ be compact quantum groups. We say H is a *quantum subgroup* (or sometimes simply called a subgroup) if there exists a surjective $*$-homomorphism $r : C(G) \to C(H)$ such that r satisfies $\delta_H \circ r = (r \otimes r) \circ \delta_G$. This surjection is often called the restriction homomorphism. In general, a choice of r is not unique. The *(left) quotient space* is defined by

$$C(H \setminus G) = \big\{x \in C(G) \bigm| (r \otimes \mathrm{id}) \circ \delta_G(x) = 1 \otimes x\big\}.$$

We have to be careful of r when we consider a quotient space. Actually it depends on r and we shall denote it by ${}^H C(G)$, however, we abuse the notation of $H \setminus G$. Note that the left quotient space has a right G-action.

3. Ergodic systems

A compact quantum covariant system $\{A, G, \alpha\}$ with $A^\alpha = \mathbb{C}1$ is called a *compact quantum ergodic system* (or simply an ergodic system). We investigate ergodic systems in this section. The faithful conditional expectation $\varphi = (\mathrm{id} \otimes h) \circ \alpha$ onto $A^\alpha = \mathbb{C}1$ is the unique invariant state in this case, where invariance means: $(\varphi \otimes \mathrm{id})(\alpha(x)) = \varphi(x)1$ for any $x \in A$. Note that φ becomes a trace if G is a compact group [10, Theorem 4.1]. In [4, Theorem 17], it has been proved that the dimension of A_π is less than or equal to D_π^2 for any $\pi \in \widehat{G}$. We will make a sharper estimate of these dimensions. In order to do this, we have to explain the existence of the modular automorphism group with respect to the state φ. By [4, Proposition 18], there exists the unital multiplicative linear map $\Theta : \mathcal{A} \to \mathcal{A}$ such that $\varphi(ab) = \varphi(\Theta(b)a)$ for any $a, b \in \mathcal{A}$. Let M be the von Neumann algebra $\pi_\varphi(A)''$ associated to the state φ via GNS-representation $\{H_\varphi, \pi_\varphi, \xi_\varphi\}$ and the extension of φ to M is also denoted by φ. We often identify A with $\pi_\varphi(A)$. Let $p \in M$ be a projection with $\varphi(p) = 0$. Then for any $a \in \mathcal{A}$ we have $\varphi(a^*pa) = \varphi(\Theta(a)a^*p)$. This is equal to 0 by the Cauchy–Schwarz inequality. From this we see $p\mathcal{A}\xi_\varphi = 0$ and $p = 0$, because the linear subspace $\mathcal{A}\xi_\varphi$ is dense in H_φ. Hence φ is a faithful normal state on M and there exists the modular automorphism group $\{\sigma^\varphi\}_{t\in\mathbb{R}}$ on M. Since φ is an invariant state on A for the action of G, the action α extends to the action on M. An action of a compact quantum group on a von Neumann algebra is defined similarly to the case of a C^*-algebra. Note the following useful equality about the modular automorphism group and the scaling automorphism group:

$$\alpha \circ \sigma_t^\varphi = \left(\sigma_t^\varphi \otimes \tau_{-t}\right) \circ \alpha \quad \text{for all } t \in \mathbb{R}.$$

For its proof, readers are referred to [7, Théorème 2.9]. The spectral subspace M_π is the σ-weak closure of A_π, however, they coincide because of the finite-dimensionality of A_π. The proof of the following lemma is straightforward by the above formula.

Lemma 3.1. *Let S be an intertwiner of H_π and A. For any $t \in \mathbb{R}$, we define the map $S_t : H_\pi \to A$ by $S_t\xi_j^\pi = (F_\pi)_{j,j}^{-it}\sigma_t^\varphi(S\xi_j^\pi)$ for any $j \in I_\pi$. Then S_t is also an intertwiner of H_π and A for any $t \in \mathbb{R}$, in particular we have $\sigma_t^\varphi(A_\pi) = A_\pi$ for any $t \in \mathbb{R}$ and we see any element of A_π is analytic for $\{\sigma_t^\varphi\}_{t\in\mathbb{R}}$.*

From this lemma we see that $\sigma_t^\varphi(A) = A$ for $t \in \mathbb{R}$ and the following proposition holds.

Proposition 3.2. *Let $\{A, G, \alpha\}$ be an ergodic system and φ be the invariant state on A. Then there exists the modular automorphism group $\{\sigma_t^\varphi\}_{t\in\mathbb{R}}$ on A.*

Definition 3.3. Let $\{A, G, \alpha\}$ be an ergodic system and π be an element of $\widehat{G}$. A vector $\xi = (\xi_j)_{j\in I_\pi} \in \bigoplus_{j\in I_\pi} A_\pi$ is called a π-eigenvector if $\alpha(\xi_j) = \sum_{k\in I_\pi} \xi_k \otimes w(\pi)_{k,j}$ for any $j \in I_\pi$. The set of π-eigenvectors is called the π-eigenvector space and denoted by $X_{w(\pi)}$.

Let $\{A, G, \alpha\}$ and $\{\xi^\lambda\}_{\lambda\in\Lambda}$ be a covariant system and a set of its π_λ-eigenvectors, respectively. A C^*-subalgebra generated by ξ_r^λ for all $r \in I_{\pi_\lambda}$ and $\lambda \in \Lambda$ is denoted by $C^*(\{\xi^\lambda\}_{\lambda\in\Lambda})$. The quantum group G acts on it invariantly. We say that it is a G-invariant C^*-subalgebra generated by $\{\xi^\lambda\}_{\lambda\in\Lambda}$. We make a π-eigenvector space to a Hilbert space by defining its inner product with

$$(\xi \mid \eta) = \sum_{k\in I_\pi} \xi_k\eta_k^*, \quad \text{for all } \xi, \eta \in X_{w(\pi)}.$$

Indeed, the above right-hand side sits in the fixed point algebra and therefore becomes a scalar by ergodicity.

Definition 3.4. Let $\{A, G, \alpha\}$ be an ergodic system and π be an element of $\widehat{G}$. We define the following two operations on eigenvector spaces.

(1) For any $t \in \mathbb{R}$, we define the linear map $U_t^{w(\pi)} : X_{w(\pi)} \to X_{w(\pi)}$ by

$$\big(U_t^{w(\pi)}\xi\big)_k = (F_\pi)_{k,k}^{-it}\sigma_t^\varphi(\xi_k) \quad \text{for all } \xi \in X_\pi, k \in I_\pi.$$

(2) We define the conjugate linear map $T_{w(\pi)} : X_{w(\pi)} \to X_{\overline{w(\pi)}}$.

$$(T_{w(\pi)}\xi)_k = (F_\pi)_{k,k}^{-\frac{1}{2}}\xi_k^* \quad \text{for all } \xi \in X_\pi, k \in I_\pi.$$

The well-definiteness of the above first operation has been already checked in Lemma 3.1. For the second one, we shall justify the equality $\alpha((T_{w(\pi)}\xi)_k) = \sum_{\ell \in I_\pi}(T_{w(\pi)}\xi)_\ell \otimes \overline{w(\pi)}_{\ell,k}$ for any $\xi \in X_{w(\pi)}$ and $k \in I_\pi$. Indeed,

$$\begin{aligned}
\alpha\big((T_{w(\pi)}\xi)_k\big) &= (F_\pi)_{k,k}^{-\frac{1}{2}}\alpha\big(\xi_k^*\big) \\
&= (F_\pi)_{k,k}^{-\frac{1}{2}}\sum_{\ell \in I_\pi}\xi_\ell^* \otimes w(\pi)_{\ell,k}^* \\
&= (F_\pi)_{k,k}^{-\frac{1}{2}}\sum_{\ell \in I_\pi}\xi_\ell^* \otimes (F_\pi)_{k,k}^{\frac{1}{2}}(F_\pi)_{\ell,\ell}^{-\frac{1}{2}}\overline{w(\pi)}_{\ell,k} \\
&= \sum_{\ell \in I_\pi}(T_{w(\pi)}\xi)_\ell \otimes \overline{w(\pi)}_{\ell,k}.
\end{aligned}$$

Proposition 3.5. *Let $\{A, G, \alpha\}$ be an ergodic system and π be an element of $\widehat{G}$. Then the map* $\mathrm{Hom}_G(H_\pi, A) \ni S \to (S(\xi_k^\pi))_{k \in I_\pi} \in X_{w(\pi)}$ *is a linear isomorphism.*

Proof. For $\xi \in X_{w(\pi)}$, define the element $S \in \mathrm{Hom}_G(H_\pi, A)$ by $S(\xi_k^\pi) = \xi_k$ for any $k \in I_\pi$. It is easy to see that this map is well defined and gives the desired inverse map. □

This proposition shows a spectral subspace A_π is spanned by entries of π-eigenvectors. The next lemma has already appeared in the proof of [4, Proposition 18], however, we give a proof for readers' convenience.

Lemma 3.6. *Let ξ and η be π-eigenvectors of an ergodic system $\{A, G, \alpha\}$. Then the following equalities hold*:

(1) $\varphi(\xi_k\eta_\ell^*) = D_\pi^{-1}(F_\pi)_{k,k}\delta_{k,\ell}(\xi \mid \eta)$, *for all* $k, \ell \in I_\pi$,
(2) $\varphi(\xi_k^*\eta_\ell) = 0$, *if* $k \neq \ell \in I_\pi$, *and* $\varphi(\xi_k^*\eta_k) = \varphi(\xi_\ell^*\eta_\ell)$, *for all* $k, \ell \in I_\pi$.

Proof. (1) It is a straightforward calculation as follows:

$$\begin{aligned}
\varphi\big(\xi_k \eta_\ell^*\big) &= \sum_{r,s\in I_\pi} \xi_r \eta_s^* h\big(w(\pi)_{r,k} w(\pi)_{s,\ell}^*\big) \\
&= \sum_{r,s\in I_\pi} \xi_r \eta_s^* D_\pi^{-1} (F_\pi)_{k,k} \delta_{r,s} \delta_{k,\ell} \\
&= D_\pi^{-1} (F_\pi)_{k,k} \delta_{k,\ell} \sum_{r\in I_\pi} \xi_r \eta_r^* \\
&= D_\pi^{-1} (F_\pi)_{k,k} \delta_{k,\ell} (\xi \mid \eta).
\end{aligned}$$

(2) Let $Z = (\varphi(\xi_k^* \eta_\ell))_{k,\ell \in I_\pi}$ be a matrix on the representation space H_π. Then we have:

$$\begin{aligned}
\big(w(\pi)^* Z w(\pi)\big)_{i,j} &= \sum_{k,\ell\in I_\pi} w(\pi)_{k,i}^* Z_{k,\ell} w(\pi)_{\ell,j} \\
&= \sum_{k,\ell\in I_\pi} w(\pi)_{k,i}^* \varphi\big(\xi_k^* \eta_\ell\big) w(\pi)_{\ell,j} \\
&= \sum_{k,\ell\in I_\pi} (\varphi \otimes \mathrm{id})\big(\big(\xi_k^* \otimes w(\pi)_{k,i}^*\big)\big(\eta_\ell \otimes w(\pi)_{\ell,j}\big)\big) \\
&= (\varphi \otimes \mathrm{id})\alpha\big(\xi_i^* \eta_j\big) \\
&= \varphi\big(\xi_i^* \eta_j\big) \\
&= Z_{i,j}.
\end{aligned}$$

Therefore the operator Z commutes with the irreducible unitary representation $w(\pi)$, so we have $Z \in \mathbb{C}$. □

Lemma 3.7. *Let $\{A, G, \alpha\}$ be an ergodic system and π be an element of $\widehat{G}$. Then it follows $T_{w(\pi)}^* T_{w(\pi)} = U_i^{w(\pi)}$.*

Proof. Take π-eigenvectors ξ and η. Then we have

$$\begin{aligned}
(T_{w(\pi)}\xi \mid T_{w(\pi)}\eta) &= \sum_{k\in I_\pi} (T_{w(\pi)}\xi)_k (T_{w(\pi)}\eta)_k^* \\
&= \sum_{k\in I_\pi} (F_\pi)_{k,k}^{-\frac{1}{2}} (F_\pi)_{k,k}^{-\frac{1}{2}} \xi_k^* \eta_k \\
&= \sum_{k\in I_\pi} (F_\pi)_{k,k}^{-1} \varphi\big(\xi_k^* \eta_k\big) \\
&= \sum_{k\in I_\pi} (F_\pi)_{k,k}^{-1} \varphi\big(\sigma_i^\varphi(\eta_k) \xi_k^*\big) \\
&= \sum_{k\in I_\pi} (F_\pi)_{k,k}^{-1} \varphi\big((F_\pi)_{k,k}^{-1} \big(U_i^{w(\pi)} \eta\big)_k \xi_k^*\big)
\end{aligned}$$

$$= \sum_{k\in I_\pi} (F_\pi)_{k,k}^{-2} \cdot D_\pi^{-1} (F_\pi)_{k,k} \big(U_i^{w(\pi)}\eta \bigm| \xi\big)$$
$$= \big(U_i^{w(\pi)}\eta \bigm| \xi\big),$$

where we use the result of the previous lemma and $\sum_{k\in I_\pi}(F_\pi^{-1})_{k,k} = D_\pi$. Hence we obtain the desired equality. □

Let $T_{w(\pi)} = J_{w(\pi)}|T_{w(\pi)}|$ be the polar decomposition of the conjugate linear map $T_{w(\pi)}: X_{w(\pi)} \to X_{\overline{w(\pi)}}$. Since it is easy to see $T_{\overline{w(\pi)}} \circ T_{w(\pi)} = \mathrm{id}_{X_{w(\pi)}}$, the map $J_{w(\pi)}$ is a conjugate unitary map and satisfies $J_{\overline{w(\pi)}} = J_{w(\pi)}^{-1}$, and $J_{\overline{w(\pi)}}|T_{\overline{w(\pi)}}|J_{w(\pi)} = |T_{w(\pi)}|^{-1}$. In particular, we obtain

$$\mathrm{Sp}\big(U_{\frac{i}{2}}^{w(\pi)}\big) = \mathrm{Sp}\big(U_{\frac{i}{2}}^{\overline{w(\pi)}}\big)^{-1}.$$

With these preparations, the following result about bounds of multiplicities holds.

Theorem 3.8. *Let $\{A, G, \alpha\}$ be an ergodic system and π be an element of $\widehat{G}$. Then we have*

$$\dim \mathrm{Hom}_G(H_\pi, A) \leqslant D_\pi.$$

Proof. The proof is essentially due to [4, Theorem 17] or [20, Theorems 1, 2]. As we have proved, in Proposition 3.5, the hom-space $\mathrm{Hom}_G(H_\pi, A)$ is naturally isomorphic to $X_{w(\pi)}$. Hence it suffices to show $d := \dim(X_{w(\pi)}) \leqslant D_\pi$. The unitary $\mathbb{R}$-action $\{U_t^{w(\pi)}\}_{t\in\mathbb{R}}$ enables us to take an orthonormal basis of $X_{w(\pi)}$ $\{\xi^p\}_{1\leqslant p\leqslant d}$ as $U_t^{w(\pi)}\xi^p = \lambda_p^{it}\xi^p$ for all $t \in \mathbb{R}$ where λ_p is a positive real number. Then we define an operator entry matrix M by

$$M = \begin{pmatrix} \xi^1 \\ \xi^2 \\ \vdots \\ \xi^d \end{pmatrix} \in \mathbb{M}_{d,d_\pi}(A_\pi),$$

where each ξ^p is treated as a row vector. Adding 0 entries, we embed the matrix M to a larger square matrix (still denoted by M) of size n ($\geqslant d, d_\pi$), if necessary. By orthonormality, we have $MM^* = 1_d$. Let $k, \tilde{F}$ be positive matrices $\mathrm{diag}(\lambda_1^{-1}, \lambda_2^{-1}, \dots, \lambda_d^{-1}, 1, \dots, 1)$ and $\mathrm{diag}(F_{w(\pi)}, 1, \dots, 1)$ in $\mathbb{M}_n(\mathbb{C})_+$, respectively. Then we have

$$\begin{aligned}\sum_{p=1}^d \lambda_p^{-1} &= \mathrm{Tr}_k(1_d) \\ &= (\varphi \otimes \mathrm{Tr}_k)(MM^*) \\ &= (\varphi \otimes \mathrm{Tr}_k)\big(M^*\big(\sigma_{-i}^\varphi \otimes \mathrm{Ad}(k)\big)(M)\big) \\ &= (\varphi \otimes \mathrm{Tr}_k)\big(M^*M\tilde{F}k^{-1}\big) \\ &= (\varphi \otimes \mathrm{Tr})(M^*M\tilde{F})\end{aligned}$$

$$\leqslant (\varphi \otimes \mathrm{Tr})(1 \otimes \tilde{F} 1_{d_\pi})$$
$$= D_\pi,$$

where we use $(\sigma_t^\varphi \otimes \mathrm{Ad}(k^{it}))(M) = M \tilde{F}^{it} k^{-it}$ for $t \in \mathbb{R}$. Hence we obtain $\mathrm{Tr}_{X_{w(\pi)}}(U_i^{w(\pi)}) \leqslant D_\pi$ and similarly $\mathrm{Tr}_{X_{\overline{w(\pi)}}}(U_i^{\overline{w(\pi)}}) \leqslant D_{\bar{\pi}} = D_\pi$. Since we have already known $\mathrm{Sp}(U_i^{w(\pi)}) = \mathrm{Sp}(U_i^{\overline{w(\pi)}})^{-1}$, it follows $2d \leqslant \mathrm{Tr}_{X_{w(\pi)}}(U_i^{w(\pi)}) + \mathrm{Tr}_{X_{\overline{w(\pi)}}}(U_i^{\overline{w(\pi)}})$. Therefore we obtain $2d \leqslant 2D_\pi$. □

From this result, the dimension of the π-eigenspace A_π is less than or equal to $d_\pi D_\pi$. We can derive the quantum version of [20, Theorem 2].

Theorem 3.9. *Let $\{A, G, \alpha\}$ be a compact quantum covariant system and π be an element of $\widehat{G}$. If p and q are minimal projections in A^α, then we have*:

$$\dim \mathrm{Hom}_G(H_\pi, pAq) \leqslant D_\pi.$$

Proof. This proof is also essentially due to [4, Theorem 17] or [20, Theorems 1, 2], however, we cannot use the trace property as in the case of a compact group, we have to prove the finite-dimensionality of $\mathrm{Hom}_G(H_\pi, pAq)$ at first. If the linear space $pA^\alpha q$ is not 0, we can take non-zero norm 1 element x from $pA^\alpha q$. Then we see $xx^* \in pA^\alpha p = \mathbb{C}p$ and $x^*x \in qA^\alpha q = \mathbb{C}q$ and have $xx^* = p$ and $x^*x = q$. It follows $(pAq)_\pi = (pAp)_\pi x$. Therefore the assertion of this theorem follows from applying the previous theorem to pAp. Next we assume $pA^\alpha q = 0$. Because of $p \perp q$, we may assume that A is unital and $1 = p + q$ by considering $(p+q)A(p+q)$. Let $\varphi(x) = \varphi_p(pxp) + \varphi_q(qxq)$ where φ_p and φ_q are the invariant states on pAp and qAq, respectively. Since the conditional expectation with respect to φ $E_\alpha : A \to A^\alpha$ is given by $E_\alpha(x) = pxp + qxq$, φ is the invariant positive functional on A. Let $d \leqslant \infty$ be the dimension of the linear space $\mathrm{Hom}_G(H_\pi, A)$ and fix a natural number $0 \leqslant d' \leqslant d$. Note that we can take π-eigenvectors $\{\xi^j\}_{1 \leqslant j \leqslant d'}$ from $(pAq)_\pi = pA_\pi q$, which satisfy $\varphi((\xi_r^j)^* \xi_s^k) = \delta_{j,k}\delta_{r,s}$ for $1 \leqslant j, k \leqslant d'$ and $r, s \in I_\pi$. In fact, if we have $\{\xi^j\}_{1 \leqslant j \leqslant k-1}$ as before, we can take a π-eigenvector ξ^k which satisfies $\varphi((\xi_1^j)^* \xi_1^k) = \delta_{j,k}$ for $1 \leqslant j \leqslant p-1$. Then modifying Lemma 3.6 to the case of pAq, we achieve to take desired vectors. Let M be the matrix

$$M = \begin{pmatrix} T_{w(\pi)}\xi^1 \\ T_{w(\pi)}\xi^2 \\ \vdots \\ T_{w(\pi)}\xi^{d'} \end{pmatrix}.$$

Adding 0-entries, we embed M to a larger square matrix (still denoted by M) of size n ($\geqslant d', d_\pi$). Note that $T_{w(\pi)}\xi^j$ is a $\bar{\pi}$-eigenvector for qAp: $T_{w(\pi)}\xi^j = ((F_\pi)_{r,r}^{-\frac{1}{2}} \xi_r^{j*})_{r \in I_\pi}$. By the computation in the proof of Lemma 3.7, it follows $MM^* = D_\pi q \otimes 1_{d'}$. Hence we have $M^*M \leqslant D_\pi p \otimes 1_{d_\pi}$ in $pAp \otimes \mathbb{B}(H_\pi)$. Then it yields:

$$D_\pi^2 \geqslant (\varphi \otimes \mathrm{Tr}_{F_\pi})(M^*M)$$

$$= \sum_{r \in I_\pi} \left((F_\pi)_{r,r} \left(F_\pi^{-\frac{1}{2}}\right)_{r,r} \left(F_\pi^{-\frac{1}{2}}\right)_{r,r} \sum_{1 \leqslant j \leqslant d'} \varphi\left(\xi_r^j \xi_r^{j*}\right) \right)$$
$$= \sum_{1 \leqslant j \leqslant d'} \left(\xi^j \mid \xi^j\right),$$

where $(\xi^j \mid \xi^j)$ means the inner product of π-eigenvectors for pAq. Let us define another inner product $(\cdot \mid \cdot)_2$ of π-eigenvectors for pAq by $(\xi \mid \eta)_2 = \varphi(\eta_r^* \xi_r)$ for any π-eigenvectors ξ, η. Note that this value does not depend on the choice of r by Lemma 3.6. Let W be a linear space spanned by $\{\xi^j\}_{1 \leqslant j \leqslant d'}$. There are two inner products $(\cdot \mid \cdot)$ and $(\cdot \mid \cdot)_2$ on W. Let A_W be the matrix which satisfies $(A_W \xi \mid \eta)_2 = D_\pi^{-1}(\xi \mid \eta)$. Then from the above inequality we have $\mathrm{Tr}_W(A_W) \leqslant D_\pi$, where Tr_W is the non-normalized trace associated to W. Similarly considering $T_{w(\pi)}(W)$ for W, we also obtain $\mathrm{Tr}_{T_{w(\pi)}(W)}(A_{T_{w(\pi)}(W)}) \leqslant D_\pi$. Moreover, we see $A_W = T_{w(\pi)}^* T_{w(\pi)}$, where the involution $*$ comes from the conjugate linear map $T_{w(\pi)} : W \to T_{w(\pi)}(W)$ between $(\cdot \mid \cdot)_2$-inner product spaces. In fact, we have

$$\begin{aligned}
(T_{w(\pi)}\xi \mid T_{w(\pi)}\eta)_2 &= \varphi\left((T\eta)_1{}^*(T\xi)_1\right) \\
&= \left(F_\pi^{-1}\right)_{1,1} \varphi\left(\eta_1 \xi_1^*\right) \\
&= \left(F_\pi^{-1}\right)_{1,1} D_\pi^{-1} (F_\pi)_{1,1} (\eta \mid \xi) \\
&= D_\pi^{-1} (\eta \mid \xi).
\end{aligned}$$

Let us denote $T_{w(\pi)}(W)$ by $\overline{W}$. Let $T_{w(\pi)} = J_W |T_{w(\pi)}|$ and $T_{\overline{w(\pi)}} = J_{\overline{W}} |T_{\overline{w(\pi)}}|$ be the polar decomposition of the conjugate linear maps $T_{w(\pi)}$ and $T_{\overline{w(\pi)}}$ between W and $T_{w(\pi)}(W)$. The equality $T_{w(\pi)} \circ T_{\overline{w(\pi)}} = \mathrm{id}_{\overline{W}}$ yields $J_W A_W^{\frac{1}{2}} J_{\overline{W}} = A_{\overline{W}}^{-\frac{1}{2}}$. Then we have $J_W A_W J_W^* = A_{\overline{W}}^{-1}$ because J_W and $J_{\overline{W}}$ are conjugate unitary. Hence we obtain $\mathrm{Tr}(A_W) = \mathrm{Tr}(A_{\overline{W}}^{-1})$ and it follows $2d' \leqslant 2\,\mathrm{Tr}(A_W) \leqslant 2D_\pi$. □

Finally we state the modified version of Theorems 3.8 and 3.9.

Proposition 3.10. *Let $\{A, G, \alpha\}$ be an ergodic system and π be an element of $\widehat{G}$. Assume that A has a tracial state. Then we have*:

$$\dim \mathrm{Hom}(H_\pi, A) \leqslant d_\pi.$$

Proof. We have proved finite-dimensionality of $X_{w(\pi)}$. Let d be its dimension and $\{\xi^p\}_{1 \leqslant p \leqslant d}$ be an orthonormal basis of $X_{w(\pi)}$ for the inner product $(\cdot \mid \cdot)$. Take a tracial state τ on A. Define the matrix M by

$$M = \begin{pmatrix} \xi^1 \\ \xi^2 \\ \vdots \\ \xi^d \end{pmatrix}.$$

As the proof of Theorem 3.8, we look at M in a bigger square matrix algebra. Then we have $MM^* = 1_d$ and it yields

$$\begin{aligned} d_\pi &\geqslant (\tau \otimes \mathrm{Tr})(M^*M) \\ &= (\tau \otimes \mathrm{Tr})(MM^*) \\ &= d. \qquad \square \end{aligned}$$

Proposition 3.11. *Let $\{A, G, \alpha\}$ be a compact quantum covariant system and π be an element of $\widehat{G}$. Assume that A has a tracial state. If p and q are minimal projections in A^α and they are not annihilated by a trace on A, then we have*

$$\dim \mathrm{Hom}(H_\pi, pAq) \leqslant d_\pi.$$

Proof. Let d ($\leqslant D_\pi$) be the dimension of $\dim \mathrm{Hom}(H_\pi, pAq)$. Let $\{\xi^p\}_{1\leqslant p\leqslant d}$ be an orthonormal π-eigenvector for pAq where we define the inner product by

$$(\xi \mid \eta)p = \sum_{r\in I_\pi} \xi_r \eta_r^*$$

for all π-eigenvector ξ and η. Take a tracial state τ on A with $\tau(p)\tau(q) \neq 0$. Define the matrix M by

$$M = \begin{pmatrix} \xi^1 \\ \xi^2 \\ \vdots \\ \xi^d \end{pmatrix}$$

and embed it into the larger square matrix as in previous theorems. Then we have $MM^* = p \otimes 1_d$. This yields

$$\begin{aligned} d\tau(p) &= (\tau \otimes \mathrm{Tr})(MM^*) \\ &= (\tau \otimes \mathrm{Tr})(M^*M) \\ &\leqslant (\tau \otimes \mathrm{Tr})(q \otimes 1_{d_\pi}) \\ &= \tau(q)d_\pi. \end{aligned}$$

Hence we have $d \leqslant d_\pi \frac{\tau(q)}{\tau(p)}$. Considering $qAp = (pAq)^*$ and $d = \dim \mathrm{Hom}(H_\pi, qAp)$, we also have $d \leqslant d_{\bar{\pi}} \frac{\tau(p)}{\tau(q)} = d_\pi \frac{\tau(p)}{\tau(q)}$. Finally we obtain $d \leqslant d_\pi \min(\frac{\tau(q)}{\tau(p)}, \frac{\tau(p)}{\tau(q)}) \leqslant d_\pi$. $\square$

Remark 3.12. Assume that the linear space pAq is not 0. Then $\tau(p) \neq 0$ if and only if $\tau(q) \neq 0$. Indeed, take $\pi \in \widehat{G}$ as $pA_\pi q \neq 0$ and assume $\tau(q) = 0$, then we obtain $\tau(p) = 0$ by the proof of the above proposition. The converse assertion holds by applying the involution $*$ from pAq to qAp.

4. Equivariant K-theory

We follow [20] and use equivariant K-theory to obtain a multiplicity map. Readers are referred to, for example, [1] and [19] for equivariant K-theory. Let $\{A, G, \alpha\}$ be a compact quantum covariant system with A unital. Let E be a finitely generated projective Hilbert A-module and $\delta_E : E \to E \otimes C(G)$ be a linear map. The pair $\{E, \delta_E\}$ is called a G-equivariant A-module if they satisfy the following:

(1) $(\mathrm{id} \otimes \delta) \circ \delta_E = (\delta_E \otimes \mathrm{id}) \circ \delta_E$,
(2) $\delta_E(ea) = \delta_E(e)\alpha(a)$ for all $e \in E$ and $a \in a$,
(3) $\langle \delta_E(e) \mid \delta_E(e') \rangle = \alpha(\langle e \mid e' \rangle)$ for all $e, e' \in E$,
(4) the linear subspace $\delta_E(E)(\mathbb{C} \otimes C(G))$ is dense in $E \otimes C(G)$.

Actually the second equality follows from the third one. Inner products of Hilbert modules are conjugate-linear for the first variable. Two G-equivariant A-modules are equivalent if there exists a bijective linear map intertwining of the actions of G and A. The set of these equivalence classes becomes an abelian semigroup by the direct sum and its Grothendieck group is denoted by $K_0^G(A)$ and this is called an *equivariant K-group*. For a G-equivariant A-module $\{E, \delta_E\}$ we define its crossed product Hilbert $A \rtimes_\alpha G$-module by $E \rtimes_\alpha G = E \otimes_A (A \rtimes_\alpha G)$. This module becomes a (left) $C_r^*(G)$-module by an appropriate way [19, Lemme 5.2]. Here we take a non-rigorous picture for its action. Let x be an element of $C_r^*(G)$ and we assume its dual coproduct has an expansion $\hat{\delta}(x) = \sum x_{(0)} \otimes x_{(1)}$. It is considered that $C_r^*(G)$ acts on E through the differential representation with respect to δ_E. Hence we have an action $x \cdot (e \otimes_A y) = \sum x_{(0)}(e) \otimes_A x_{(1)} y$. This action is compatible with the right action of A, because in $A \rtimes_\alpha G$ we have $z(\alpha(y)) = \sum (z_{(0)} \cdot y) z_{(1)}$ for $z \in C_r^*(G)$ and $y \in A$ where $C_r^*(G)$ acts on A with the differential representation about α. We next recall the Green–Julg isomorphism. If E_π is a tensor product module $H_\pi \otimes A$ with a finite-dimensional irreducible G-module H_π, then we have $E_\pi \otimes_A (A \rtimes_\alpha G) = H_\pi \otimes (A \rtimes_\alpha G)$. We apply $p_0 \in C_r^*(G)$ to this space. Since this action is obtained by the dual coproduct, we have $p_0(E_\pi \otimes_A A \rtimes_\alpha G) = p_0(H_\pi \otimes C_r^*(G)) \otimes_{C_r^*(G)} A \rtimes_\alpha G \cong p_{\overline{\pi}} C_r^*(G) \otimes_{C_r^*(G)} A \rtimes_\alpha G = (1 \otimes p_{\overline{\pi}}) A \rtimes_\alpha G$. In general, E is a direct summand of a direct sum of E_π. Hence the module $p_0(E \rtimes_\alpha G)$ corresponds to a projection in $M_n(A \rtimes_\alpha G)$ for some n. We also notice that $A \rtimes_\alpha G$ is stably unital. In fact we have an approximate unit of $\mathbb{K}(L^2(G))$ which consists of projections in $C_r^*(G)$. In this way, we obtain the Green–Julg isomorphism (see [19, Théorème 5.10] for its proof of KK-version):

$$\Phi_1 : K_0^G(A) \to K_0(A \rtimes_\alpha G)$$

defined by $[E] \mapsto [p_0(E \rtimes_\alpha G)]$. The inverse map of Φ_1 is given by

$$\Phi_2 : K_0(A \rtimes_\alpha G) \to K_0^G(A)$$

which sends $[q]$ to $[q(A \otimes L^2(G))^n]$, where $q \in M_n(A \rtimes_\alpha G)$ is a projection and $A \otimes L^2(G)$ is a G-equivariant A-module with $\delta_{A \otimes L^2(G)}(a \otimes \xi) = (1 \otimes W(\xi \otimes 1))\alpha(a)_{13}$. We define the usual $R(G)$-module structures on $K_0^G(A)$ and $K_0(A \rtimes_\alpha G)$ as follows. Let π be an element of $\widehat{G}$ and E be a G-equivariant A-module and $q \in M_n(A \rtimes_\alpha G)$ be a projection. For them, define the action of π:

$$\pi \cdot [E] = [H_\pi \otimes E],$$
$$\pi \cdot [q] = \bigl[(\mathrm{id}_{M_n(\mathbb{C})} \otimes \hat{\alpha}_\pi)(q)\bigr],$$

where the isomorphism $K_0(A \rtimes_\alpha G) \cong K_0(A \rtimes_\alpha G \otimes \mathrm{End}(H_\pi))$ is used. We observe the Green–Julg isomorphism is an $R(G)$-module map as in the compact group case.

Lemma 4.1. *Let B be a C^*-algebra and E be a Hilbert B-module which has a $C_r^*(G)$-action $\phi : C_r^*(G) \to \mathbb{B}(E)$. Then for any $\pi \in \widehat{G}$ there is an isomorphism as Hilbert B-modules between $p_0(H_\pi \otimes E)$ and $p_0(E \otimes H_\pi)$.*

Proof. First we compare $\hat{\delta}(p_0)$ and its opposite $\hat{\delta}^{\mathrm{op}}(p_0)$. Using $p_0 = (\mathrm{id} \otimes h)(V)$ and the pentagonal identity, we obtain $\hat{\delta}(p_0) = (\mathrm{id} \otimes \mathrm{id} \otimes h)(V_{13}V_{23})$ and $\hat{\delta}^{\mathrm{op}}(p_0) = (\mathrm{id} \otimes \mathrm{id} \otimes h)(V_{23}V_{13})$. Since V has the expansion $\sum_{\pi \in \widehat{G}, i,j \in I_\pi} E^\pi_{i,j} \otimes w(\pi)_{i,j}$, we have the following equalities:

$$\hat{\delta}(p_0) = \sum_{\pi \in \widehat{G},\, i,j \in I_\pi,\, k,\ell \in I_{\overline{\pi}}} h\bigl(w(\pi)_{i,j} w(\overline{\pi})_{k,\ell}\bigr) E^\pi_{i,j} \otimes E^{\overline{\pi}}_{k,\ell},$$
$$\hat{\delta}^{\mathrm{op}}(p_0) = \sum_{\pi \in \widehat{G},\, i,j \in I_\pi,\, k,\ell \in I_{\overline{\pi}}} h\bigl(w(\overline{\pi})_{k,\ell} w(\pi)_{i,j}\bigr) E^\pi_{i,j} \otimes E^{\overline{\pi}}_{k,\ell}.$$

We use $\sigma^h_{-i}(w(\overline{\pi})_{k,\ell}) = (F_{\overline{\pi}})_{k,k}(F_{\overline{\pi}})_{\ell,\ell} w(\overline{\pi})_{k,\ell}$ in order to get $h(w(\overline{\pi})_{k,\ell} w(\pi)_{i,j}) = (F_{\overline{\pi}})_{k,k}(F_{\overline{\pi}})_{\ell,\ell} h(w(\pi)_{i,j} w(\overline{\pi})_{k,\ell})$. Define a densely defined unbounded positive operator $F = \sum_{\pi \in \widehat{G}} F_\pi$. With this operator we get the identity

$$\hat{\delta}^{\mathrm{op}}(p_0) = (1 \otimes F)\hat{\delta}(p_0)(1 \otimes F).$$

Then we define the map $\chi : H_\pi \otimes E \to E \otimes H_\pi$ by $\chi(\xi \otimes e) = e \otimes F_\pi^{-1}\xi$. This is a bijective adjointable map for Hilbert B-modules. Then we get

$$\begin{aligned}\chi \circ (\mathrm{id} \otimes \phi)\bigl(\hat{\delta}(p_0)\bigr)(\xi \otimes e) &= \bigl(1 \otimes F_\pi^{-1}\bigr)(\phi \otimes \mathrm{id})\bigl(\hat{\delta}^{\mathrm{op}}(p_0)\bigr)(e \otimes \xi) \\ &= (\phi \otimes \mathrm{id})\bigl(\hat{\delta}(p_0)\bigr)(e \otimes F_\pi \xi).\end{aligned}$$

Hence we see χ maps $p_0(H_\pi \otimes E)$ onto $p_0(E \otimes H_\pi)$. □

Lemma 4.2. *The Green–Julg isomorphism Φ_1 is an $R(G)$-module map.*

Proof. Take $\pi \in \widehat{G}$ and a G-equivariant A-module E. We shall prove $\Phi_1(\pi \cdot [E]) = (\hat{\alpha}_\pi)_*(\Phi_1([E]))$. For the left-hand side we have

$$\bigl[p_0\bigl((H_\pi \otimes E) \otimes_A A \rtimes_\alpha G\bigr)\bigr] = \bigl[p_0(H_\pi \otimes E \rtimes_\alpha G)\bigr].$$

For the right one we have

$$\begin{aligned}(\hat{\alpha}_\pi)_*\bigl(\bigl[p_0(E \rtimes_\alpha G)\bigr]\bigr) &= \bigl[p_0(E \rtimes_\alpha G) \otimes_{\hat{\alpha}_\pi} \bigl(A \rtimes_\alpha G \otimes \mathrm{End}(H_\pi)\bigr)\bigr] \\ &= \bigl[p_0\bigl(E \rtimes_\alpha G \otimes \mathrm{End}(H_\pi)\bigr)\bigr].\end{aligned}$$

Moving it to $K_0(A \rtimes_\alpha G)$ by tensoring $A \rtimes_\alpha G \otimes H_\pi$ over $A \rtimes_\alpha G \otimes \mathrm{End}(H_\pi)$, we get $(\hat{\alpha}_\pi)_*(\Phi_1([E])) = [p_0(E \rtimes_\alpha G \otimes H_\pi)]$. Hence we obtain the desired equality by applying the previous lemma to $E \rtimes_\alpha G$ and $B = A \rtimes_\alpha G$. $\square$

Let $\{A, G, \alpha\}$ be a compact quantum ergodic system. Then there exist an index set I and a Hilbert space H_i for each $i \in I$ such that the crossed product $A \rtimes_\alpha G$ is isomorphic to $\bigoplus_{i\in I} \mathbb{K}(H_i)$ (see [4, Theorem 19], [20, Corollary 2] for its proof). Let us fix minimal projections $e_i \in \mathbb{K}(H_i)$ for all $i \in I$. Hence we have $K_0^G(A) \cong \bigoplus_{i\in I} \mathbb{Z}[e_i]$ by the Green–Julg theorem. Using the isomorphism $\Phi_2 : K_0(A \rtimes_\alpha G) \to K_0^G(A)$, we can easily check that in $K_0^G(A)$, $[e_i]$ becomes a G-equivariant A-module $[e_i(A \otimes L^2(G))]$. Now we consider the $R(G)$-module structure of $K_0^G(A)$. Let π be an element of $\widehat{G}$. We define the (not necessarily finite size) matrix $\mathbb{M}(\pi) = (\mathbb{M}(\pi)_{i,j})_{i,j\in I}$ by

$$\pi \cdot [e_j] = \sum_{i\in I} \mathbb{M}(\pi)_{i,j}[e_i].$$

This equality holds in $K_0^G(A) \cong K_0(A \rtimes_\alpha G)$. If we treat $[e_i] = [e_i(A \otimes L^2(G))]$ in $K_0^G(A)$, we get the equality $\mathbb{M}(\pi)_{i,j} = \dim \mathrm{Hom}_{(G,A)}(H_\pi \otimes e_j(A \otimes L^2(G)), e_i(A \otimes L^2(G)))$.

Lemma 4.3. *We have an isomorphism between linear spaces* $\mathrm{Hom}_{(G,A)}(H_\pi \otimes e_j(A \otimes L^2(G)), e_i(A \otimes L^2(G)))$ *and* $\mathrm{Hom}_G(H_\pi, e_i(A \otimes \mathbb{K}(L^2(G)))e_j)$.

Proof. We may assume projections e_i and e_j are majored by $1 \otimes p$ where p is a projection in $C_r^*(G) \subset \mathbb{K}(L^2(G))$. Let H_0 be a closed subspace $pL^2(G)$. It is finite-dimensional and G-invariant, that is, $W(H \otimes \mathbb{C}) \subset H \otimes C(G)$. Let $\{\eta_p\}_{p\in J}$ be an orthonormal basis of H_0. With this basis we give a matrix representation $(w_{p,q})_{p,q\in J}$ of W by $W(\eta_p \otimes 1) = \sum_{q\in J} \eta_q \otimes w_{q,p}$. From now we write Γ_0 and Γ_1 for $C(G)$-comodule maps $H_\pi \otimes A \otimes H_0 \to H_\pi \otimes A \otimes H_0 \otimes C(G)$ and $A \otimes H_0 \to A \otimes H_0 \otimes C(G)$, respectively. Take an intertwiner S from $\mathrm{Hom}_{(G,A)}(H_\pi \otimes A \otimes H_0, A \otimes H_0)$. Then we can choose $a_{k,q,p} \in A$ for all $k \in I_\pi$ and $p, q \in J$ such that they satisfy $S(\xi_k \otimes a \otimes \eta_p) = \sum_{q\in J} a_{k,q,p} a \otimes \eta_q$ for all $a \in A$ by A-linearity. Since S is a G-homomorphism, we have $\Gamma_1 S(\xi_k \otimes 1 \otimes \eta_p) = (S \otimes 1)(\Gamma_0(\xi_k \otimes 1 \otimes \eta_p))$. The left-hand side is equal to

$$\sum_{q\in J} \Gamma_1(a_{k,q,p} \otimes \eta_q) = \sum_{q,r\in J} (1 \otimes \eta_r \otimes w_{r,q})\alpha(a_{k,q,p})_{13}.$$

The right-hand side is equal to

$$\sum_{\ell\in I_\pi, s\in J} (S \otimes 1)\big(\xi_\ell \otimes 1 \otimes \eta_s \otimes w(\pi)_{\ell,k} w_{s,p}\big) = \sum_{\ell\in I_\pi, r,s\in J} a_{\ell,r,s} \otimes \eta_r \otimes w(\pi)_{\ell,k} w_{s,p}.$$

Hence we get

$$\sum_{q\in J} (1 \otimes w_{r,q})\alpha(a_{k,q,p}) = \sum_{\ell\in I_\pi, s\in J} a_{\ell,r,s} \otimes w(\pi)_{\ell,k} w_{s,p}. \tag{4.1}$$

Next take an intertwiner T from $\mathrm{Hom}_G(H_\pi, A \otimes \mathbb{B}(H_0))$. Then we can choose $b_{k,q,p} \in A$ for all $k \in I_\pi$ and $p, q \in J$ such that they satisfy $T\xi_k = \sum_{q,p\in J} b_{k,q,p} \otimes \eta_q \odot \eta_p$. Since T is a

G-homomorphism, we have $\tilde{\alpha} \circ T(\xi_k) = (T \otimes 1) \circ w(\pi)(\xi_k \otimes 1)$ for all $k \in I_\pi$. The left-hand side is equal to

$$\sum_{q,t \in J} \tilde{\alpha}(b_{k,q,t} \otimes \eta_q \odot \eta_t) = \sum_{q,t \in J} W_{23}\alpha(b_{k,q,t})_{13}(1 \otimes 1 \otimes \eta_q \odot \eta_t)W_{23}^*$$
$$= \sum_{q,r,s,t \in J} (1 \otimes \eta_r \odot \eta_s \otimes 1)(1 \otimes 1 \otimes w_{r,q})\alpha(b_{k,q,t})_{13}\bigl(1 \otimes 1 \otimes w_{s,t}^*\bigr).$$

The right-hand side is equal to

$$\sum_{\ell \in I_\pi} T\xi_\ell \otimes w(\pi)_{\ell,k} = \sum_{\ell,r,s} b_{\ell,r,s} \otimes \eta_r \odot \eta_s \otimes w(\pi)_{\ell,k}.$$

Hence we obtain the following equality:

$$\sum_{q,t \in J} (1 \otimes w_{r,q})\alpha(b_{k,q,t})\bigl(1 \otimes w_{s,t}^*\bigr) = \sum_{\ell} b_{\ell,r,s} \otimes w(\pi)_{\ell,k}.$$

Multiplying $w_{s,p}$ and summing up with s, we get

$$\sum_{q \in J}(1 \otimes w_{r,q})\alpha(b_{k,q,p}) = \sum_{\ell \in I_\pi, s \in J} b_{\ell,r,s} \otimes w(\pi)_{\ell k} w_{s,p}. \tag{4.2}$$

Now we construct maps $\nu : \mathrm{Hom}_{(G,A)}(H_\pi \otimes A \otimes H_0, A \otimes H_0) \to \mathrm{Hom}_G(H_\pi, A \otimes \mathbb{B}(H_0))$ and $\nu^{-1} : \mathrm{Hom}_G(H_\pi, A \otimes \mathbb{B}(H_0)) \to \mathrm{Hom}_{(G,A)}(H_\pi \otimes A \otimes H_0, A \otimes H_0)$ by the following equalities. For all $k \in I_\pi$ and $p \in J$,

$$\nu(S)(\xi_k) = \sum_{q,p \in J} a_{k,q,p} \otimes \eta_q \odot \eta_p,$$
$$\nu^{-1}(T)(\xi_k \otimes a \otimes \eta_p) = \sum_{q \in J} b_{k,q,p} a \otimes \eta_q.$$

They are actually linear operators in each hom-space because equalities (4.1) and (4.2) have the same form. Since clearly we have $\nu \circ \nu^{-1} = \mathrm{id}$ and $\nu^{-1} \circ \nu = \mathrm{id}$, they give an isomorphism between $\mathrm{Hom}_{(G,A)}(H_\pi \otimes A \otimes H_0, A \otimes H_0)$ and $\mathrm{Hom}_G(H_\pi, A \otimes \mathbb{B}(H_0))$. Then we also easily check that ν maps the subspace $\mathrm{Hom}_{(G,A)}(H_\pi \otimes e_j(A \otimes L^2(G)), e_i(A \otimes L^2(G))) \subset \mathrm{Hom}_{(G,A)}(H_\pi \otimes A \otimes H_0, A \otimes H_0)$ to the subspace $\mathrm{Hom}_G(H_\pi, e_i(A \otimes \mathbb{K}(L^2(G)))e_j) \subset \mathrm{Hom}_G(H_\pi, A \otimes \mathbb{B}(H_0))$. □

Therefore, as in the classical case we can derive the following result by A. Wassermann [20, p. 304].

Corollary 4.4. *Let $\{A, G, \alpha\}$ be a compact quantum ergodic system and $\mathbb{M} : R(G) \to M_{|I|}(\mathbb{Z})$ be the multiplicity map. For all i, j in the index set I, we have*

$$\mathbb{M}(\pi)_{i,j} = \dim \mathrm{Hom}\bigl(H_\pi, e_i\bigl(A \otimes \mathbb{K}\bigl(L^2(G)\bigr)\bigr)e_j\bigr).$$

It yields the identity $\mathbb{M}(\overline{\pi})_{i,j} = \mathbb{M}(\pi)_{j,i}$ for all $\pi \in \widehat{G}$ and $i, j \in I$ and therefore the multiplicity map $\mathbb{M}: R(G) \to M_{|I|}(\mathbb{Z})$ is a $*$-homomorphism. As we have seen, some properties of multiplicity maps also hold in the quantum case. However, we have to be careful of the existence of its common eigenvector (see, [20, Theorem 17] for classical case). In order to study this problem we need several lemmas.

Lemma 4.5. *For all $i, j \in I$, the reduced G-space $e_i(A \otimes \mathbb{K}(L^2(G)))e_j$ is not* 0.

Proof. Let M be the σ-weak closure of A in $\mathbb{B}(H_\varphi)$ where H_φ is the GNS Hilbert space of the invariant state φ. Define the relation $\sim$ in the index set I as $i \sim j$ if and only if $e_i(A \otimes \mathbb{K}(L^2(G)))e_j \neq 0$. We show this is an equivalence relation. A non-trivial part is transitivity of $\sim$. Assume $i \sim j$ and $j \sim k$. Then there exists $\pi, \rho \in \widehat{G}$ which satisfy $\mathbb{M}(\pi)_{i,j} > 0$ and $\mathbb{M}(\rho)_{j,k} > 0$. Then we have $\mathbb{M}(\pi \cdot \rho)_{i,k} \geqslant \mathbb{M}(\pi)_{i,j}\mathbb{M}(\rho)_{j,k} > 0$ and the left-hand side is equal to $\sum_{\tau \in \widehat{G}} N_\tau^{\pi\rho}\mathbb{M}(\tau)_{i,k}$. Hence there exists $\tau \in \widehat{G}$ with $\mathbb{M}(\tau)_{i,k} > 0$. Therefore the reduced space $e_i(A \otimes \mathbb{K}(L^2(G)))e_k$ is not 0, that is, $i \sim k$. Let $I = \bigsqcup_{\lambda \in \Lambda} I_\lambda$ be the decomposition with respect to the equivalence relation $\sim$. Let z_i be the central projection in $Z(M)$ with $z(e_i) = z_i \otimes 1$ for all $i \in I$ where $z(e_i)$ means the central support projection of e_i in $M \otimes \mathbb{B}(L^2(G))$. Since for $\lambda \neq \mu \in \Lambda$ we have $e_i(A \otimes \mathbb{K}(L^2(G)))e_j = 0$ for all $i \in I_\lambda$ and $j \in I_\mu$, we see $z_i z_j = 0$ for all $i \in I_\lambda$ and $j \in I_\mu$. We claim $1 = \bigvee_{i \in I} z_i = \sum_{\lambda \in \Lambda} \bigvee_{i \in I_\lambda} z_i$. This follows because $A \rtimes_\alpha G \cong \bigoplus_{i \in I} \mathbb{K}(H_i)$ and the sum of all diagonal atoms is equal to 1 in strong operator topology. By definition of a central support projection we obtain $Mz_i \otimes \mathbb{B}(L^2(G)) = M \otimes \mathbb{B}(L^2(G))e_i M \otimes \mathbb{B}(L^2(G))$ for all $i \in I$. Then we see $\tilde{\alpha}(Mz_i \otimes \mathbb{B}(L^2(G))) \subset (M \otimes \mathbb{B}(L^2(G))e_i M \otimes \mathbb{B}(L^2(G))) \otimes L^\infty(G) = Mz_i \otimes \mathbb{B}(L^2(G)) \otimes L^\infty(G)$ and hence $\tilde{\alpha}(z_i \otimes 1) \leqslant z_i \otimes 1 \otimes 1$ for all $i \in I$. Set $z_\lambda = \bigvee_{i \in I_\lambda} z_i$. Then a family $\{z_\lambda\}_{\lambda \in \Lambda}$ is a partition of unity and satisfies $\tilde{\alpha}(z_\lambda \otimes 1) \leqslant z_\lambda \otimes 1 \otimes 1$ for all $\lambda \in \Lambda$. Therefore we obtain $\tilde{\alpha}(z_\lambda \otimes 1) = z_\lambda \otimes 1 \otimes 1$ for all $\lambda \in \Lambda$. Since the left-hand side is equal to $W_{23}\alpha(z_\lambda)_{13}W_{23}^*$, we have $\alpha(z_\lambda) = z_\lambda \otimes 1$ for all $\lambda \in \Lambda$. Ergodicity of α yields $z_\lambda = 1$ or 0 for all $\lambda \in \Lambda$. Hence Λ is a singleton and it completes the proof. $\square$

From this lemma we have the following irreducibility criterion which has been studied in [20, Section 10].

Lemma 4.6. *Let π be an element of $\widehat{G}$. If for any $\rho \in \widehat{G}$ there exists $k \in \mathbb{Z}_{\geqslant 0}$ such that ρ is contained in π^k, then for any $i, j \in I$ there exists $\ell \in \mathbb{Z}_{\geqslant 0}$ with* $(\mathbb{M}(\pi)^\ell)_{i,j} > 0$.

Proof. Since G-space $e_i(A \otimes \mathbb{K}(L^2(G)))e_j$ is not 0, there exists an element ρ in I such that $\mathbb{M}(\rho)_{i,j} = \dim \operatorname{Hom}(H_\rho, e_i(A \otimes \mathbb{K}(L^2(G)))e_j) > 0$. By assumption there exists $\ell \in \mathbb{N}$ such that π^ℓ contains ρ. Then we have $(\mathbb{M}(\pi)^\ell)_{i,j} = \mathbb{M}(\pi^\ell)_{i,j} \geqslant \mathbb{M}(\rho)_{i,j} > 0$. $\square$

In this case, we want to show the existence of a Perron–Frobenius eigenvector, however, we cannot use the dual weight $\hat{\varphi}$ as in the compact group case because it is not necessarily a trace. We will give an eigenvector under some assumptions later.

Lemma 4.7. *Let $m: R(G) \to \mathbb{Z}$ be a $*$-homomorphism such that it satisfies $m(\pi_0) = 1$ and $m(\widehat{G}) \subset \mathbb{Z}_{>0}$. Assume $\mathbb{M}(\pi)_{i,j} \leqslant m(\pi)$ for all $\pi \in \widehat{G}$ and $i, j \in I$. Then the matrix $\mathbb{M}(\rho)$ is a bounded operator on $l^2(I)$ and its norm is less than or equal to $m(\rho)^2$ for all $\rho \in R(G)_+$.*

Proof. Take ρ from $R(G)_+$. Then inequality $\sum_{j\in I}\mathbb{M}(\rho)_{i,j}^2 \leqslant m(\rho)^2$ is checked by the following calculation,

$$\begin{aligned}\sum_{j\in I}\mathbb{M}(\rho)_{i,j}^2 &= \big(\mathbb{M}(\rho)\mathbb{M}(\overline{\rho})\big)_{i,i}\\ &= \mathbb{M}(\rho\cdot\overline{\rho})_{i,i}\\ &= \sum_{\sigma\in\widehat{G}} N_\sigma^{\rho\overline{\rho}}\mathbb{M}(\sigma)_{i,i}\\ &\leqslant \sum_{\sigma\in\widehat{G}} N_\sigma^{\rho\overline{\rho}} m(\sigma)\\ &= m(\rho)m(\overline{\rho})\\ &= m(\rho)^2.\end{aligned}$$

Take a finitely supported vector ξ in $l^2(\mathbb{Z})$. Let $C(i,\rho)$ be a finite set $\{j\in I \mid \mathbb{M}(\rho)_{i,j}\neq 0\}$. From the above inequality we particularly obtain $|C(i,\rho)|\leqslant m(\rho)^2$. We assume $\rho=\overline{\rho}\in R(G)_+$. Then we have the desired inequality

$$\begin{aligned}\big\|\mathbb{M}(\rho)\xi\big\|^2 &= \sum_{i\in I}\bigg|\sum_{j\in I}\mathbb{M}(\rho)_{i,j}\xi_j\bigg|^2\\ &= \sum_{i\in I}\bigg|\sum_{j\in C(i,\rho)}\mathbb{M}(\rho)_{i,j}\xi_j\bigg|^2\\ &= \sum_{i\in I}\sum_{j\in C(i,\rho)}\mathbb{M}(\rho)_{i,j}^2\cdot\sum_{j\in C(i,\rho)}|\xi_j|^2\\ &\leqslant m(\rho)^2\sum_{i\in I}\sum_{j\in C(i,\rho)}|\xi_j|^2\\ &= m(\rho)^2\sum_{j\in I}\big|C(j,\rho)\big||\xi_j|^2\\ &\leqslant m(\rho)^4\sum_{j\in I}|\xi_j|^2.\end{aligned}$$

For general $\rho\in R(G)_+$ we apply the above result to a positive self-conjugate $\overline{\rho}\rho$ and get

$$\big\|\mathbb{M}(\rho)\big\|^2 = \big\|\mathbb{M}(\overline{\rho}\rho)\big\| \leqslant m(\overline{\rho}\rho)^2 = m(\rho)^4. \qquad \square$$

Lemma 4.8. *If a self-conjugate element $\pi\in\widehat{G}$ has the property in Lemma* 4.6*, then $\mathbb{M}(\pi)$ has an eigenvector with the eigenvalue $\|\mathbb{M}(\pi)\|\leqslant D_\pi^2$. Moreover if A has a tracial state, then $\|\mathbb{M}(\pi)\|\leqslant d_\pi^2$.*

Proof. Since we know the matrix $\mathbb{M}(\pi)$ is irreducible and bounded, there exists an eigenvector with the eigenvalue $\|\mathbb{M}(\pi)\|$. For all $i,j\in I$ we have proved $e_i(A\otimes\mathbb{K}(L^2(G)))e_j\neq 0$. By

Theorem 3.9 we have $\mathbb{M}(\pi)_{i,j} \leqslant D_\pi$. Hence we obtain the first assertion by Lemma 4.7. If A has a tracial state τ, then the tracial weight $\tilde{\tau} = \tau \otimes \mathrm{Tr}$ is semifinite on $A \rtimes_\alpha G$ and $\tilde{\tau}(e_i) > 0$ for all $i \in I$. In fact if $\tilde{\tau}(e_i)$ is equal to 0 for some $i \in I$, we have $\tilde{\tau}(e_j) = 0$ for all $j \in I$ by Remark 3.12 and Lemma 4.5. It shows $\tilde{\tau} = 0$ and this is contradiction. Now we apply Lemma 4.7 with $m(\pi) = d_\pi$ and obtain the second assertion. $\square$

Now we consider the special case $G = SU_q(2)$. We refer to [13,22] or the next section for its representation theory of $SU_q(2)$. Its irreducible representations are labeled as $\{\pi_\nu\}_{\nu \in \frac{1}{2}\mathbb{Z}_{>0}}$ and their fusion rules are determined by

$$\pi_{\frac{1}{2}} \cdot \pi_\nu = \pi_{\nu+\frac{1}{2}} + \pi_{\nu-\frac{1}{2}}$$

for all $\nu \in \frac{1}{2}$.

Lemma 4.9. *Let $m : R(SU_q(2)) \to \mathbb{Z}$ be a dimension function with $m(\pi_\nu) \leqslant d_{\pi_\nu}^2$ for all $\nu \in \frac{1}{2}$. Then we obtain $m(\pi_\nu) = d_{\pi_\nu}$ for all $\nu \in \frac{1}{2}$.*

Proof. Write t_0 for $m(\pi_{\frac{1}{2}})$. Define polynomials $\{p_\nu\}_{\nu \in \frac{1}{2}\mathbb{Z}}$ recursively by $p_{\nu+\frac{1}{2}}(s) = sp_\nu(s) - p_{\nu-\frac{1}{2}}(s)$ and $p_0 = 1$, $p_{\frac{1}{2}} = s$. Then we get $m(\pi_\nu) = p_\nu(t_0)$. From the positivity of $m(\pi_\nu)$ we have $t_0 \geqslant 2$. Assume $t_0 > 2$. For $s > 2$, the polynomials $p_\nu(s)$ are strictly increasing and we get a lower bound by an affine line $p_\nu(2)'(s - 2) + p_\nu(2)$ where $p_\nu(s)'$ means the derivative at s. Putting $s = t_0$, we obtain an equality $m(\pi_\nu) \geqslant (t - 2)p_\nu(2)' + 2\nu + 1$. The left-hand side has upper bound by $d_{\pi_\nu}^2 = (2\nu + 1)^2$. In order to derive contradiction, it suffices to show the right-hand side has a polynomial degree 3 with respect to ν. By the definition of p_ν, we get $p_{\nu+\frac{1}{2}}(2)' - p_\nu(2)' = p_\nu(2)' - p_{\nu-\frac{1}{2}}(2)' + 2\nu + 1$. This immediately gives $p_\nu(2)' = \frac{2}{3}\nu(\nu + 1)(2\nu + 1)$. $\square$

We show the following main result in this section which asserts the existence of multiplicity vector $\boldsymbol{c}$ under a tracial condition on A.

Theorem 4.10. *Let $\{A, SU_q(2), \alpha\}$ be a compact quantum ergodic system. Assume that A has a tracial state. Then there exists a positive entry vector $\boldsymbol{c} = (c_i)_{i \in I}$ such that $\mathbb{M}(\pi_\nu)\boldsymbol{c} = d_{\pi_\nu}\boldsymbol{c}$ for all $\nu \in \frac{1}{2}$.*

Proof. By Lemma 4.8 there exists an eigenvector $\boldsymbol{c}$ such that $\mathbb{M}(\pi_{\frac{1}{2}})\boldsymbol{c} = t\boldsymbol{c}$ where $t = \|\mathbb{M}(\pi_{\frac{1}{2}})\|$. Since any π_ν is written by the polynomial P_ν of $\pi_{\frac{1}{2}}$, we can define the dimension function $m : R(SU_q(2)) \to \mathbb{Z}$ with $\mathbb{M}(\pi_\nu)\boldsymbol{c} = m(\pi_\nu)\boldsymbol{c}$ for all $\nu \in \frac{1}{2}$. We show $m(\pi_\nu) = \|\mathbb{M}(\pi_\nu)\|$ for all $\nu \in \frac{1}{2}$. The self-adjoint operator $\mathbb{M}(\pi_\nu)$ is written by the polynomial P_ν of $\mathbb{M}(\pi_{\frac{1}{2}})$. Hence we have $\|\mathbb{M}(\pi_\nu)\| \geqslant P_\nu(\|\mathbb{M}(\pi_{\frac{1}{2}})\|) = P_\nu(t) = m(\pi_\nu)$. The converse inequality is obtained by applying the Schur test to $\mathbb{M}(\pi_\nu)\boldsymbol{c} = m(\pi_\nu)\boldsymbol{c}$. Therefore we have $m(\pi_\nu) = \|\mathbb{M}(\pi_\nu)\| \leqslant d_{\pi_\nu}^2$. By the previous lemma, we have $m(\pi_\nu) = d_{\pi_\nu}$ for all $\nu \in \frac{1}{2}$. $\square$

The next lemma has already been proved in [21, Lemma 1, Theorem 2] for $q = 1$, that is, the $SU(2)$ case.

Lemma 4.11. *Let $\{A, SU(2), \alpha\}$ be an ergodic system. Then its $\pi_{\frac{1}{2}}$-eigenvector space $X_{\pi_{\frac{1}{2}}}$ is not one-dimensional.*

Proof. Let $\xi = (a, b)$ be a $\pi_{\frac{1}{2}}$-eigenvector. Then we have

$$\alpha(a) = a \otimes x + b \otimes v, \qquad \alpha(b) = a \otimes u + b \otimes y.$$

Consider a vector $\eta = (b^*, -q^{-1}a^*)$. By an easy calculation, we see η is also a $\pi_{\frac{1}{2}}$-eigenvector. Assume that $X_{\pi_{\frac{1}{2}}}$ is one-dimensional. Then there exists a complex number μ such that $\eta = \mu\xi$. So we get $b^* = \mu a$ and $-q^{-1}a^* = \mu b$. It follows $-q^{-1}a = |\mu|^2 a$. Hence we obtain $0 > -q^{-1} = |\mu|^2 > 0$. This is a contradiction. $\square$

Actually, we can show $\pi_{\nu/2}$-eigenvector space (ν is odd) is even-dimensional. Hence we obtain the result corresponding to [21, Theorem 1] for positive q.

Corollary 4.12. *Let $\{A, SU(2), \alpha\}$ be an ergodic system. Assume that A has a tracial state and q is positive. Then its multiplicity diagram is one of type $1, \mathbb{T}_n$ $(n \geqslant 2), \mathbb{T}, SU(2), D_n^*$ $(n \geqslant 2), D_\infty^*, A_4^*, S_4^*$ and A_5^*.*

Example 4.13. We consider the multiplicity diagrams of quantum spheres $C(S^2_{q,\lambda})$. If $\lambda_0 = c(n)$, then its spectral pattern (or finite-dimensionality) allows only the diagram of type $SU(2)$ in Fig. 6 (see Appendix A). If $0 \leqslant \lambda_0 \leqslant 1$, then its spectral pattern derives the diagram of type $\mathbb{T}$ in Fig. 5. If $\{A, SU_q(2), \alpha\}$ is an ergodic system and A is finite-dimensional, then its multiplicity diagram of $\pi_{\frac{1}{2}}$ must be of type $SU(2)$ by its finite-dimensionality. Hence the classification by Podleś [16] shows that A is G-isomorphic to $\mathrm{End}(H_{\pi_\nu})$ for some $\nu \in \mathbb{Z}_{\geqslant \frac{1}{2}}$.

We recall the definition of the McKay diagrams. Let $H \subset G$ be a quantum subgroup with the restriction map r_H and $w \in \mathbb{B}(H_w) \otimes C(G)$ be a unitary representation of G. We denote the restricted representation $(\mathrm{id} \otimes r_H)(w)$ by $w|_H$. Prepare the vertices $\{\sigma\}_{\sigma \in \widehat{H}}$. Let σ_0 be the one-dimensional trivial representation of H. First we consider the irreducible decomposition $w|_H \cdot \sigma_0 = \bigoplus_{\sigma \in \widehat{H}} N_\sigma^{w|_H} \sigma$ where the scalar $N_\sigma^{w|_H}$ means a multiplicity. Then we draw arrows from σ_0 to σ as above with $N_\sigma^{w|_H}$-times, respectively. Second for each σ in the above decomposition we consider the irreducible decomposition of $w|_H \cdot \sigma$ and we draw arrows from σ to the irreducible representations in a similar way. By repeating this procedure, we get the McKay diagram for $H \subset G$ with respect to w. If w is self-conjugate, then the diagram is non-oriented because of the symmetry $N_\sigma^{w|_H \tau} = N_{w|_H \sigma}^{\tau}$. When we treat quantum subgroups of $SU_q(2)$, the McKay diagrams are drawn with respect to the fundamental representation $\pi_{\frac{1}{2}}$. Now we consider a G-covariant system $\{C(H \setminus G), \delta\}$ where H is a quantum subgroup with the restriction map r_H. It is well known that the multiplicity diagram and the McKay diagram coincide in the classical case, however, we give the proof of the general quantum group case for readers' convenience. We denote $(r_H \otimes \mathrm{id})(V_\ell)$ by $V_\ell|_H$. The C^*-algebra $C(G) \otimes \mathbb{K}(L^2(G))$ has the left and right actions α_ℓ and β_r defined by $\alpha_\ell(x) = \mathrm{Ad}\, V_{\ell}{}_{13}^* V_{\ell}{}_{12}^* (1 \otimes x)$ and $\beta_r(x) = \mathrm{Ad}\, V_{13}(x \otimes 1)$ for all $x \in C(G) \otimes \mathbb{K}(L^2(G))$, respectively. We define the left H-action α_ℓ^H by the composition $(r_H \otimes \mathrm{id}) \circ \alpha_\ell$. Consider the map $\mathrm{Ad}\, V_\ell : C(G) \otimes \mathbb{K}(L^2(G)) \to C(G) \otimes \mathbb{K}(L^2(G))$.

Lemma 4.14.

(1) *The map* $\operatorname{Ad} V_\ell$ *gives an isomorphism between* $C(H \setminus G) \otimes \mathbb{K}(L^2(G))$ *and* ${}^H(C(G) \otimes \mathbb{K}(L^2(G)))$ *where the latter one is the fixed point algebra of the left* H*-action* α_ℓ^H*. Moreover, it maps* $C(H \setminus G) \rtimes_\delta G$ *to* $\mathbb{C} \otimes {}^H\mathbb{K}(L^2(G))$.
(2) $\operatorname{Ad} V_\ell$ *intertwines the right* G*-actions* $\tilde{\alpha}$ *and* β_r.

Proof. (1) It suffices to show that the map $\operatorname{Ad} V_\ell$ intertwines left H-actions $\operatorname{Ad} V_\ell|_H \otimes \mathrm{id}$ and α_ℓ^H. This is immediately verified by the equality $V_{\ell 23} V_\ell|_H{}_{12}^* = V_\ell|_H{}_{13}^* V_\ell|_H{}_{12}^* V_{\ell 23}$.

(2) Similarly the equality $V_{\ell 12} W_{23} V_{13} = V_{13} V_{\ell 12}$ gives the desired intertwining property. □

Now we choose a left irreducible H-module K_σ^ℓ for each $\sigma \in \widehat{H}$. Then we have the irreducible decomposition $L^2(G) = \bigoplus_{\sigma \in \widehat{H}} K_\sigma^\ell \otimes L_\sigma$ where L_σ is the multiplicity space for σ. Note that all the L_σ are non-zero, because there exists $\pi \in \widehat{G}$ such that σ is contained in $\pi|_H$. Then we get ${}^H\mathbb{K}(L^2(G)) = \bigoplus_{\sigma \in \widehat{H}} \mathbb{C}1_\sigma \otimes \mathbb{K}(L_\sigma)$. Let $e_\sigma = 1_\sigma \otimes p_\sigma$ be a minimal projection of $\mathbb{C}1_\sigma \otimes \mathbb{K}(L_\sigma)$. By the previous lemma we have $K_0(C(H \setminus G) \rtimes_\delta G) = \bigoplus_{\sigma \in \widehat{H}} \mathbb{Z}[e_\sigma]$. Hence the vertices of multiplicity maps are represented by the elements of $\widehat{H}$. Then the following proposition holds.

Proposition 4.15. *Let* π *be an element of* $\widehat{G}$*. Then we have* $\mathbb{M}(\pi)_{\rho,\sigma} = N_\sigma^{\pi|_H \rho}$ *for all* ρ*,* σ *in* $\widehat{H}$.

Proof. We use Corollary 4.4. The G-module $e_\rho C(H \setminus G) \otimes \mathbb{K}(L^2(G)) e_\sigma$ is isomorphic to ${}^H(C(G) \otimes e_\rho \mathbb{K}(L^2(G)) e_\sigma)$ by the previous lemma. Since the quantum group G acts on the first tensor component, the multiplicity space for π is ${}^H(H_\pi^\ell \otimes e_\rho \mathbb{K}(L^2(G)) e_\sigma)$ which is linearly isomorphic to $\operatorname{Hom}_H(H_{\overline{\pi|_H}}^\ell, K_\rho \otimes K_{\bar{\sigma}})$. Hence its dimension is equal to $N_{\overline{\pi|_H}}^{\rho\bar{\sigma}} = N_\sigma^{\pi|_H \rho}$. □

Define the vector $\boldsymbol{d} = (d_\sigma)_{\sigma \in \widehat{H}}$ and we have $\mathbb{M}(\pi)\boldsymbol{d} = d_\pi \boldsymbol{d}$. Hence we have solved the eigenvector problem in the case of the quotient spaces.

Corollary 4.16. *Let* H *be a quantum subgroup of* G *and consider the ergodic system* $\{C(H \setminus G), \delta\}$*. Then the multiplicity diagram for* $\pi \in \widehat{G}$ *coincides with the McKay diagram of* $H \subset G$ *with respect to* π.

Next we study reduced ergodic systems associated to ergodic systems and show the heredity of information of multiplicity maps.

Definition 4.17. Let $\{A, G, \alpha\}$ and $\{B, G, \alpha\}$ be compact quantum ergodic systems and take index sets I, J with $A \rtimes_\alpha G \cong \bigoplus_{i \in I} \mathbb{K}(H_i)$ and $B \rtimes_\beta G \cong \bigoplus_{j \in J} \mathbb{K}(H_i{}')$. We say two ergodic systems are of the same type if there exists a bijective map $\theta : I \to J$ satisfying $\mathbb{M}^B(\pi) \circ \theta = \theta \circ \mathbb{M}^A(\pi)$ for all $\pi \in G$, where $\mathbb{M}^A$ and $\mathbb{M}^B$ are multiplicity maps for ergodic systems $\{A, G, \alpha\}$ and $\{B, G, \alpha\}$, respectively.

Theorem 4.18. *Let* $\{A, G, \alpha\}$ *be a compact quantum ergodic system and take an index set* I *with* $A \rtimes_\alpha G \cong \bigoplus_{i \in I} \mathbb{K}(H_i)$*. Take minimal projections* $\{e_i\}_{i \in I}$ *from* $\{\mathbb{K}(H_i)\}_{i \in I}$*. Then for all* $i \in I$*, two ergodic systems* $\{A, G, \alpha\}$ *and* $\{e_i(A \otimes \mathbb{K}(L^2(G)))e_i, G, \tilde{\alpha}\}$ *are of the same type.*

Proof. Denote $\mathbb{K}(L^2(G))$ and $\mathbb{B}(L^2(G))$ simply by $\mathbb{K}$ and $\mathbb{B}$. We claim that the covariant system $\{A\otimes\mathbb{K}\otimes\mathbb{K}, G, \tilde{\tilde{\alpha}}\}$ is strongly equivalent to $\{\mathbb{K}\otimes A\otimes\mathbb{K}, G, \mathrm{id}\otimes\tilde{\alpha}\}$. Let us consider the map $\mathrm{Ad}(W_{23}^*): A\otimes\mathbb{K}\otimes\mathbb{K}\to A\otimes\mathbb{K}\otimes\mathbb{K}$. For $x\in A$, $k\in\mathbb{K}\otimes\mathbb{K}$ we have:

$$\begin{aligned} W_{23}^*\tilde{\tilde{\alpha}}\bigl(W_{23}(x\otimes k)W_{23}^*\bigr)W_{23} &= W_{23}^*W_{34}W_{24}W_{23}\alpha(x)_{14}(1\otimes k\otimes 1)W_{23}^*W_{24}^*W_{34}^*W_{23} \\ &= W_{34}\alpha(x)_{14}(1\otimes k\otimes 1)W_{34}^*, \end{aligned}$$

where we use the pentagonal identity for W, $W_{23}W_{13}W_{12}=W_{12}W_{23}$. Next flip the first and second tensor component of $A\otimes\mathbb{K}\otimes\mathbb{K}$ and the claim is proved. Hence we get an isomorphism $(A\otimes\mathbb{K})\rtimes_{\tilde{\alpha}} G\cong\mathbb{K}\otimes(A\rtimes_\alpha G)$. A projection $e_i\otimes 1$ is in the fixed point algebra of the multiplier algebra $M(A\otimes\mathbb{K}\otimes\mathbb{K})$ for $\tilde{\tilde{\alpha}}$. So it is mapped to a projection f_i in the multiplier algebra $M(\mathbb{K}\otimes A\rtimes_\alpha G)$. Hence the system $\{e_i(A\otimes\mathbb{K})e_i\otimes\mathbb{K}, G, \tilde{\tilde{\alpha}}\}$ is conjugate to $\{f_i(\mathbb{K}\otimes A\otimes\mathbb{K})f_i, G, \mathrm{id}\otimes\tilde{\alpha}\}$. The projection f_i is decomposed into a direct sum of projections $\{p_j\}_{j\in I}$ which are in $\{\mathbb{B}\otimes\mathbb{B}(H_j)\}_{j\in I}$. We claim that they are non-zero projections. It suffices to prove the central support of f_i in $\mathbb{B}\otimes M\rtimes_\alpha G$ is equal to 1 by passing to von Neumann algebras, that is, isomorphic covariant systems $\{e_i(M\otimes\mathbb{B})e_i\otimes\mathbb{B}, G, \tilde{\tilde{\alpha}}\}$ and $\{f_i(\mathbb{B}\otimes M\otimes\mathbb{B})f_i, G, \mathrm{id}\otimes\tilde{\alpha}\}$. Hence we show the central support of $e_i\otimes 1$ in $(M\otimes\mathbb{B}\otimes\mathbb{B})^{\tilde{\tilde{\alpha}}}=(M\otimes\mathbb{B})\rtimes_{\tilde{\alpha}} G$ is equal to 1. By duality theorem, we get the isomorphism of inclusions, $M\rtimes_\alpha G\otimes\mathbb{C}\subset(M\otimes\mathbb{B})\rtimes_{\tilde{\alpha}} G\cong\hat{\alpha}(M\rtimes_\alpha G)\subset M\rtimes_\alpha G\otimes\mathbb{B}$. We study the central support of $\hat{\alpha}(e_i)$ in $M\rtimes_\alpha G\otimes\mathbb{B}$. Fix $j\in I$ and we can take $\pi\in\widehat{G}$ with $\mathbb{M}(\pi)_{j,i}>0$ by Lemma 4.5. By the definition of $\mathbb{M}(\pi)$, we have $[\hat{\alpha}_\pi(e_i)]=\sum_{k\in I}\mathbb{M}(\pi)_{k,i}[e_k]$. Hence the jth component of $\hat{\alpha}_\pi(e_i)$ is a non-zero projection, in particular, the central support of $\hat{\alpha}(e_i)$ in $M\rtimes_\alpha G\otimes\mathbb{B}$ is equal to 1. Therefore the claim is proved. Let H_j' be a Hilbert space $p_j(L^2(G)\otimes H_j)$ and we have an isomorphism $e_i(A\otimes\mathbb{K})e_i\rtimes_{\tilde{\alpha}} G\cong\bigoplus_{j\in I}\mathbb{K}(H_j')$. Next we choose minimal projections $\{q_j\}_{j\in I}$ from $\mathbb{K}(H_j')$. Let p be a minimal projection in $\mathbb{K}$. For any $j\in I$, the projection q_j is equivalent to $p\otimes e_j$ in $\mathbb{K}\otimes\mathbb{K}(H_j)\subset(\mathbb{K}\otimes A\otimes\mathbb{K})^{\mathrm{id}\otimes\tilde{\alpha}}$. Then we have a G-isomorphism between reduced spaces $q_j(\mathbb{K}\otimes A\otimes\mathbb{K})q_k$ and $(p\otimes e_j)(\mathbb{K}\otimes A\otimes\mathbb{K})(p\otimes e_k)=\mathbb{C}p\otimes e_j(A\otimes\mathbb{K})e_k$ for all $j,k\in I$. Therefore we have proved the equality $\dim\mathrm{Hom}_G(H_\pi, q_j(\mathbb{K}\otimes A\otimes\mathbb{K})q_k)=\dim\mathrm{Hom}_G(H_\pi, e_j(A\otimes\mathbb{K})e_k)$ for all $\pi\in\widehat{G}$ and $j,k\in I$. □

Definition 4.19. Let $\{A,G,\alpha\}$ be a compact quantum ergodic system. A unital C^*-subalgebra $B\subset A$ is called a G-invariant C^*-subalgebra if it satisfies $\alpha(B)\subset B\otimes C(G)$. Then $\{B,G,\alpha\}$ is called a subsystem of $\{A,G,\alpha\}$ and denote this situation by $\{B,G,\alpha\}\subset\{A,G,\alpha\}$. A subsystem of $\{C(G),G,\delta\}$ is called a right coideal.

Let B be a G-invariant C^*-subalgebra of A. The inclusion G-homomorphism $\iota: B\to A$ induces the inclusion of crossed products, $B\rtimes_\alpha G\subset A\rtimes_\alpha G$. Taking index sets I, J as $B\rtimes_\alpha G\cong\bigoplus_{i\in I}\mathbb{K}(H_i)$ and $A\rtimes_\alpha G\cong\bigoplus_{j\in J}\mathbb{K}(K_i)$. Let Λ be an inclusion matrix of $B\rtimes_\alpha G\subset A\rtimes_\alpha G$, that is, $\mathbb{K}(H_i)$ is amplified into $\mathbb{K}(K_j)$ by $\Lambda_{j,i}$ times. We can easily show the next proposition by the definition of multiplicity maps.

Proposition 4.20. *Let $\{B,G,\alpha\}\subset\{A,G,\alpha\}$ be an inclusion of compact quantum ergodic systems. Let $\mathbb{M}^B$ and $\mathbb{M}^A$ be the multiplicity maps. Then we have $\Lambda\mathbb{M}^B(\pi)=\mathbb{M}^A(\pi)\Lambda$ for all $\pi\in\widehat{G}$.*

We recall an observation by A. Wassermann, which has an important role in his classification program in [21]. Let $p_0 = (\mathrm{id} \otimes h)(V)$ be a minimal projection of $C_r^*(G)$. Note that it is central in not only $C_r^*(G)$ but also $C_\ell^*(G)$. It is also a minimal projection on $L^2(G)$. Then we have

$$(1 \otimes p_0) A \rtimes_\alpha G (1 \otimes p_0) = \big(A \otimes p_0 \mathbb{K}\big(L^2(G)\big) p_0\big)^{\tilde{\alpha}} = A^\alpha \otimes \mathbb{C} p_0 = \mathbb{C}(1 \otimes p_0).$$

This shows that p_0 is a minimal projection in $A \rtimes_\alpha G$. Hence if we take minimal projections in $\{e_i\}_{i \in I}$ as before, then there exists unique i_0 such that e_{i_0} is equivalent to p_0 in $A \rtimes_\alpha G$. We often say this index corresponds to p_0. Then we obtain an isomorphism as covariant systems between $\{A, G, \alpha\}$ and $\{e_{i_0} A \otimes \mathbb{K}(L^2(G)) e_{i_0}, G, \tilde{\alpha}\}$. From this we get the following result.

Corollary 4.21. *Let $A \subset C(G)$ be a right coideal. Then there exists an eigenvector $\boldsymbol{c} = (c_i)_{i \in I}$ which satisfies the following conditions.*

(1) *All entries of $\boldsymbol{c}$ are strictly positive integers and there exists an index $i_0 \in I$ which satisfies $c_{i_0} = 1$.*
(2) *It is a common eigenvector of the multiplicity map*; $\mathbb{M}(\pi)\boldsymbol{c} = d_\pi \boldsymbol{c}$ *for all* $\pi \in \widehat{G}$.
(3) *Two covariant systems $\{A, G, \alpha\}$ and $\{e_{i_0} A \otimes \mathbb{K}(L^2(G)) e_{i_0}, G, \tilde{\alpha}\}$ are isomorphic.*

Proof. Take a minimal projection p_π of $\mathrm{End}(H_\pi) \subset C_r^*(G)$ for $\pi \in \widehat{G}$. Let π be an element of $\widehat{G}$. Consider the dual coaction $\hat{\delta}_\pi : C_r^*(G) \to C_r^*(G) \otimes \mathrm{End}(H_\pi)$. We want to compute $\hat{\delta}_\pi(p_0)$ in $K_0(C_r^*(G))$. Since we have the isomorphism $K_0(C_r^*(G)) \cong R(G) = K_0^G(\mathbb{C})$ by $[p_\pi] \to \pi$, we see $[\hat{\delta}_\pi(p_0)] = [p_\pi]$. Now let Λ be an inclusion matrix for $A \rtimes_\alpha G \subset C(G) \rtimes_\delta G$. It is actually a row vector because $C(G) \rtimes_\alpha G$ is isomorphic to $\mathbb{K}(L^2(G))$. By the previous proposition we have $\mathbb{M}'(\pi)\Lambda = \Lambda \mathbb{M}(\pi)$ for all $\pi \in \widehat{G}$, where $\mathbb{M}'$ and $\mathbb{M}$ are the multiplicity maps for $\{A, G, \alpha\}$ and $\{C(G), G, \delta\}$, respectively. Let us take $1 \otimes p_0$ for a minimal projection of $C(G) \rtimes_\delta G \cong \mathbb{K}(L^2(G))$. From the first part of this proof we have $\pi[p_0] = [p_\pi] = d_\pi [p_0]$ in $K_0(C(G) \rtimes_\delta G)$. Hence the action $\mathbb{M}'(\pi)$ is multiplication by d_π and we get $\Lambda \mathbb{M}(\pi) = d_\pi \Lambda$. Transposing it, we have $\mathbb{M}(\bar{\pi})^T \Lambda = d_{\bar{\pi}} {}^T\Lambda$. Define a vector $\boldsymbol{c} = {}^T\Lambda$. Let us take an index i_0 which corresponds to p_0 in $A \rtimes_\alpha G$. Since $1 \otimes p_0$ is also minimal in $C(G) \rtimes_\delta G$, we get $c_{i_0} = \Lambda_{i_0} = 1$. □

We end this section with the following proposition. We use the same notations as before.

Proposition 4.22. *Let $A \subset C(G)$ be a right coideal and take an eigenvector $\boldsymbol{c} = {}^T\Lambda$ for its multiplicity map. If there exists an index $j \in I$ with $c_j = 1$, then the reduced ergodic system $e_j(A \otimes \mathbb{K}(L^2(G)))e_j$ is G-equivariantly embedded into $C(G)$. In particular, it becomes a right coideal of $C(G)$.*

Proof. Let p_0 be a minimal projection in $\mathbb{K}(L^2(G))$ associated to the trivial representation. Since the ranks of e_j and p_0 in $C(G) \rtimes_\delta G \cong \mathbb{K}(L^2(G))$ are equal, we have a partial isometry v in $C(G) \rtimes_\delta G$ such that it satisfies $v^*v = e_j$ and $vv^* = p_0$. Note that v belongs to $C(G) \rtimes_\delta G = (C(G) \otimes \mathbb{K}(L^2(G)))^G$. Then we have a desired G-equivariant $*$-homomorphism $e_j(A \otimes \mathbb{K}(L^2(G)))e_j \to C(G) \otimes \mathbb{C} p_0$ defined by $a \mapsto vav^*$. □

Remark 4.23. Let $\{A, G, \alpha\}$ and $\{B, G, \beta\}$ be two ergodic systems and $\theta : A \to B$ be a G-equivariant $*$-homomorphism. Then it must be faithful as we see below. Let φ_A and φ_B be

the unique faithful invariant states. Since θ is G-equivariant, $\varphi_B \circ \theta$ is an invariant state. By its uniqueness we obtain $\varphi_A = \varphi_B \circ \theta$. Hence θ is faithful.

5. Elementary results for $SU_q(2)$

In this section, we summarize basic facts for $SU_q(2)$ for readers' convenience. Readers are referred to, for example, [13,22] for its basic theory and [6,11,12,15–17] for the results on quantized universal enveloping algebras, the quantum spheres and quantum subgroups in $SU_q(2)$. We adopt the same notation as [13] in this paper. We treat a real number q in $[-1,1] \setminus \{0\}$.

5.1. $SU_q(2)$

The smooth function algebra $A(SU_q(2))$ is the universal $*$-algebra generated by four elements x, u, v and y with the following relations:

$$ux = qxu, \qquad vx = qxv, \qquad yu = quy, \qquad yv = qvy,$$
$$uv = vu, \qquad xy - q^{-1}uv = yx - quv = 1, \qquad x^* = y, \qquad u^* = -q^{-1}v.$$

Its universal C^*-algebra is denoted by $C(SU_q(2))$. We often use a positive operator $\zeta = -q^{-1}uv$. We make $A(SU_q(2))$ a Hopf $*$-algebra by defining coproduct δ, counit ε and antipode κ as follows:

$$\begin{pmatrix} \delta(x) & \delta(u) \\ \delta(v) & \delta(y) \end{pmatrix} = \begin{pmatrix} x \otimes 1 & u \otimes 1 \\ v \otimes 1 & y \otimes 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \otimes x & 1 \otimes u \\ 1 \otimes v & 1 \otimes y \end{pmatrix},$$
$$\begin{pmatrix} \varepsilon(x) & \varepsilon(u) \\ \varepsilon(v) & \varepsilon(y) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} \kappa(x) & \kappa(u) \\ \kappa(v) & \kappa(y) \end{pmatrix} = \begin{pmatrix} y & -qu \\ -q^{-1}v & x \end{pmatrix}.$$

Then the pair $SU_q(2) = (C(SU_q(2)), \delta)$ becomes a compact quantum group and it is often called the *twisted $SU(2)$ group*. The maps δ and ε are extended to $C(SU_q(2))$ norm continuously and the map κ is extended to $C(SU_q(2))$ as a closed operator. The Woronowicz characters $\{f_z\}_{z\in\mathbb{C}}$ are given by

$$\begin{pmatrix} f_z(x) & f_z(u) \\ f_z(v) & f_z(y) \end{pmatrix} = \begin{pmatrix} |q|^z & 0 \\ 0 & |q|^{-z} \end{pmatrix} \quad \text{for all } z \in \mathbb{C}.$$

5.2. Irreducible representations

The equivalence classes of irreducible representations $\widehat{SU_q(2)}$ are indexed by spin numbers $\nu \in \frac{1}{2}\mathbb{Z}_{\geqslant 0}$ and we fix a selection of irreducible representations $\{w(\pi_\nu)\}_\nu$ corresponding to spins ν as follows. The representation space $H_\nu = H_{w(\pi_\nu)}$ is $(2\nu+1)$-dimensional and fix the orthonormal basis $\{\xi_r^\nu\}_{r\in I_\nu}$ where the index set I_ν is $\{-\nu, -\nu+1, \dots, \nu-1, \nu\}$. Then we define the matrix $w(\pi_\nu) \in \mathbb{B}(H_\nu) \otimes A(SU_q(2))$ by setting $w(\pi_\nu)_{i,j}$ as follows.

(1) Case $i+j \leqslant 0, i \geqslant j$:

$$x^{-i-j} v^{i-j} q^{(\nu+j)(j-i)} \begin{bmatrix} \nu+i \\ i-j \end{bmatrix}_{q^2}^{\frac{1}{2}} \begin{bmatrix} \nu-j \\ i-j \end{bmatrix}_{q^2}^{\frac{1}{2}} P_{\nu+j}^{(i-j,-i-j)}\big(\zeta; q^2\big),$$

(2) Case $i+j \leqslant 0, i \leqslant j$:

$$x^{-i-j} u^{j-i} q^{(\nu+i)(i-j)} \begin{bmatrix} \nu-i \\ j-i \end{bmatrix}_{q^2}^{\frac{1}{2}} \begin{bmatrix} \nu+j \\ j-i \end{bmatrix}_{q^2}^{\frac{1}{2}} P_{\nu+i}^{(j-i,-i-j)}\big(\zeta; q^2\big),$$

(3) Case $i+j \geqslant 0, i \leqslant j$:

$$q^{(j-i)(j-\nu)} \begin{bmatrix} \nu-i \\ j-i \end{bmatrix}_{q^2}^{\frac{1}{2}} \begin{bmatrix} \nu+j \\ j-i \end{bmatrix}_{q^2}^{\frac{1}{2}} P_{\nu-j}^{(j-i,i+j)}\big(\zeta; q^2\big) u^{j-i} y^{i+j},$$

(4) Case $i+j \geqslant 0, i \geqslant j$:

$$q^{(i-j)(i-\nu)} \begin{bmatrix} \nu+i \\ i-j \end{bmatrix}_{q^2}^{\frac{1}{2}} \begin{bmatrix} \nu-j \\ i-j \end{bmatrix}_{q^2}^{\frac{1}{2}} P_{\nu-i}^{(i-j,i+j)}\big(\zeta; q^2\big) v^{i-j} y^{i+j},$$

where we have used the *q-binomial coefficients* and the *little q-Jacobi polynomials*:

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{(q;q)_m}{(q;q)_n (q;q)_{m-n}}, \quad (t;q)_m = \prod_{s=0}^{m-1} \big(1 - tq^s\big),$$

$$P_n^{(\alpha,\beta)}(z;q) = \sum_{r \geqslant 0} \frac{(q^{-n};q)_r (q^{\alpha+\beta+n+1};q)_r}{(q;q)_r (q^{\alpha+1};q)_r} (qz)^r.$$

This yields the following formula:

$$w(\pi_\nu)_{i,j}^* = (-q)^{i-j} w(\pi_\nu)_{-i,-j}$$

for all $\pi_\nu \in \widehat{SU_q(2)}$ and $i, j \in I_\nu$. For an integer n, we define the *q-integer* and its factorial by

$$(n)_q = \frac{|q|^n - |q|^{-n}}{|q| - |q|^{-1}}, \qquad (n)_q! = (n)_q (n-1)_q \cdots (1)_q.$$

We summarize useful formulae as follows:

(1) $f_z(w(\pi_\nu)_{r,s}) = \delta_{r,s} |q|^{-2rz}$,
(2) $\sigma_t^h(w(\pi_\nu)_{r,s}) = |q|^{-2(r+s)it} w(\pi_\nu)_{r,s}$,
(3) $\tau_t(w(\pi_\nu)_{r,s}) = |q|^{-2(r-s)it} w(\pi_\nu)_{r,s}$,
(4) $R(w(\pi_\nu)_{r,s}) = (-1)^{r-s} w(\pi_\nu)_{-s,-r}$,

for all $z \in \mathbb{C}$, $t \in \mathbb{R}$, $\nu \in \frac{1}{2}\mathbb{Z}_{\geqslant 0}$ and $r, s \in I_\nu$. The F-matrix $F_{\pi_\nu} = (\mathrm{id} \otimes f_1)(w(\pi_\nu))$ is equal to $\mathrm{diag}(q^{2\nu}, q^{2\nu-2}, \dots, q^{-(2\nu-2)}, q^{-2\nu})$. Hence we obtain $D_{\pi_\nu} = (2\nu+1)_q$. In the case of spin $\nu = \frac{1}{2}$, we have the following matrices:

$$\begin{pmatrix} \sigma_t^h(x) & \sigma_t^h(u) \\ \sigma_t^h(v) & \sigma_t^h(y) \end{pmatrix} = \begin{pmatrix} |q|^{2it}x & u \\ v & |q|^{-2it}y \end{pmatrix},$$

$$\begin{pmatrix} \tau_t(x) & \tau_t(u) \\ \tau_t(v) & \tau_t(y) \end{pmatrix} = \begin{pmatrix} x & |q|^{2it}u \\ |q|^{-2it}v & y \end{pmatrix},$$

$$\begin{pmatrix} R(x) & R(u) \\ R(v) & R(y) \end{pmatrix} = \begin{pmatrix} y & -u \\ -v & x \end{pmatrix}.$$

On the calculation about the Haar state, we have

$$h(\zeta^n) = \frac{1-q^2}{1-q^{2(n+1)}} \quad \text{for all } n \in \mathbb{Z}_{\geqslant 0}.$$

5.3. Quantum subgroups of $SU_q(2)$

We recall the embedding of $\mathbb{T}$ into $SU_q(2)$ for $-1 \leqslant q < 1$. The torus group $\mathbb{T}$ is identified with the set $\{z \in \mathbb{C} \mid |z| = 1\}$ and its cyclic subgroup $\mathbb{T}_m$ ($m \geqslant 2$) is generated by $\exp(\frac{2\pi\sqrt{-1}}{m})$. For $-1 \leqslant q < 1$ if $\theta : C(SU_q(2)) \to C(\mathbb{T})$ is a restriction map, then it must be as follows:

$$\begin{pmatrix} \theta(x) & \theta(u) \\ \theta(v) & \theta(y) \end{pmatrix} = \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \bar{z} & 0 \\ 0 & z \end{pmatrix},$$

where $z \in \mathbb{T} \subset \mathbb{C}$ is a usual coordinating map. Let $\pi_{\mathbb{T}}$ and $\pi'_{\mathbb{T}}$ be the first and second restriction maps in the above equalities, respectively. The group $\mathbb{T}$ is called the *maximal torus subgroup* of $SU_q(2)$ and we always treat it with the restriction map $\pi_{\mathbb{T}}$. Note that quotient space $\mathbb{T} \setminus SU_q(2)$ or $\mathbb{T}_m \setminus SU_q(2)$ do not depend on the choice of $\pi_{\mathbb{T}}$ or $\pi'_{\mathbb{T}}$. The quotient space $C(\mathbb{T} \setminus SU_q(2))$ is often called the *canonical homogeneous sphere*, which is generated by $\{w(\pi_1)_{0,r}\}_{r \in I_1}$. The quotient space $C(\mathbb{T}_2 \setminus SU_q(2))$ is denoted by $C(SO_q(3))$ which is also a compact quantum group by restricting the coproduct δ. We can easily see $C(SO_q(3))$ has the spectral pattern: $\bigoplus_{k \in \mathbb{Z}_{\geqslant 0}} (2k+1)\pi_k$ and hence it does not depend on positivity or negativity of the parameter q. In fact, we have an isomorphism $\Xi_q : C(SO_q(3)) \to C(SO_{-q}(3))$ as compact quantum groups defined by the following equality, where 2 by 2 matrices $\left(\begin{smallmatrix} x & u \\ v & y \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ are fundamental representations of $SU_q(2)$ and $SU_{-q}(2)$, respectively:

$$\begin{aligned} &\begin{pmatrix} \Xi_q(x^2) & \sqrt{1+q^2}\,\Xi_q(xu) & \Xi_q(u^2) \\ \sqrt{1+q^2}\,\Xi_q(xv) & \Xi_q(1+(q+q^{-1})uv) & \sqrt{1+q^2}\,\Xi_q(uy) \\ \Xi_q(v^2) & \sqrt{1+q^2}\,\Xi_q(vy) & \Xi_q(y^2) \end{pmatrix} \\ &\quad = \begin{pmatrix} a^2 & -i\sqrt{1+q^2}ab & b^2 \\ i\sqrt{1+q^2}ac & 1-(q+q^{-1})bc & i\sqrt{1+q^2}bd \\ c^2 & -i\sqrt{1+q^2}cd & d^2 \end{pmatrix}. \end{aligned}$$

With this isomorphism, we often identify $SO_q(3)$ with $SO_{-q}(3)$.

Now we recall classical results on closed subgroups of $SU(2)$. Let us use the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

They give an orthonormal basis for a three-dimensional real vector space $\mathbb{R}^3 = \mathbb{R}\sigma_1 + \mathbb{R}\sigma_2 + \mathbb{R}\sigma_3$. We define a double covering $\pi_1 : SU(2) \to SO(3)$ by the adjoint action $\pi_1(g)(a) = gag^{-1}$ for all $g \in SU(2)$ and $a \in \mathbb{R}^3$. For a subgroup $H \subset SO(3)$ we write H^* for its inverse image by π_1 and such a subgroup is called a *binary subgroup*. A symmetric group and alternative groups A_4, S_4 and A_5 are embedded into $SO(3)$ as the *tetrahedral group*, the *octahedral group* and the *icosahedral group*, respectively. The group D_m $(2 \leqslant m \leqslant \infty)$ is a *dihedral group*, and $\mathbb{T}$ and $\mathbb{T}_m$ $(2 \leqslant m)$ are the ordinary torus and the cyclic groups of order m. In $SU(2)$ all the closed subgroups are conjugate to one of the *trivial group* 1, the *cyclic group* $\mathbb{T}_n$, $SU(2)$, the maximal torus $\mathbb{T}$, the *binary dihedral group* D_m^* $(2 \leqslant m \leqslant \infty)$, the *binary tetrahedral group* A_4^*, the *binary octahedral group* S_4^* and the *binary icosahedral group* A_5^*. Since we need explicit embedding of $\mathbb{T}_n$ and D_n in the final section, we prepare a few notations. Let us define rotation matrices

$$r^{12}(\theta) = \begin{pmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{pmatrix}, \qquad r^{23}(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix},$$
$$r^{13}(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix},$$

where $\pi_1(r^{ij}(\theta))$ gives the rotation of angle θ in σ_i–σ_j plane. The cyclic group $\mathbb{T}_n$ is specified by two angles χ and ψ and denoted by $\mathbb{T}_n^{\chi,\psi}$. It consists of rotations of the angle $\frac{2\pi}{n}$ around the axis; $-\cos\chi\sin\psi\sigma_1 - \sin\chi\sin\psi\sigma_2 + \cos\psi\sigma_3$. The dihedral group D_n is specified by three angles ϕ, χ and ψ and denoted by $D_n^{\phi,\chi,\psi}$. It is generated by $\mathbb{T}_n^{\chi,\psi}$ and the rotation of the angle π around the axis; $\cos\phi\cos\chi\cos\psi\sigma_1 + \sin\phi\sin\chi\sigma_2 + \cos\phi\sin\psi\sigma_3$. By definition we have $\mathbb{T}_n^{\chi,\psi} = \mathrm{Ad}(\pi_1(r^{12}(\chi)r^{13}(\psi)))(\mathbb{T}_n^{0,0})$ and $D_n^{\phi,\chi,\psi} = \mathrm{Ad}(\pi_1(r^{12}(\chi)r^{13}(\psi)r^{12}(\phi)))(D_n^{0,0,0})$.

In [17] he has classified all the quantum subgroups of $SU_q(2)$ for $-1 \leqslant q < 1$. We summarize it for readers' convenience and analysis on $SU_{-1}(2)$ later.

(1) $0 < |q| < 1$ *case*. Its quantum subgroup is one of 1, $\mathbb{T}_m$ $(m \geqslant 2)$, $\mathbb{T}$, and $SU_q(2)$. For $\mathbb{T}_m$ $(m \geqslant 2)$ and $\mathbb{T}$, their restriction map is $\pi_{\mathbb{T}}$ or $\pi'_{\mathbb{T}}$.

(2) $q = -1$ *case*. For $g = \begin{pmatrix} \alpha & -\bar{\gamma} \\ \gamma & \bar{\alpha} \end{pmatrix} \in SU(2)$, define the $*$-homomorphism $\upsilon_g : C(SU_{-1}(2)) \to \mathbb{B}(\mathbb{C}^2)$ by

$$\begin{pmatrix} \upsilon_g(x) & \upsilon_g(u) \\ \upsilon_g(v) & \upsilon_g(y) \end{pmatrix} = \begin{pmatrix} \alpha\sigma_1 & \bar{\gamma}\sigma_2 \\ \gamma\sigma_2 & \bar{\alpha}\sigma_1 \end{pmatrix}.$$

Now we define the irreducible representation τ_g of $C(SU_{-1}(2))$ as a C^*-algebra. They give all irreducible representations of $C(SU_{-1}(2))$.

(i) $\alpha, \gamma \neq 0$ *case*. $\tau_g = \upsilon_g$.

(ii) $\alpha = 0$ *case*. $\tau_g : C(SU_{-1}(2)) \to \mathbb{C}$ is

$$\begin{pmatrix} \tau_g(x) & \tau_g(u) \\ \tau_g(v) & \tau_g(y) \end{pmatrix} = \begin{pmatrix} 0 & \bar{\gamma} \\ \gamma & 0 \end{pmatrix}.$$

(iii) $\gamma = 0$ *case.* $\tau_g : C(SU_{-1}(2)) \to \mathbb{C}$ is

$$\begin{pmatrix} \tau_g(x) & \tau_g(u) \\ \tau_g(v) & \tau_g(y) \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix}.$$

For a compact subset $Z \subset SU(2)$ we consider the direct sum representation $\pi_Z = \bigoplus_{g \in Z} \pi_g$. In [17, Proposition 2.4], it is characterized when $\pi_Z : C(SU_{-1}(2)) \to \pi_Z(C(SU_{-1}(2)))$ gives a restriction map to a quantum subgroup. We write $C(G_Z) = \pi_Z(C(SU_{-1}(2)))$. Before a summary we study the embedding $\mathbb{T}_n \subset SU_{-1}(2)$ a little. For $a \in C(SU_{-1}(2))$ and $g \in \mathbb{T}_n$ we write simply $a(g)$ for $r(a)(g)$ where r is a restriction map. Let g be a generator of $\mathbb{T}_n$. Since we have $xv = -vx$, $x(g) = 0$ or $v(g) = 0$. We consider $v(g) \neq 0$. By the definition of the restriction map we have $\left(\begin{smallmatrix} x(g^2) & u(g^2) \\ v(g^2) & y(g^2) \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. This shows n must be 2. We denote the restriction by π_{D_1}. Hence if $n \geqslant 3$, the embedding of $\mathbb{T}_n$ is unique and if $n = 2$, two embedding arises. Since in this paper notations $\mathbb{T}$ and $\mathbb{T}_n$ are used for the maximal torus and its cyclic subgroups, we prepare the notation D_1 for a subgroup of order 2 which is embedded by π_{D_1}. Of course subgroups D_1 and $\mathbb{T}_2$ are isomorphic, however, right coideals by them are not isomorphic.

(a) $C(G_Z)$ is an abelian C^*-algebra case (that is, G_Z is an ordinary group), then G_Z is (isomorphic to) one of trivial group 1, the maximal torus $\mathbb{T}$, the cyclic groups $\mathbb{T}_n$ ($\subset \mathbb{T}$) ($n \geqslant 2$), D_1 and dihedral groups D_n ($2 \leqslant n \leqslant \infty$) containing $\mathbb{T}_n$. (This isomorphism gives the conjugation β_z^L on right coideals by them.) About D_n ($1 \leqslant n \leqslant \infty$) its corresponding subset Z_{D_n} is $\{r^{12}(\frac{2\pi k}{n}), r^{12}(\frac{2\pi k}{n}) r^{23}(\pi) \mid 0 \leqslant k \leqslant n-1\}$. Note that Z_{D_n} is not a subgroup in $SU(2)$ if odd n. If G_Z is not $\mathbb{T}_n$ or D_n (odd n), Z is a binary subgroup and it shows the spectral pattern of the right coideal $C(G_Z \setminus SU_{-1}(2))$ consists of integer spins. If G_Z is, then it also has half-integer spin parts.

(b) If $C(G_Z)$ is a non-abelian C^*-algebra, Z is one of binary subgroups whose image by π_1 is listed in [17, p. 11]. Note that it depends on the embedding of a closed subgroup into $SO(3)$. Then it is known that $C(G_Z \setminus SU_{-1}(2)) \subset C(SO_{-1}(3))$ and $\Xi_{-1}(C(G_Z \setminus SU_{-1}(2))) = C(H \setminus SO_1(3))$.

5.4. $U_q(su_2)$

The quantum universal enveloping algebra $U_q(su_2)$ is generated by four elements k, k^{-1}, e and f which satisfy the following relations:

$$kk^{-1} = 1 = k^{-1}k,$$
$$kek^{-1} = qe, \qquad kfk^{-1} = q^{-1}f,$$
$$ef - fe = \frac{k^2 - k^{-2}}{q - q^{-1}}.$$

$U_q(su_2)$ is realized as a Hopf $*$-subalgebra of the algebraic functional space $A(SU_q(2))^*$ as follows.

(1) $q > 0$ *case.*

$$k^{\pm 1}\big(w(\pi_\nu)_{r,s}\big) = q^{\mp r} \delta_{r,s},$$
$$e\big(w(\pi_\nu)_{r,s}\big) = \delta_{r+1,s} \sqrt{(\nu + s)_q (\nu - s + 1)_q},$$

$$f\big(w(\pi_\nu)_{r,s}\big)=\delta_{r-1,s}\sqrt{(\nu-s)_q(\nu+s+1)_q}.$$

(2) $q<0$ *case.*

$$\begin{aligned}k^{\pm1}\big(w(\pi_\nu)_{r,s}\big)&=q^{\mp r}\delta_{r,s},\\ e\big(w(\pi_\nu)_{r,s}\big)&=\delta_{r+1,s}\sqrt{-1}^{-2\nu+1}\sqrt{(\nu+s)_q(\nu-s+1)_q},\\ f\big(w(\pi_\nu)_{r,s}\big)&=\delta_{r-1,s}\sqrt{-1}^{-2\nu+1}\sqrt{(\nu-s)_q(\nu+s+1)_q},\end{aligned}$$

where $q^n=\sqrt{-1}^{2n}(-q)^n$ for a half-integer n. The Hopf $*$-algebra structure of $U_q(su_2)$ is given by

(1) $k^*=k$, $e^*=f$, $f^*=e$,
(2) $\hat\delta(k)=k\otimes k$, $\hat\delta(e)=e\otimes k+k^{-1}\otimes e$, $\hat\delta(f)=f\otimes k+k^{-1}\otimes f$,
(3) $\hat\kappa(k)=k^{-1}$, $\hat\kappa(e)=-qe$, $\hat\kappa(f)=-q^{-1}f$.

Hence $U_q(su_2)$ is a Hopf $*$-subalgebra of $A(SU_q(2))$. For a smooth corepresentation of $A(SU_q(2))$, $\alpha:K\to K\otimes A(SU_q(2))$, we prepare $U_q(su_2)$-module structure on K by $\theta\cdot\xi=(\mathrm{id}\otimes\theta)(w\xi)$ for all $\theta\in U_q(su_2)$ and $\xi\in K$. This representation of $U_q(su_2)$ is called a differential representation of a corepresentation (α,K). Consider irreducible representation $(w(\pi_\nu),H_\nu)$ and we have the following formulae about its $U_q(su_2)$-module structure.

(1) $q>0$ *case.*

$$\begin{aligned}k^{\pm1}\cdot\xi_r^\nu&=q^{\mp r}\xi_r^\nu,\\ e\cdot\xi_r^\nu&=\sqrt{(\nu+r)_q(\nu-r+1)_q}\,\xi_{r-1}^\nu,\\ f\cdot\xi_r^\nu&=\sqrt{(\nu-r)_q(\nu+r+1)_q}\,\xi_{r+1}^\nu.\end{aligned}$$

(2) $q<0$ *case.*

$$\begin{aligned}k^{\pm1}\cdot\xi_r^\nu&=q^{\mp r}\xi_r^\nu,\\ e\cdot\xi_r^\nu&=\sqrt{-1}^{-2\nu+1}\sqrt{(\nu+r)_q(\nu-r+1)_q}\,\xi_{r-1}^\nu,\\ f\cdot\xi_r^\nu&=\sqrt{-1}^{-2\nu+1}\sqrt{(\nu-r)_q(\nu+r+1)_q}\,\xi_{r+1}^\nu\end{aligned}$$

for all $r\in I_\nu$. For a half-integer n a vector $\xi\in K$ is called a *highest weight vector* of *weight* n if it satisfies $k\cdot\xi=q^n\xi$ and $e\cdot\xi=0$. For example, the vector $\xi_{-\nu}^\nu\in H_\nu$ is a highest weight vector of weight ν. It is well known that a tensor product $U_q(su_2)$-module $H_\mu\otimes H_\nu$ is isomorphic to the direct sum $U_q(su_2)$-module $\bigoplus_{|\mu-\nu|\leqslant\ell\leqslant\mu+\nu}H_\ell$, where ℓ runs through half-integers. If we want to make a highest weight vector of H_ℓ from those of H_μ and H_ν, the following well-known lemmas are useful.

Lemma 5.1. *For positive q, consider the tensor product $U_q(su_2)$-module $H_\mu \otimes H_\nu$. Let ℓ be a half-integer with $|\mu - \nu| \leqslant \ell \leqslant \mu + \nu$. Define the coefficients $(C^\ell_{\mu,\nu})_r$ for $0 \leqslant r \leqslant \mu + \nu - \ell$ by*

$$\left(C^\ell_{\mu,\nu}\right)_r = q^{-\frac{1}{2}(\ell+1)(\mu+\nu-\ell)}\left(-q^{\ell+1}\right)^r \prod_{t=1}^{r} \sqrt{\frac{(\mu+\nu-\ell+1-t)_q(\mu-\nu+\ell+t)_q}{(t)_q(2\nu-t+1)_q}}.$$

Then a vector $\eta^\ell = \sum_{r=0}^{\mu+\nu-\ell} (C^\ell_{\mu,\nu})_r \xi^\mu_{-\ell+\nu-r} \otimes \xi^\nu_{-\nu+r}$ is a highest weight vector of weight ℓ in $H_\mu \otimes H_\nu$.

Proof. The action of $U_q(su_2)$ on $H_\mu \otimes H_\nu$ is given via coproduct. With this, we can easily justify $k \cdot \eta^\ell = q^{-\ell}\eta^\ell$ and $e \cdot \eta^\ell = 0$. □

Lemma 5.2. *For negative q, consider the tensor product $U_q(su_2)$-module $H_\mu \otimes H_\nu$. Let ℓ be a half-integer with $|\mu - \nu| \leqslant \ell \leqslant \mu + \nu$ and ξ^μ and ξ^ν be vectors of copy of π_μ and π_ν, respectively. Define the coefficients $(C^\ell_{\mu,\nu})_r$ for $0 \leqslant r \leqslant \mu + \nu - \ell$ by*

$$\left(C^\ell_{\mu,\nu}\right)_r = q_0^{-\frac{1}{2}(\ell+1)(\mu+\nu-\ell)}(-1)^{(-\mu+\nu+\ell)r} q_0^{r(\ell+1)} \prod_{t=1}^{r} \sqrt{\frac{(\mu+\nu-\ell+1-t)_q(\mu-\nu+\ell+t)_q}{(t)_q(2\nu+1-t)_q}}.$$

Then a vector $\eta^\ell = \sum_{r=0}^{\mu+\nu-\ell} (C^\ell_{\mu,\nu})_r \xi^\mu_{-\ell+\nu-r} \otimes \xi^\nu_{-\nu+r}$ is a highest weight vector of weight ℓ in $H_\mu \otimes H_\nu$.

5.5. Eigenvectors and their products

Let Y_μ be a linear space of π_μ-eigenvectors of $C(SU_q(2))$. We prepare the notation of eigenvectors $\boldsymbol{w}^\mu_r := (w(\pi_\mu)_{r,t})_{t\in I_\mu}$ for $r \in I_\mu$. They give an orthonormal basis of Y_μ. Let us consider a covariant system $(A, SU_q(2), \alpha)$. For its eigenvector spaces $\{X_\nu\}_{\nu\in\frac{1}{2}\mathbb{Z}_{\geqslant 0}}$ we define the *product of eigenvectors* $\Psi_\ell : X_\mu \times X_\nu \to X_\ell$ by using Lemma 5.1:

$$\Psi_\ell\left(\xi^\mu, \xi^\nu\right)_{-\ell} = \sum_{r=0}^{\mu+\nu-\ell} \left(C^\ell_{\mu,\nu}\right)_r \xi^\mu_{-\ell+\nu-r}\xi^\nu_{-\nu+r}$$

for all eigenvectors ξ^μ and ξ^ν, where the coefficients $(C^\ell_{\mu,\nu})_r$ are given in Lemmas 5.1 and 5.2.

5.6. Quantum spheres

In [16], ergodic systems $\{A, SU_q(2), \alpha\}$ with $\dim A_{\pi_1} = 1$ and $A = C^*(A_{\pi_1})$ are classified for $|q| < 1$. They are $SU_q(2)$-isomorphic to one of the *quantum spheres*. We summarize his classification. Let X_{π_1} be the π_1-eigenvector space. Since this is one-dimensional, we can take the (unique) π_1-eigenvector ξ with the following properties,

$$T\xi_\lambda = \xi_\lambda, \qquad (\xi_\lambda, \xi_\lambda) = 1, \qquad \Psi_1(\xi_\lambda, \xi_\lambda) = \lambda\xi_\lambda,$$

where λ is a non-negative real constant. From the second assumption the C^*-algebra A is generated by $(\xi_\lambda)_{-1}$, $(\xi_\lambda)_0$ and $(\xi_\lambda)_1$. Write λ_0 for $(q^{-1}-q)^{-1}\lambda$. For $n \in \mathbb{Z}_{\geqslant 1}$ we define a positive number $c(n) = \frac{q^{n+1}+q^{-n-1}}{\sqrt{(n)_q(n+2)_q}}$. Then the classification is done as follows.

Case 1. If $\lambda_0 > 1$, then there exists $n \in \mathbb{Z}_{\geqslant 1}$ such that $\lambda = c(n)$ and we obtain a G-isomorphism $A \cong \mathrm{End}(H_{\pi_{\frac{n}{2}}})$. In this case the spectral pattern of A is $\pi_0 \oplus \pi_1 \oplus \cdots \oplus \pi_n$.

Case 2. If $0 \leqslant \lambda_0 \leqslant 1$ and $q > 0$, then the map $\xi_\lambda = ((\xi_\lambda)_{-1}, (\xi_\lambda)_0, (\xi_\lambda)_1) \mapsto (q^{\frac{1}{2}}\sqrt{\frac{1-\lambda_0^2}{(2)_q}}, \lambda_0, -q^{-\frac{1}{2}}\sqrt{\frac{1-\lambda_0^2}{(2)_q}}) \cdot w(\pi_1)$ gives a G-equivariant embedding $A \hookrightarrow C(SU_q(2))$. In this case the spectral pattern of A is $\bigoplus_{\ell \in \mathbb{Z}_{\geqslant 0}} \pi_\ell$.

Case 3. If $0 \leqslant \lambda_0 \leqslant 1$ and $q < 0$, then the map $\xi_\lambda = ((\xi_\lambda)_{-1}, (\xi_\lambda)_0, (\xi_\lambda)_1) \mapsto ((-q)^{\frac{1}{2}}\sqrt{\frac{1-\lambda_0^2}{(2)_q}}, \lambda_0, (-q)^{-\frac{1}{2}}\sqrt{\frac{1-\lambda_0^2}{(2)_q}}) \cdot w(\pi_1)$ gives a G-equivariant embedding $A \hookrightarrow C(SU_q(2))$. In this case the spectral pattern of A is $\bigoplus_{\ell \in \mathbb{Z}_{\geqslant 0}} \pi_\ell$.

In this paper, we use the notation $C(S^2_{q,\lambda})$ for the right coideal which is defined in the above cases 2 and 3. The quantum sphere $C(S^2_{q,\lambda})$ is considered as a q-deformation of $C(\mathbb{T}^{0,\psi} \setminus SO(3))$ with the parameter $\lambda_0 = \cos\psi$ (see Lemma 8.2). If $\lambda_0 = 1$, $C(S_{q,(q^{-1}-q)^{-1}})$ becomes the canonical homogeneous sphere $C(\mathbb{T} \setminus SU_q(2))$. Let β_z^L be a left action of the maximal torus on $C(SU_q(2))$ defined by $(\mathrm{ev}_z \circ \pi_{\mathbb{T}} \otimes \mathrm{id}) \circ \delta$ for all $z \in \mathbb{T}$. It satisfies $(\beta_z^L \otimes \mathrm{id}) \circ \delta = \delta \circ \beta_z^L$, that is, β_z^L gives an $SU_q(2)$-isomorphism. Note that all embeddings of quantum spheres into $C(SU_q(2))$ are obtained by rotations of β^L for the above given embedding. Similarly we define the right action β^R of the maximal torus by $\beta^R = (\mathrm{id} \otimes \pi_{\mathbb{T}}) \circ \delta$.

5.7. Connected graphs and McKay diagrams

We show the list of all the connected graphs of norm 2 in Appendix A. For their classification, readers are referred to [9, Lemma 1.4.1]. The labels except for A'_m correspond to closed subgroups of $SU(2)$ or $SU_{-1}(2)$ by their McKay diagrams about fundamental representation $\pi_{\frac{1}{2}}$ or multiplicity diagrams of the right coideals obtained by quotients. We will see later that type A'_m does not occur even if we investigate right coideals.

Remark 5.3. We give a simple remark on the absence of A'_m $(3 \leqslant m \leqslant \infty)$ for a special case. Let $C(G)$ be a compact quantum group which has an irreducible unitary representation w generating $R(G)$ (that is, $C(G)$ is a compact matrix pseudogroup [23]). We consider a compact quantum subgroup $C(H)$ whose McKay diagram about w is of type A'_m $(m \geqslant 3)$. We simply denote the restriction of w onto H by $w|_H$. Now vertices in A'_m corresponds to the irreducible representations of H and let $\rho_0, \rho_1, \ldots$ be the irreducible representations from the left-hand side in Figs. 14, 15. Since the entries of the Perron–Frobenius eigenvector are equal to the dimensions of the corresponding irreducible modules, $\widehat{H}$ must be a group. By definition of the McKay diagram, $w|_H \cdot \rho_0 = \rho_0 + \rho_1$ holds. Hence we have $w|_H = 0 + \rho_1\rho_0^{-1}$ where 0 is the trivial representation. Then it yields $w|_H \cdot \rho_1 = \rho_1 + \rho_1\rho_0^{-1}\rho_1$, and this shows there must exist a single loop at ρ_1. Hence in this case, the McKay diagram of type A'_m $(m \geqslant 3)$ does not appear.

6. Classification of right coideals of $C(SU_q(2))$: $0 < q < 1$ case

In the case of $q = 1$, the main theorem of [21, p. 309] asserts that any ergodic system of $SU(2)$ is an induced system $\{\mathrm{End}(W), H, \beta\}$ where H is a closed subgroup of $SU(2)$ and W is a finite-dimensional H-module. A closed subgroup H is conjugate to one of $1, \mathbb{T}_n$ $(n \geqslant 2)$, $\mathbb{T}$, $SU(2)$, D_n^* $(n \geqslant 2)$, D_∞^*, A_4^*, S_4^* and A_5^* and the multiplicity diagrams of the induced systems from them are the Mckay diagrams (see Appendix A). However, in the case of $0 < q < 1$, there are a little quantum subgroups of $SU_q(2)$ as is proved in [17, Theorem 2.1]. So we are interested in the absence of some types of right coideals for $0 < q < 1$. Finally, we obtain the following classification result for right coideals.

Theorem 6.1. *Let $A \subset C(SU_q(2))$ be a right coideal. Then its multiplicity diagram is one of types $1, \mathbb{T}_n$ $(n \geqslant 2)$, $\mathbb{T}$, $SU(2)$ and D_∞^*. If it is of type $\mathbb{T}$, then it is one of the quantum spheres. Otherwise it is unique up to conjugation by β^L.*

Now we start the proof of Theorem 6.1. In cases (I)–(III), A_4^*, S_4^* and A_5^* types are rejected. In case (IV), we show that D_∞^* type survives and it is unique. In case (V), D_m^* $(m \geqslant 2)$ types are rejected. In case (VI), we show that $\mathbb{T}_m$ $(m \geqslant 2)$ types are quotient ones.

(I) A_4^* *case.* A has a spectral pattern $\pi_0 \oplus \pi_3 \oplus \pi_4 \oplus 2\pi_6 \oplus \pi_7 \oplus \cdots$. Like the discussion of [21, p. 321], we focus on spectral gaps: π_1, π_2 and π_5. We will soon notice the importance of using the both of even and odd spin. Let $\eta = (\eta_r)_{r\in I_3}$ be a self-conjugate π_3-eigenvector of A. Take scalars $\{c_r\}_{r\in I_3}$ such that $\eta_r = \sum_{s\in I_3} c_s w(\pi_3)_{s,r}$ for all $r \in I_3$. Applying Lemma 5.1, we obtain the following highest weight vectors:

$$\begin{aligned}
\Psi_5(\eta,\eta)_{-5} &= q^{-3}\eta_{-2}\eta_{-3} - q^{3}\eta_{-3}\eta_{-2},\\
\Psi_2(\eta,\eta)_{-2} &= q^{-6}\eta_1\eta_{-3} - q^{-3}\sqrt{\frac{(4)_q(3)_q}{(6)_q}}\eta_0\eta_{-2} + \frac{(4)_q(3)_q}{\sqrt{(6)_q(5)_q(2)_q}}\eta_{-1}\eta_{-1}\\
&\quad - q^{3}\sqrt{\frac{(4)_q(3)_q}{(6)_q}}\eta_{-2}\eta_0 + q^{6}\eta_{-3}\eta_1,\\
\Psi_1(\eta,\eta)_{-1} &= q^{-5}\eta_2\eta_{-3} - q^{-3}\sqrt{\frac{(5)_q(2)_q}{(6)_q}}\eta_1\eta_{-2} + q^{-1}\sqrt{\frac{(4)_q(3)_q}{(6)_q}}\eta_0\eta_{-1}\\
&\quad - q\sqrt{\frac{(4)_q(3)_q}{(6)_q}}\eta_{-1}\eta_0 + q^{3}\sqrt{\frac{(5)_q(2)_q}{(6)_q}}\eta_{-2}\eta_1 - q^{5}\eta_{-3}\eta_2.
\end{aligned}$$

They are actually 0 vectors because of the absence of spectra. Applying the lowering operator $f \in U_q(su_2)$ to these vectors, we obtain

$$\begin{aligned}
f\cdot\Psi_5(\eta,\eta)_{-5} &= \sqrt{(5)_q(2)_q}\eta_{-1}\eta_{-3} + \sqrt{(6)_q}\big(q^{-5} - q^{5}\big)\eta_{-2}^2 - \sqrt{(5)_q(2)_q}\eta_{-3}\eta_{-1},\\
f^2\cdot\Psi_5(\eta,\eta)_{-5} &= q^{3}\sqrt{(4)_q(3)_q}\eta_0\eta_{-3} + \big(q^{-3} + q^{-1} - q^{7}\big)\sqrt{(6)_q(5)_q(2)_q}\eta_{-1}\eta_{-2}\\
&\quad + \big(q^{-7} - q - q^{3}\big)\sqrt{(6)_q(5)_q(2)_q}\eta_{-2}\eta_{-1} + q^{-3}\sqrt{(4)_q(3)_q}\eta_{-3}\eta_0
\end{aligned}$$

and

$$\begin{aligned}\sqrt{(6)_q} f\cdot\Psi_1(\eta,\eta)_{-1} &= q^{-2}(6)_q\eta_3\eta_{-3}+\big(-q^{-1}(5)_q(2)_q+q^{-3}(6)_q\big)\eta_2\eta_{-2}\\ &\quad+\big((4)_q(3)_q-q^{-2}(5)_q(2)_q\big)\eta_1\eta_{-1}+\big(q^{-1}-q\big)(4)_q(3)_q\eta_0^2\\ &\quad+\big(q^2(5)_q(2)_q-(4)_q(3)_q\big)\eta_{-1}\eta_1+\big(q(5)_q(2)_q-q^3(6)_q\big)\eta_{-2}\eta_2\\ &\quad-q^2(6)_q\eta_{-3}\eta_3.\end{aligned}$$

Recall a restriction map $\pi_{\mathbb{T}}: C(SU_q(2))\to C(\mathbb{T})$. It sends $w(\pi_\nu)_{r,s}$ to $\delta_{r,s}z^{-2r}$ for $\nu\in\frac{1}{2}\mathbb{Z}_{\geqslant 0}$ and $r,s\in I_\nu$. Then we have $\pi_{\mathbb{T}}(\eta_r)=z^{-2r}c_r$ for $r\in I_3$. From $0=\pi_{\mathbb{T}}(f\cdot\Psi_5(\eta,\eta)_{-5})$, we obtain $c_{-2}=0$. Then from $0=\pi_{\mathbb{T}}(\Psi_1(\eta,\eta)_{-1})=\pi_{\mathbb{T}}(f^2\cdot\Psi_5(\eta,\eta)_{-5})$, we have $c_{-3}c_0=c_{-1}c_0=0$. If c_0 is not 0, then c_{-3} and c_{-1} are equal to 0. Hence $\{c_r\}_{r\in I_3}$ are all zero except for $r=0$. From $0=\pi_{\mathbb{T}}(f\cdot\Psi_1(\eta,\eta)_{-1})$ we have $c_0=0$, this is contradiction. Therefore c_0 must be 0. Then the last equality deduces to $(6)_q c_3c_{-3}=(5)_q(2)_q c_1c_{-1}$. Because of the self-conjugacy of η we have $c_{-3}=-q^3\overline{c_3}$ and $c_{-1}=-q\overline{c_1}$. Then we have

$$(6)_q|c_3|^2=q^{-2}(5)_q(2)_q|c_1|^2.$$

From the second one we get

$$\big(q^{-6}+q^6\big)c_1c_{-3}+\frac{(4)_q(3)_q}{\sqrt{(6)_q(5)_q(2)_q}}c_{-1}^2=0.$$

We can easily see that there does not exist a solution to the above two equalities except for the case $c_3=c_1=0$. Therefore we have rejected existence of right coideal of type A_4^*.

(II) S_4^* *case*. A has a spectral pattern $\pi_0\oplus\pi_4\oplus\pi_6\oplus\pi_8\oplus\pi_9\oplus\pi_{10}\oplus\cdots$. There are spectral gaps of π_ν for $\nu=1,2,3,5,7$. Let $\eta=(\eta_s)_{s\in I_4}=\sum_{s\in I_4}c_s\boldsymbol{w}_s^4$ be a non-zero self-conjugate π_4-eigenvector of A. We derive contradiction by showing the coefficients c_s are all zero. Let us consider eigenvectors

$$\Psi_7(\eta,\eta)_{-7}=q^{-4}\eta_{-3}\eta_{-4}-q^4\eta_{-4}\eta_{-3},$$

$$\begin{aligned}\Psi_5(\eta,\eta)_{-5}&=q^{-9}\eta_{-1}\eta_{-4}-q^{-3}\sqrt{\frac{(6)_q(3)_q}{(8)_q}}\eta_{-2}\eta_{-3}+q^3\sqrt{\frac{(6)_q(3)_q}{(8)_q}}\eta_{-3}\eta_{-2}\\ &\quad-q^9\eta_{-4}\eta_{-1},\end{aligned}$$

$$\begin{aligned}\Psi_3(\eta,\eta)_{-3}&=q^{-10}\eta_1\eta_{-4}-q^{-6}\sqrt{\frac{(5)_q(4)_q}{(8)_q}}\eta_0\eta_{-3}+q^{-2}\frac{(5)_q(4)_q}{\sqrt{(8)_q(7)_q(2)_q}}\eta_{-1}\eta_{-2}\\ &\quad-q^2\frac{(5)_q(4)_q}{\sqrt{(8)_q(7)_q(2)_q}}\eta_{-2}\eta_{-1}+q^6\sqrt{\frac{(5)_q(4)_q}{(8)_q}}\eta_{-3}\eta_0-q^{10}\eta_{-4}\eta_1,\end{aligned}$$

$$\Psi_2(\eta,\eta)_{-2}=q^{-9}\eta_2\eta_{-4}-q^{-6}\sqrt{\frac{(6)_q(3)_q}{(8)_q}}\eta_1\eta_{-3}$$

$$
\begin{aligned}
&+ q^{-3}\sqrt{\frac{(6)_q(5)_q(4)_q(3)_q}{(8)_q(7)_q(2)_q}}\eta_0\eta_{-2} - \frac{(5)_q(4)_q}{(8)_q(7)_q(2)_q}\eta_{-1}^2 \\
&+ q^{3}\sqrt{\frac{(6)_q(5)_q(4)_q(3)_q}{(8)_q(7)_q(2)_q}}\eta_{-2}\eta_0 - q^{6}\sqrt{\frac{(6)_q(3)_q}{(8)_q}}\eta_1\eta_{-3} + q^{9}\eta_{-4}\eta_2,
\end{aligned}
$$

$$
\begin{aligned}
\Psi_1(\eta,\eta)_{-1} = {} & q^{-7}\eta_3\eta_{-4} - q^{-5}\sqrt{\frac{(7)_q(2)_q}{(8)_q}}\eta_2\eta_{-3} + q^{-3}\sqrt{\frac{(6)_q(3)_q}{(8)_q}}\eta_1\eta_{-2} \\
& - q^{-1}\sqrt{\frac{(5)_q(4)_q}{(8)_q}}\eta_0\eta_{-1} + q\sqrt{\frac{(5)_q(4)_q}{(8)_q}}\eta_{-1}\eta_0 - q^{3}\sqrt{\frac{(6)_q(3)_q}{(8)_q}}\eta_{-2}\eta_1 \\
& + q^{5}\sqrt{\frac{(7)_q(2)_q}{(8)_q}}\eta_{-3}\eta_2 - q^{7}\eta_{-4}\eta_3.
\end{aligned}
$$

They are in fact 0 vectors by the gap at each spectrum. Applying lowering operator f to these vectors, we get

$$
\begin{aligned}
f\cdot\Psi_7(\eta,\eta)_{-7} = {} & \sqrt{(7)_q(2)_q}\,\eta_{-2}\eta_{-4} + \left(q^{-7}-q^{7}\right)\sqrt{(8)_q}\,\eta_{-3}^2 - \sqrt{(7)_q(2)_q}\,\eta_{-4}\eta_{-2}, \\
f\cdot\Psi_5(\eta,\eta)_{-5} = {} & q^{-5}\sqrt{(5)_q(4)_q}\,\eta_0\eta_{-4} + \frac{q^{-10}(8)_q-(6)_q(3)_q}{\sqrt{(8)_q}}\eta_{-1}\eta_{-3} \\
& + \left(q^{5}-q^{-5}\right)\sqrt{\frac{(7)_q(6)_q(3)_q(2)_q}{(8)_q}}\eta_{-2}\eta_{-2} \\
& + \frac{-q^{10}(8)_q+(6)_q(3)_q}{\sqrt{(8)_q}}\eta_{-3}\eta_{-1} - q^{5}\sqrt{(5)_q(4)_q}\,\eta_{-4}\eta_0, \\
f\cdot\Psi_3(\eta,\eta)_{-3} = {} & q^{-6}\sqrt{(6)_q(3)_q}\,\eta_2\eta_{-4} + \frac{q^{-9}\sqrt{(8)_q}-q^{-3}(5)_q(4)_q}{\sqrt{(8)_q}}\eta_1\eta_{-3} \\
& + \left((5)_q(4)_q - q^{-6}(7)_q(2)_q\right)\frac{(5)_q(4)_q}{(8)_q(7)_q(2)_q}\eta_0\eta_{-2} \\
& + \left(q^{-3}-q^{3}\right)(5)_q(4)_q\frac{(6)_q(3)_q}{(8)_q(7)_q(2)_q}\eta_{-1}\eta_{-1} \\
& + \left(q^{6}(7)_q(2)_q - (5)_q(4)_q\right)\frac{(5)_q(4)_q}{(8)_q(7)_q(2)_q}\eta_{-2}\eta_0 \\
& + \frac{q^{3}(5)_q(4)_q - q^{9}\sqrt{(8)_q}}{\sqrt{(8)_q}}\eta_{-3}\eta_1 - q^{6}\sqrt{(6)_q(3)_q}\,\eta_{-4}\eta_2, \\
f\cdot\Psi_1(\eta,\eta)_{-1} = {} & q^{-3}\sqrt{(8)_q}\,\eta_4\eta_{-4} + \frac{q^{-4}(8)_q - q^{-2}(7)_q(2)_q}{\sqrt{(8)_q}}\eta_3\eta_{-3}
\end{aligned}
$$

$$+\frac{q^{-1}(6)_q(3)_q - q^{-3}(7)_q(2)_q}{\sqrt{(8)_q}}\eta_2\eta_{-2} + \frac{q^{-2}(6)_q(3)_q - (5)_q(4)_q}{\sqrt{(8)_q}}\eta_1\eta_{-1}$$

$$+\frac{(q-q^{-1})(5)_q(4)_q}{\sqrt{(8)_q}}\eta_0\eta_0 + \frac{(5)_q(4)_q - q^2(6)_q(3)_q}{\sqrt{(8)_q}}\eta_{-1}\eta_1$$

$$+\frac{q^3(7)_q(2)_q - q(6)_q(3)_q}{\sqrt{(8)_q}}\eta_{-2}\eta_2 + \frac{q^2(7)_q(2)_q - q^4(8)_q}{\sqrt{(8)_q}}\eta_{-3}\eta_3$$

$$- q^3\sqrt{(8)_q}\,\eta_{-4}\eta_4.$$

Applying $\pi_{\mathbb{T}}$ to these vectors, we obtain the following equations:

$$c_{-3} = 0, \tag{6.1}$$

$$\bigl(q^{-5}-q^5\bigr)\sqrt{(5)_q(4)_q}\,c_0c_{-4} + \bigl(q^5-q^{-5}\bigr)\sqrt{\frac{(7)_q(6)_q(3)_q(2)_q}{(8)_q}}\,c_{-2}^2 = 0, \tag{6.2}$$

$$\bigl(q^{-6}-q^6\bigr)\sqrt{(6)_q(3)_q}\,c_2c_{-4} + \bigl(q^6-q^{-6}\bigr)\frac{(5)_q(4)_q}{(8)_q}c_0c_{-2}$$
$$+\bigl(q^{-3}-q^3\bigr)(5)_q(4)_q\frac{(6)_q(3)_q}{(8)_q(7)_q(2)_q}c_{-1}^2 = 0, \tag{6.3}$$

$$\bigl(q^{-3}-q^3\bigr)\sqrt{(8)_q}\,c_4c_{-4} + \frac{(q^{-1}-q)(6)_q(3)_q - (q^3-q^{-3})(7)_q(2)_q}{\sqrt{(8)_q}}c_2c_{-2}$$
$$+\frac{(q^{-2}-q^2)(6)_q(3)_q}{\sqrt{(8)_q}}c_1c_{-1} + \frac{(q-q^{-1})(5)_q(4)_q}{\sqrt{(8)_q}}c_0^2 = 0. \tag{6.4}$$

Also noticing $c_{-3} = 0$, we get the following equations via original eigenvectors $\Psi_\nu(\eta,\eta)$ for $\nu = 5, 3, 2$:

$$\bigl(q^{-9}-q^9\bigr)c_{-1}c_{-4} = 0, \tag{6.5}$$

$$\bigl(q^{-10}-q^{10}\bigr)c_1c_{-4} + \bigl(q^{-2}-q^2\bigr)\frac{(5)_q(4)_q}{(8)_q(7)_q(2)_q}c_{-1}c_{-2} = 0, \tag{6.6}$$

$$\bigl(q^{-9}+q^9\bigr)c_2c_{-4} + \bigl(q^3+q^{-3}\bigr)\sqrt{\frac{(6)_q(5)_q(4)_q(3)_q}{(8)_q(7)_q(2)_q}}\,c_0c_{-2} - \frac{(5)_q(4)_q}{(8)_q(7)_q(2)_q}c_{-1}^2 = 0. \tag{6.7}$$

From (6.5) c_{-1} or c_{-4} are 0. If c_{-4} is 0, then we have $c_{-2} = c_{-1} = c_0 = 0$ by (6.2)–(6.4). This is a contradiction. Hence c_{-4} is not 0 and c_{-1} is equal to 0. Next we can derive $c_2c_{-4} = c_0c_{-2} = 0$ by (6.3) and (6.7). Then we have $c_{-2} = 0$. This yields $c_0 = 0$ through (6.2). Finally we get $c_{-4} = 0$ by (6.4). However this is a contradiction.

(III) A_5^* *case*. A C^*-algebra A has a spectral pattern $\pi_0 \oplus \pi_6 \oplus \pi_{10} \oplus \pi_{12} \oplus \cdots$. Let $\eta = (\eta_s)_{s\in I_6} = \sum_{s\in I_6} c_s \boldsymbol{w}_s^6$ be a non-zero self-conjugate π_6-eigenvector. Equality $c_{-s} = (-q)^s\overline{c_s}$ follows from the self-conjugacy. We show coefficients c_s are all zero and derive contradiction. In order to do it we make use of spectral gaps at π_ν for $\nu = 11, 9, 8, 7, 4, 2$ and 1.

First we see the gap at 11. We construct a π_{11}-eigenvector $\Psi_{11}(\eta,\eta)$ and apply the lowering operator f to $\Psi_{11}(\eta,\eta)_{-11}$. Then we have the following equalities:

$$\Psi_{11}(\eta,\eta)_{-11} = q^{-6}\eta_{-5}\eta_{-6} - q^{6}\eta_{-6}\eta_{-5},$$
$$f\cdot\Psi_{11}(\eta,\eta)_{-11} = \sqrt{(11)_q(2)_q}\,\eta_{-4}\eta_{-6} + \left(q^{-11}-q^{11}\right)\sqrt{(12)_q}\,\eta_{-5}^2 - \sqrt{(11)_q(2)_q}\,\eta_{-6}\eta_{-4}.$$

Applying $\pi_{\mathbb{T}}$ to the second one, we get $c_{-5}=0$.

Next we see the gap at 9. We make $\Psi_9(\eta,\eta)_{-9}$ and $f\cdot\Psi_9(\eta,\eta)_{-9}$ as follows:

$$\begin{aligned}
\sqrt{(12)_q}\,\Psi_9(\eta,\eta)_{-9} &= q^{-15}\sqrt{(12)_q}\,\eta_{-3}\eta_{-6} - q^{-5}\sqrt{(10)_q(3)_q}\,\eta_{-4}\eta_{-5}\\
&\quad + q^{5}\sqrt{(10)_q(3)_q}\,\eta_{-5}\eta_{-4} - q^{15}\sqrt{(12)_q}\,\eta_{-6}\eta_{-3},\\
\sqrt{(12)_q}\,f\cdot\Psi_9(\eta,\eta)_{-9} &= q^{-9}\sqrt{(12)_q(9)_q(4)_q}\,\eta_{-2}\eta_{-6} + \left(q^{-18}(12)_q - (10)_q(3)_q\right)\eta_{-3}\eta_{-5}\\
&\quad + \left(q^{9}-q^{-9}\right)\sqrt{(11)_q(10)_q(3)_q(2)_q}\,\eta_{-4}\eta_{-4}\\
&\quad + \left((10)_q(3)_q - q^{18}(12)_q\right)\eta_{-5}\eta_{-3} - q^{9}\sqrt{(12)_q(9)_q(4)_q}\,\eta_{-6}\eta_{-2}.
\end{aligned}$$

Applying $\pi_{\mathbb{T}}$ to the first one, we get $c_{-3}c_{-6}=0$. Similarly from the second one we get the following equality:

$$\sqrt{(12)_q(9)_q(4)_q}\,c_{-2}c_{-6} - \sqrt{(11)_q(10)_q(3)_q(2)_q}\,c_{-4}^2 = 0. \tag{6.8}$$

Next we look at the gap at 8. In this case we need two operators:

$$\begin{aligned}
&\sqrt{(12)_q(11)_q(2)_q}\,\Psi_8(\eta,\eta)_{-8}\\
&\quad = q^{-18}\sqrt{(12)_q(11)_q(2)_q}\,\eta_{-2}\eta_{-6} - q^{-9}\sqrt{(11)_q(9)_q(4)_q(2)_q}\,\eta_{-3}\eta_{-5}\\
&\qquad + \sqrt{(10)_q(9)_q(4)_q(3)_q}\,\eta_{-4}\eta_{-4} - q^{9}\sqrt{(11)_q(9)_q(4)_q(2)_q}\,\eta_{-5}\eta_{-3}\\
&\qquad + q^{18}\sqrt{(12)_q(11)_q(2)_q}\,\eta_{-6}\eta_{-2},\\
&\sqrt{(12)_q(11)_q(2)_q}\,f^2\cdot\Psi_8(\eta,\eta)_{-8}\\
&\quad = q^{-6}\sqrt{(12)_q(11)_q(8)_q(7)_q(6)_q(5)_q(2)_q}\,\eta_0\eta_{-6}\\
&\qquad + \left(\left(q^{-13}+q^{-15}\right)(12)_q - q(9)_q(4)_q\right)\sqrt{(11)_q(8)_q(5)_q(2)_q}\,\eta_{-1}\eta_{-5}\\
&\qquad + \left\{q^{8}(10)_q(9)_q(4)_q(3)_q + q^{-22}(12)_q(11)_q(2)_q\right.\\
&\qquad \left. - \left(q^{-6}+q^{-8}\right)(11)_q(8)_q(4)_q(2)_q\right\}\eta_{-2}\eta_{-4}\\
&\qquad + \left((10)_q(3)_q - \left(q^{-15}+q^{15}\right)(11)_q\right)(2)_q\sqrt{(10)_q(9)_q(4)_q(3)_q}\,\eta_{-3}\eta_{-3}
\end{aligned}$$

$$+\{q^{-8}(10)_q(9)_q(4)_q(3)_q+q^{22}(12)_q(11)_q(2)_q-(q^6+q^8)(11)_q(8)_q(4)_q(2)_q\}\eta_{-4}\eta_{-2}$$
$$+((q^{13}+q^{15})(12)_q-q^{-1}(9)_q(4)_q)\sqrt{(11)_q(8)_q(5)_q(2)_q}\,\eta_{-5}\eta_{-1}$$
$$+q^6\sqrt{(12)_q(11)_q(8)_q(7)_q(6)_q(5)_q(2)_q}\,\eta_{-6}\eta_0.$$

From the first equality we get the following one:

$$(q^{-18}+q^{18})\sqrt{(12)_q(11)_q(2)_q}\,c_{-2}c_{-6}+\sqrt{(10)_q(9)_q(4)_q(3)_q}\,c_{-4}^2=0. \tag{6.9}$$

Equalities (6.8) and (6.9) shows $c_{-2}c_{-6}=0$ and $c_{-4}=0$. Hence by the above second equality we get

$$(q^{-6}+q^6)\sqrt{(12)_q(11)_q(8)_q(7)_q(6)_q(5)_q(2)_q}\,c_0c_{-6}$$
$$+((10)_q(3)_q-(q^{-15}+q^{15})(11)_q)(2)_q\sqrt{(10)_q(9)_q(4)_q(3)_q}\,c_{-3}^2=0. \tag{6.10}$$

Next we see the gap at 7. The operators $\sqrt{(12)_q(11)_q(2)_q}\,\Psi_7(\eta,\eta)_{-7}$ and $\sqrt{(12)_q(11)_q(2)_q}\,f\cdot\Psi_7(\eta,\eta)_{-7}$ are as follows:

$$\sqrt{(12)_q(11)_q(2)_q}\,\Psi_7(\eta,\eta)_{-7}$$
$$=q^{-20}\sqrt{(12)_q(11)_q(2)_q}\,\eta_{-1}\eta_{-6}-q^{-12}\sqrt{(11)_q(8)_q(5)_q(2)_q}\,\eta_{-2}\eta_{-5}$$
$$+q^{-4}\sqrt{(9)_q(8)_q(5)_q(4)_q}\,\eta_{-3}\eta_{-4}-q^{4}\sqrt{(9)_q(8)_q(5)_q(4)_q}\,\eta_{-4}\eta_{-3}$$
$$+q^{12}\sqrt{(11)_q(8)_q(5)_q(2)_q}\,\eta_{-5}\eta_{-2}-q^{20}\sqrt{(12)_q(11)_q(2)_q}\,\eta_{-6}\eta_{-1},$$

$$\sqrt{(12)_q(11)_q(2)_q}\,f\cdot\Psi_7(\eta,\eta)_{-7}=q^{-14}\sqrt{(12)_q(11)_q(7)_q(6)_q(2)_q}\,\eta_0\eta_{-6}$$
$$+(q^{-21}(12)_q-q^{-7}(8)_q(5)_q)\sqrt{(11)_q(2)_q}\,\eta_{-1}\eta_{-5}$$
$$+((9)_q(4)_q-q^{-14}(11)_q(2)_q)\sqrt{(8)_q(5)_q}\,\eta_{-2}\eta_{-4}$$
$$+(q^{-7}-q^7)\sqrt{(10)_q(9)_q(8)_q(5)_q(4)_q(3)_q}\,\eta_{-3}\eta_{-3}$$
$$+((9)_q(4)_q-q^{-14}(11)_q(2)_q)\sqrt{(8)_q(5)_q}\,\eta_{-4}\eta_{-2}$$
$$+(q^7(8)_q(5)_q-q^{21}(12)_q)\sqrt{(11)_q(2)_q}\,\eta_{-5}\eta_{-1}$$
$$-q^{14}\sqrt{(12)_q(11)_q(7)_q(6)_q(2)_q}\,\eta_{-6}\eta_0.$$

From the second one we get the following equation:

$$\begin{aligned}&\left(q^{-14}-q^{14}\right)\sqrt{(12)_q(11)_q(7)_q(6)_q(2)_q}c_0c_{-6}\\&\quad+\left(q^{-7}-q^{7}\right)\sqrt{(10)_q(9)_q(8)_q(5)_q(4)_q(3)_q}c_{-3}^2=0.\end{aligned}\tag{6.11}$$

On Eqs. (6.10) and (6.11) we can easily show that the determinant of the following matrix is not 0 for $0<q<1$

$$\begin{pmatrix}(q^{-6}+q^6) & ((10)_q(3)_q-(q^{-15}+q^{15})(11)_q)(2)_q\\(q^{-14}-q^{14}) & (q^{-7}-q^7)(8)_q(5)_q\end{pmatrix}.$$

Hence we obtain $c_0c_{-6}=0$ and $c_{-3}=0$.

Next we see the gap at 4. We use an eigenvector $\Psi_4(\eta,\eta)$,

$$\begin{aligned}&\sqrt{(12)_q(11)_q(10)_q(9)_q(4)_q(3)_q(2)_q}\Psi_4(\eta,\eta)_{-4}\\&\quad=q^{-20}\sqrt{(12)_q(11)_q(10)_q(9)_q(4)_q(3)_q(2)_q}\eta_2\eta_{-6}\\&\qquad-q^{-15}\sqrt{(11)_q(10)_q(9)_q(8)_q(5)_q(4)_q(3)_q(2)_q}\eta_1\eta_{-5}\\&\qquad+q^{-10}\sqrt{(10)_q(9)_q(8)_q(7)_q(6)_q(5)_q(4)_q(3)_q}\eta_0\eta_{-4}\\&\qquad-q^{-5}(7)_q(6)_q\sqrt{(9)_q(8)_q(5)_q(4)_q}\eta_{-1}\eta_{-3}\\&\qquad+(8)_q(7)_q(6)_q(5)_q\eta_{-2}^2\\&\qquad-q^{5}(7)_q(6)_q\sqrt{(9)_q(8)_q(5)_q(4)_q}\eta_{-3}\eta_{-1}\\&\qquad+q^{10}\sqrt{(10)_q(9)_q(8)_q(7)_q(6)_q(5)_q(4)_q(3)_q}\eta_{-4}\eta_0\\&\qquad-q^{15}\sqrt{(11)_q(10)_q(9)_q(8)_q(5)_q(4)_q(3)_q(2)_q}\eta_{-5}\eta_1\\&\qquad+q^{20}\sqrt{(12)_q(11)_q(10)_q(9)_q(4)_q(3)_q(2)_q}\eta_{-6}\eta_2.\end{aligned}$$

We know $c_2c_{-6}=0$ because of $c_{-2}c_{-6}=0$. Hence the above equality derives $c_{-2}=0$.

Next we see the gap at 2. The operator $\Psi_2(\eta,\eta)_{-2}$ is

$$\begin{aligned}&\sqrt{(12)_q(11)_q(2)_q},\Psi_2(\eta,\eta)_{-2}\\&\quad=q^{-15}\sqrt{(12)_q(11)_q(2)_q}\eta_4\eta_{-6}-q^{-12}\sqrt{(11)_q(10)_q(3)_q(2)_q}\eta_3\eta_{-5}\\&\qquad+q^{-9}\sqrt{(10)_q(9)_q(4)_q(3)_q}\eta_2\eta_{-4}-q^{-6}\sqrt{(9)_q(8)_q(5)_q(4)_q}\eta_1\eta_{-3}\\&\qquad+q^{-3}\sqrt{(8)_q(7)_q(6)_q(5)_q}\eta_0\eta_{-2}-(7)_q(6)_q\eta_{-1}^2\\&\qquad+q^{3}\sqrt{(8)_q(7)_q(6)_q(5)_q}\eta_{-2}\eta_0-q^{6}\sqrt{(9)_q(8)_q(5)_q(4)_q}\eta_{-3}\eta_1\end{aligned}$$

$$+ q^9\sqrt{(10)_q(9)_q(4)_q(3)_q}\,\eta_{-4}\eta_2 - q^{12}\sqrt{(11)_q(10)_q(3)_q(2)_q}\,\eta_{-5}\eta_3$$
$$+ q^{15}\sqrt{(12)_q(11)_q(2)_q}\,\eta_{-6}\eta_4.$$

This shows $c_{-1} = 0$.

Finally we see the gap at 1. The operators $\Psi_1(\eta,\eta)_{-1}$ and $f \cdot \Psi_1(\eta,\eta)_{-1}$ are

$$\begin{aligned}\sqrt{(12)_q}\Psi_1(\eta,\eta)_{-1} &= q^{-11}\sqrt{(12)_q}\,\eta_5\eta_{-6} - q^{-9}\sqrt{(11)_q(2)_q}\,\eta_4\eta_{-5} + q^{-7}\sqrt{(10)_q(3)_q}\,\eta_3\eta_{-4}\\ &\quad - q^{-5}\sqrt{(9)_q(4)_q}\,\eta_2\eta_{-3} + q^{-3}\sqrt{(8)_q(5)_q}\,\eta_1\eta_{-2} - q^{-1}\sqrt{(7)_q(6)_q}\,\eta_0\eta_{-1}\\ &\quad + q\sqrt{(6)_q(7)_q}\,\eta_{-1}\eta_0 - q^3\sqrt{(5)_q(8)_q}\,\eta_{-2}\eta_1 + q^5\sqrt{(4)_q(9)_q}\,\eta_{-3}\eta_2\\ &\quad + q^7\sqrt{(10)_q(3)_q}\,\eta_{-4}\eta_3 - q^9\sqrt{(11)_q(2)_q}\,\eta_{-5}\eta_4 - q^{11}\sqrt{(12)_q}\,\eta_{-6}\eta_5,\end{aligned}$$

$$\begin{aligned}&\sqrt{(12)_q}f\cdot\Psi_1(\eta,\eta)_{-1}\\ &\quad= q^{-5}(12)_q\eta_6\eta_{-6} + \big(q^{-6}(12)_q - q^{-4}(11)_q(2)_q\big)\eta_5\eta_{-5}\\ &\qquad + \big(q^{-3}(10)_q(3)_q - q^{-5}(11)_q(2)_q\big)\eta_4\eta_{-4} + \big(q^{-4}(10)_q(3)_q - q^{-2}(9)_q(4)_q\big)\eta_3\eta_{-3}\\ &\qquad + \big(q^{-1}(8)_q(5)_q - q^{-3}(9)_q(4)_q\big)\eta_2\eta_{-2} + \big(q^{-2}(8)_q(5)_q - (7)_q(6)_q\big)\eta_1\eta_{-1}\\ &\qquad + \big(q - q^{-1}\big)(7)_q(6)_q\eta_0\eta_0 + \big((7)_q(6)_q - q^2(8)_q(5)_q\big)\eta_{-1}\eta_1\\ &\qquad + \big(q^3(9)_q(4)_q - q(8)_q(5)_q\big)\eta_{-2}\eta_2 + \big(q^2(9)_q(4)_q - q^4(10)_q(3)_q\big)\eta_{-3}\eta_3\\ &\qquad + \big(q^5(11)_q(2)_q - q^3(10)_q(3)_q\big)\eta_{-4}\eta_4 + \big(q^4(11)_q(2)_q - q^6(12)_q\big)\eta_{-5}\eta_5\\ &\qquad - q^5(12)_q\eta_{-6}\eta_6.\end{aligned}$$

Since we have $c_s = 0$ for $s = 5, 4, 3, 2$ and 1, the above equality yields

$$\big(q^{-5} - q^5\big)(12)_q c_6 c_{-6} + \big(q - q^{-1}\big)(7)_q(6)_q c_0^2 = 0.$$

We also know $c_0c_6 = 0$ because of $c_0c_{-6} = 0$. Hence we obtain $c_{-6} = c_0 = 0$.

(IV) D_∞^* *case*. We conclude that this case actually occurs and a right coideal of this type is unique up to conjugation by β_z^L.

Lemma 6.2. *Consider the quantum sphere $C(S_{q,0}^2)$. Then a vector*

$$\big(q\sqrt{(3)_q!}, 0, -\sqrt{(4)_q}, 0, q^{-1}\sqrt{(3)_q!}\big)w(\pi_2)$$

is a π_2-eigenvector of $C(S_{q,0}^2)$.

Proof. A highest weight vector of weight 1, $(\xi_0^1)_{-1}$ in $C(S_{q,0}^2)$ is given by $(\xi_0^1)_{-1} = q^{\frac12}\sqrt{(2)_q}^{-1}x^2 - q^{-\frac12}\sqrt{(2)_q}^{-1}v^2$. Then it is easy to see that the vector in the above statement is a highest weight vector of weight 2, $(2)_q\sqrt{(3)_q!}(\xi_0^1)_{-1}^2$. □

Let $\xi = (\xi_{-1}, \xi_0, \xi_1)$ be the canonical π_1-eigenvector of $C(S^2_{q,0})$. By its definition, $\{\xi_r\}_{r\in I_1}$ satisfies the following conditions,

$$\xi^*_{-1} = -q\xi_1, \qquad \xi^*_0 = \xi_0, \qquad \xi_{\pm1}\xi_0 = q^{\pm2}\xi_0\xi_{\pm1}, \tag{6.12}$$

$$\xi_1\xi_{-1} = \left(q+q^{-1}\right)^{-1}\left(q^2\xi_0^2 - 1\right), \tag{6.13}$$

$$\xi_{-1}\xi_1 = \left(q+q^{-1}\right)^{-1}\left(q^{-2}\xi_0^2 - 1\right). \tag{6.14}$$

An embedding of $C(S^2_{q,0})$ into $C(SU_q(2))$ is given by

$$\xi = \left(\sqrt{1+q^{-2}}^{\,-1}, 0, -\sqrt{1+q^2}^{\,-1}\right)w(\pi_1),$$

or more precisely

$$\xi_{-1} = \sqrt{1+q^{-2}}^{\,-1}x^2 - \sqrt{1+q^2}^{\,-1}v^2, \qquad \xi_0 = qxu - vy,$$

$$\xi_{-1} = \sqrt{1+q^{-2}}^{\,-1}u^2 - \sqrt{1+q^2}^{\,-1}y^2.$$

Consider the smooth part of $C(S^2_{q,0})$, that is, the $*$-algebra generated by $\{\xi_r\}_{r\in I_1}$ and denote it by $\mathcal{A}$. Let $\mathcal{A}_0$ be a $SU_q(2)$-invariant $*$-subalgebra of $\mathcal{A}$ generated by $\{\xi_r\xi_s\}_{r,s\in I_1}$. In [5, Proposition 2.9], it is shown that $\{\xi_0^m\xi_{-1}^n\}_{m,n\in\mathbb{Z}_{\geqslant0}}$ and $\{\xi_0^m\xi_1^n\}_{m,n\in\mathbb{Z}_{\geqslant0}}$ are basis for a vector space $\mathcal{A}$. Since the $*$-subalgebra $\mathcal{A}_0$ is generated by words of even length because of the previous equalities, we see that ξ_r is not contained in $\mathcal{A}_0$ for $r\in I_1$. In particular, we have $\mathcal{A}_0 \subsetneq \mathcal{A}$. Let A_0 be the norm closure of $\mathcal{A}_0$. By $SU_q(2)$-invariance of $\mathcal{A}_0$ we see that A_0 is also $SU_q(2)$-invariant and $A_0 \subsetneq C(S^2_{q,0})$. Next we prove that π_3-spectral subspace of A_0 is 0. Since $\xi^3_{-1} \neq 0$ is the highest weight vector in $C(S^2_{q,0})_{\pi_3}$, it is not contained in A_0. Again $SU_q(2)$-invariance yields that ξ^3_{-1} is not contained in A_0. In a similar way, we can prove that all the odd spin spectral subspaces of A_0 are 0. Of course, even spin spectral subspaces of A_0 are not 0 because of $\xi^{2n}_{-1} \neq 0$ for all $n\in\mathbb{Z}_{\geqslant0}$. The spectral pattern $\bigoplus_{k\in\mathbb{Z}_{\geqslant0}}\pi_{2k}$ follows from the spectral pattern of $C(S^2_{q,0})$ $\bigoplus_{k\in\mathbb{Z}_{\geqslant0}}\pi_k$. Therefore, we have shown that A_0 is of type D^*_∞. Next lemma says that a right coideal of D^*_∞-type is unique up to conjugation of the automorphism β^L_z.

Lemma 6.3. *Let $A \subset C(SU_q(2))$ be a right coideal of type D^*_∞. Then there exists $z\in\mathbb{C}$ such that A is $\beta^L_z(B_0)$.*

Proof. By assumption A has the spectral pattern $\pi_0\oplus\pi_2\oplus\pi_4\oplus\cdots$. In the following discussion we do not need to use the gap at π_3. Let $\eta = (\eta_s)_{s\in I_2} = \sum_{s\in I_2} c_s \boldsymbol{w}^2_s$ be a self-conjugate π_2-eigenvector of A. We consider zero vectors $\Psi_1(\eta,\eta)_{-1}$ and $f\cdot\Psi_1(\eta,\eta)_{-1}$. They become as follows:

$$\Psi_1(\eta,\eta)_{-1} = q^{-3}\eta_1\eta_{-2} - q^{-1}\sqrt{\frac{(3)_q(2)_q}{(4)_q}}\eta_0\eta_{-1} + q\sqrt{\frac{(3)_q(2)_q}{(4)_q}}\eta_{-1}\eta_0 - q^3\eta_{-2}\eta_1,$$

$$\begin{aligned}
f \cdot \Psi_1(\eta, \eta)_{-1} = {} & q^{-1}\sqrt{(4)_q}\eta_2\eta_{-2} + \sqrt{(4)_q}^{-1}\big(q^{-2}(4)_q - (3)_q!\big)\eta_1\eta_{-1} \\
& + \sqrt{(4)_q}^{-1}\big(q - q^{-1}\big)(3)_q!\eta_0^2 + \sqrt{(4)_q}^{-1}\big((3)_q! - q^2(4)_q\big)\eta_{-1}\eta_1 \\
& - q\sqrt{(4)_q}\eta_{-2}\eta_2.
\end{aligned}$$

Apply the map $\pi_{\mathbb{T}}$ to the above equalities and we obtain:

$$\big(q^{-3} - q^3\big)c_1c_{-2} + \big(q - q^{-1}\big)\sqrt{\frac{(3)_q!}{(4)_q}}c_{-1}c_0 = 0,$$

$$\big(q^{-1} - q\big)\sqrt{(4)_q}c_2c_{-2} + \big(q^{-2} - q^2\big)\sqrt{(4)_q}c_1c_{-1} + \big(q - q^{-1}\big)\sqrt{(4)_q}^{-1}(3)_q!c_0^2 = 0.$$

Self-conjugacy of η yields $c_{-1} = -q\overline{c_1}$ and $c_{-2} = q^2c_2$. Assume c_1 is not 0. Then from the first equality we get $c_{-2} = -q\sqrt{\frac{(2)_q}{(4)_q(3)_q}}\frac{c_{-1}}{c_1}c_0$. If we put this one to the second equality, we get

$$\frac{(2)_q - (3)_q^2(2)_q}{(3)_q}c_0^2 + (2)_q(4)_qc_1c_{-1} = 0.$$

Since we know $(3)_q > 1$ and $c_1c_{-1} = -q|c_1|^2 < 0$, the left-hand side is strictly negative unless $c_0 = c_1 = 0$. This also shows $c_2 = 0$ and this is a contradiction. Hence we get $c_1 = 0$. Then from the second equality we get $|c_2| = q^{-1}\sqrt{\frac{(3)_q!}{(4)_q}}|c_0|$. We may assume $c_0 = -\sqrt{(4)_q}$ by scalar multiplication and $c_2 = q^{-1}\sqrt{(3)_q!}$ by a conjugation of β_z^L. $\square$

Let A_0 be the right coideal of type D_∞^* generated by $q\sqrt{(3)_q!}w(\pi_2)_{-2,s} - \sqrt{(4)_q}w(\pi_2)_{0,s} + q^{-1}\sqrt{(3)_q!}w(\pi_2)_{2,s}$ for $s \in I_2$ as before. We remark that $\boldsymbol{w}_0^2$ does not generate a right coideal of type D_∞^*. In fact, it generates the canonical homogeneous sphere $C(\mathbb{T} \setminus SU_q(2))$. This is essentially due to the effect of $q \neq 1$ (see also Lemma 7.5). The C^*-algebra A_0 is actually the Toeplitz algebra as we see below. By [14, Lemma 3.2] there exists the matrix units $\{e_{i,j}^+\}_{i,j\in\mathbb{Z}_{\geqslant 0}}$ and $\{e_{i,j}^-\}_{i,j\in\mathbb{Z}_{\geqslant 0}}$ in $C(S_{q,0}^2)$ which satisfy the following equalities:

$$\xi_{-1}e_{k,k}^\pm = \pm\gamma_ke_{k+1,k}^\pm, \qquad \xi_1e_{k,k}^\pm = \mp q^{-1}\gamma_{k-1}e_{k-1,k}^\pm, \tag{6.15}$$

$$\xi_0 = \sum_{k=0}^\infty -q^{2k+1}e_{k,k}^- + \sum_{k=0}^\infty q^{2k+1}e_{k,k}^+ \tag{6.16}$$

for all $k \in \mathbb{Z}_{\geqslant 0}$, where $\gamma_k = q^{\frac{1}{2}}(q + q^{-1})^{-\frac{1}{2}}(1 - q^{4k+4})^{\frac{1}{2}}$. Note the equality $1 = \sum_{k=0}^\infty e_{k,k}^- + \sum_{k=0}^\infty e_{k,k}^+$ in $C(SU_q(2))$ where the summation converges in the strong operator topology in $\mathbb{B}(L^2(SU_q(2)))$. In order to verify it, it suffices to show $1 = \sum_{k=0}^\infty h(e_{k,k}^-) + \sum_{k=0}^\infty h(e_{k,k}^+)$, because the Haar state h is normal and faithful. This equality holds by the following lemma.

Lemma 6.4. *We have* $h(e_{k,k}^\pm) = 2^{-1}q^{2k}(1 - q^2)$ *for all* $k \in \mathbb{Z}_{\geqslant 0}$.

Proof. Let $X_{\pi_1} = \mathbb{C}\xi$ be the π_1-eigenvector space of $C(S^2_{q,0})$. Recall the conjugation map T and the one-parameter unitary group U_t on X_{π_1} which are defined by $(T\xi)_r = (-q)^{-r}\xi^*_{-r}$ and $(U_t\xi)_r = (F_{\pi_1})^{-it}_{r,r}\sigma^h_t(\xi_r)$ for all $r \in I_{\pi_1}$ and $t \in \mathbb{R}$, where σ^h_t is the modular automorphism group with respect to the invariant state h on $C(S^2_{q,0})$. As is proved in Lemma 3.7, we have $IT^*TI^* = U_i$ where the unitary map I is defined by $(I\xi)_r = \xi_{-r}$ for $r \in I_{\pi_1}$. It shows that the spectrum of U_i is inverse-closed and hence $U_i = \mathrm{id}$. It yields the formula $\sigma^h_t(\xi_r) = q^{-2rit}\xi_r$. We have the equality $h(\xi_1 e^{\pm}_{k,k}\xi_{-1}) = -q^{-1}\gamma^2_{k-1}h(e^{\pm}_{k-1,k-1})$ by (6.15). The left-hand side is equal to

$$\begin{aligned} h(\xi_1 e^{\pm}_{k,k}\xi_{-1}) &= h(\sigma^h_i(\xi_{-1})\xi_1 e^{\pm}_{k,k}) \\ &= q^{-2}h(\xi_{-1}\xi_1 e^{\pm}_{k,k}) \\ &= q^{-2}(q+q^{-1})^{-1}(q^{4k}-1)h(e^{\pm}_{k,k}), \end{aligned}$$

where we used (6.13) for the third equality. Hence we have $h(e^{\pm}_{k,k}) = q^{2k}h(e^{\pm}_{0,0})$ for $k \geqslant 1$. Since the equality $h(\xi_0) = 0$ holds, we get $h(e^{+}_{0,0}) = h(e^{-}_{0,0})$. Using Lemma 3.6, we see $h(\xi_0^2) = (3)^{-1}$ and obtain $h(e^{\pm}_{0,0}) = 2^{-1}(1-q^2)$. $\square$

Now we consider the invariant subalgebra $A_0 \subset C(S^2_{q,0})$. Let us define the matrix units $e_{i,j} = e^{-}_{i,j} + e^{+}_{i,j}$ for $i, j \in \mathbb{Z}_{\geqslant 0}$. The C^*-algebra A_0 is generated by them and the unilateral shift. It also concludes that the right coideal von Neumann algebra A_0'' is isomorphic to $\mathbb{B}(\ell^2(\mathbb{Z}_{\geqslant 0}))$.

(V) D^*_m $(m \geqslant 2)$ *case*. We will show their non-existences. We study a C^*-algebra A whose spectral pattern is $\bigoplus_{k\in\mathbb{Z}_{\geqslant 0}}(\frac{1+(-1)^k}{2} + [\frac{k}{m}])\pi_k$. Making use of a gap at π_1 as Lemma 6.3, we can similarly prove that elements of one π_2-eigenvector generates a right coideal of type D^*_∞. (The π_2-eigenvector space is two-dimensional when m is equal to 2.) So we may assume there is a π_2-eigenvector ξ^2_0 in A. Recall that ξ^2_0 is defined by $\xi^2_0 = (q\sqrt{(3)_q(2)_q}, 0, -\sqrt{(4)_q}, 0, q^{-1}\sqrt{(3)_q(2)_q})w(\pi_2)$. Note that the index 0 of ξ^2_0 means $\lambda = 0$ of $C(S^2_{q,\lambda})$.

Lemma 6.5. *Let n be a half-integer with $n \geqslant 2$. Then we have the following equalities for all $t \in I_n$:*

$$\Psi_{n-2}(\boldsymbol{w}^2_{-2}, \boldsymbol{w}^n_t) = q^{-2t+6}\sqrt{\frac{(n+t)_q(n+t-1)_q(n+t-2)_q(n+t-3)_q}{(2n)_q(2n-1)_q(2n-2)_q(2n-3)_q}}\,\boldsymbol{w}^{n-2}_{t-2},$$

$$\Psi_{n-2}(\boldsymbol{w}^2_{0}, \boldsymbol{w}^n_t) = q^{-2n-2t+4}\sqrt{\frac{(4)_q(3)_q}{(2)_q}}\sqrt{\frac{(n+t)_q(n+t-1)_q(n-t)_q(n-t-1)_q}{(2n)_q(2n-1)_q(2n-2)_q(2n-3)_q}}\,\boldsymbol{w}^{n-2}_{t},$$

$$\Psi_{n-2}(\boldsymbol{w}^2_{2}, \boldsymbol{w}^n_t) = q^{-4n-2t+2}\sqrt{\frac{(n-t)_q(n-t-1)_q(n-t-2)_q(n-t-3)_q}{(2n)_q(2n-1)_q(2n-2)_q(2n-3)_q}}\,\boldsymbol{w}^{n-2}_{t+2}.$$

Proof. For the first equality we use $w(\pi_2)_{-2,2-r} = \begin{bmatrix}4\\r\end{bmatrix}^{\frac{1}{2}}_{q^2} x^r u^{4-r}$ for $r \in I_2$. From this we obtain

$$\Psi_{n-2}(\boldsymbol{w}^2_{-2}, \boldsymbol{w}^n_t)_{-(n-2)} = \sum_{r=0}^{4}(C^{2,n}_{n-2})_r\begin{bmatrix}4\\r\end{bmatrix}^{\frac{1}{2}}_{q^2} x^r u^{4-r} w(\pi_n)_{t,-n+r}$$

for all $t \in I_n$ where the coefficients $(C^{2,n}_{n-2})_r$ are given as follows:

$$\big(C^{2,n}_{n-2}\big)_0 = q^{-2(n-1)}, \qquad \big(C^{2,n}_{n-2}\big)_1 = -q^{-(n-1)}\sqrt{\frac{(4)_q}{(2n)_q}},$$

$$\big(C^{2,n}_{n-2}\big)_2 = \sqrt{\frac{(4)_q(3)_q}{(2n)_q(2n-1)_q}}, \qquad \big(C^{2,n}_{n-2}\big)_3 = -q^{n-1}\sqrt{\frac{(4)_q(3)_q(2)_q}{(2n)_q(2n-1)_q(2n-2)_q}},$$

$$\big(C^{2,n}_{n-2}\big)_4 = q^{2(n-1)}\sqrt{\frac{(4)_q(3)_q(2)_q}{(2n)_q(2n-1)_q(2n-2)_q(2n-3)_q}}.$$

This must be a scalar multiple of $x^{n-t}v^{n+t-4}$. So we may compute only the $r=4$ term $(C^{2,n}_{n-2})_4 x^4 w(\pi_n)_{t,-n+4}$. If $w(\pi_n)_{t,-n+4}$ contains the word u such that $t < -n+4$, we can disregard the effect of this term. In order to simplify calculations we use the symbol $\sim$ which ignores terms with a power of u. Assume $t \geqslant -n+4$ and then we have

$$w(\pi_n)_{t,-n+4} \sim q^{4(-n-t+4)}\begin{bmatrix} n+t \\ 4 \end{bmatrix}_{q^2}^{\frac{1}{2}}\begin{bmatrix} 2n-4 \\ n-t \end{bmatrix}_{q^2}^{\frac{1}{2}} x^{n-t-4}v^{n+t-4}, \quad \text{if } t-n+4 \leqslant 0,$$

$$w(\pi_n)_{t,-n+4} \sim q^{(n+t-4)(t-n)}\begin{bmatrix} n+t \\ 4 \end{bmatrix}_{q^2}^{\frac{1}{2}}\begin{bmatrix} 2n-4 \\ n-t \end{bmatrix}_{q^2}^{\frac{1}{2}} v^{n+t-4}y^{-n+t+4}, \quad \text{if } t-n+4 \geqslant 0.$$

In both cases, the following result holds:

$$\begin{aligned} x^4 w(\pi_n)_{t,-n+4} &\sim q^{4(-n-t+4)}\begin{bmatrix} n+t \\ 4 \end{bmatrix}_{q^2}^{\frac{1}{2}}\begin{bmatrix} 2n-4 \\ n-t \end{bmatrix}_{q^2}^{\frac{1}{2}} x^{n-t}v^{n+t-4} \\ &= q^{4(-n-t+4)}\begin{bmatrix} n+t \\ 4 \end{bmatrix}_{q^2}^{\frac{1}{2}} w(\pi_{n-2})_{t-2,-(n-2)}. \end{aligned}$$

Hence we obtain the first equality:

$$\begin{aligned} \Psi_{n-2}\big(\boldsymbol{w}^2_{-2}, \boldsymbol{w}^n_t\big)_{-(n-2)} &= q^{2(n-1)}q^{4(-n-t+4)}\sqrt{\frac{(4)_q!}{(2n)_q\cdots(2n-3)_q}} \\ &\quad \times \begin{bmatrix} n+t \\ 4 \end{bmatrix}_{q^2}^{\frac{1}{2}} w(\pi_{n-2})_{t-2,-(n-2)} \\ &= q^{-2t+6}\sqrt{\frac{(n+t)_q\cdots(n+t-3)_q}{(2n)_q\cdots(2n-3)_q}}\, w(\pi_{n-2})_{t-2,-(n-2)}. \end{aligned}$$

Next we show the second equality. The matrix element $w(\pi_2)_{0,2-r}$ is as follows.

$$w(\pi_2)_{0,2-r} \sim q^{(4-r)(2-r)} \begin{bmatrix} 2 \\ r-2 \end{bmatrix}_{q^2}^{\frac{1}{2}} \begin{bmatrix} r \\ 2 \end{bmatrix}_{q^2}^{\frac{1}{2}} x^{r-2} v^{r-2}, \quad \text{if } r \geqslant 2,$$
$$w(\pi)_{0,2-r} \sim 0, \quad \text{if } r \leqslant 1.$$

Hence we may only compute the coefficient of $x^{n-t-2} v^{n+t-2}$ of the next one

$$\sum_{r \geqslant 2} \left(C_{n-2}^{2,n}\right)_r q^{(4-r)(2-r)} \begin{bmatrix} 2 \\ r-2 \end{bmatrix}_{q^2}^{\frac{1}{2}} \begin{bmatrix} r \\ 2 \end{bmatrix}_{q^2}^{\frac{1}{2}} x^{r-2} v^{r-2} w(\pi_n)_{t,-n+r}.$$

If $t \geqslant -n+4$, then we have

$$w(\pi_2)_{0,2-r} w(\pi_n)_{t,-n+r} \sim q^{r^2-(2t+4)r-2n+2t+8} \begin{bmatrix} n+t \\ r \end{bmatrix}_{q^2}^{\frac{1}{2}} \begin{bmatrix} 2n-r \\ n-t \end{bmatrix}_{q^2}^{\frac{1}{2}}$$
$$\times \begin{bmatrix} 2 \\ r-2 \end{bmatrix}_{q^2}^{\frac{1}{2}} \begin{bmatrix} r \\ 2 \end{bmatrix}_{q^2}^{\frac{1}{2}} \begin{bmatrix} 2n-4 \\ n+t-2 \end{bmatrix}_{q^2}^{-\frac{1}{2}} w(\pi_{n-2})_{t,-(n-2)}.$$

When $t < -n+4$, then $w(\pi_n)_{t,-n+r}$ contains the word u if and only if $2n-r < n-t$. In this case the binomial $\left[\begin{smallmatrix} 2n-r \\ n-t \end{smallmatrix}\right]_{q^2}^{\frac{1}{2}}$ is 0 by definition. From this observation, we may continue the computation under $t \geqslant -n+4$ in order to get the result about general t. Then we can carry out the following calculation:

$$\begin{aligned}
&\sum_{r=2}^{4} \left(C_{n-2}^{2,n}\right)_r q^{r^2-(2t+4)r-2n+2t+8} \begin{bmatrix} n+t \\ r \end{bmatrix}_{q^2}^{\frac{1}{2}} \begin{bmatrix} 2n-r \\ n-t \end{bmatrix}_{q^2}^{\frac{1}{2}} \begin{bmatrix} 2 \\ r-2 \end{bmatrix}_{q^2}^{\frac{1}{2}} \begin{bmatrix} r \\ 2 \end{bmatrix}_{q^2}^{\frac{1}{2}} \begin{bmatrix} 2n-4 \\ n+t-2 \end{bmatrix}_{q^2}^{-\frac{1}{2}} \\
&= \left(C_{n-2}^{2,n}\right)_2 q^{-2n-2t+4} \begin{bmatrix} n+t \\ 2 \end{bmatrix}_{q^2}^{\frac{1}{2}} \begin{bmatrix} 2n-2 \\ n-t \end{bmatrix}_{q^2}^{\frac{1}{2}} \begin{bmatrix} 2n-4 \\ n+t-2 \end{bmatrix}_{q^2}^{-\frac{1}{2}} \\
&\quad + \left(C_{n-2}^{2,n}\right)_3 q^{-2n-4t+5} \begin{bmatrix} n+t \\ 3 \end{bmatrix}_{q^2}^{\frac{1}{2}} \begin{bmatrix} 2n-3 \\ n-t \end{bmatrix}_{q^2}^{\frac{1}{2}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}_{q^2}^{\frac{1}{2}} \begin{bmatrix} 3 \\ 2 \end{bmatrix}_{q^2}^{\frac{1}{2}} \begin{bmatrix} 2n-4 \\ n+t-2 \end{bmatrix}_{q^2}^{-\frac{1}{2}} \\
&\quad + \left(C_{n-2}^{2,n}\right)_4 q^{-2n-4t+5} \begin{bmatrix} n+t \\ 4 \end{bmatrix}_{q^2}^{\frac{1}{2}} \begin{bmatrix} 2n-4 \\ n-t \end{bmatrix}_{q^2}^{\frac{1}{2}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}_{q^2}^{\frac{1}{2}} \begin{bmatrix} 4 \\ 2 \end{bmatrix}_{q^2}^{\frac{1}{2}} \begin{bmatrix} 2n-4 \\ n+t-2 \end{bmatrix}_{q^2}^{-\frac{1}{2}} \\
&= \begin{bmatrix} 2n-4 \\ n+t-2 \end{bmatrix}_{q^2}^{-\frac{1}{2}} \sqrt{\frac{(4)_q (3)_q}{(2)_q}} \sqrt{\frac{(2n-4)_q!}{(2n)_q \cdots (2n-3)_q}} \sqrt{\frac{(n+t)_q!}{(n-t)_q!}} q^{\frac{1}{2}n^2-\frac{1}{2}t^2-n-t+1} \\
&\quad \times \left\{ \frac{(2n-2)_q (2n-3)_q}{(n+t-2)_q!} q^{-n+t+1} - \frac{(2n-3)_q (2)_q}{(n+t-3)_q!} + \frac{q^{n-t-1}}{(n+t-4)_q!} \right\} \\
&= q^{-2n-2t+4} \sqrt{\frac{(4)_q (3)_q}{(2)_q}} \sqrt{\frac{(n+t)_q (n+t-1)_q (n-t)_q (n-t-1)_q}{(2n)_q (2n-1)_q (2n-2)_q (2n-3)_q}}.
\end{aligned}$$

Hence we have proved the second equality.

Finally we prove the third equality. The matrix element $w(\pi_2)_{2,2-r}$ is equal to $\begin{bmatrix}4\\r\end{bmatrix}_{q^2}^{\frac{1}{2}} v^r y^{4-r}$ for all $0 \leqslant r \leqslant 4$. If $t \geqslant -n+4$, we have

$$v^r y^{4-r} w(\pi_n)_{t,-n+r} \sim q^{r(r-2t-4)} \begin{bmatrix} n+t \\ r \end{bmatrix}_{q^2}^{\frac{1}{2}} \begin{bmatrix} 2n-r \\ n-t \end{bmatrix}_{q^2}^{\frac{1}{2}} x^{n-t-4} v^{n+t}$$
$$= q^{r(r-2t-4)} \begin{bmatrix} n+t \\ r \end{bmatrix}_{q^2}^{\frac{1}{2}} \begin{bmatrix} 2n-r \\ n-t \end{bmatrix}_{q^2}^{\frac{1}{2}} \begin{bmatrix} 2n-4 \\ t+2 \end{bmatrix}_{q^2}^{-\frac{1}{2}} w(\pi_{n-2})_{t+2,-(n-2)}.$$

For $t < -n+4$, $w(\pi_n)_{t,-n+r}$ contains the word u if and only if $n+t < r$. Then the binomial $\begin{bmatrix}n+t\\r\end{bmatrix}_{q^2}$ is equal to 0. As in the proof of the second equality, we may treat only $t \geqslant -n+4$. We compute the coefficient of the desired equality as follows:

$$\sum_{r=0}^{4} \bigl(C_{n-2}^{2,n}\bigr)_r \begin{bmatrix} 4 \\ r \end{bmatrix}_{q^2}^{\frac{1}{2}} q^{r(r-2t-4)} \begin{bmatrix} n+t \\ r \end{bmatrix}_{q^2}^{\frac{1}{2}} \begin{bmatrix} 2n-r \\ n-t \end{bmatrix}_{q^2}^{\frac{1}{2}} \begin{bmatrix} 2n-4 \\ t+2 \end{bmatrix}_{q^2}^{-\frac{1}{2}}$$
$$= q^{-2(n-1)} \begin{bmatrix} 2n \\ n-t \end{bmatrix}_{q^2}^{\frac{1}{2}} \begin{bmatrix} 2n-4 \\ t+2 \end{bmatrix}_{q^2}^{-\frac{1}{2}}$$
$$- q^{-(n-1)} q^{-2t-3} \sqrt{\frac{(4)_q}{(2)_q}} \begin{bmatrix} 4 \\ 1 \end{bmatrix}_{q^2}^{\frac{1}{2}} \begin{bmatrix} n+t \\ 1 \end{bmatrix}_{q^2}^{\frac{1}{2}} \begin{bmatrix} 2n-1 \\ n-t \end{bmatrix}_{q^2}^{\frac{1}{2}} \begin{bmatrix} 2n-4 \\ t+2 \end{bmatrix}_{q^2}^{-\frac{1}{2}}$$
$$+ q^{2(-2t-2)} \sqrt{\frac{(4)_q (3)_q}{(2n)_q (2n-1)_q}} \begin{bmatrix} 4 \\ 2 \end{bmatrix}_{q^2}^{\frac{1}{2}} \begin{bmatrix} n+t \\ 2 \end{bmatrix}_{q^2}^{\frac{1}{2}} \begin{bmatrix} 2n-2 \\ n-t \end{bmatrix}_{q^2}^{\frac{1}{2}} \begin{bmatrix} 2n-4 \\ t+2 \end{bmatrix}_{q^2}^{-\frac{1}{2}}$$
$$- q^{n-1} q^{3(-2t-1)} \sqrt{\frac{(4)_q!}{(2n)_q \cdots (2n-2)_q}} \begin{bmatrix} 4 \\ 3 \end{bmatrix}_{q^2}^{\frac{1}{2}} \begin{bmatrix} n+t \\ 3 \end{bmatrix}_{q^2}^{\frac{1}{2}} \begin{bmatrix} 2n-3 \\ n-t \end{bmatrix}_{q^2}^{\frac{1}{2}} \begin{bmatrix} 2n-4 \\ t+2 \end{bmatrix}_{q^2}^{-\frac{1}{2}}$$
$$+ q^{2(n-1)} q^{-8t} \sqrt{\frac{(4)_q!}{(2n)_q \cdots (2n-3)_q}} \begin{bmatrix} n+t \\ 4 \end{bmatrix}_{q^2}^{\frac{1}{2}} \begin{bmatrix} 2n-4 \\ n-t \end{bmatrix}_{q^2}^{\frac{1}{2}} \begin{bmatrix} 2n-4 \\ t+2 \end{bmatrix}_{q^2}^{-\frac{1}{2}}$$
$$= q^{\frac{1}{2}n^2-\frac{1}{2}t^2-2t-4} \sqrt{\frac{(n+t)_q!}{(n-t)_q!}} (n+t)_q!^{-1} \frac{\sqrt{(2n)_q!}}{(2n)_q \cdots (2n-3)_q} \begin{bmatrix} 2n-4 \\ t+2 \end{bmatrix}_{q^2}^{-\frac{1}{2}}$$
$$\times \Bigl\{ q^{-2n+2t+6} (2n)_q \cdots (2n-3)_q - q^{-n+t+3} (4)_q (2n-1)_q \cdots (2n-3)_q (n+t)_q$$
$$+ \frac{(4)_q (3)_q}{(2)_q} (2n-2)_q (2n-3)_q (n+t)_q (n+t-1)_q$$
$$- q^{n-t-3} (4)_q (2n-3)_q (n+t)_q \cdots (n+t-2)_q$$
$$+ q^{2n-2t-6} (n+t)_q \cdots (n+t-3)_q \Bigr\}$$
$$= q^{\frac{1}{2}n^2-\frac{1}{2}t^2-2t-4} \sqrt{\frac{(n+t)_q!}{(n-t)_q!}} \frac{1}{(n+t)_q!} \frac{\sqrt{(2n)_q!}}{(2n)_q \cdots (2n-3)_q} \begin{bmatrix} 2n-4 \\ t+2 \end{bmatrix}_{q^2}^{-\frac{1}{2}}$$

$$
\begin{aligned}
&\times q^{-6n-2t+6}(n-t)_q\cdots(n-t-3)_q\\
&=q^{-4n-2t+2}\sqrt{\frac{(n-t)_q\cdots(n-t-3)_q}{(2n)_q\cdots(2n-3)_q}}. \qquad \square
\end{aligned}
$$

Let n be the smallest odd integer which satisfies $[\frac{n}{m}]=1$. Then π_n appears in the spectral pattern of A once and π_{n-2} does not. Therefore $\Psi_{n-2}: X_2\times X_n\to X_{n-2}$ is a 0-map. Let $\eta=\sum_{t\in I_n} d_t \boldsymbol{w}_t^n$ be a self-conjugate π_n-eigenvector. The self-conjugacy yields $d_{-t}=(-q)^t\overline{d_t}$ for all $t\in I_n$. We shall show the complex numbers $\{d_t\}_{t\in I_n}$ are all 0. It derives non-existence of type D_m^* for $m\geqslant 2$.

Lemma 6.6. *The complex numbers $\{d_t\}_{t\in I_n}$ satisfy the following recurrence formula for $t\in I_n=\{-n,-n+1,\dots,n-1,n\}$:*

$$
\begin{aligned}
&q^{2n-1}(2)_q\sqrt{(n+t+2)_q(n+t+1)_q(n+t)_q(n+t-1)_q}\, d_{t+2}\\
&\quad-(4)_q\sqrt{(n+t)_q(n+t-1)_q(n-t)_q(n-t-1)_q}\, d_t\\
&\quad+q^{-2n+1}(2)_q\sqrt{(n-t+2)_q(n-t+1)_q(n-t)_q(n-t-1)_q}\, d_{t-2}=0,
\end{aligned}
$$

where $d_t=0$ if $|t|\geqslant n+1$.

Proof. Take the π_2-eigenvector $\xi_0^2=\sum_{s\in I_2} c_s\boldsymbol{w}_s$ where $c_{\pm 2}=q^{\mp 1}\sqrt{(3)_q!}$, $c_{\pm 1}=0$ and $c_0=-\sqrt{(4)_q}$. Then we have $\Psi_{n-2}(\xi_0^2,\zeta)=0$. Multiplying $q^{2n-4}\sqrt{(2n)_q\cdots(2n-3)_q}$ to the left-hand side, we can derive

$$
\begin{aligned}
&\sum_{t\in I_n}\Bigg(c_{-2}q^{2n-2t-2}\sqrt{(n+t+2)_q\cdots(n+t-1)_q}\, d_{t+2}\\
&\quad+c_0q^{-2t}\sqrt{\frac{(4)_q(3)_q}{(2)_q}}\sqrt{(n+t)_q(n+t-1)_q(n-t)_q(n-t-1)_q}\, d_t\\
&\quad+c_2q^{-2n-2t+2}\sqrt{(n-t+2)_q\cdots(n-t-1)_q}\, d_{t-2}\Bigg)\boldsymbol{w}_t^{n-2}=0.
\end{aligned}
$$

Then we obtain the desired formula. $\square$

Let us write

$$
\begin{aligned}
\alpha_t&=q^{2n-1}(2)_q\sqrt{(n+t+2)_q(n+t+1)_q(n+t)_q(n+t-1)_q},\\
\beta_t&=-(4)_q\sqrt{(n+t)_q(n+t-1)_q(n-t)_q(n-t-1)_q},\\
\gamma_t&=q^{-2n+1}(2)_q\sqrt{(n-t+2)_q(n-t+1)_q(n-t)_q(n-t-1)_q}.
\end{aligned}
$$

Then $\{d_t\}_{t\in I_n}$ satisfies

$$\alpha_t d_{t+2} + \beta_t d_t + \gamma_t d_{t-2} = 0. \tag{6.17}$$

By definition, we have $\alpha_{-t} = q^{4n-2}\gamma_t$ and $\beta_{-t} = \beta_t$. From the above recurrence formula at $-t$, we obtain

$$q^{4n-2}\gamma_t d_{-t+2} + \beta_t d_{-t} + q^{-4n+2}\alpha_t d_{-t-2} = 0.$$

The self-conjugacy $d_{-t} = (-q)^t \overline{d_t}$ yields the following equality,

$$q^{-4n+4}\alpha_t d_{t+2} + \beta_t d_t + q^{4n-4}\gamma_t d_{t-2} = 0. \tag{6.18}$$

Through formulae (6.17) and (6.18), we obtain

$$d_{t+2} = q^{-2}\sqrt{\frac{(n-t+2)_q \cdots (n-t-1)_q}{(n+t+2)_q \cdots (n+t-1)_q}}\, d_{t-2} \tag{6.19}$$

for $t \geqslant -n+2$. Again from (6.17), we have $\beta_t d_t = (-1-q^{4n-4})\gamma_t d_{t-2}$. Hence for $|t| \leqslant n-2$, we obtain the following formula.

$$d_t = q^{-1}\big(q^{-2n+2} + q^{2n-2}\big)\frac{(2)_q}{(4)_q}\sqrt{\frac{(n-t+2)_q(n-t+1)_q}{(n+t)_q(n+t-1)_q}}\, d_{t-2}. \tag{6.20}$$

Then we consider the above equalities (6.19) and (6.20). For $-n+2 \leqslant t \leqslant n-4$ we obtain by using (6.20) twice

$$d_{t+2} = q^{-2}\big(q^{-2n+2} + q^{2n-2}\big)^2\frac{(2)_q^2}{(4)_q^2}\sqrt{\frac{(n-t+2)_q \cdots (n-t-1)_q}{(n+t+2)_q \cdots (n+t-1)_q}}\, d_{t-2}. \tag{6.21}$$

By (6.19) and (6.21) we have a equality

$$d_{t-2} = \big(q^{-2n+2} + q^{2n-2}\big)^2\frac{(2)_q^2}{(4)_q^2} d_{t-2} \tag{6.22}$$

for $-n+2 \leqslant t \leqslant n-4$. We easily see that $1 = (q^{-2n+2} + q^{2n-2})^2\frac{(2)_q^2}{(4)_q^2}$ holds if and only if $n = 0$ or 2, however, this does not occur because n is an odd number. Thus we have $d_{t-2} = 0$ for $-n+2 \leqslant t \leqslant n-4$. It follows $d_t = 0$ for $-n \leqslant t \leqslant n-6$. For $n \geqslant 7$ we can derive $d_t = 0$ immediately. So we have to consider the cases $n = 3$ and $n = 5$.

(1) $n = 5$ *case*. We have already known $d_t = 0$ for $-5 \leqslant t \leqslant -1$. Using (6.20), we have $d_0 = 0$. Hence we have $d_t = 0$ for all $t \in I_5$.

(2) $n = 3$ *case*. We have already known $d_{-3} = 0$. Using (6.20) for $t = -1, 1$, we have $d_t = 0$ for odd t. From (6.19) and (6.20) for $t = 0$ we obtain

$$d_2 = q^{-2} d_{-2},$$
$$d_0 = q^{-1}\big(q^{-4} + q^4\big)\frac{(2)_q}{(3)_q}\sqrt{\frac{(5)_q(4)_q}{(3)_q(2)_q}}\,d_{-2}.$$

This shows d_{-2} and d_2 are real numbers. Now we consider a π_1-eigenvector $\Psi_1(\eta,\eta)=0$. As in the case A_4^*, we obtain

$$\begin{aligned}\sqrt{(6)_q}\, f\cdot\Psi_1(\eta,\eta)_{-1} &= q^{-2}(6)_q\eta_3\eta_{-3} + \big(-q^{-1}(5)_q(2)_q + q^{-3}(6)_q\big)\eta_2\eta_{-2}\\ &\quad + \big((4)_q(3)_q - q^{-2}(5)_q(2)_q\big)\eta_1\eta_{-1} + \big(q^{-1}-q\big)(4)_q(3)_q\eta_0^2\\ &\quad + \big(q^2(5)_q(2)_q - (4)_q(3)_q\big)\eta_{-1}\eta_1 + \big(q(5)_q(2)_q - q^3(6)_q\big)\eta_{-2}\eta_2\\ &\quad - q^2(6)_q\eta_{-3}\eta_3.\end{aligned}$$

Applying $\pi_{\mathbb{T}}$ to the above both sides, we have

$$0 = \big(\big(q-q^{-1}\big)(5)_q(2)_q + \big(q^{-3}-q^3\big)(6)_q\big)d_{-2}d_2 + \big(q^{-1}-q\big)(4)_q(3)_q d_0^2.$$

Since d_{-2} and d_0 are real numbers, we obtain

$$q^{-2}\big((6)_q(3)_q - (5)_q(2)_q\big)d_{-2}^2 + (4)_q(3)_q d_0^2 = 0,$$

however, the left-hand side is positive because the above d_t are real. Hence we can get $d_{-2} = d_0 = d_2 = 0$.

(VI) $\mathbb{T}_m$ $(m \geqslant 2)$ *case*. We treat a C^*-algebra A whose spectral pattern is one of the following:

- $\mathbb{T}_{2\ell-1}$ $(\ell \geqslant 2)$: $\bigoplus_{k\in\mathbb{Z}_{\geqslant 0}}(1+2[\frac{k}{2\ell-1}])\pi_k \oplus \bigoplus_{k\in\mathbb{Z}_{\geqslant 0}} 2[\frac{k+\ell}{2\ell-1}]\pi_{k+\frac{1}{2}}$,
- $\mathbb{T}_{2\ell}$ $(\ell \geqslant 1)$: $\bigoplus_{k\in\mathbb{Z}_{\geqslant 0}}(1+2[\frac{k}{\ell}])\pi_k$.

If A is of type $\mathbb{T}_2$, then we can easily derive $A = C(\mathbb{T}_2\setminus SU_q(2))$. Hence we study the case $\mathbb{T}_m$ for $m \geqslant 3$. In the cases, the π_1-multiplicity are one, so the linear subspace A_{π_1} generates a quantum sphere $C(S^2_{q,\lambda})$. We shall prove this is the canonical homogeneous sphere $C(\mathbb{T}\setminus SU_q(2))$, that is, the parameter $\lambda_0 = (q^{-1}-q)^{-1}\lambda$ is equal to 1. In order to do this, we take the same strategy as in the case of type D_m^*. We have to prepare the following lemma which is proved in similar way to Lemma 6.5 and we omit the proof. Recall a π_1-eigenvector of $C(S^2_{q,\lambda})$,

$$\xi_\lambda^1 = q^{\frac{1}{2}}\sqrt{\frac{1-\lambda_0^2}{(2)_q}}\,\boldsymbol{w}_{-1}^1 + \lambda_0\boldsymbol{w}_0^1 - q^{-\frac{1}{2}}\sqrt{\frac{1-\lambda_0^2}{(2)_q}}\,\boldsymbol{w}_1^1.$$

Lemma 6.7. *Let n be a half-integer with $n \geqslant 1$. Then we have the following equalities for all $t \in I_n$:*

$$\Psi_{n-1}\big(\boldsymbol{w}_{-1}^1, \boldsymbol{w}_t^n\big) = q^{-t+2}\sqrt{\frac{(n+t)_q(n+t-1)_q}{(2n)_q(2n-1)_q}}\,\boldsymbol{w}_{t-1}^{n-1},$$

$$\Psi_{n-1}\bigl(\boldsymbol{w}_0^1, \boldsymbol{w}_t^n\bigr) = -q^{-n-t+1}\sqrt{\frac{(2)_q(n-t)_q(n+t)_q}{(2n)_q(2n-1)_q}}\,\boldsymbol{w}_t^{n-1},$$

$$\Psi_{n-1}\bigl(\boldsymbol{w}_1^1, \boldsymbol{w}_t^n\bigr) = q^{-2n-t}\sqrt{\frac{(n-t)_q(n-t-1)_q}{(2n)_q(2n-1)_q}}\,\boldsymbol{w}_{t+1}^{n-1}.$$

The following lemma is easily proved.

Lemma 6.8. *Let n be a half-integer with $n \geqslant 1$. Then we have the following equalities for all $t \in I_n$:*

$$\begin{aligned} q^{n-1}\sqrt{(2)_q(2n)_q(2n-1)_q}\,\Psi_{n-1}\bigl(\xi_\lambda^1, \boldsymbol{w}_t^n\bigr) = {}& q^{n-t+\frac{3}{2}}\sqrt{1-\lambda_0^2}\sqrt{(n+t)_q(n+t-1)_q}\,\boldsymbol{w}_{t-1}^{n-1} \\ & - q^{-t}(2)_q\lambda_0\sqrt{(n-t)_q(n+t)_q}\,\boldsymbol{w}_t^{n-1} \\ & - q^{-n-t-\frac{3}{2}}\sqrt{1-\lambda_0^2}\sqrt{(n-t)_q(n-t-1)_q}\,\boldsymbol{w}_{t+1}^{n-1}. \end{aligned}$$

We need another lemma whose proof is also the similar as for Lemma 6.6.

Lemma 6.9. *Let n be a half-integer with $n \geqslant 1$ and $\eta = \sum_{t\in I_n} d_t \boldsymbol{w}_t^n$ be a π_n-eigenvector of $C(SU_q(2))$ such that $\Psi_{n-1}(\xi_\lambda^1, \eta) = 0$. Then $\{d_t\}_{t\in I_n}$ satisfies the following recurrence equation for all $t \in I_n = \{-n, -n+1, \ldots, n-1, n\}$:*

$$\begin{aligned} & q^{n+\frac{1}{2}}\sqrt{1-\lambda_0^2}\sqrt{(n+t+1)_q(n+t)_q}\,d_{t+1} \\ & - (2)_q\lambda_0\sqrt{(n+t)_q(n-t)_q}\,d_t \\ & - q^{-n-\frac{1}{2}}\sqrt{1-\lambda_0^2}\sqrt{(n-t+1)_q(n-t)_q}\,d_{t-1} = 0 \end{aligned}$$

where we define $d_t = 0$ if $|t| \geqslant n+1$.

We show $A = C(\mathbb{T}_m \setminus SU_q(2))$ for in each case of odd and even order cyclic groups.

(1) $\mathbb{T}_{2\ell-1}$ ($\ell \geqslant 2$) *case*. We focus on $\pi_{\ell-\frac{1}{2}}$-spectral subspace. Write n for $\ell - \frac{1}{2}$. Since π_n-eigenvector space is two-dimensional and π_{n-1}-eigenvector space is 0, the map $\Psi_{n-1}(\xi_\lambda^1, \cdot) : X_{\pi_n} \to X_{\pi_{n-1}}$ is 0-map. Let $\eta = \sum_{t\in I_n} d_t \boldsymbol{w}_t^n$ be a non-zero π_n-eigenvector for A. Recall its conjugate eigenvector $T\eta = \sum_{t\in I_n}(-q)^{-t}\overline{d_{-t}}\boldsymbol{w}_t^n$, where we have defined $(-q)^s = \sqrt{-1}^{2s}|q|^s$ for all real number s. Then η and $T\eta$ are in the kernel of $\Psi_{n-1}(\xi_\lambda^1, \cdot)$. By Lemma 6.9 we obtain the following recurrence equations:

$$\alpha_t d_{t+1} + \beta_t d_t + \gamma_t d_{t-1} = 0, \tag{6.23}$$

$$q^{-2n}\alpha_t d_{t+1} + \beta_t d_t + q^{2n}\gamma_t d_{t-1} = 0, \tag{6.24}$$

where we put

$$\alpha_t = q^{n+\frac{1}{2}}\sqrt{1-\lambda_0^2}\sqrt{(n+t+1)_q(n+t)_q},$$
$$\beta_t = -(2)_q\lambda_0\sqrt{(n+t)_q(n-t)_q},$$
$$\gamma_t = -q^{-n-\frac{1}{2}}\sqrt{1-\lambda_0^2}\sqrt{(n-t+1)_q(n-t)_q}.$$

Assume that λ_0 is not equal to 1. Then we know $\alpha_t = 0$ if and only if $t = -n$ and $\gamma_t = 0$ if and only if $t = n$. By (6.23) and (6.24) we obtain

$$\alpha_t d_{t+1} = q^{2n}\gamma_t d_{t-1}. \tag{6.25}$$

Hence again by (6.23), we have

$$\beta_t d_t = -\left(1+q^{2n}\right)\gamma_t d_{t-1}.$$

Assume λ_0 is equal to 0. Then the above equality shows $d_{t-1} = 0$ for $-n \leqslant t \leqslant n-1$. By (6.25) d_{n-1} and d_n are also equal to 0, however, this is a contradiction to non-triviality of η. Next we consider the case $0 < \lambda_0 < 1$. From the above equality d_t for $-n+1 \leqslant t \leqslant n-1$ are uniquely determined by d_{-n}. The number d_n is determined by the previous equation as $d_n = q^{2n}\alpha_{n-1}^{-1}\gamma_{n-1}d_{n-2}$ where we use the fact that α_{n-1} is never 0 for $n \geqslant \frac{3}{2}$. Therefore $(d_t)_{t\in I_n}$ is a scalar multiple of a vector. In particular the π_n-eigenvector space X_{π_n} becomes one-dimensional. This is a contradiction. It concludes λ_0 is equal to 1 in this case. From simple computation we obtain $X_n = \mathbb{C}\boldsymbol{w}_{-n}^n + \mathbb{C}\boldsymbol{w}_n^n$. We show $A = C(\mathbb{T}_{2\ell-1} \setminus SU_q(2))$. The C^*-algebra $C(\mathbb{T}_{2\ell-1} \setminus SU_q(2))$ is generated by the matrix elements $w(\pi_\nu)_{ns,t}$ for all $\nu \in \frac{1}{2}\mathbb{Z}_{\geqslant 0}$ and $t \in I_\nu$, where s is an integer such that $ns \in I_\nu$. If A contains $C(\mathbb{T}_{2\ell-1} \setminus SU_q(2))$, then they coincide because of its common spectral pattern. By $SU_q(2)$-invariance of A, it suffices to show $w(\pi_\nu)_{ns,-\nu} = \left[{2\nu \atop \nu+ns}\right]_{q^2}^{\frac{1}{2}} x^{\nu+ns}v^{\nu-ns}$ are contained in A for non-negative s. This is equal to $\left[{2\nu \atop \nu+ns}\right]_{q^2}^{\frac{1}{2}} x^{2ns}x^{\nu-ns}v^{\nu-ns}$. Since A contains $xv \in C(\mathbb{T} \setminus SU_q(2))$ and $x^{2n} = \left[{2\nu \atop \nu+ns}\right]_{q^2}^{-\frac{1}{2}} w(\pi_n)_{-n,-n}$, the element $x^{\nu+ns}v^{\nu-ns}$ is in A. This shows A is in fact $C(\mathbb{T}_{2\ell-1} \setminus SU_q(2))$.

(2) $\mathbb{T}_{2\ell}$ ($\ell \geqslant 2$) *case*. Write n for ℓ in this case. We analyze the recurrence equation in Lemma 6.9 under the condition that a π_n-eigenvector $\eta = \sum_{t\in I_n} d_t\boldsymbol{w}_t^n$ is self-conjugate. Assume that λ_0 is not equal to 1. When λ_0 is equal to 0, η is the zero vector as in the previous case. In the case $0 < \lambda_0 < 1$ we know that the space of its solution is one-dimensional and we can assume that d_t are all real number. Then we have

$$d_t = -\left(1+q^{2n}\right)\beta_t^{-1}\gamma_t d_{t-1}$$

for all $|t| \leqslant n-1$. For $-n+1 \leqslant t \leqslant n-2$ we obtain the following equality:

$$\begin{aligned} d_{-t} &= -\left(1+q^{2n}\right)\frac{\gamma_{-t}}{\beta_{-t}}d_{-t-1} \\ &= -(-q)^{t+1}\left(1+q^{2n}\right)\frac{\gamma_{-t}}{\beta_{-t}}d_{t+1} \end{aligned}$$

$$= (-q)^{t+1}\left(1+q^{2n}\right)^2 \frac{\gamma_{-t}\gamma_{t+1}}{\beta_{-t}\beta_{t+1}} d_t$$
$$= -q\left(1+q^{2n}\right)^2 \frac{\gamma_{-t}\gamma_{t+1}}{\beta_{-t}\beta_{t+1}} d_{-t}.$$

Since its coefficient $\frac{\gamma_{-t}\gamma_{t+1}}{\beta_{-t}\beta_{t+1}}$ is strictly positive, we have $d_t = 0$ for $-n+2 \leqslant t \leqslant n-1$. We now treat a positive integer $n \geqslant 2$. It deduces $-n+2 \leqslant n-2$ and we obtain $d_n = 0$ by (6.25). Hence we have $d_t = 0$ for all $t \in I_n$ because of $-n+2 \leqslant 0$ and the self-conjugacy of η. Under the condition $\lambda_0 < 1$ we have shown $\eta = 0$ if η is self-conjugate and $\Psi_{n-1}(\xi_\lambda^1, \eta) = 0$. Now we consider the map $\Psi_{n-1}(\xi_\lambda^1, \cdot): X_n \to X_{n-1}$. We know X_n is three-dimensional and X_{n-1} is one-dimensional. Take two linearly independent π_n-eigenvectors ζ_1 and ζ_2 from X_n which satisfy $\Psi_{n-1}(\xi_\lambda^1, \zeta_1) = \Psi_{n-1}(\xi_\lambda^1, \zeta_2) = 0$. For conjugate eigenvectors $T\zeta_1$ and $T\zeta_2$, we can take complex numbers μ_1 and μ_2 which are defined by $\Psi_{n-1}(\xi_\lambda^1, T\zeta_1) = \mu_1 \Psi_{n-1}(\xi_\lambda^1, \xi_\lambda^n)$ and $\Psi_{n-1}(\xi_\lambda^1, T\zeta_2) = \mu_2 \Psi_{n-1}(\xi_\lambda^1, \xi_\lambda^n)$, where ξ_λ^ν is a self-conjugate π_ν-eigenvector for $C(S_{q,\lambda}^2)$. Note $\Psi_{n-1}(\xi_\lambda^1, \xi_\lambda^n)$ is not the zero vector. Multiplying complex numbers to ζ_1 and ζ_2, respectively, we may assume μ_1 and μ_2 are real numbers. If μ_1 is equal to 0, then the self-conjugate vectors $\zeta_1 + T\zeta_1$ and $\sqrt{-1}(\zeta_1 - T\zeta_1)$ are in the kernel of $\Psi_{n-1}(\xi_\lambda^1, \cdot)$. From the above discussion they are 0. This shows immediately $\zeta_1 = 0$. This is a contradiction. Hence we have a non-zero real number μ_1. Similarly we show μ_2 is non-zero. We may assume they are equal to 1. For $i = 1, 2$ we have $\Psi_{n-1}(\xi_\lambda^1, \zeta_i + T\zeta_i - \xi_\lambda^n) = 0$. Then it yields $\zeta_i + T\zeta_i = \xi_\lambda^n$ because of the self-conjugacy of $\zeta_i + T\zeta_i - \xi_\lambda^n$ for each i. It follows $\zeta_1 - \zeta_2 = -T(\zeta_1 - \zeta_2)$. In particular $T(\zeta_1 - \zeta_2)$ is in the kernel of $\Psi_{n-1}(\xi_\lambda^1, \cdot)$. This shows $\zeta_1 - \zeta_2 = 0$, however this is a contradiction to linear independence of ζ_1 and ζ_2. Therefore we can derive $\lambda_0 = 1$. Then we obtain $X_n = \mathbb{C}\boldsymbol{w}_{-n}^n + \mathbb{C}\boldsymbol{w}_0^n + \mathbb{C}\boldsymbol{w}_n^n$. As in the proof of the case $\mathbb{T}_{2\ell-1}$, we can prove $A = C(\mathbb{T}_{2\ell} \setminus SU_q(2))$.

7. Classification of right coideals of $C(SU_q(2))$: $-1 < q < 0$ case

We use the same strategy as in the previous section in order to classify right coideals of $C(SU_q(2))$ for negative q. We prepare a positive parameter q_0 defined by $q_0 = -q$. Recall a q-integer $(n)_q = (n)_{q_0}$ and $q^n = \sqrt{-1}^{2n} q_0^n$ for a half-integer n. Now we investigate each type of multiplicity diagrams. Contrary to the case of positive q, we have to treat other graphs which have a single loop at a vertex as listed in Appendix A. Recall the π_1-eigenvector of $C(S_{q,\lambda}^2)$, $\xi_\lambda^1 = (q_0^{\frac{1}{2}}\sqrt{\frac{1-\lambda_0^2}{(2)_q}}, \lambda_0, q_0^{-\frac{1}{2}}\sqrt{\frac{1-\lambda_0^2}{(2)_q}}) w(\pi_1)$. If a right coideal A is of type $\mathbb{T}$, then it actually becomes $\beta_z^L(C(S_{q,\lambda}^2))$ for some $z \in \mathbb{T}$ by Podleś' classification results on the quantum spheres [16, Theorem 1]. We know that $C(SO_q(3))$ and $C(SO_{-q}(3))$ are isomorphic as compact quantum groups. Hence we have the complete collection of right coideals which have only spectral subspaces with integer spins. In this way, we can reject the existence of type A_4^*, S_4^*, A_5^* and D_m^* $(m \geqslant 2)$. For right coideals of D_∞^* and $\mathbb{T}_{2n}$ $(n \geqslant 1)$ we have shown their uniqueness up to conjugation by β^L. Hence we have to investigate right coideals of type $\mathbb{T}_n$ (odd $n \geqslant 3$), D_n (odd $n \geqslant 1$), A_m' $(m \geqslant 3)$. The main classification result for $-1 < q < 0$ is as follows.

Theorem 7.1. *Let $A \subset C(SU_q(2))$ be a right coideal. Then its multiplicity diagram is one of type $1, \mathbb{T}_n$ $(n \geqslant 2)$, $\mathbb{T}, SU(2), D_\infty^*$ and D_1. If it is of type $\mathbb{T}$, it is one of the quantum spheres. Otherwise it is unique up to conjugation by β^L.*

In case (I), it is shown that $\mathbb{T}_{2\ell-1}$ ($\ell \geqslant 2$) types are quotient type. In case (II), D_n (odd $n \geqslant 3$) types are rejected. In case (III), we show that D_1 type survives and its uniqueness. In case (IV), A'_m ($m \geqslant 3$) types are rejected.

(I) $\mathbb{T}_{2\ell-1}$ ($\ell \geqslant 2$) *case*. We treat a C^*-algebra A whose spectral pattern is $\bigoplus_{k\in\mathbb{Z}_{\geqslant 0}}(1+2[\frac{k}{2\ell-1}])\pi_k \oplus \bigoplus_{k\in\mathbb{Z}_{\geqslant 0}} 2[\frac{k+\ell}{2\ell-1}]\pi_{k+\frac{1}{2}}$. As in the previous section we study the C^*-subalgebra generated by π_1-spectral subspace and assert that is the canonical homogeneous sphere. We begin classifying this case by stating a negative q version of Lemma 6.7.

Lemma 7.2. *Let n be a half-integer with $n \geqslant 1$. Then we have the following equalities for all $t \in I_n$:*

$$\Psi_{n-1}\big(\boldsymbol{w}^1_{-1}, \boldsymbol{w}^n_t\big) = q_0^{-t+2}\sqrt{\frac{(n+t)_q(n+t-1)_q}{(2n)_q(2n-1)_q}}\,\boldsymbol{w}^{n-1}_{t-1},$$

$$\Psi_{n-1}\big(\boldsymbol{w}^1_{0}, \boldsymbol{w}^n_t\big) = -(-1)^{n-t}q_0^{-n-t+1}\sqrt{\frac{(2)_q(n-t)_q(n+t)_q}{(2n)_q(2n-1)_q}}\,\boldsymbol{w}^{n-1}_{t},$$

$$\Psi_{n-1}\big(\boldsymbol{w}^1_{1}, \boldsymbol{w}^n_t\big) = q_0^{-2n-t}\sqrt{\frac{(n-t)_q(n-t-1)_q}{(2n)_q(2n-1)_q}}\,\boldsymbol{w}^{n-1}_{t+1}.$$

If necessary, we consider the conjugation by β^L and may assume $C(S^2_{q,\lambda}) \subset A$.

Lemma 7.3. *Let n be a half-integer with $n \geqslant 1$. Then we get the following equality for $t \in I_n$:*

$$\begin{aligned}\Psi_{n-1}\big(\xi^1_\lambda, \boldsymbol{w}^n_t\big) = {}& q_0^{-t+\frac{5}{2}}\sqrt{1-\lambda_0^2}\sqrt{\frac{(n+t)_q(n+t-1)_q}{(2)_q(2n)_q(2n-1)_q}}\,\boldsymbol{w}^{n-1}_{t-1}\\ &-(-1)^{n-t}q_0^{-n-t+1}\lambda_0\sqrt{\frac{(2)_q(n-t)_q(n+t)_q}{(2n)_q(2n-1)_q}}\,\boldsymbol{w}^{n-1}_{t}\\ &+q_0^{-2n-t-\frac{1}{2}}\sqrt{1-\lambda_0^2}\sqrt{\frac{(n-t)_q(n-t-1)_q}{(2)_q(2n)_q(2n-1)_q}}\,\boldsymbol{w}^{n-1}_{t+1}.\end{aligned}$$

Lemma 7.4. *Let n be a half-integer with $n \geqslant 1$ and $\eta = \sum_{t\in I_n} d_t\boldsymbol{w}^n_t$ be a π_n-eigenvector with $\Psi_{n-1}(\xi^1_\lambda, \eta) = 0$. Then we get the following recurrence equation for $t \in I_n$:*

$$\begin{aligned}&q_0^{n+\frac{1}{2}}\sqrt{1-\lambda_0^2}\sqrt{(n+t)_q(n+t+1)_q}\,d_{t+1}\\ &\quad-(-1)^{n-t}\lambda_0(2)_q\sqrt{(n-t)_q(n+t)_q}\,d_t\\ &\quad+q_0^{-n-\frac{1}{2}}\sqrt{1-\lambda_0^2}\sqrt{(n-t)_q(n-t+1)_q}\,d_{t-1} = 0,\end{aligned}$$

where we define $d_t = 0$ for $|t| \geqslant n+1$.

We write n for $\ell-\frac{1}{2}$. Then π_n-eigenvector space of A is two-dimensional. We may assume π_1-part generates a quantum sphere $C(S^2_{q,\lambda})$ as usual. We derive $\lambda_0=1$ and it immediately derives $A=C(\mathbb{T}_{2\ell-1}\setminus C(SU_q(2)))$. Now let $\eta=\sum_{t\in I_n} d_t \boldsymbol{w}^n_t$ be a π_n-eigenvector of A. By the absence of π_{n-1}-part we have $\Psi_{n-1}(\xi^1_\lambda,\eta)=0=\Psi_{n-1}(\xi^1_\lambda,T\eta)$. From the previous lemmas we get

$$\alpha_t d_{t+1}+\beta_t d_t+\gamma_t d_{t-1}=0,$$
$$q_0^{-2n}\alpha_t d_{t+1}-\beta_t d_t+q_0^{2n}\gamma_t d_{t-1}=0,$$

where we put

$$\alpha_t=q_0^{n+\frac{1}{2}}\sqrt{1-\lambda_0^2}\sqrt{(n+t)_q(n+t+1)_q},$$
$$\beta_t=-(-1)^{n-t}\lambda_0(2)_q\sqrt{(n-t)_q(n+t)_q},$$
$$\gamma_t=q_0^{-n-\frac{1}{2}}\sqrt{1-\lambda_0^2}\sqrt{(n-t)_q(n-t+1)_q}.$$

Then we can prove $(d_t)_{t\in I_n}$ is a scalar multiple of a vector as in the $\mathbb{T}_{2\ell-1}$ case in the previous section. This is a contradiction. Hence we have proved $\lambda_0=1$ and $A=C(\mathbb{T}_{2\ell-1}\setminus SU_q(2))$.

(II) D_n (*odd* $n\geqslant 3$) *case*. Before proof of the non-existence of this case, we shall state some basic lemmas without proofs which are proved similarly by direct calculations as before. We state a negative q and Ψ_{n-1} version of Lemma 6.5.

Lemma 7.5. *Let n be a half-integer with $n\geqslant\frac{3}{2}$. Then we have the following equalities for all $t\in I_n$:*

$$\Psi_{n-1}\big(\boldsymbol{w}^2_{-2},\boldsymbol{w}^n_t\big)=(-1)^{n-t}q_0^{-\frac{1}{2}n-2t+6}\frac{1}{(2n-2)_q}\times\sqrt{\frac{(4)_q(n-t+1)_q(n+t)_q(n+t-1)_q(n+t-2)_q}{(2n)_q(2n-1)_q}}\,\boldsymbol{w}^{n-1}_{t-2},$$

$$\Psi_{n-1}\big(\boldsymbol{w}^2_{0},\boldsymbol{w}^n_t\big)=(-1)^{n-t}q_0^{-\frac{3}{2}n-2t+3}\frac{q_0^{-n-1}(n-t-1)_q-q_0^{n+1}(n+t-1)_q}{(2n-2)_q}\times\sqrt{\frac{(3)_q!(n+t)_q(n-t)_q}{(2n)_q(2n-1)_q}}\,\boldsymbol{w}^{n-1}_{t},$$

$$\Psi_{n-1}\big(\boldsymbol{w}^2_{2},\boldsymbol{w}^n_t\big)=-(-1)^{n-t}q_0^{-\frac{5}{2}n-2t}\frac{1}{(2n-2)_q}\times\sqrt{\frac{(4)_q(n+t+1)_q(n-t)_q(n-t-1)_q(n-t-2)_q}{(2n)_q(2n-1)_q}}\,\boldsymbol{w}^{n-1}_{t+2}.$$

We take a vector $\xi^2_0=(q_0\sqrt{(3)_q!},0,-\sqrt{(4)_q},0,q_0^{-1}\sqrt{(3)_q!})w(\pi_2)$ as a π_2-eigenvector of $C(S^2_{q,0})$. Recall its entries generate the right coideal of type D^*_∞. We denote it by $A_{D^*_\infty}$.

Lemma 7.6. *Let n be a half-integer with $n \geqslant \frac{3}{2}$. Then we have the following equalities for all $t \in I_n$:*

$$\begin{aligned}
&-q_0^{\frac{3}{2}n-3}\sqrt{\frac{(2n)_q(2n-1)_q}{(4)_q!}}\Psi_{n-1}\big(\xi_0^2, \boldsymbol{w}_t^n\big)\\
&\quad= -(-1)^{n-t}q_0^{n-2t+4}\sqrt{(n-t+1)_q(n+t)_q(n+t-1)_q(n+t-2)_q}\,\boldsymbol{w}_{t-2}^{n-1}\\
&\qquad+(-1)^{n-t}q_0^{-2t}\big(q_0^{-n-1}(n-t-1)_q - q_0^{n+1}(n+t-1)_q\big)\sqrt{(n+t)_q(n-t)_q}\,\boldsymbol{w}_t^{n-1}\\
&\qquad+(-1)^{n-t}q_0^{-n-2t-4}\sqrt{(n+t+1)_q(n-t)_q(n-t-1)_q(n-t-2)_q}\,\boldsymbol{w}_{t+2}^{n-1}.
\end{aligned}$$

Lemma 7.7. *Let n be a half-integer with $n \geqslant \frac{3}{2}$ and $\eta = \sum_{t\in I_n} d_t \boldsymbol{w}_t^n$ be a π_n-eigenvector of A such that it satisfies $\Psi_{n-1}(\xi_0^2, \eta) = 0$. Then we have the following recurrence equation for $t \in I_n$:*

$$\begin{aligned}
&-q_0^n\sqrt{(n-t-1)_q(n+t+2)_q(n+t+1)_q(n+t)_q}\,d_{t+2}\\
&\quad+\big(q_0^{-n-1}(n-t-1)_q - q_0^{n+1}(n+t-1)_q\big)\sqrt{(n+t)_q(n-t)_q}\,d_t\\
&\quad+q_0^{-n}\sqrt{(n+t-1)_q(n-t+2)_q(n-t+1)_q(n-t)_q}\,d_{t-2} = 0,
\end{aligned}$$

where we define $d_t = 0$ for $|t| \geqslant n+1$.

Lemma 7.8. *Let n be a half-integer in $\frac{3}{2} + \mathbb{Z}_{\geqslant 0}$ and A be a right coideal in $C(SU_q(2))$ such that A contains $A_{D_\infty^*}$. If a π_n-eigenvector η of A satisfies $\Psi_{n-1}(\xi_0^2, \eta) = 0$ and $\Psi_{n-1}(\xi_0^2, T\eta) = 0$, then η is the zero vector.*

Proof. Take complex numbers $\{d_t\}_{t\in I_n}$ with $\eta = \sum_{t\in I_n} d_t \boldsymbol{w}_t^n$. Note that $T\eta = \sum_{t\in I_n} (-q)^{-t}\overline{d_{-t}}\,\boldsymbol{w}_t^n$. Let us prepare the following notations α_t, β_t and γ_t:

$$\begin{aligned}
\alpha_t &= -q_0^n\sqrt{(n-t-1)_q(n+t+2)_q(n+t+1)_q(n+t)_q},\\
\beta_t &= \big(q_0^{-n-1}(n-t-1)_q - q_0^{n+1}(n+t-1)_q\big)\sqrt{(n+t)_q(n-t)_q},\\
\gamma_t &= q_0^{-n}\sqrt{(n+t-1)_q(n-t+2)_q(n-t+1)_q(n-t)_q}.
\end{aligned}$$

From our assumption on η and $T\eta$, we get the following recurrence equations for $t \in I_n$ by the previous lemma:

$$\begin{aligned}
\alpha_t d_{t+2} + \beta_t d_t + \gamma_t d_{t-2} &= 0,\\
q_0^{-2n+2}\alpha_t d_{t+2} - \beta_{-t} d_t + q_0^{2n-2}\gamma_t d_{t-2} &= 0.
\end{aligned} \tag{7.1}$$

In the above equations put $t = -n+1$ and we get

$$-q_0^n\sqrt{(2n-2)_q(3)_q!}\,d_{-n+3}+q_0^{-n-1}(2n-2)_q\sqrt{(2n-1)_q}\,d_{-n+1}=0,$$
$$-q_0^{-n+2}\sqrt{(2n-2)_q(3)_q!}\,d_{-n+3}+q_0^{n+1}(2n-2)_q\sqrt{(2n-1)_q}\,d_{-n+1}=0.$$

This derives $d_{-n+1}=d_{-n+3}=0$ because $0<q_0<1$. We know $\alpha_t=0$ if and only if $t=-n, n-1$. By using (7.1) inductively, we get $d_{-n+2k-1}=0$ for $k=1,\dots,n+\frac{1}{2}$. Similarly we get $d_{n-2k+1}=0$ for $k=1,\dots,n+\frac{1}{2}$. Hence we have proved $\eta=0$. □

Let A be a right coideal of type D_n (odd $n\geqslant 3$). Then its spectral pattern is

$$\bigoplus_{k\in\mathbb{Z}_{\geqslant 0}}\left(\left[\frac{k}{n}\right]+\frac{1+(-1)^k}{2}\right)\pi_k\oplus\bigoplus_{k\in\mathbb{Z}_{\geqslant 0}}\left(\left[\frac{2k+1}{n}\right]-\left[\frac{k}{n}\right]\right)\pi_{k+\frac{1}{2}}.$$

If we look at the integer spin spectral pattern, then we have $\pi_0\oplus\pi_2\oplus\cdots$. Therefore the π_2-spectral subspace A_{π_2} generates a right coideal of type D_∞^*. Considering $\beta_z^L(A)$ for some $z\in\mathbb{T}$, we may assume A contains $A_{D_\infty^*}$. Next look at the half-integer part and we see that $\pi_{\frac{n}{2}}$ appears once and $\pi_{\frac{n}{2}-1}$ does not there. Then we can make use of Lemma 7.8 and conclude such a right coideal does not exist.

(III) D_1 *case*. We show the existence of this case and its uniqueness up to conjugation. In order to do we need some elementary lemmas.

Lemma 7.9. *Take a positive integer n. Then we have the following equalities*:

$$f\cdot x^n=\sqrt{-1}^{\,-n+1}q_0^{\frac{n-1}{2}}(n)_q x^{n-1}u,\qquad f\cdot v^n=\sqrt{-1}^{\,-n+1}q_0^{\frac{n-1}{2}}(n)_q v^{n-1}y.$$

Proof. It is easily proved by using

$$x^n=w(\pi_{\frac{n}{2}})_{-\frac{n}{2},-\frac{n}{2}}\quad\text{and}\quad f\cdot x^n=\sqrt{-1}^{\,-n+1}\sqrt{(n)_q}\,w(\pi_{\frac{n}{2}})_{-\frac{n}{2},-\frac{n}{2}+1}.$$

It is similar for v^n. □

Lemma 7.10. *For $r,s\geqslant 0$ we have the following equality*:

$$h\big(x^ry^ru^sv^s\big)=q_0^{r(s+1)}\frac{(r)_q!(s)_q!}{(r+s+1)_q!}.$$

Proof. Recall $f\cdot y^r=f\cdot u^{s-1}=0$ and we get

$$\begin{aligned}f\cdot x^{r+1}y^ru^{s-1}v^s&=\big(f\cdot x^{r+1}\big)\cdot\big(k\cdot y^ru^{s-1}v^s\big)+\big(k^{-1}\cdot x^{r+1}\big)\cdot\big(f\cdot y^ru^{s-1}v^s\big)\\&=\sqrt{-1}^{\,-r}q_0^{\frac{r}{2}}(r+1)_q\cdot q^{-\frac{r}{2}-\frac{s-1}{2}+\frac{s}{2}}x^ruy^ru^{s-1}v^s\\&\quad+q^{\frac{-(r+1)}{2}}\cdot q^{\frac{r}{2}+\frac{s-1}{2}}\sqrt{-1}^{\,-s+1}q_0^{\frac{s}{2}}(s)_qx^{r+1}y^ru^{s-1}v^{s-1}y\\&=\sqrt{-1}^{\,-r}q_0^{\frac{r}{2}}(r+1)_q\cdot q^{-\frac{3r}{2}+\frac{1}{2}}x^ry^ru^sv^s\end{aligned}$$

$$+ q^{\frac{s}{2}-1}\sqrt{-1}^{-s+1} q_0^{\frac{s}{2}}(s)_q q^{-2(s-1)} x^{r+1} y^{r+1} u^{s-1} v^{s-1}$$
$$= \sqrt{-1} q_0^{-r+\frac{1}{2}}(r+1)_q \cdot x^r y^r u^s v^s - \sqrt{-1} q_0^{-s+\frac{1}{2}}(s)_q x^{r+1} y^{r+1} u^{s-1} v^{s-1}.$$

The Haar state h has the property that $h(f \cdot a) = 0$ for $a \in A(SU_q(2))$. Apply h to the above both sides and we have

$$h\left(x^{r+1} y^{r+1} u^{s-1} v^{s-1}\right) = q_0^{s-r} \frac{(r+1)_q}{(s)_q} h\left(x^r y^r u^s v^s\right).$$

Inductively we can calculate as desired. $\square$

Recall the $\pi_{\frac{1}{2}}$-spectral subspace $Y_{\frac{1}{2}} = \mathbb{C}x + \mathbb{C}v + \mathbb{C}u + \mathbb{C}y$. Let $P_{\frac{1}{2}} : C(SU_q(2)) \to Y_{\frac{1}{2}}$ be the projection introduced in the second section. It is an orthogonal projection with respect to the Haar state.

Lemma 7.11. *For $r, s \geqslant 0$ we have the following equalities*:

$$P_{\frac{1}{2}}\left(x^r y^r u^s v^s u\right) = q_0^{r(s+2)} \frac{(2)_q (r)_q!(s+1)_q!}{(r+s+2)_q!} u,$$
$$P_{\frac{1}{2}}\left(x^r y^r u^s v^s y\right) = q_0^{r(s+1)-s} \frac{(2)_q (r+1)_q!(s)_q!}{(r+s+2)_q!} u.$$

Proof. Recall two maps $\beta^L = (\pi_{\mathbb{T}} \otimes \mathrm{id}) \circ \delta$ and $\beta^R = (\mathrm{id} \otimes \pi_{\mathbb{T}}) \circ \delta$ which act on x, v, u and y as follows:

$$\begin{pmatrix} \beta_z^L(x) & \beta_z^L(u) \\ \beta_z^L(v) & \beta_z^L(y) \end{pmatrix} = \begin{pmatrix} zx & zu \\ \bar{z}v & \bar{z}y \end{pmatrix},$$
$$\begin{pmatrix} \beta_z^R(x) & \beta_z^R(u) \\ \beta_z^R(v) & \beta_z^R(y) \end{pmatrix} = \begin{pmatrix} zx & \bar{z}u \\ zv & \bar{z}y \end{pmatrix},$$

where z runs on the torus $\mathbb{T} \subset \mathbb{C}$. Hence the above four elements have the different spectrums with respect to β^L and β^R. It is easy to see that the element $x^r y^r u^s v^s u$ has the same spectrum as u. Hence it has the expansion in $L^2(SU_q(2))$ $x^r y^r u^s v^s u = \lambda u + \cdots$ where λ is a complex number. By orthogonality of u and $w(\pi_\nu)_{i,j}$ for $\nu \geqslant 1$ and $i, j \in I_\nu$, we get

$$\begin{aligned} \lambda &= h(uu^*)^{-1} h\left(x^r y^r u^s v^s uu^*\right) \\ &= q_0(2)_q\left(-q^{-1}\right) h\left(x^r y^r u^{s+1} v^{s+1}\right) \\ &= q_0^{r(s+2)} \frac{(2)_q (r)_q!(s+1)_q!}{(r+s+2)_q!}. \end{aligned}$$

We also get the desired result on y in the same way. $\square$

We propose the following lemma. Later we see this equality guarantees the existence of the right coideal of type D_1. Let us write $p_r = \prod_{t=1}^{r}(1+q^{-t})$.

Lemma 7.12. *For $k \geqslant 0$ we have the following equality*:

$$\sum_{r=0}^{k} \begin{bmatrix} k \\ r \end{bmatrix}_q (-1)^{k-r} q^{r(r-k+1)} p_r p_{k-r+1} = q^{-k} \sum_{r=0}^{k} \begin{bmatrix} k \\ r \end{bmatrix}_q (-1)^{k-r} q^{r(r-k+2)} p_{r+1} p_{k-r}.$$

Moreover, this is precisely equal to $(1+q^{-1})(q^{-2};q^{-1})_k$.

Proof. Denote the above left- and right-hand sides by a_k and b_k, respectively. We want to lead the recurrence formula of them. In computations below, we use the following equalities:

$$\begin{bmatrix} n+1 \\ r \end{bmatrix}_q = \begin{bmatrix} n \\ r-1 \end{bmatrix}_q + q^r \begin{bmatrix} n \\ r \end{bmatrix}_q, \qquad p_{n+1} = p_n + q^{-n-1} p_n.$$

Change variables r to $k-r$ in the formula of a_k and b_k and we get

$$a_k = q^k \sum_{r=0}^{k} \begin{bmatrix} k \\ r \end{bmatrix}_q (-1)^r q^{r(r-k-1)} p_{r+1} p_{k-r},$$
$$b_k = q^k \sum_{r=0}^{k} \begin{bmatrix} k \\ r \end{bmatrix}_q (-1)^r q^{r(r-k-2)} p_r p_{k-r+1}.$$

For a_k we compute as follows:

$$\begin{aligned}
q^{-k-1} a_{k+1} &= \sum_{r=0}^{k+1} \begin{bmatrix} k+1 \\ r \end{bmatrix}_q (-1)^r q^{r(r-k-2)} p_{r+1} p_{k-r+1} \\
&= \sum_{r=0}^{k+1} \left(\begin{bmatrix} k \\ r-1 \end{bmatrix}_q + q^r \begin{bmatrix} k \\ r \end{bmatrix}_q \right) (-1)^r q^{r(r-k-2)} p_{r+1} p_{k-r+1} \\
&= \sum_{r=0}^{k} \begin{bmatrix} k \\ r \end{bmatrix}_q (-1)^{r+1} q^{(r+1)(r-k-1)} p_{r+2} p_{k-r} + \sum_{r=0}^{k} \begin{bmatrix} k \\ r \end{bmatrix}_q (-1)^r q^{r(r-k-1)} p_{r+1} p_{k-r+1} \\
&= -q^{-k-1} \left(\sum_{r=0}^{k} \begin{bmatrix} k \\ r \end{bmatrix}_q (-1)^{r+1} q^{r(r-k)} \left(1+q^{-r-2}\right) p_{r+1} p_{k-r} \right) \\
&\quad + \sum_{r=0}^{k} \begin{bmatrix} k \\ r \end{bmatrix}_q (-1)^r q^{r(r-k-1)} \left(1+q^{-k+r-1}\right) p_{r+1} p_{k-r} \\
&= \left(1-q^{-k-3}\right) \sum_{r=0}^{k} \begin{bmatrix} k \\ r \end{bmatrix}_q (-1)^r q^{r(r-k-1)} p_{r+1} p_{k-r} \\
&= q^{-k}\left(1-q^{-k-3}\right) a_k.
\end{aligned}$$

Hence we get $a_{k+1} = (q - q^{-k-2}) a_k$. Similarly we compute b_{k+1} as follows:

$$
\begin{aligned}
q^{-k-1}b_{k+1} &= \sum_{r=0}^{k+1} \begin{bmatrix} k+1 \\ r \end{bmatrix}_q (-1)^r q^{r(r-k-3)} p_r p_{k-r+2} \\
&= \sum_{r=0}^{k+1} \left(\begin{bmatrix} k \\ r-1 \end{bmatrix}_q + q^r \begin{bmatrix} k \\ r \end{bmatrix}_q \right) (-1)^r q^{r(r-k-3)} p_r p_{k-r+2} \\
&= \sum_{r=0}^{k} \begin{bmatrix} k \\ r \end{bmatrix}_q (-1)^{r+1} q^{(r+1)(r-k-2)} p_{r+1} p_{k-r+1} + \sum_{r=0}^{k} \begin{bmatrix} k \\ r \end{bmatrix}_q (-1)^r q^{r(r-k-2)} p_r p_{k-r+2} \\
&= -q^{-k-2} \left(\sum_{r=0}^{k} \begin{bmatrix} k \\ r \end{bmatrix}_q (-1)^r q^{r(r-k-1)} \big(p_r + q^{-r-1} p_r \big) p_{k-r+1} \right) \\
&\quad + \sum_{r=0}^{k} \begin{bmatrix} k \\ r \end{bmatrix}_q (-1)^r q^{r(r-k-2)} \big(1 + q^{-k+r-2} \big) p_r p_{k-r+1} \\
&= \big(-q^{-k-3} + 1 \big) \sum_{r=0}^{k} \begin{bmatrix} k \\ r \end{bmatrix}_q (-1)^r q^{r(r-k-2)} p_r p_{k-r+1} \\
&= q^{-k} \big(1 - q^{-k-3} \big) b_k.
\end{aligned}
$$

Hence we get $b_{k+1} = (q - q^{-k-2}) b_k$, which is the same form as a_{k+1}. We can easily check $a_0 = p_1 = b_0$ and it deduces the desired equality. $\square$

At last, we prove the existence of a right coideal A of type D_1. It has a spectral pattern, $\bigoplus_{k \in \mathbb{Z}_{\geqslant 0}} (k + \frac{1+(-1)^k}{2}) \pi_k \oplus \bigoplus_{k \in \mathbb{Z}_{\geqslant 0}} (k+1) \pi_{k+\frac{1}{2}}$. If A really exists, its spectral subspace $A_{\pi_{\frac{1}{2}}}$ generates the whole C^*-algebra A. In fact, the generated C^*-algebra contains $\pi_{\frac{1}{2}}$ and the only D_1 case admits it. We have to clarify a self-conjugate $\pi_{\frac{1}{2}}$-eigenvector $\eta = \sum_{s \in I_{\frac{1}{2}}} d_s \boldsymbol{w}_s^{\frac{1}{2}}$. By the self-conjugacy, we get $d_{-\frac{1}{2}} = (-q)^{\frac{1}{2}} \overline{d_{\frac{1}{2}}}$. We may assume $d_{\frac{1}{2}}$ is real number by applying β_z^L for some $z \in \mathbb{T}$. Uniqueness up to conjugation follows from this observation. Hence we conclude $\eta = (\sqrt{q_0} x + v, \sqrt{q_0} u + y)$ is a desired $\pi_{\frac{1}{2}}$-vector. Let us write $a = \sqrt{q_0} x + v$ and $b = \sqrt{q_0} u + y$. We study the C^*-algebra A, which is generated by a and b, and derive the result that it is really of type D_1. By direct calculation, we obtain $a^* = \sqrt{q_0} b$ and $\sqrt{q_0} ab + \sqrt{q_0}^{-1} ba = 1 + q_0$. Because of this equality, the smooth part of A is linearly spanned by $a^k b^\ell$ for $k, \ell \geqslant 0$. We shall show $P_{\frac{1}{2}}(a^k b^\ell) \in \mathbb{C}a + \mathbb{C}b$. If it is done, then it shows the $\pi_{\frac{1}{2}}$-multiplicity is exactly one. Since other types which have a single loop at a vertex do not occur, we can conclude A is of type D_1. Now we start a proof. Applying β^R to a and b, we get $\beta_z^R(a) = za$ and $\beta_z^L(b) = \bar{z} b$ for $z \in \mathbb{T}$. Hence the element $P_{\frac{1}{2}}(a^k b^\ell)$ must have the following form.

$$
\begin{cases}
P_{\frac{1}{2}}\big(a^{\ell+1} b^\ell\big) = \lambda a, \\
P_{\frac{1}{2}}\big(a^k b^{k+1}\big) = \mu b, \\
P_{\frac{1}{2}}\big(a^k b^\ell\big) = 0 \quad \text{if } |k - \ell| \neq 1,
\end{cases}
$$

where λ and μ are complex number. If we show $a^k b^{k+1} = \mu b + \cdots$, then we have $a^{k+1} b^k = \overline{\mu} a + \cdots$. Hence it suffices to show $P_{\frac{1}{2}}(a^k b^{k+1}) = \mu b$. Recall the following well-known binomial equality

$$(c+d)^n = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix}_q c^r d^{n-r} \quad \text{if } dc = qcd.$$

Then we have

$$a^k b^{k+1} = \sum_{\substack{0 \leqslant r \leqslant k \\ 0 \leqslant s \leqslant k+1}} \begin{bmatrix} k \\ r \end{bmatrix}_q \begin{bmatrix} k+1 \\ s \end{bmatrix}_q \sqrt{q_0}^{\,s+r} x^r v^{k-r} u^s y^{k+1-s}.$$

We want to take out the coefficient of u and y. For the sake of this, we make use of β^L. Since we have $\beta_z^L(x^r v^{k-r} u^s y^{k+1-s}) = z^{2r+2s-2k-1} x^r v^{k-r} u^s y^{k+1-s}$, we obtain

$$\begin{aligned}
P_{\frac{1}{2}}\bigl(a^k b^{k+1}\bigr) &= \sum_{r=0}^{k} \begin{bmatrix} k \\ r \end{bmatrix}_q \begin{bmatrix} k+1 \\ k+1-r \end{bmatrix}_q \sqrt{q_0}^{\,k+1} P_{\frac{1}{2}}\bigl(x^r v^{k-r} u^{k+1-r} y^r\bigr) \\
&\quad + \sum_{r=0}^{k} \begin{bmatrix} k \\ r \end{bmatrix}_q \begin{bmatrix} k+1 \\ k-r \end{bmatrix}_q \sqrt{q_0}^{\,k} P_{\frac{1}{2}}\bigl(x^r v^{k-r} u^{k-r} y^{r+1}\bigr) \\
&= \sum_{r=0}^{k} \begin{bmatrix} k \\ r \end{bmatrix}_q \begin{bmatrix} k+1 \\ r \end{bmatrix}_q \sqrt{q_0}^{\,k+1} q^{-r(-2r+2k+1)} P_{\frac{1}{2}}\bigl(x^r y^r u^{k-r} v^{k-r} u\bigr) \\
&\quad + \sum_{r=0}^{k} \begin{bmatrix} k \\ r \end{bmatrix}_q \begin{bmatrix} k+1 \\ k-r \end{bmatrix}_q \sqrt{q_0}^{\,k} q^{-r(-2r+2k)} P_{\frac{1}{2}}\bigl(x^r y^r u^{k-r} v^{k-r} y\bigr) \\
&= \frac{\sqrt{q_0}^{\,k+1} (2)_q}{(k+2)_q} \sum_{r=0}^{k} \begin{bmatrix} k \\ r \end{bmatrix}_q \begin{bmatrix} k+1 \\ r \end{bmatrix}_q (-1)^{r(k-r)} q^{r(r-k+1)} \frac{(r)_q!(k-r+1)_q!}{(k+1)_q!} u \\
&\quad + \frac{\sqrt{q_0}^{\,k} q_0^{-k} (2)_q}{(k+2)_q} \sum_{r=0}^{k} \begin{bmatrix} k \\ r \end{bmatrix}_q \begin{bmatrix} k+1 \\ k-r \end{bmatrix}_q (-1)^{r(k-r)+k} q^{r(r-k+2)} \frac{(r+1)_q!(k-r)_q!}{(k+1)_q!} y,
\end{aligned} \tag{7.2}$$

where we have used Lemma 7.11 in the last equality. Moreover we use the following formula:

$$\frac{(n)_q!}{(r)_q!(n-r)_q!} = (-1)^{r(n-r)} \frac{p_n}{p_r p_{n-r}} \begin{bmatrix} n \\ r \end{bmatrix}_q.$$

Then (7.2) is equal:

$$(7.2) = \frac{(-1)^k \sqrt{q_0}^{\,k+1} (2)_q}{(k+2)_q p_{k+1}} \sum_{r=0}^{k} \begin{bmatrix} k \\ r \end{bmatrix}_q (-1)^{k-r} q^{r(r-k+1)} p_r p_{k-r+1} u$$

$$+\frac{(-1)^k\sqrt{q_0}^k q_0^{-k}(2)_q}{(k+2)_q!p_{k+1}}\sum_{r=0}^{k}\begin{bmatrix}k\\r\end{bmatrix}_q(-1)^{k-r}q^{r(r-k+2)}p_{r+1}p_{k-r}y$$

$$=\frac{(-1)^k\sqrt{q_0}^k(2)_q}{(k+2)_q p_{k+1}}\big(1+q^{-1}\big)\big(q^{-2};q^{-1}\big)_k(\sqrt{q_0}u+y),$$

where we have used Lemma 7.12. Hence we have obtained $P_{\frac{1}{2}}(a^k b^{k+1})\in\mathbb{C}(\sqrt{q_0}u+y)$ and this completes the proof of the existence of a right coideal of type D_1.

(IV) A'_m ($3\leqslant m\leqslant\infty$) *case*. We assume that there exists a right coideal of type A'_m. Although we only treat the case of finite m here, we can similarly derive a contradiction in the case of $m=\infty$. Look at the first vertex from the left of Fig. 14. Since the entry of the Perron–Frobenius eigenvector of this vertex is 1, the corresponding ergodic system becomes a right coideal of $C(SU_q(2))$ by Proposition 4.22. Let us denote the right coideal by A. It is also of type A'_m by Theorem 4.18. The spectral subspace $A_{\pi_{\frac{1}{2}}}$ generates a right coideal of type D_1, which is denoted by B. Now we consider the subsystem $B\subset A$. Let Λ be the inclusion matrix of $B\rtimes_\delta SU_q(2)\subset A\rtimes_\delta SU_q(2)$. Apply Proposition 4.20 and we get the equality $\Lambda\mathbb{M}^B(\pi_{\frac{1}{2}})=\mathbb{M}^A(\pi_{\frac{1}{2}})\Lambda$, where $\mathbb{M}^B$ and $\mathbb{M}^A$ are the multiplicity maps of B and A, respectively. They have the following form:

$$\mathbb{M}^B(\pi_{\frac{1}{2}})=\begin{pmatrix}1&1\\1&1\end{pmatrix},\qquad \mathbb{M}^A(\pi_{\frac{1}{2}})=\begin{pmatrix}1&1&0&0&\dots&0&0\\1&0&1&\ddots&\ddots&0&0\\0&1&0&\ddots&\ddots&\vdots&\vdots\\0&0&\ddots&\ddots&\ddots&0&0\\\vdots&\ddots&\ddots&\ddots&\ddots&1&0\\0&0&\dots&0&1&0&1\\0&0&\dots&0&0&1&1\end{pmatrix}.$$

We know that the minimal projection $p_0\in C_r^*(SU_q(2))$ is also minimal in $B\rtimes_\delta SU_q(2)$ and $A\rtimes_\delta SU_q(2)$. We discuss the corresponding vertex of p_0. About $B\rtimes_\delta SU_q(2)$ we may assume p_0 sits at the left vertex of Fig. 12. About $A\rtimes_\delta SU_q(2)$ we may assume p_0 sits at the first vertex from the left in Fig. 14 because the reduced system $p_0(A\otimes\mathbb{K}(L^2(SU_q(2))))p_0$ is canonically isomorphic to A. Hence the inclusion matrix Λ must be as the following form:

$$\Lambda=\begin{pmatrix}1&\lambda_1\\0&\lambda_2\\\vdots&\vdots\\0&\lambda_{m-1}\\0&\lambda_m\end{pmatrix}.$$

Then we want to solve the equation $\Lambda\mathbb{M}^B(\pi_{\frac{1}{2}})=\mathbb{M}^A(\pi_{\frac{1}{2}})\Lambda$ about the non-negative integers $\lambda_1\dots\lambda_m$, however, we immediately see it has no solutions. This is a contradiction and hence there is not a right coideal of type A'_m.

8. Classification of right coideals of $C(SU_{-1}(2))$

In this final section we complete the classification program of right coideals associated to the quantum $SU(2)$ group. Its representation theory such as actions of $U_{-1}(su_2)$ on $C(SU_{-1}(2))$ or the Clebsh–Gordan coefficients is obtained by the limit of $q_0 \to 1$. The continuous function algebra $C(SU_{-1}(2))$ is generated by x, u, v and y which satisfy

$$ux = -xu, \qquad vx = -xv, \qquad uy = -yu, \qquad vy = -yv, \qquad uv = vu,$$
$$xy + uv = yx + uv = 1, \qquad x^* = y, \qquad u^* = v.$$

By simple calculation, we see $\kappa^2 = \mathrm{id}$. Hence $C(SU_{-1}(2))$ is a compact Kac algebra. Refer the theory of general Kac algebras to [8]. One of differences between cases of $q = -1$ and $q^2 \neq 1$ is amount of the quantum subgroups (or right coideals). Indeed, as we have seen there are not a lot of right coideals in $C(SU_q(2))$, on the contrary, in [17] he completely collects plentiful quantum subgroups of $SU_{-1}(2)$. The main result in this section is as follows. The definition of $\eta^{\frac{n}{2}}$ and $\hat{\eta}^{\frac{n}{2}}$ is given in the case of type $\mathbb{T}_n$ with odd $n \geqslant 3$.

Theorem 8.1. *If a right coideal A is not of type $\mathbb{T}_n$ (odd $n \geqslant 3$) or D_n (odd $n \geqslant 1$), there exists a closed subgroup H in $SO_1(3)$ such that A is $C(H \setminus SO_{-1}(3))$. If a right coideal A is of type $\mathbb{T}_n$ (odd $n \geqslant 3$), A is conjugated to $C(\mathbb{T}_n \setminus SU_{-1}(2))$ or $C^*(\eta^{\frac{n}{2}}, \hat{\eta}^{\frac{n}{2}})$. If a right coideal A is of type D_1, then A is conjugated to $C(D_1 \setminus SU_{-1}(2))$. If a right coideal A is of type D_n (odd $n \geqslant 3$), A is conjugated to $C(D_n \setminus SU_{-1}(2))$ or $C^*(\eta^{\frac{n}{2}})$. Here conjugation is given by β_z^L for some $z \in \mathbb{T}$.*

Although $C(\mathbb{T}_n \setminus SU_{-1}(2))$ is not isomorphic to $C^*(\eta^{\frac{n}{2}}, \hat{\eta}^{\frac{n}{2}})$ as a $SU_{-1}(2)$-covariant system, in Proposition 8.11 we show that $C(D_n \setminus SU_{-1}(2))$ is isomorphic to $C^*(\eta^{\frac{n}{2}})$ as a $SU_{-1}(2)$-covariant system. As we have said in the previous section, we have the isomorphism between $C(SO_{-1}(3))$ and $C(SO_1(3))$ as compact quantum groups. Hence we can conclude that right coideals are quotient by subgroups when they are one of type A_4^*, S_4^*, A_5^*, D_m^* ($m \geqslant 2$), D_∞^*, $\mathbb{T}_{2n}$ ($n \geqslant 1$). Hence we have to study other types: $\mathbb{T}_n$ (odd $n \geqslant 3$), A'_m ($m \geqslant 3$), D_n (odd $n \geqslant 1$). Contents of proofs are as follows. In (I), A'_m ($m \geqslant 3$) types are rejected. In (II), we show D_1 type is unique and therefore quotient type. In (III), $\mathbb{T}_n$ (odd $n \geqslant 3$) types are classified. In the last (IV), D_n (odd $n \geqslant 1$) types are classified.

(I) A'_m ($m \geqslant 3$) *case*. The entirely same proof as in the previous section derives a contradiction well.

(II) D_1 *case*. If we consider the limit $q_0 \to 1$ in the previous section, we can show its existence. In that procedure we do not need any lemma stated in that section, because $\pi_{\frac{1}{2}}$-eigenvector is uniquely determined and in [17, Proposition 3.8] it has been shown that there exists a right coideal by subgroup D_1 whose $\pi_{\frac{1}{2}}$-multiplicity is one.

(III) $\mathbb{T}_n$ (*odd* $n \geqslant 3$) *case*. A has the spectral pattern

$$\bigoplus_{k \in \mathbb{Z}_{\geqslant 0}} \left(1 + 2\left[\frac{k}{n}\right]\right)\pi_k \oplus \bigoplus_{k \in \mathbb{Z}_{\geqslant 0}} 2\left[\frac{2k+n+1}{2n}\right]\pi_{k+\frac{1}{2}}.$$

Since its integer part becomes a right coideal in $C(SO_{-1}(3))$, it must be quotient by a subgroup $H \subset SO_{-1}(3)$. From the spectral pattern there exist angles $0 \leqslant \chi < 2\pi$ and $0 \leqslant \psi < \pi$ such that $H = \mathbb{T}_n^{\chi,\psi}$. We begin to determine a π_1-eigenvector of $C(\mathbb{T}_n^{\chi,\psi} \setminus SO_{-1}(3))$.

Lemma 8.2. *The following vector is a π_1-eigenvector for $C(\mathbb{T}_n^{\chi,\psi} \setminus SO_{-1}(3))$:*

$$\xi^{\chi,\psi} = \sqrt{2}^{-1}\big(-ie^{i\chi}\sin\psi, \sqrt{2}\cos\psi, ie^{-i\chi}\sin\psi\big)w(\pi_1).$$

Proof. First we consider a π_1-eigenvector $\zeta^{\chi,\psi} = \sum_{t\in I_1} c_t \boldsymbol{w}_t^1$ of $C(\mathbb{T}_n^{\chi,\psi} \setminus SO(3))$. The left action of $\pi_1(g) \in \mathbb{T}_n^{\chi,\psi}$ is given by multiplication of $w(\pi_1)(g)$ to $w(\pi_1)$ from the left. Hence we have to get a vector $(c_{-1}, c_0, c_1)w(\pi_1)(g) = (c_{-1}, c_0, c_1)$ for all $g \in SU(2)$ with $\pi_1(g) \in \mathbb{T}_n^{\chi,\psi}$. If χ and ψ are equal to 0, we can easily get $c_{\pm 1} = 0$ and $c_0 = 1$. Since we know $\mathbb{T}_n^{\chi,\psi} = \mathrm{Ad}(\pi_1(r^{12}(\chi)r^{13}(\psi)))(\mathbb{T}_n^{0,0})$, we obtain $(c_{-1}, c_0, c_1) = (0,1,0)w(\pi_1)(r^{13}(-\psi)r^{12}(-\chi))$. Using two matrices

$$w(\pi_1)\big(r^{13}(-\psi)\big) = \begin{pmatrix} \cos^2\frac{\psi}{2} & -\sqrt{2}\sin\frac{\psi}{2}\cos\frac{\psi}{2} & \sin^2\frac{\psi}{2} \\ \sqrt{2}\sin\frac{\psi}{2}\cos\frac{\psi}{2} & 1-2\sin^2\frac{\psi}{2} & -\sqrt{2}\sin\frac{\psi}{2}\cos\frac{\psi}{2} \\ \sin^2\psi & -\sqrt{2}\sin\frac{\psi}{2}\cos\frac{\psi}{2} & \cos^2\frac{\psi}{2} \end{pmatrix}$$

and

$$w(\pi_1)\big(r^{12}(-\chi)\big) = \begin{pmatrix} e^{i\chi} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\chi} \end{pmatrix},$$

we have $c_{\pm 1} = \mp\sqrt{2}^{-1}e^{\mp i\chi}\sin\psi$ and $c_0 = \cos\psi$. Next we make use of $\Xi_1 : C(SO_1(3)) \to C(SO_{-1}(3))$ and get the desired eigenvector. $\square$

Next we determine the $\pi_{\frac{n}{2}}$-eigenvector space which is two-dimensional. Let us write m for $\frac{n}{2}$. For $\eta = \sum_{t\in I_m} d_t \boldsymbol{w}_t^m$ we use $q = -1$ version of Lemma 7.2 and we obtain

$$\begin{aligned}\sqrt{2m(2m-1)}\,\Psi_{m-1}\big(\xi^{\chi,\psi}, \eta\big) = \sum_{t\in I_m} \Big\{ & -ie^{i\chi}\sin\psi\sqrt{(m+t)(m+t+1)}\,d_{t+1} \\ & - (-1)^{m-t}2\cos\psi\sqrt{(m-t)(m+t)}\,d_t \\ & + ie^{-i\chi}\sin\psi\sqrt{(m-t)(m-t+1)}\,d_{t-1}\Big\}\boldsymbol{w}_t^m.\end{aligned}$$

Define α_t, β_t and γ_t by

$$\begin{aligned}\alpha_t &= -ie^{i\chi}\sin\psi\sqrt{(m+t)(m+t+1)}, \\ \beta_t &= -(-1)^{m-t}2\cos\psi\sqrt{(m-t)(m+t)}, \\ \gamma_t &= ie^{-i\chi}\sin\psi\sqrt{(m-t)(m-t+1)}.\end{aligned}$$

Then we have $\alpha_{-t} = \overline{\gamma_t}$ and $\beta_{-t} = -\beta_t$. If η is a π_m-eigenvector for A, then we have the following recurrence formula for $t \in I_m$:

$$\alpha_t d_{t+1} + \beta_t d_t + \gamma_t d_{t-1} = 0. \tag{8.1}$$

If we assume η is self-conjugate, we have

$$\alpha_t d_{t+1} - \beta_t d_t + \gamma_t d_{t-1} = 0$$

where we define $d_t = 0$ for $|t| \geqslant m+1$. From (8.1) we get $\beta_t d_t = 0$ and $\alpha_t d_{t+1} + \gamma_t d_{t-1} = 0$.

(a) $\psi \neq 0, \frac{\pi}{2}$ *case*. Since β_t is equal to zero if $|t| = m$, we have $d_t = 0$ for $|t| \leqslant m-1$. We also have $\alpha_{m-1} d_m + \gamma_{m-1} d_{m-2} = 0$. Then we get $d_m = 0$ by $\alpha_{m-1} \neq 0$ and $m-2 \geqslant -m+1$. This shows d_t are all equal to zero. This is not appropriate.

(b) $\psi = 0$ *case*. Then similarly we have $d_t = 0$ for $|t| \leqslant m-1$. Since π_m-eigenvector space is two-dimensional, it is spanned by $\boldsymbol{w}^m_{\pm m}$. This shows $A = C(\mathbb{T}_n \setminus SU_{-1}(2))$.

(c) $\psi = \frac{\pi}{2}$ *case*. We have only a non-trivial equation $\alpha_t d_{t+1} + \gamma_t d_{t-1} = 0$. Its solution space is two-dimensional. We give an explicit solution as follows. The proof is straightforward.

Lemma 8.3. *Let m be a half-integer in $\frac{3}{2} + \mathbb{Z}_{\geqslant 0}$ and $0 \leqslant \chi < 2\pi$. Consider the recurrence equation,*

$$-ie^{i\chi}\sqrt{(m+t)(m+t+1)}\,d_{t+1} + ie^{-i\chi}\sqrt{(m-t)(m-t+1)}\,d_{t-1} = 0.$$

Then its solution is a linear combination of

$$d_t = e^{i(m-t)\chi}\begin{bmatrix} 2m \\ m-t \end{bmatrix}^{\frac{1}{2}} \quad \text{and} \quad d_t = (-1)^{m-t} e^{i(m-t)\chi}\begin{bmatrix} 2m \\ m-t \end{bmatrix}^{\frac{1}{2}}.$$

Define two π_m-eigenvectors of $C(SU_{-1}(2))$ η^m_χ and $\hat{\eta}^m_\chi$ by

$$\eta^{m,\chi} = \sum_{t \in I_m} e^{i(m-t)\chi}\begin{bmatrix} 2m \\ m-t \end{bmatrix}^{\frac{1}{2}} \boldsymbol{w}^m_t, \qquad \hat{\eta}^{m,\chi} = \sum_{t \in I_m} (-1)^{m-t} e^{i(m-t)\chi}\begin{bmatrix} 2m \\ m-t \end{bmatrix}^{\frac{1}{2}} \boldsymbol{w}^m_t.$$

Since the π_m-eigenvector space of A is two-dimensional, it is spanned by the above vectors. Conversely we consider the right coideal B which is generated by $\eta^{m,\chi}$ and $\hat{\eta}^{m,\chi}$. We want to conclude that B is a right coideal of type $\mathbb{T}_n$ and hence B coincides with A as a result. For the sake of this, it suffices to show that the π_ℓ-eigenvector space of B is trivial for all $\ell \in \{\frac{1}{2}, \ldots, m-1\}$. We prepare the element $g^{\theta,\chi}$ of $SU(2)$ defined by

$$g^{\theta,\chi} = \begin{pmatrix} \cos\frac{\theta}{2} & -ie^{-i\chi}\sin\frac{\theta}{2} \\ -ie^{i\chi}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}.$$

Recall the $*$-homomorphism $\upsilon_{g^{\theta,\chi}} : C(SU_{-1}(2)) \to \mathbb{B}(\mathbb{C}^2)$:

$$\begin{pmatrix} \upsilon_{g^{\theta,\chi}}(x) & \upsilon_{g^{\theta,\chi}}(u) \\ \upsilon_{g^{\theta,\chi}}(v) & \upsilon_{g^{\theta,\chi}}(y) \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2}\sigma_1 & ie^{-i\chi}\sin\frac{\theta}{2}\sigma_2 \\ -ie^{i\chi}\sin\frac{\theta}{2}\sigma_2 & \cos\frac{\theta}{2}\sigma_1 \end{pmatrix}.$$

Define a $*$-homomorphism $\rho_{g^{\theta,\chi}} = (\upsilon_{g^{\theta,\chi}} \otimes \mathrm{id}) \circ \delta$ and we have

$$\begin{pmatrix} \rho_{g^{\theta,\chi}}(x) & \rho_{g^{\theta,\chi}}(u) \\ \rho_{g^{\theta,\chi}}(v) & \rho_{g^{\theta,\chi}}(y) \end{pmatrix}$$
$$= \begin{pmatrix} \cos\frac{\theta}{2}\sigma_1 \otimes x + ie^{-i\chi}\sin\frac{\theta}{2}\sigma_2 \otimes v & \cos\frac{\theta}{2}\sigma_1 \otimes u + ie^{-i\chi}\sin\frac{\theta}{2}\sigma_2 \otimes y \\ -ie^{i\chi}\sin\frac{\theta}{2}\sigma_2 \otimes x + \cos\frac{\theta}{2}\sigma_1 \otimes v & -ie^{i\chi}\sin\frac{\theta}{2}\sigma_2 \otimes u + \cos\frac{\theta}{2}\sigma_1 \otimes y \end{pmatrix}.$$

Lemma 8.4. *Let m be a half-integer in $\frac{1}{2} + \mathbb{Z}_{\geqslant 0}$. For all $s \in I_m$ we have the following equalities*:

$$\rho_{g^{\theta,\chi}}\big(\big(\eta^{m,\chi}\big)_s\big) = \sum_{r\in I_m} \big(\cos m\theta\sigma_1 + (-1)^{m-r} i \sin m\theta\sigma_2\big) \otimes e^{i(m-r)\chi} \begin{bmatrix} 2m \\ m-r \end{bmatrix}^{\frac{1}{2}} w(\pi_m)_{r,s},$$

$$\rho_{g^{\theta,\chi}}\big(\big(\hat{\eta}^{m,\chi}\big)_s\big) = \sum_{r\in I_m} \big((-1)^{m-r} \cos m\theta\sigma_1 - i \sin m\theta\sigma_2\big) \otimes e^{i(m-r)\chi} \begin{bmatrix} 2m \\ m-r \end{bmatrix}^{\frac{1}{2}} w(\pi_m)_{r,s}.$$

Proof. Since $\rho_{g^{\theta,\chi}}$ commutes with the lowering operator f, it is enough to show those equalities with $s = -m$. Here it is proved by induction on m. We can easily show $\rho_{g^{\theta,\chi}} \circ \beta_z^L = (\mathrm{id} \otimes \beta_z^L) \circ \rho_{g_{\theta,0}}$ where z is equal to $e^{i\frac{\chi}{2}}$. Moreover we have $\eta_{-m}^{m,\chi} = e^{im\chi}\beta_z^L(\eta_{-m}^{m,0})$ and $\hat{\eta}_{-m}^{m,\chi} = e^{im\chi}\beta_z^L(\hat{\eta}_{-m}^{m,0})$. Hence it yields $\rho_{g^{\theta,\chi}}(\eta_{-m}^{m,\chi}) = e^{im\chi}(\mathrm{id} \otimes \beta_z^L) \circ \rho_{g_{\theta,0}}(\eta_{-m}^{m,0})$ and $\rho_{g^{\theta,\chi}}(\hat{\eta}_{-m}^{m,\chi}) = e^{im\chi}(\mathrm{id} \otimes \beta_z^L) \circ \rho_{g_{\theta,0}}(\hat{\eta}_{-m}^{m,0})$. Hence we may assume χ is equal to 0. For simplicity of notations we write g^θ, η^m and $\hat{\eta}^m$ for $g_{\theta,0}$, $\eta^{m,0}$ and $\hat{\eta}^{m,0}$, respectively. When m is equal to $\frac{1}{2}$, the desired equalities are easily obtained. We assume that the desired equalities hold for m. We can justify an equality about $\rho_{g^\theta}(\eta_{-(m+1)}^{m+1})$ as follows. Making use of the equality $\eta_{-(m+1)}^{m+1} = x^2\eta_{-m}^m + 2x\eta_{-m}^m v + \eta_{-m}^m v^2$, the computation of $\rho_{g^\theta}(x^2\eta_{-m}^m)$, $\rho_{g^\theta}(2x\eta_{-m}^m v)$ and $\rho_{g^\theta}(\eta_{-m}^m v^2)$ is carried out in the following way:

$$\begin{aligned}
\rho_{g^\theta}\big(x^2\eta_{-m}^m\big) &= \left(\cos^2\frac{\theta}{2} \otimes x^2 - 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\sigma_3 \otimes xv - \sin^2\frac{\theta}{2} \otimes v^2\right) \\
&\quad \times \sum_{r\in I_m} \big(\cos m\theta\sigma_1 + (-1)^{m-r} i \sin m\theta\sigma_2\big) \otimes \begin{bmatrix} 2m \\ m-r \end{bmatrix} x^{m-r} v^{m+r} \\
&= \sum_{r\in I_{m+1}} \left(\cos^2\frac{\theta}{2}\big(\cos m\theta\sigma_1 + (-1)^{m+1-r} i \sin m\theta\sigma_2\big)\begin{bmatrix} 2m \\ m-r-1 \end{bmatrix}\right. \\
&\quad + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\big(-\sin m\theta\sigma_1 + (-1)^{m+1-r} i \cos m\theta\sigma_2\big)\begin{bmatrix} 2m \\ m-r \end{bmatrix} \\
&\quad \left. - \sin^2\frac{\theta}{2}\big(\cos m\theta\sigma_1 + (-1)^{m+1-r} i \sin m\theta\sigma_2\big)\begin{bmatrix} 2m \\ m-r+1 \end{bmatrix}\right) \\
&\quad \otimes x^{m+1-r} v^{m+1+r}, \\
\rho_{g^\theta}\big(x\eta_{-m}^m v\big) &= \sum_{r\in I_{m+1}} \left(\sin\frac{\theta}{2}\cos\frac{\theta}{2}\big(-\sin m\theta\sigma_1 + (-1)^{m+r} i \cos m\theta\sigma_2\big)\begin{bmatrix} 2m \\ m-r-1 \end{bmatrix}\right.
\end{aligned}$$

$$+\left(\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}\right)\left(\cos m\theta\sigma_1-(-1)^{m-r}i\sin m\theta\sigma_2\right)\begin{bmatrix}2m\\m-r\end{bmatrix}$$
$$+\sin\frac{\theta}{2}\cos\frac{\theta}{2}\left(-\sin m\theta\sigma_1-(-1)^{m-r}i\cos m\theta\sigma_2\right)\begin{bmatrix}2m\\m-r+1\end{bmatrix}\Bigg)$$
$$\otimes x^{m+1-r}v^{m+1+r}$$

and

$$\rho_{g^\theta}\left(\eta^m_{-m}v^2\right)=\sum_{r\in I_{m+1}}\left(-\sin^2\frac{\theta}{2}\left(\cos m\theta\sigma_1-(-1)^{m-r}i\sin m\theta\sigma_2\right)\begin{bmatrix}2m\\m-r-1\end{bmatrix}\right.$$
$$+2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\left(-\sin m\theta\sigma_1+(-1)^{m+r}i\cos m\theta\sigma_2\right)\begin{bmatrix}2m\\m-r\end{bmatrix}$$
$$\left.+\cos^2\frac{\theta}{2}\left(\cos m\theta\sigma_1-(-1)^{m-r}i\sin m\theta\sigma_2\right)\begin{bmatrix}2m\\m-r+1\end{bmatrix}\right)\otimes x^{m+1-r}v^{m+1+r}.$$

Then we can derive $\rho_{g_\theta}(\eta^{m+1}_{-(m+1)})=\sum_{r\in I_{m+1}}(\cos(m+1)\theta+(-1)^{m+1-r}i\sin(m+1)\theta)\otimes\left[\begin{smallmatrix}2m+2\\m+1-r\end{smallmatrix}\right]x^{m+1-r}v^{m+1+r}$ where we use the formula $\left[\begin{smallmatrix}2m+2\\m+1-r\end{smallmatrix}\right]=\left[\begin{smallmatrix}2m\\m-r-1\end{smallmatrix}\right]+2\left[\begin{smallmatrix}2m\\m-r\end{smallmatrix}\right]+\left[\begin{smallmatrix}2m\\m-r+1\end{smallmatrix}\right]$. By induction the assertion about $\rho_{g^\theta}(\eta^m_{-m})$ is justified. About $\rho_{g^\theta}(\hat\eta^m_{-m})$ we make use of $\rho_{g^\theta}\circ\beta^L_i=(\mathrm{id}\otimes\beta^L_i)\circ\rho_{g^{-\theta}}$ and $\hat\eta^m_{-m}=i^{2m}\beta^L_i(\eta^m_{-m})$. $\square$

Especially putting $\theta=\frac{\pi}{m}k$ $(k=0,\dots,n-1)$ and $z=e^{-i\frac{\chi}{2}}$, we get

$$\rho_{g_{\frac{\pi}{m}k,\chi}}\left(\eta^{m,\chi}_s\right)=(-1)^k\sigma_1\otimes\eta^{m,\chi}_s,$$
$$\rho_{g_{\frac{\pi}{m}k,\chi}}\left(\hat\eta^{m,\chi}_s\right)=(-1)^k\sigma_1\otimes\hat\eta^{m,\chi}_s.$$

Therefore B is contained in a C^*-algebra

$$C^{n,\chi}=\left\{a\in C\left(SU_{-1}(2)\right)\;\middle|\;\rho_{g_{\frac{\pi}{m}k,\chi}}(a)\in(\mathbb{C}+\mathbb{C}\sigma_1)\otimes\mathbb{C}a\text{ for all }0\leqslant k<n\right\}.$$

Actually, $C^{n,\chi}$ is a right coideal because of $(\rho_{g_{\frac{\pi}{m}k,\chi}}\otimes\mathrm{id})\circ\delta=(\mathrm{id}\otimes\delta)\circ\rho_{g_{\frac{\pi}{m}k,\chi}}$.

Lemma 8.5. *The following equalities hold*:

$$\Psi_1\left(\eta^{m,\chi},\hat\eta^{m,\chi}\right)=-i\sqrt{m}2^{2m}e^{2im\chi}\xi^{\chi,\frac{\pi}{2}},$$
$$\Psi_1\left(\hat\eta^{m,\chi},\eta^{m,\chi}\right)=-i\sqrt{m}2^{2m}e^{2im\chi}\xi^{\chi,\frac{\pi}{2}},$$
$$\Psi_1\left(\eta^{m,\chi},\eta^{m,\chi}\right)=\Psi_1\left(\hat\eta^{m,\chi},\hat\eta^{m,\chi}\right)=0.$$

Proof. We give a proof for only the first one. Others are proved similarly. By definition we have $\Psi_1(\eta^{m,\chi},\hat\eta^{m,\chi})=\sum_{r=0}^{2m-1}(C^1_{m,m})_r\eta^{m,\chi}_{m-r-1}\hat\eta^{m,\chi}_{-m+r}$ where $(C^1_{m,m})_r$ is equal to $(-1)^r\sqrt{2m}^{-1}\sqrt{(r+1)(2m-r)}$. Define complex numbers $c_{\pm1}$ and c_0 by $\Psi_1(\eta^{m,\chi},\hat\eta^{m,\chi})=$

$c_{-1}\boldsymbol{w}^1_{-1} + c_0\boldsymbol{w}^1_0 + c_1\boldsymbol{w}^1_1$. In order to obtain their values we use the restriction homomorphism $\pi_{\mathbb{T}} : C(SU_{-1}(2)) \to C(\mathbb{T})$. In fact we get $\pi_{\mathbb{T}}(\Psi_1(\eta^{m,\chi}, \hat{\eta}^{m,\chi})_s) = z^{2s}c_s$ for all $z \in \mathbb{T}$ and $1 \leqslant s \leqslant 1$. First we have

$$\Psi_1\left(\eta^{m,\chi}, \hat{\eta}^{m,\chi}\right)_{-1} = \sum_{r=0}^{2m-1} \sqrt{2m}^{-1}(-1)^r\sqrt{(r+1)(2m-r)}\left(\eta^{m,\chi}\right)_{m-1-r}\hat{\eta}^{m,\chi}_{-m+r}. \tag{8.2}$$

Applying $\pi_{\mathbb{T}}$ to the both side, we have

$$\begin{aligned} c_{-1} &= \sum_{r=0}^{2m-1} \sqrt{2m}^{-1}(-1)^r\sqrt{(r+1)(2m-r)}\begin{bmatrix} 2m \\ r+1 \end{bmatrix}^{\frac{1}{2}}\begin{bmatrix} 2m \\ 2m-r \end{bmatrix}^{\frac{1}{2}} \\ &\quad \times (-1)^{2m-r}e^{i(r+1)\chi}e^{i(2m-r)\chi} \\ &= (-1)^{2m}\sqrt{2m}^{-1}e^{i(2m+1)\chi}\sum_{r=0}^{2m-1}\frac{2m!}{r!(2m-1-r)!} \\ &= -2^{2m-1}\sqrt{2m}e^{i(2m+1)\chi}. \end{aligned}$$

If we apply the lowering operator f to (8.2), then we can easily obtain

$$\Psi_1\left(\eta^{m,\chi}, \hat{\eta}^{m,\chi}\right)_0 = -\sqrt{m}^{-1}\sum_{r=0}^{2m}(m-r)\left(\eta^{m,\chi}\right)_{m-r}\hat{\eta}^{m,\chi}_{-m+r}, \tag{8.3}$$

where we use the fact of $f \cdot \xi_p = i^{-2\nu+1}\sqrt{(\nu-p)(\nu+p+1)}\,\xi_{p+1}$ for a π_ν-eigenvector ξ. Applying $\pi_{\mathbb{T}}$ to (8.3), we obtain $c_0 = -\sqrt{m}^{-1}\sum_{r=0}^{2m}(m-r)\left[\begin{smallmatrix} 2m \\ r \end{smallmatrix}\right](-1)^{2m-r} = 0$. Since we also have

$$\Psi_1\left(\eta^{m,\chi}, \hat{\eta}^{m,\chi}\right)_1 = \sum_{r=0}^{2m-1} \sqrt{2m}^{-1}(-1)^r\sqrt{(r+1)(2m-r)}\left(\eta^{m,\chi}\right)_{m-r}\hat{\eta}^{m,\chi}_{-m+r+1},$$

similarly we can derive $c_1 = 2^{2m-1}\sqrt{2m}e^{i(2m-1)\chi}$. $\square$

Now we start to prove that B is a right coideal of type $\mathbb{T}_n$. Let n' be a smallest odd integer such that the $\pi_{\frac{n'}{2}}$-eigenvector space of B is not trivial. Put $m' = \frac{n'}{2}$. We claim $m' \geqslant \frac{3}{2}$. This is because there exist no non-zero complex numbers $c_{-\frac{1}{2}}$ and $c_{\frac{1}{2}}$ such that $c_{-\frac{1}{2}}x + c_{\frac{1}{2}}v$ is an element of $C^{m,\chi}$. From the previous lemma, B contains π_1-part and it shows B is not of type D_ℓ for some odd $\ell \geqslant 3$ but of $\mathbb{T}_{n'}$. Again by Lemma 8.3 for m', the $\pi_{m'}$-eigenvector space is spanned by $\eta^{m',\chi}$ and $\hat{\eta}^{m',\chi}$. In particular, $\eta^{m',\chi}_{-m'}$ is an element of $C^{n,\chi}$. Since we have for $0 \leqslant k \leqslant n-1$

$$\begin{aligned} \rho_{g_{\frac{2\pi}{n}k,\chi}}\left(\eta^{m',\chi}_{-m'}\right) &= \sum_{r\in I_{m'}}\left(\cos\frac{m'}{m}\pi k\sigma_1 - (-1)^{m'-r}i\sin\frac{m'}{m}\pi k\sigma_2\right) \\ &\quad \otimes e^{i(m'-r)\chi}\begin{bmatrix} 2m' \\ m'-r \end{bmatrix}^{\frac{1}{2}} w(\pi_{m'})_{r,-m'} \end{aligned}$$

by Lemma 8.4, $\frac{m'}{m}$ must be equal to 1. Hence the π_ℓ-eigenvector space is trivial for $\ell \in \{\frac{1}{2}, \dots, m-1\}$. This case is only admitted as type $\mathbb{T}_n$. As a result we also see that B coincides with $C^{n,\chi}$.

We have proved that a right coideal of type $\mathbb{T}_n$ (odd $n \geqslant 3$) arises from the cyclic group in the maximal torus $\mathbb{T}$ or the cyclic group $\mathbb{T}_n^{\frac{\pi}{2},\chi}$. We have to check two right coideals $C(D_n \setminus SU_{-1}(2))$ and $C^{n,\chi}$ are not $SU_{-1}(2)$-isomorphic. It suffices to prove it in the case of $\chi = 0$ by considering β^L. Assume that there exists an $SU_{-1}(2)$-isomorphism $\vartheta : C(D_n \setminus SU_{-1}(2)) \to C^{n,0}$. It induces the map between eigenvector spaces defined by $\vartheta(\xi) = (\vartheta(\xi_r))_{r \in I_\mu}$ for π_μ-eigenvector ξ. On the eigenvector spaces, ϑ preserves the inner product, the conjugation operation T and moreover the product map Ψ. Take complex numbers λ and μ which satisfy $\vartheta(\boldsymbol{w}_{-m}^m) = \lambda\eta^m + \mu\hat{\eta}^m$. By Lemma 8.5 and $\Psi_1(\boldsymbol{w}_{-m}^m, \boldsymbol{w}_{-m}^m) = 0$, we have $-i\lambda\mu\sqrt{m}2^{2m+1}\xi^{0,\frac{\pi}{2}} = 0$. Hence $\lambda = 0$ or $\mu = 0$. By using β^L we may assume $\mu = 0$. Now we have $\vartheta(\boldsymbol{w}_{-m}^m) = \lambda\eta^m$. If we apply the conjugation T to the both side, we obtain $\vartheta(\boldsymbol{w}_m^m) = \bar{\lambda}\eta^m$. This shows the range of ϑ onto π_m-eigenvector space is one-dimensional, however, this is contradiction. Therefore $C(\mathbb{T}_n \setminus SU_{-1}(2))$ and $C^{n,0}$ are not $SU_{-1}(2)$-isomorphic. Finally we will make a discussion on characterizing a right coideal $C^{n,\chi}$ by a quantum subgroup of $SU_{-1}(2)$. Let us recall a closed subgroup of $SU_{-1}(2)$ which corresponds to $\mathbb{T}_n^{\frac{\pi}{2},0}$, that is, we consider the compact set $Z = \pi_1^{-1}(\mathbb{T}_n^{\frac{\pi}{2},0})$ and define $C(G_Z) = \pi_Z(C(SU_{-1}(2)))$. We denote a subgroup G_Z by $(\mathbb{T}_n^{\frac{\pi}{2},0})_\#$. The set Z is actually a binary subgroup $(\mathbb{T}_n^{\frac{\pi}{2},0})^*$ and consists of $\{g^{\frac{2\pi}{n}k} \mid 0 \leqslant k \leqslant 2n-1\}$. Hence $C((\mathbb{T}_n^{\frac{\pi}{2},0})_\#) \subset \bigoplus_{0 \leqslant k \leqslant 2n-1} \mathbb{M}_{s(k)}(\mathbb{C})$ where $s(k) = 1$ if $k = 0, n$ and $s(k) = 2$ otherwise. We denote the coproduct of $C((\mathbb{T}_n^{\frac{\pi}{2},0})_\#)$ by δ_Z. Now we look at the equality $\rho_{g^{\frac{2\pi}{n}k}}(\eta_s^m) = (-1)^k\sigma_1 \otimes \eta_s^m$ for $0 \leqslant k \leqslant 2n-1$ and $s \in I_m$. Define a self-adjoint unitary in $C((\mathbb{T}_n^{\frac{\pi}{2},0})_\#)$ by $w = \bigoplus_{k \neq 0,n}(-1)^k\sigma_1 \oplus 1 \oplus -1$. Then the above equality is equivalent to $(\pi_Z \otimes \mathrm{id}) \circ \delta(\eta_s^m) = w \otimes \eta_s^m$. Moreover, we have

$$\begin{aligned}
\delta_Z(w) \otimes \eta_s^m &= (\delta_Z \circ \pi_Z \otimes \mathrm{id}) \circ \delta(\eta_s^m) \\
&= (\pi_Z \otimes \pi_Z \otimes \mathrm{id}) \circ (\delta \otimes \mathrm{id})\delta(\eta_s^m) \\
&= (\pi_Z \otimes \pi_Z \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta)\delta(\eta_s^m) \\
&= w \otimes w \otimes \eta_s^m.
\end{aligned}$$

Hence we have proved the following proposition.

Proposition 8.6. *A self-adjoint unitary w is a group-like element in $C((\mathbb{T}_n^{\frac{\pi}{2},0})_\#)$.*

A C^*-algebra generated by w becomes a Hopf $*$-algebra which is isomorphic to $C(\mathbb{Z}_2)$. With this identification we have $C^{n,0} = \{a \in C(SU_{-1}(2)) \mid (\pi_Z \otimes \mathrm{id}) \circ \delta(a) \in C(\mathbb{Z}_2) \otimes \mathbb{C}a\}$. For general $C^{n,\chi}$ we also get similar results by using β_z^L. We study all the irreducible representations of $(\mathbb{T}_n^{\frac{\pi}{2},0})_\#$ as follows. Let us denote the restriction of the fundamental representation to $(\mathbb{T}_n^{\frac{\pi}{2},0})_\#$ simply by $w(\pi_{\frac{1}{2}})|$. Then we have $w(\pi_{\frac{1}{2}})| = \begin{pmatrix} \pi_Z(x) & \pi_Z(u) \\ \pi_Z(v) & \pi_Z(y) \end{pmatrix}$ where we have

$$\pi_Z(x) = \bigoplus_{k \neq 0,n} \cos\frac{\pi}{n}k\sigma_1 \oplus 1 \oplus -1,$$

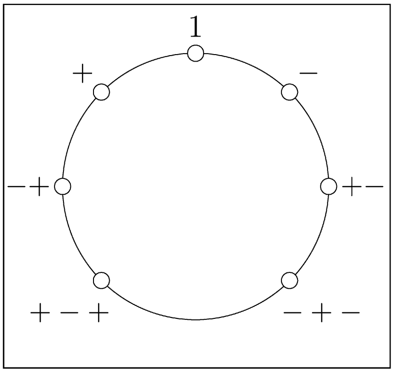

Fig. 1. The McKay diagram of $(\mathbb{T}_n^{0,\frac{\pi}{2}})_\#$, $2n$ nodes.

$$\pi_Z(u) = \bigoplus_{k\neq 0,n} i \sin\frac{\pi}{n}k\sigma_2 \oplus 0 \oplus 0,$$

$$\pi_Z(v) = \bigoplus_{k\neq 0,n} -i \sin\frac{\pi}{n}k\sigma_2 \oplus 0 \oplus 0,$$

$$\pi_Z(y) = \bigoplus_{k\neq 0,n} \cos\frac{\pi}{n}k\sigma_1 \oplus 1 \oplus -1.$$

The unitary representation $w(\pi_{\frac{1}{2}})|$ invariantly acts on two lines $\mathbb{C}\xi_\pm = \mathbb{C}\binom{1}{\pm i}$ and their one-dimensional actions are given by w_+ and w_- with

$$w_\pm = \bigoplus_{k\neq 0,n} \begin{pmatrix} 0 & e^{\pm i\frac{\pi}{n}k} \\ e^{\mp i\frac{\pi}{n}k} & 0 \end{pmatrix} \oplus 1 \oplus -1.$$

Hence we have $w(\pi_{\frac{1}{2}})| = w_+ \oplus w_-$. Notice that $w_\pm$ are the self-conjugate unitary representations. This shows the difference from the fusion rules of the ordinary cyclic group $\mathbb{T}_{2n}$. And define the tensor product representations $w_{+-} = w_+w_-$ and $w_{-+} = w_-w_+$. Then they are self-conjugate and non-equivalent to each other, where non-equivalence comes from the non-commutativity of the underlying space $(\mathbb{T}_n^{\frac{\pi}{2},0})_\#$. If we want to write the McKay diagram, we compute $w(\pi_{\frac{1}{2}})| \cdot w_+ = 1 \oplus w_{-+}$ and bond the vertices $+$ and $-+$ by a single line. We continue these procedures to get all the irreducible representations. As a result, they are represented by the reduced words $(+-+-$, etc.) whose lengths are less than or equal to n (Fig. 1). For length n word, we can easily check $w = +-\cdots-+ = -+\cdots+-$. Hence w sits at the bottom vertex. The fusion rules are given by this equality, $+^2 = 1$, $-^2 = 1$ and $\overline{(\pm)} = \pm$. In the above proposition, we get a self-adjoint unitary w. It is the nth irreducible representation $+-\cdots-+$. Let us denote $g = +-$ and $s = +$. Then we have $s^2 = 1$, $g^n = 1$ and $sgs = g^{-1}$. This shows that the dual group $\widehat{(\mathbb{T}_n^{\frac{\pi}{2},0})_\#}$ is isomorphic to D_n, and hence $C((\mathbb{T}_n^{\frac{\pi}{2},0})_\#) \cong C_r^*(D_n)$ as the Hopf algebras.

(IV) D_n (*odd* $n \geqslant 3$) *case*. A has the spectral pattern

$$\bigoplus_{k\in\mathbb{Z}_{\geqslant 0}}\left(\left[\frac{k}{n}\right]+\frac{1+(-1)^k}{2}\right)\pi_k \oplus \bigoplus_{k\in\mathbb{Z}_{\geqslant 0}}\left(\left[\frac{2k+1}{n}\right]-\left[\frac{k}{n}\right]\right)\pi_{k+\frac{1}{2}}.$$

Let B be a right coideal of the integer parts of A. Since B is a right coideal of $C(SO_{-1}(3))$, it must be a quotient by subgroup of $SO_{-1}(3)$ for some angles $0 \leqslant \phi < \frac{2\pi}{n}$, $0 \leqslant \chi < 2\pi$ and $-\pi \leqslant \psi \leqslant \pi$. Look at the spectral pattern for B and we see that $B = C(D_n^{\phi,\chi,\psi} \setminus SO_{-1}(3))$ We have to clarify its self-conjugate π_2-eigenvector.

Lemma 8.7. *The following vector is a π_2-eigenvector for $C(D_n^{\phi,\chi,\psi} \setminus SO_{-1}(3))$:*

$$\zeta^{\phi,\chi,\psi} = \left(e^{i2\chi}\frac{\sqrt{6}}{4}\sin^2\psi, ie^{i\chi}\frac{\sqrt{6}}{2}\sin\psi\cos\psi, 1-\frac{6}{4}\sin^2\psi,\right.$$
$$\left. -ie^{-i\chi}\frac{\sqrt{6}}{2}\sin\psi\cos\psi, e^{-i2\chi}\frac{\sqrt{6}}{4}\sin^2\psi\right)w(\pi_2).$$

Proof. The proof is done as Lemma 8.2 with considering $w(\pi_2)$ and $D_{n,\phi,\chi,\psi} = \mathrm{Ad}(\pi_1(r^{12}(\chi)r^{13}(\psi)r^{12}(\phi)))(D_n^{0,0,0})$. $\square$

Next we use the following lemma in order to get a $\pi_{\frac{n}{2}}$-eigenvector of A, which is proved by $q_0 \to 1$ in Lemma 7.5 and an elementary calculation for $\boldsymbol{w}^2_{\pm 1}$.

Lemma 8.8. *Let m be a half-integer with $m \in \frac{1}{2}+\mathbb{Z}_{\geqslant 0}$. Then we have the following equalities for all $t \in I_m$:*

$$\Psi_{m-1}\left(\boldsymbol{w}^2_{-2}, \boldsymbol{w}^m_t\right) = (-1)^{m-t}\frac{1}{2m-2}\sqrt{\frac{4(m-t+1)(m+t)(m+t-1)(m+t-2)}{2m(2m-1)}}\boldsymbol{w}^{m-1}_{t-2},$$

$$\Psi_{m-1}\left(\boldsymbol{w}^2_{-1}, \boldsymbol{w}^m_t\right) = \frac{2(m-2t+2)}{2m-2}\sqrt{\frac{(m+t)(m+t-1)}{2m(2m-1)}}\boldsymbol{w}^{m-1}_{t-1},$$

$$\Psi_{m-1}\left(\boldsymbol{w}^2_{0}, \boldsymbol{w}^m_t\right) = -(-1)^{m-t}\frac{2t}{(2m-2)}\sqrt{\frac{3!(m+t)(m-t)}{2m(2m-1)}}\boldsymbol{w}^{m-1}_{t},$$

$$\Psi_{m-1}\left(\boldsymbol{w}^2_{1}, \boldsymbol{w}^m_t\right) = -\frac{2(m+2t+2)}{2m-2}\sqrt{\frac{(m-t)(m-t-1)}{2m(2m-1)}}\boldsymbol{w}^{m-1}_{t+1},$$

$$\Psi_{m-1}\left(\boldsymbol{w}^2_{2}, \boldsymbol{w}^m_t\right) = -(-1)^{m-t}\frac{1}{2m-2}\sqrt{\frac{4(m+t+1)(m-t)(m-t-1)(m-t-2)}{2m(2m-1)}}\boldsymbol{w}^{m-1}_{t+2}.$$

Let us define $\alpha_t, \beta_t, \gamma_t, \delta_t$ and ε_t for $t \in I_m$ by

$$\alpha_t = \frac{\sqrt{6}}{4} e^{i2\chi} \sin^2\psi (-1)^{m-t} \sqrt{4(m-t-1)(m+t+2)(m+t+1)(m+t)},$$
$$\beta_t = \sqrt{6} e^{i\chi} \sin\psi \cos\psi (m-2t) \sqrt{(m+t+1)(m+t)},$$
$$\gamma_t = 2\left(-1 + \frac{6}{4}\sin^2\psi\right)(-1)^{m-t} t \sqrt{3!(m+t)(m-t)},$$
$$\delta_t = -\sqrt{6} e^{-i\chi} \sin\psi \cos\psi (m+2t) \sqrt{(m-t+1)(m-t)},$$
$$\varepsilon_t = -\frac{\sqrt{6}}{4} e^{-i2\chi} \sin^2\psi (-1)^{m-t} \sqrt{4(m+t-1)(m-t+2)(m-t+1)(m-t)},$$

with $\alpha_m = 0 = \varepsilon_{-m}$. Now we put $m = \frac{n}{2}$. Take a self-conjugate π_m-eigenvector $\eta = \sum_{t \in I_m} d_t \boldsymbol{w}_t^m$ of A. The self-conjugacy means $d_{-t} = \overline{d_t}$. Since the π_{m-1}-part of A is zero, we obtain $\Psi_{m-1}(\zeta_\psi, \eta) = 0$ and this is equivalent to the following recurrence formula:

$$\alpha_t d_{t+2} + \beta_t d_{t+1} + \gamma_t d_t + \delta_t d_{t-1} + \varepsilon_t d_{t-2} = 0 \tag{8.4}$$

for $t \in I_m$, where we define $d_t = 0$ for $|t| \geqslant m+1$ as usual. Making use of $d_{-t} = \overline{d_t}$, $\alpha_{-t} = \overline{\varepsilon_t}$, $\beta_{-t} = -\overline{\delta_t}$ and $\gamma_{-t} = \gamma_t$, where we use $(-1)^{2t} = -1$, we also have

$$\alpha_t d_{t+2} - \beta_t d_{t+1} + \gamma_t d_t - \delta_t d_{t-1} + \varepsilon_t d_{t-2} = 0.$$

From (8.4) and this we get

$$\alpha_t d_{t+2} + \gamma_t d_t + \varepsilon_t d_{t-2} = 0, \tag{8.5}$$
$$\beta_t d_{t+1} + \delta_t d_{t-1} = 0. \tag{8.6}$$

We analyze them as follows.

(1) $0 < \psi < \frac{\pi}{2}$ *case*. We want to derive a contradiction to non-triviality of η. If we give a number d_{-m}, then by (8.5) or (8.6) we can inductively determine $d_{-m+2}, \ldots, d_{m-1}$. By the self-conjugacy, we obtain the whole numbers d_t. Hence η is uniquely determined by d_{-m} or d_m. Because of non-triviality of solutions $\{d_t\}_{t \in I_m}$, the determinant of the following matrix must be zero for all $t \in I_m$:

$$\begin{pmatrix} \alpha_t & \gamma_t & \varepsilon_t \\ \beta_{t+1} & \delta_{t+1} & 0 \\ 0 & \beta_{t-1} & \delta_{t-1} \end{pmatrix}.$$

Hence we have the equation for t, m and ψ,

$$\alpha_t \delta_{t-1} \delta_{t+1} - \beta_{t+1} \gamma_t \delta_{t-1} + \beta_{t-1} \beta_{t+1} \varepsilon_t = 0 \tag{8.7}$$

for $t \in I_m$. If we put $t = m-1$, then we easily get $\sin^2\psi = \frac{3m-4}{5m-8}$.

(1a) $n = 3$ *case*. Then m is equal to $\frac{3}{2}$ and we have $\sin^2\psi = -1 < 0$. This is a contradiction.

(1b) $n \geqslant 5$ *case*. If we put $t = m-2$ ($\geqslant -m+1$), then we easily get $\sin^2\psi = \frac{12(m-2)^2}{5m^2-24m+24}$. This is a contradiction to $\sin^2\psi = \frac{3m-4}{5m-8}$.

Therefore ψ must be equal to 0 or $\frac{\pi}{2}$.

(2) $\psi = 0$ *case*. We have $\eta = \boldsymbol{w}_0^m$ and $\gamma_t = -2(-1)^{m-t} t\sqrt{3!(m+t)(m-t)}$ and $\alpha_t = \beta_t = \delta_t = \varepsilon_t = 0$. Then we get $d_t = 0$ for $|t| \leqslant m-1$. This case A is $C(D_n \setminus SU_{-1}(2))$.

(3) $\psi = \frac{\pi}{2}$ *case*. We have $\zeta^{\phi,\chi,\frac{\pi}{2}} = \frac{\sqrt{6}}{4} e^{i2\chi} \boldsymbol{w}_{-2}^2 - \frac{1}{2}\boldsymbol{w}_0^2 + \frac{\sqrt{6}}{4} e^{-i2\chi} \boldsymbol{w}_2^2$ and $\beta_t = \delta_t = 0$ and others are

$$\alpha_t = \frac{\sqrt{6}}{4} e^{i2\chi} (-1)^{m-t} \sqrt{4(m-t-1)(m+t+2)(m+t+1)(m+t)},$$
$$\gamma_t = (-1)^{m-t} t\sqrt{3!(m+t)(m-t)},$$
$$\varepsilon_t = -\frac{\sqrt{6}}{4} e^{-i2\chi} (-1)^{m-t} \sqrt{4(m+t-1)(m-t+2)(m-t+1)(m-t)}.$$

Recall vectors $\eta^{m,\chi}$ and $\hat{\eta}^{m,\chi}$ which are introduced in the previous case $\mathbb{T}_n$. We easily confirm that they are independent solutions of $\alpha_t d_{t+2} + \gamma_t d_t + \varepsilon_t d_{t-2} = 0$. Let $\eta = \lambda_0 \eta^{m,\chi} + \lambda_1 \hat{\eta}^{m,\chi}$ be a self-conjugate π_m-eigenvector of A where λ_0 and λ_1 are complex numbers. Since the conjugation operation T satisfies $T\eta^{m,\chi} = e^{-i2m\chi}\eta^{m,\chi}$ and $T\hat{\eta}^{m,\chi} = -e^{-i2m\chi}\hat{\eta}^{m,\chi}$, there exist real numbers μ_0 and μ_1 with $\lambda_0 = \mu_0 e^{-im\chi}$ and $\lambda_1 = i\mu_1 e^{-im\chi}$. The π_1-eigenvector $\Psi_1(\eta,\eta)$ must be zero because of the absence of π_1-part in A. By Lemma 8.5 we have $\Psi_1(\eta,\eta) = 2\mu_0\mu_1\sqrt{2m}2^{2m-1}\xi^{\chi,\frac{\pi}{2}}$. It shows μ_0 or μ_1 is equal to zero. Hence the π_m-eigenvector space of B consists of scalar multiples of $\eta^{m,\chi}$ or $\hat{\eta}^{m,\chi}$. We can easily check $\beta_i^L(\eta_r^{m,\chi}) = i^{-2m}\hat{\eta}_r^{m,\chi}$ for all $r \in I_m$ and hence may assume that the π_m-eigenvector space of A is spanned by $\eta^{m,\chi}$ if necessary by applying β_i^L. Moreover we may also assume $\chi = 0$ by applying β_z^L where $z = e^{i\frac{\chi}{2}}$. Let B be a right coideal generated by $\eta_r^m = \eta_r^{m,0}$ with all $r \in I_m$. Notice that B is contained in $C^{n,0}$ which is defined in the discussion on the type $\mathbb{T}_n$. Hence B is of type $\mathbb{T}_n$ or D_n. We want to show B is actually of type D_n. This is proved by Proposition 8.11 or the following direct calculation which takes us the similar characterization on the right coideal as Proposition 8.6. In order to study it we have to clarify the type of the right coideal C which is the integer part of B. Whether C is of type $\mathbb{T}_n$ or D_n, its π_2-multiplicity is one. Hence the π_2-eigenvector of C is a scalar multiple of $\zeta^{\phi,0,\frac{\pi}{2}}$. We show that there exists an angle $0 \leqslant \phi < \frac{2\pi}{n}$ such that $C = C(D_n^{\phi,0,\frac{\pi}{2}} \setminus SO_{-1}(3))$. We already know all η_r^m for $r \in I_m$ are fixed by the rotation of $\mathbb{T}_n^{0,\frac{\pi}{2}}$. It suffices to show $\eta_r^m \eta_s^m$ is fixed by the rotation of angle π around the axis $\pi_1(r^{23}(-\phi)) \cdot \sigma_3$ for any $r, s \in I_m$. This rotation is obtained by the matrix $k^\phi \in SU(2)$ $k^\phi = \left(\begin{smallmatrix} -i\cos\phi & -\sin\phi \\ \sin\phi & i\cos\phi \end{smallmatrix}\right)$. Define the $*$-homomorphism $\rho_{k^\phi} = (\upsilon_{k^\phi} \otimes \mathrm{id}) \circ \delta$.

Lemma 8.9. *For any $m \in \frac{1}{2} + \mathbb{Z}_{\geqslant 0}$ and $s \in I_m$ the following equalities hold*:

$$\rho_{k^\phi}(\eta_s^m) = i^{2m} \sum_{r \in I_m} \left(\cos 2m\phi\sigma_1 - (-1)^{m-r} i \sin 2m\phi\sigma_2\right) \otimes (-1)^{m-r} \begin{bmatrix} 2m \\ m-r \end{bmatrix}^{\frac{1}{2}} w(\pi_m)_{r,s},$$

$$\rho_{k^\phi}(\hat{\eta}_s^m) = i^{2m} \sum_{r \in I_m} \left(\cos 2m\phi\sigma_1 + (-1)^{m-r} i \sin 2m\phi\sigma_2\right) \otimes \begin{bmatrix} 2m \\ m-r \end{bmatrix}^{\frac{1}{2}} w(\pi_m)_{r,s}.$$

Proof. k^ϕ is equal to $r^{12}(\pi)r^{23}(2\phi) = r^{12}(\pi)g^{2\phi,0}$ where $g^{\theta,\chi}$ is a matrix defined in the study of $\mathbb{T}_n$ case. Since we have $\rho_{k^\phi}(x) = -i\rho_{g^{-2\phi,0}}(x)$ and $\rho_{k^\phi}(v) = i\rho_{g^{-2\phi,0}}(v)$, the equalities

$\rho_{k^\phi}(\eta^m_{-m}) = i^{2m}\rho_{g^{2\phi},0}(\hat{\eta}^m_{-m})$ and $\rho_{k^\phi}(\hat{\eta}^m_{-m}) = i^{2m}\rho_{g^{2\phi},0}(\eta^m_{-m})$ hold. By Lemma 8.4 we obtain the desired equalities. $\square$

Now we solve the equation for ϕ; $\rho_{k^\phi}(\eta^m_s\eta^m_t) = 1 \otimes \eta^m_s\eta^m_t$ for all $s,t \in I_m$. First we consider $s = t = -m$. We focus on the term of x^{4m} in the both sides. In the left-hand side, we have $i^{4m}(\cos 2m\phi\sigma_1 - i\sin 2m\phi\sigma_2)^2 \otimes x^{4m}$. In the right one, we have $1 \otimes x^4$. Hence we have $i^{4m}(\cos 2m\phi\sigma_1 - i\sin 2m\phi\sigma_2)^2 = 1$. It yields $\cos 4m\phi = -1$ and hence ϕ must be equal to $\frac{\pi}{2n}$ or $\frac{3\pi}{2n}$. Then we obtain $\rho_{k^\phi}(\eta^m_s) = \pm i^{2m+1}\sigma_2 \otimes \eta^m_s$ for all $s \in I_m$ with respect to $\phi = \frac{\pi}{2n}$ or $\frac{3\pi}{2n}$. It shows the desired fixed element property of $\eta^m_s\eta^m_t$ by ρ_{k^ϕ} for all $s,t \in I_m$ because of $i^{2m+1} \in \{-1,1\}$. Therefore we have proved B is a right coideal of type D_n. Note that B does not depend on the choice of $\phi = \frac{\pi}{2n}$ or $\frac{3\pi}{2n}$. In summary, a right coideal of type D_n is made from the quotient $D_n \setminus SU_{-1}(2)$ or generated by a π_m-eigenvector η^m.

As in the previous case of $\mathbb{T}_n$ (odd $n \geqslant 3$) we can also observe the similar results about a group-like unitary. Here we treat only the case $\chi = 0, \phi = \frac{\pi}{2n}$, that is, Z is $\pi_1^{-1}(D_n^{\frac{\pi}{2n},0,\frac{\pi}{2}})$ in order to avoid similar arguments. Denote the subgroup G_Z by $(D_n^{\frac{\pi}{2n},0,\frac{\pi}{2}})_\#$. Now Z consists of $\{g^{\frac{2\pi}{n}k}\}_{0\leqslant k\leqslant 2n-1} \cup \{k^\phi g^{\frac{2\pi}{n}\ell}\}_{0\leqslant \ell\leqslant 2n-1}$. Hence the subgroup $(D_n^{\frac{\pi}{2n},0,\frac{\pi}{2}})_\#$ contains $(\mathbb{T}_n^{0,\frac{\pi}{2}})_\#$. We call the subsets $\{g^{\frac{2\pi}{n}k}\}_{0\leqslant k\leqslant 2n-1}$ and $\{k^\phi g^{\frac{2\pi}{n}\ell}\}_{0\leqslant \ell\leqslant 2n-1}$ the cyclic component and the reflective component, respectively. By definition of the restriction map, we have

$$C\big((D_n^{\frac{\pi}{2n},0,\frac{\pi}{2}})_\#\big) \subset \bigoplus_{0\leqslant k\leqslant 2n-1} \mathbb{M}_{s(k)}(\mathbb{C}) \oplus \bigoplus_{0\leqslant \ell\leqslant 2n-1} \mathbb{M}_{t(\ell)}(\mathbb{C})$$

where $s(k) = 2$ except for $k = 0, 2$ and $t(\ell) = 2$ except for $\ell = \frac{n-1}{2}, \frac{3n-1}{2}$. First note the equality $k^\phi g^\theta = k^{\phi+\frac{\theta}{2}}$. Then we get

$$\rho_{k^\phi g^\theta}\big(\eta^m_s\big) = i^{2m}\sum_{r\in I_m}\big(-\sin m\theta\sigma_1 - (-1)^{m-r} i\cos m\theta\sigma_2\big) \otimes (-1)^{m-r}\begin{bmatrix} 2m \\ m-r\end{bmatrix}^{\frac{1}{2}} w(\pi_m)_{r,s}.$$

Hence it immediately derives

$$\rho_{k^\phi g^{\frac{2\pi}{n}k}}\big(\eta^m_s\big) = i^{2m+1}(-1)^{k+1}\sigma_2 \otimes \eta^m_s \quad \text{for } 0 \leqslant k \leqslant n-1.$$

Define a self-adjoint unitary

$$\widetilde{w} = w \oplus \bigoplus_{\ell\neq \frac{n-1}{2},\frac{3n-1}{2}} (-1)^{\ell+1} i^{2m+1}\sigma_2 \oplus 1 \oplus -1$$

where w is a self-adjoint group-like unitary defined in the previous $\mathbb{T}_n$ case.

Proposition 8.10. *A self-adjoint unitary $\widetilde{w}$ is a group-like element in $C((D_n^{\frac{\pi}{2n},0,\frac{\pi}{2}})_\#)$.*

Let $C(\mathbb{Z}_2)$ be a Hopf $*$-subalgebra $\mathbb{C} + \mathbb{C}\widetilde{w}$ in $C((D_n^{\frac{\pi}{2n},0,\frac{\pi}{2}})_\#)$. Then we get

$$B = \big\{a \in C\big(SU_{-1}(2)\big) \bigm| (\pi_Z \otimes \mathrm{id}) \circ \delta(a) \in C(\mathbb{Z}_2) \otimes \mathbb{C}a\big\}.$$

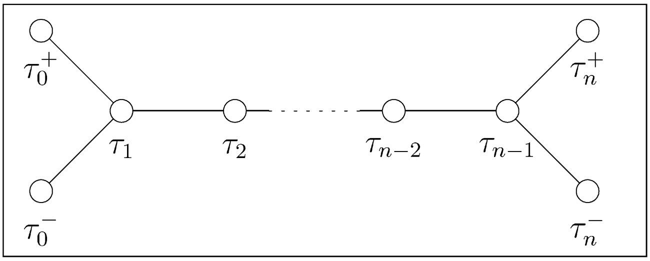

Fig. 2. The McKay diagram for $(D_n^{\frac{\pi}{2n},0,\frac{\pi}{2}})_\#$, $n+3$ nodes.

We study all the irreducible representations of $(D_n^{\frac{\pi}{2n},0,\frac{\pi}{2}})_\#$. The McKay diagram of $(D_n^{\frac{\pi}{2n},0,\frac{\pi}{2}})_\# \subset SU_{-1}(2)$ about $\pi_{\frac{1}{2}}$ is shown in Fig. 2. In Fig. 2, τ_0^+ is the trivial representation and τ_1 is the restriction of $w(\pi_{\frac{1}{2}})$ to $D_n^{\frac{\pi}{2n},0,\frac{\pi}{2}}$. We study where the above group-like unitary $\widetilde{w}$ sits. The restriction of $w(\pi_{\frac{1}{2}})$ to $(D_n^{\frac{\pi}{2n},0,\frac{\pi}{2}})_\#$ is denoted simply by $w(\pi_{\frac{1}{2}})|$. For the $\pi_{\frac{1}{2}}$-module W_1 we use an orthonormal basis $\xi_\pm = \mathbb{C}\binom{1}{\pm i}$. Then we obtain $w(\pi_{\frac{1}{2}})| \cdot \xi_\pm = \xi_\pm \otimes w_\pm$ for the cyclic component and $w(\pi_{\frac{1}{2}})| \cdot \xi_\pm = \xi_\mp \otimes v_\pm$ for the reflective component, where $w_\pm$ have been already defined in the previous $\mathbb{T}_n$ case and

$$v_\pm = \bigoplus_{\ell \neq \frac{n-1}{2}, \frac{3n-1}{2}} \left(-i\cos\left(\phi + \frac{\pi}{n}\ell\right)\sigma_1 \pm i\sin\left(\phi + \frac{\pi}{n}\ell\right)\sigma_2\right) \oplus \pm i \oplus \mp i.$$

Hence $v_\pm^* = -v_\pm$ and $v_\pm^2 = -1$. Let us consider the tensor product $W_1 \otimes W_1$. Its irreducible submodules are $W_0^\pm = \mathbb{C}(\xi_+ \otimes \xi_+ \mp \xi_- \otimes \xi_-)$ and $W_2 = \mathbb{C}\xi_+ \otimes \xi_- + \mathbb{C}\xi_- \otimes \xi_+$ which give $\tau_0^\pm$ and τ_2, respectively. Hence for τ_0^-, the cyclic component acts trivially and the reflective component acts as the multiplication of -1. Proceeding tensor products and decompositions, we get the τ_j-module $W_j = \mathbb{C}\xi_+ \otimes \xi_- \otimes \cdots \otimes \xi_- \otimes \xi_+ + \mathbb{C}\xi_- \otimes \xi_+ \otimes \cdots \otimes \xi_+ \otimes \xi_-$ for $1 \leqslant j \leqslant n-1$, where the length of the words $\xi_\pm$ is j. The cyclic component acts on W_j as the direct sum module of $w_+w_- \cdots w_-w_+$ and $w_-w_+ \cdots w_+w_-$. The reflective component acts there $v_+v_- \cdots v_-v_+$ and $v_-v_+ \cdots v_+v_-$. When j is odd, we have

$$v_\pm v_\mp \cdots v_\mp v_\pm = \bigoplus_{\ell \neq \frac{n-1}{2}, \frac{3n-1}{2}} i^j \left(-\cos j\left(\phi + \frac{\pi}{n}\ell\right)\sigma_1 \pm \sin j\left(\phi + \frac{\pi}{n}\ell\right)\sigma_2\right) \oplus \pm i \oplus \mp i.$$

Finally considering the tensor products module $W_1 \otimes W_{n-1}$, we get its one-dimensional submodules $W_n^\pm = \mathbb{C}(\xi_\pm \otimes \xi_\mp \otimes \cdots \otimes \xi_\mp \otimes \xi_\pm \pm i\xi_\mp \otimes \xi_\pm \otimes \cdots \otimes \xi_\pm \otimes \xi_\mp)$. We investigate the action of $(D_n^{\frac{\pi}{2n},0,\frac{\pi}{2}})_\#$ on them. The cyclic component acts by $w_+w_- \cdots w_-w_+ = w$. Since we have the equality $v_\pm v_\mp \cdots v_\mp v_\pm = \bigoplus_{k \neq \frac{n-1}{2}, \frac{3n-1}{2}} i^n (-1)^k (-1)^{\pm 1+1} \sigma_2 \oplus \pm i \oplus \mp i$, the reflection component acts on $W_n^\pm$ by $\bigoplus_{k \neq \frac{n-1}{2}, \frac{3n-1}{2}} i^{n+1} (-1)^k (-1)^{\frac{\pm 1+1}{2}} \sigma_2 \oplus \pm 1 \oplus \mp 1$. Hence the self-adjoint group-like unitary $\widetilde{w}$ is the unitary representation on W_n^+. Especially we get $\overline{\tau_n^\pm} = \tau_n^\pm$. Recall the classical $q = 1$ case $D_n^* \subset SU(2)$. Then it has the same McKay diagram, however, we know

$\overline{\tau_n^{\pm}} = \tau_n^{\mp}$. This shows the difference of the representation theory of $(D_n^{\frac{\pi}{2n},0,\frac{\pi}{2}})_\# \subset SU_{-1}(2)$ and $D_n^* \subset SU(2)$.

We have classified all the right coideals of type D_n (odd $n \geqslant 3$). They are $C(D_n \setminus SU_{-1}(2))$ or $C^*(\eta^m)$ up to conjugacy by the maximal torus action β^L, where $C^*(\eta^m)$ is a right coideal generated by η_r^m for all $r \in I_m$. Finally we end this section with the following result.

Proposition 8.11. *As $SU_{-1}(2)$-covariant systems, $C(D_n \setminus SU_{-1}(2))$ and $C^*(\eta^m)$ are isomorphic.*

Proof. Let g be a matrix in $SU(2)$,

$$r^{13}\left(\frac{\pi}{2}\right) = \begin{pmatrix} \sqrt{2}^{-1} & \sqrt{2}^{-1} \\ -\sqrt{2}^{-1} & \sqrt{2}^{-1} \end{pmatrix}.$$

Recall the $SU_{-1}(2)$-homomorphism $\rho_g = (\upsilon_g \otimes \mathrm{id}) \circ \delta : C(SU_{-1}(2)) \to \mathbb{B}(\mathbb{C}^2) \otimes C(SU_{-1}(2))$. υ_g satisfies $\begin{pmatrix} \upsilon_g(x) & \upsilon_g(u) \\ \upsilon_g(v) & \upsilon_g(y) \end{pmatrix} = \begin{pmatrix} \sqrt{2}^{-1}\sigma_1 & -\sqrt{2}^{-1}\sigma_2 \\ -\sqrt{2}^{-1}\sigma_2 & \sqrt{2}^{-1}\sigma_1 \end{pmatrix}$. Then we obtain

$$\upsilon_g\big(w(\pi_m)_{-m,r}\big) = \sqrt{2}^{-2m}(-1)^{m+r}\begin{bmatrix} 2m \\ m-r \end{bmatrix}^{\frac{1}{2}} \sigma_1^{m-r}\sigma_2^{m+r},$$

$$\upsilon_g\big(w(\pi_m)_{m,r}\big) = \sqrt{2}^{-2m}(-1)^{m-r}\begin{bmatrix} 2m \\ m-r \end{bmatrix}^{\frac{1}{2}} \sigma_2^{m-r}\sigma_1^{m+r}$$

for all $r \in I_m$. We also have the equality

$$(-1)^{m+r}\sigma_1^{m-r}\sigma_2^{m+r} + (-1)^{m-r}\sigma_2^{m-r}\sigma_1^{m+r} = (-1)^{2m}(\sigma_2 - \sigma_1)$$

for all $r \in I_m$. Then we have

$$\begin{aligned}
\rho_g\big(x^n + v^n\big) &= \rho_g\big(w(\pi_m)_{-m,-m} + w(\pi_m)_{m,-m}\big) \\
&= \sum_{r \in I_m}\big(\upsilon_g\big(w(\pi_m)_{-m,r}\big) + \upsilon_g\big(w(\pi_m)_{m,r}\big)\big) \otimes w(\pi_m)_{r,-m} \\
&= \sqrt{2}^{-2m}\sum_{r \in I_m}\begin{bmatrix} 2m \\ m-r \end{bmatrix}^{\frac{1}{2}}\big((-1)^{m+r}\sigma_1^{m-r}\sigma_2^{m+r} + (-1)^{m-r}\sigma_2^{m-r}\sigma_1^{m+r}\big) \\
&\quad \otimes w(\pi_m)_{r,-m} \\
&= -\sqrt{2}^{-2m}\sum_{r \in I_m}\begin{bmatrix} 2m \\ m-r \end{bmatrix}^{\frac{1}{2}}(\sigma_2 - \sigma_1) \otimes w(\pi_m)_{r,-m} \\
&= -\sqrt{2}^{-2m}(\sigma_2 - \sigma_1) \otimes \eta_{-m}^m.
\end{aligned}$$

Let us define the self-adjoint unitary $\nu = \sqrt{2}^{-1}(\sigma_2 - \sigma_1)$. Then we have $\rho_g(w(\pi_m)_{-m,s} + w(\pi_m)_{m,s}) = -\sqrt{2}^{1-2m}\nu \otimes \eta_s^m$ for all $s \in I_m$. Hence we get $\rho_g(C(D_n \setminus SU_{-1}(2))) \subset C^*(\nu) \otimes C^*(\eta^m)$. Take a $*$-homomorphism $\omega : C^*(\nu) \to \mathbb{C}$ and we have an $SU_{-1}(2)$-homomorphism

$(\omega \otimes \mathrm{id}) \circ \rho_g : C(D_n \setminus SU_{-1}(2)) \to C^*(\eta^m)$. This is clearly surjective and the injectivity follows from Remark 4.23. □

Acknowledgments

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Appendix A

We list all the connected graphs of norm 2 from Fig. 3 to Fig. 15 and the spectral patterns of ergodic systems which sit at the vertex of 1 (the entry of the Perron–Frobenius vector). Because we need not to specify the detailed pattern of type A'_m in our classification program, we do not list it in the A'_m case.

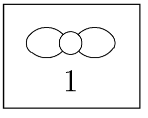

Fig. 3. 1.

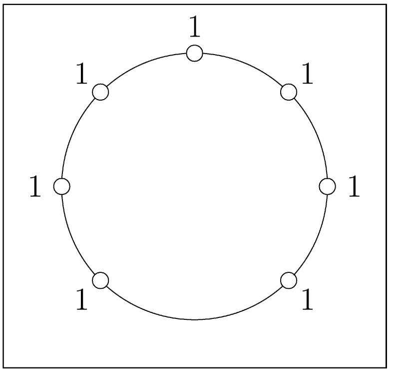

Fig. 4. $\mathbb{T}_m$, m nodes.

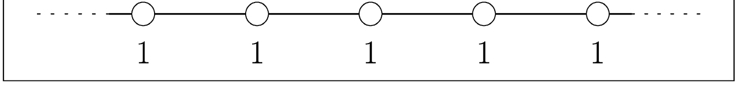

Fig. 5. $\mathbb{T}$.

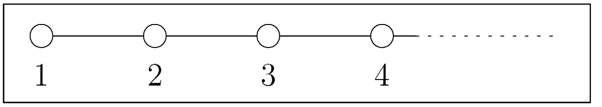

Fig. 6. $SU(2)$.

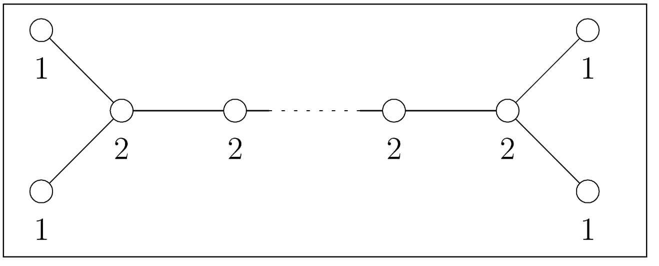

Fig. 7. D_n^* ($n \geqslant 2$), $n+3$ nodes.

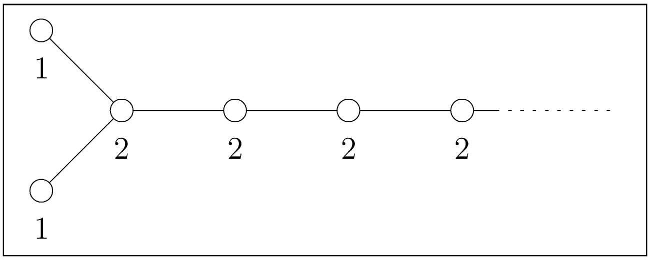

Fig. 8. D_∞^*.

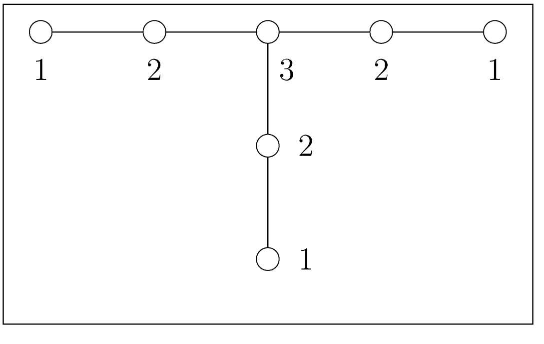

Fig. 9. A_4^*.

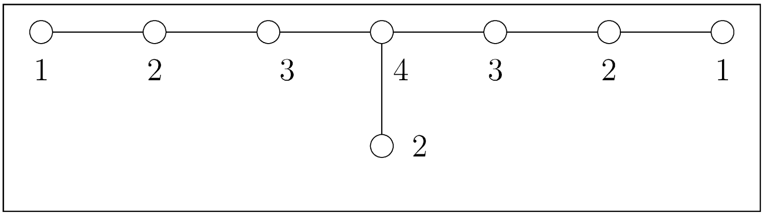

Fig. 10. S_4^*.

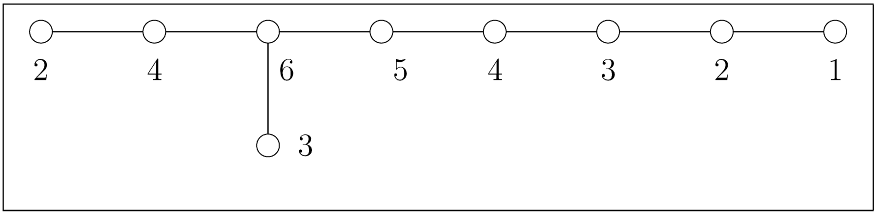

Fig. 11. A_5^*.

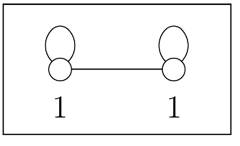

Fig. 12. D_1.

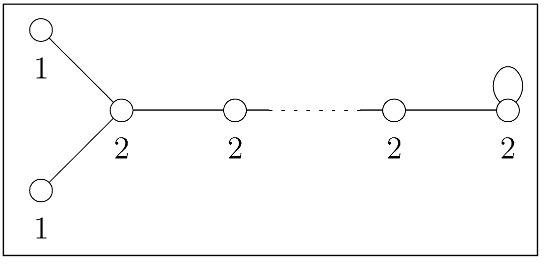

Fig. 13. D_n (odd $n \geqslant 3$), $(n+3)/2$ nodes.

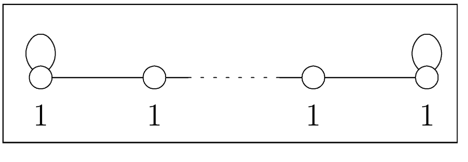

Fig. 14. A'_m ($m \geqslant 3$) nodes.

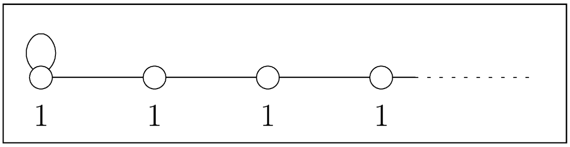

Fig. 15. A'_∞.

1	$\bigoplus_{\nu\in\frac{1}{2}}(2\nu+1)\pi_\nu$,
$\mathbb{T}_n$ (even $n\geqslant 2$)	$\bigoplus_{k\in\mathbb{Z}_{\geqslant 0}}(1+2[\frac{2k}{n}])\pi_k$,
$\mathbb{T}_n$ (odd $n\geqslant 3$)	$\bigoplus_{k\in\mathbb{Z}_{\geqslant 0}}(1+2[\frac{k}{n}])\pi_k\oplus\bigoplus_{k\in\mathbb{Z}_{\geqslant 0}}2[\frac{2k+n+1}{2n}]\pi_{k+\frac{1}{2}}$,
$\mathbb{T}$	$\bigoplus_{k\in\mathbb{Z}_{\geqslant 0}}\pi_k$,
$SU(2)$	π_0,
D_m^* ($m\geqslant 2$)	$\bigoplus_{k\in\mathbb{Z}_{\geqslant 0}}(\frac{1+(-1)^k}{2}+[\frac{k}{m}])\pi_k$,
D_∞^*	$\bigoplus_{k\in\mathbb{Z}_{\geqslant 0}}\pi_{2k}$,
A_4^*	$\pi_0\oplus\pi_3\oplus\pi_4\oplus 2\pi_6\oplus\pi_7\oplus\cdots$,
S_4^*	$\pi_0\oplus\pi_4\oplus\pi_6\oplus\pi_8\oplus\pi_9\oplus\pi_{10}\oplus\cdots$,
A_5^*	$\pi_0\oplus\pi_6\oplus\pi_{10}\oplus\pi_{12}\oplus\cdots$,
D_n (odd $n\geqslant 1$)	$\bigoplus_{k\in\mathbb{Z}_{\geqslant 0}}(\frac{1+(-1)^k}{2}+[\frac{k}{n}])\pi_k\oplus\bigoplus_{k\in\mathbb{Z}_{\geqslant 0}}([\frac{2k+1}{n}]-[\frac{k}{n}])\pi_{k+\frac{1}{2}}$.

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On infinite divisibility of positive definite functions arising from operator means

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Abstract

For a (point-wisely non-negative) positive definite function a certain criterion for its infinite divisibility (i.e., all its fractional powers are also positive definite) is obtained. This criterion enables us to show infinite divisibility for many positive definite functions appearing naturally in study of operator means. In particular, we determine when the function

$$\frac{\cosh(\nu x)+s'}{\cosh x+s} \quad \bigl(\nu \in [0,1];\ s, s' \in (-1,1]\bigr)$$

is infinitely divisible.

Keywords: Hadamard product; Infinitely divisible function; Infinitely divisible matrix; Operator mean; Positive definite function

1. Introduction

Operator means and comparison of their (unitarily invariant) norms are under active investigation (see [5,12–14,17,22] for instance), where many positive matrices with non-negative entries

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naturally appear. Typical examples are $A = [a_{ij}]_{i,j=1,2,\dots,n}$ and $B = [b_{ij}]_{i,j=1,2,\dots,n}$ (or their suitable variants) with entries

$$a_{ij} = \frac{\lambda_i^\theta - \lambda_j^\theta}{\lambda_i - \lambda_j}, \qquad b_{ij} = \frac{\lambda_i^\theta + \lambda_j^\theta}{\lambda_i + \lambda_j} \quad (0 < \theta < 1)$$

with $\lambda_1, \lambda_2, \dots, \lambda_n > 0$, and the positivity of A, B corresponds to the positive definiteness of the functions $\sinh(\theta x)/\sinh x$ and $\cosh(\theta x)/\cosh x$, respectively (see Section 2 for details). Thanks to Horn's theorem [15] the positive matrix A is actually infinitely divisible in the sense that

$$\left[a_{ij}^r\right]_{i,j=1,2,\dots,n} \geqslant 0$$

for each $r \in (0,1)$, showing that the positive definite function $f(x) = \sinh(\theta x)/\sinh x$ is indeed infinitely divisible (see (2) in Section 2), i.e., all the fractional powers $f(x)^r$ $(0 < r < 1)$ are positive definite. The reader is advised to see the recent (partly survey) article [1], where the importance of this concept is explained together with an abundance of old and new examples.

In our previous work [4] among other things the infinite divisibility of the positive definite function $\cosh(\theta x)/\cosh x$ $(0 < \theta < 1)$ was established based on a certain power series trick (see the power series (8) used in Section 4), which shows the infinite divisibility of the matrix

$$\left[\frac{\lambda_i^\theta + \lambda_j^\theta}{\lambda_i + \lambda_j}\right]_{i,j=1,2,\dots,n}.$$

From this the infinite divisibility of many "mean matrices" can be derived (as was demonstrated in [4]), and the present work can be considered as a natural continuation to this study. Let us recall that for an operator monotone function $g(x) : [0,\infty) \to [0,\infty)$ the positivity

$$\left[\frac{g(\lambda_i) + g(\lambda_j)}{\lambda_i + \lambda_j}\right]_{i,j=1,2,\dots,n} \geqslant 0$$

is known ([19], see also [5, Remark 5.2]). However, a general result on the infinite divisibility for matrices of this type (except for the special case $g(x) = x^\theta$) is unknown and seems to deserve investigation.

The classical Bochner theorem asserts that a function is positive definite if and only if its Fourier transform is non-negative. Difficulty for study on infinite divisibility lies in the fact that explicit computation for Fourier transforms of fractional powers of functions in question is almost hopeless. A notable exception is

$$\int_{-\infty}^{\infty} \frac{e^{ixy}\,dx}{\cosh^r x} = \frac{2^{r-1}|\Gamma((r+iy)/2)|^2}{\Gamma(r)}, \qquad r \in (0,1)$$

(see [10, 3.985 on p. 507] or [14, Appendix A.6]). This formula yields the positive definiteness of $1/\cosh^r x$, i.e., the infinite divisibility of $1/\cosh x$, which corresponds to the well-known fact

that the Cauchy matrix

$$\left[\frac{1}{\lambda_i+\lambda_j}\right]_{i,j=1,2,\ldots,n}$$

is infinitely divisible.

Based on some complex analysis technique we will obtain a certain criterion for infinite divisibility (see Theorem 2 and Corollary 3) in Section 3, and the Hadamard factorization theorem indeed plays a crucial role in our proof. Our criterion enables us to show the infinite divisibility for the above-mentioned functions (and many others) in a unified way. In Section 4, by combining this criterion with explicit computations of relevant Fourier transforms, we will determine when the function

$$\frac{\cosh(\nu x)+s'}{\cosh x+s} \quad \bigl(\text{with } \nu\in[0,1] \text{ and } s,s'\in(-1,1]\bigr)$$

is infinitely divisible. It is shown to be so exactly when the function is positive definite (see Theorem 11). Here is one of the most basic norm inequalities on operator means: for Hilbert space operators H, K, X with $H,K\geqslant 0$ and a unitarily invariant norm $|||\cdot|||$ we have

$$\bigl|\bigl|\bigl|H^\theta XK^{1-\theta}+H^{1-\theta}XK^\theta\bigr|\bigr|\bigr| \leqslant |||HX+XK||| \quad (0\leqslant\theta\leqslant 1).$$

It is known as the Heinz inequality [11], and can be derived from the positivity of the matrix B or equivalently from the positive definiteness of $\cosh(\theta x)/\cosh x$ (see [14] for instance). In recent years various estimates on norms of more general operator means such as

$$\bigl|\bigl|\bigl|H^\theta XK^{1-\theta}+H^{1-\theta}XK^\theta+xH^{1/2}XK^{1/2}\bigr|\bigr|\bigr|$$

(containing a parameter x) have been studied by several workers (see [5,19,22] for instance). Our analysis here gives rise to very precise information on such “generalized” Heinz-type norm inequalities. This subject (together with related topics) will be covered in our forthcoming article [18]. In the final Section 5 we will discuss infinite divisibility for miscellaneous functions such as

$$\frac{1}{\cosh z+s\cosh(\alpha z)} \quad \bigl(\text{with } \alpha\in[0,1] \text{ and } s\in(-1,1]\bigr)$$

(see Theorem 14).

2. Preliminaries

2.1. Infinitely divisible matrices

The classical Schur theorem states that the Hadamard product (or Schur product) $A\circ B$ of positive matrices A, B is positive: For $A=[a_{ij}]$, $B=[b_{ij}]\geqslant 0$ we have

$$A\circ B=[a_{ij}b_{ij}]\geqslant 0.$$

Here and throughout the positivity $A = [a_{ij}]_{i,j=1,2,\ldots,n} \geqslant 0$ means

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\overline{\xi_j} \geqslant 0$$

for each $\xi_i \in \mathbf{C}$. In particular, if $A = [a_{ij}]$ is positive, then so are the Hadamard powers $A^{\circ m} = [a_{ij}^m]$, $m \in \mathbf{N}$. When each entry is non-negative in addition, fractional Hadamard powers $A^{\circ r} = [a_{ij}^r]$ $(r > 0)$ also make sense. It is known (see [9, Theorem 2.2]) that (i) for such an $n \times n$ matrix we have $A^{\circ r} \geqslant 0$ as long as $r \geqslant n-2$ and (ii) this lower bound $n-2$ is optimal. Note that for a positive matrix $A = [a_{ij}]$ with $a_{ij} \geqslant 0$, the following three conditions are equivalent (thanks to the Schur theorem and the obvious continuity argument):

(i) $A^{\circ \frac{1}{m}} \geqslant 0$ for each $m \in \mathbf{N}$;
(ii) $A^{\circ r} \geqslant 0$ for each $r \in (0,1)$;
(iii) $A^{\circ r} \geqslant 0$ for each $r \in (0,\infty)$.

A matrix satisfying these conditions is called an *infinitely divisible* matrix. A very readable account on such matrices can be found in [16] and many examples are worked out in [1,4].

2.2. *Infinitely divisible functions*

A study on positive matrices is closely related to that of positive definite functions. Let us take $f(x) = \sinh(\theta x)/\sinh x$ with $\theta \in (0,1)$ for instance and observe

$$\begin{aligned}\sum_{i,j=1}^{n} f\left(\frac{1}{2}\log\lambda_i - \frac{1}{2}\log\lambda_j\right)\xi_i\overline{\xi_j} &= \sum_{i,j=1}^{n} \frac{((\lambda_i/\lambda_j)^{\theta/2} - (\lambda_j/\lambda_i)^{\theta/2})\xi_i\overline{\xi_j}}{(\lambda_i/\lambda_j)^{1/2} - (\lambda_j/\lambda_i)^{1/2}} \\ &= \sum_{i,j=1}^{n}\left(\frac{\lambda_i^\theta - \lambda_j^\theta}{\lambda_i - \lambda_j}\cdot \lambda_i^{\frac{1-\theta}{2}}\lambda_j^{\frac{1-\theta}{2}}\xi_i\overline{\xi_j}\right) \qquad (1)\end{aligned}$$

for $\lambda_i > 0$ and $\xi_i \in \mathbf{C}$ $(i = 1,2,\ldots,n)$. This computation obviously shows that $f(x)$ is positive definite if and only if the matrix

$$\left[\frac{\lambda_i^\theta - \lambda_j^\theta}{\lambda_i - \lambda_j}\cdot \lambda_i^{\frac{1-\theta}{2}}\lambda_j^{\frac{1-\theta}{2}}\right]$$

is positive (for each $n \in \mathbf{N}$ and $\lambda_1, \lambda_2, \ldots, \lambda_n > 0$), or equivalently so is the congruent matrix

$$\left[\frac{\lambda_i^\theta - \lambda_j^\theta}{\lambda_i - \lambda_j}\right]_{ij=1,2,\ldots,n}.$$

The Fourier transform of $f(x)$ is non-negative and $f(x)$ is a typical positive definite function (by the Bochner theorem), showing the positivity of the above matrix.

The theory of operator monotone functions (see [7] for instance) neatly fits into the current picture. Let $g:[0,\infty) \to [0,\infty)$ be an operator monotone function, i.e., we have $g(A) \geqslant g(B)$

for arbitrary positive matrices A, B (of any size) with $A \geqslant B$. The operator monotonicity is known to be characterized by (one of) the following conditions:

(i) $\left[\frac{g(\lambda)-g(\lambda_j)}{\lambda_i-\lambda_j}\right]_{ij} \geqslant 0$ for each $\lambda_1, \lambda_2, \ldots, \lambda_n > 0$ (and $n \in \mathbf{N}$);
(ii) $g(x)$ extends to an analytic function on the upper half-plane $H_+ = \{z \in \mathbf{C};\ \Im z > 0\}$ satisfying $g(H_+) \subseteq H_+$.

The operator monotonicity of x^θ is well known, i.e., $\left[\frac{\lambda_i^\theta-\lambda_j^\theta}{\lambda_i-\lambda_j}\right]_{ij} \geqslant 0$ so that (1) indeed gives us an alternative proof for the positive definiteness of $\sinh(\theta x)/\sinh x$. Actually we can do much more. Namely, Horn's theorem [15] asserts that

$$\left[\frac{g(\lambda_i)-g(\lambda_j)}{\lambda_i-\lambda_j}\right]_{ij}$$

is infinitely divisible if and only if $g(z)$ is univalent (i.e., one-to-one) on H_+. This criterion guarantees the infinite divisibility of $\left[\frac{\lambda_i^\theta-\lambda_j^\theta}{\lambda_i-\lambda_j}\right]_{ij}$ (since z^θ is univalent on H_+) while almost identical computations as (1) yield

$$\sum_{i,j=1}^{n}\left(f\left(\frac{1}{2}\log\lambda_i-\frac{1}{2}\log\lambda_j\right)\right)^r\xi_i\overline{\xi_j}=\sum_{i,j=1}^{n}\left(\left(\frac{\lambda_i^\theta-\lambda_j^\theta}{\lambda_i-\lambda_j}\right)^r\lambda_i^{\frac{(1-\theta)r}{2}}\lambda_j^{\frac{(1-\theta)r}{2}}\xi_i\overline{\xi_j}\right). \tag{2}$$

Consequently, the positive definite function $f(x)=\sinh(\theta x)/\sinh x$ ($\geqslant 0$) is actually *infinitely divisible* in the sense that $f(x)^r$ is also positive definite for each $r \in (0,1)$, or equivalently, for each $r \in (0,\infty)$.

Here are some observations:

(i) If $f(x)$, $g(x)$ are positive definite, then so are the sum $f(x)+g(x)$ and the product $f(x)g(x)$.
(ii) If $f(x)$, $g(x)$ are infinitely divisible, then so is the product $f(x)g(x)$.
(iii) When a sequence $\{f_n(x)\}$ of positive definite (respectively infinitely divisible) functions is convergent, then the limit function $\lim_{n\to\infty} f_n(x)$ is also positive definite (respectively infinitely divisible).

These are obvious consequences of respective definitions, which will be freely and repeatedly used in subsequent sections.

3. A certain criterion for infinite divisibility

In this section we will present a general criterion for infinite divisibility. Our criterion is obtained by combining the key lemma below with the classical Hadamard factorization theorem (see for [6, Chapter XI, Section 3] for instance).

Lemma 1. *We assume $a > 0$ and $b \geqslant 0$. Then, the function $\frac{1+bx^2}{1+ax^2}$ is infinitely divisible if and only if $a \geqslant b$.*

Proof. The function (whose value at $x = 0$ is 1) tends to b/a as $x \to \pm\infty$. When it is infinitely divisible, it is of course positive definite, forcing $b/a \leqslant 1$. Hence, it remains to show the non-trivial converse. For this purpose we make use of the well-known integral expression

$$x^r = \frac{\sin(\pi r)}{\pi} \int_0^\infty \frac{x}{x+\lambda} \cdot \frac{d\lambda}{\lambda^{1-r}} \quad (x \geqslant 0) \tag{3}$$

for the fractional power x^r, $0 < r < 1$, which plays an important role in the theory of operator monotone functions (see [7] for instance). Based on this formula we compute

$$\begin{aligned}
\left(\frac{1+bx^2}{1+ax^2}\right)^r &= \frac{\sin(\pi r)}{\pi} \int_0^\infty \frac{(1+bx^2)(1+ax^2)^{-1}}{(1+bx^2)(1+ax^2)^{-1}+\lambda} \frac{d\lambda}{\lambda^{1-r}} \\
&= \frac{\sin(\pi r)}{\pi} \int_0^\infty \frac{1+bx^2}{1+bx^2+\lambda(1+ax^2)} \frac{d\lambda}{\lambda^{1-r}} \\
&= \frac{\sin(\pi r)}{\pi} \int_0^\infty \frac{1+bx^2}{1+\lambda+(a\lambda+b)x^2} \frac{d\lambda}{\lambda^{1-r}} \\
&= \frac{\sin(\pi r)}{\pi} \int_0^\infty \left(\frac{b}{a\lambda+b} + \frac{\lambda(a-b)}{a\lambda+b} \cdot \frac{1}{1+\lambda+(a\lambda+b)x^2}\right) \frac{d\lambda}{\lambda^{1-r}}.
\end{aligned} \tag{4}$$

Note that the integrand here is positive definite thanks to

$$\frac{1}{1+\lambda+(a\lambda+b)x^2} = \frac{1}{2\gamma(1+\lambda)} \int_{-\infty}^\infty e^{-\frac{|y|}{\gamma}} e^{ixy}\, dy \quad \text{with } \gamma = \sqrt{\frac{a\lambda+b}{1+\lambda}} \tag{5}$$

together with the positivity $\frac{b}{a\lambda+b}, \frac{\lambda(a-b)}{a\lambda+b} \geqslant 0$. Being a "superposition" of positive definite functions, $(\frac{1+bx^2}{1+ax^2})^r$ is also positive definite, i.e., $\frac{1+bx^2}{1+ax^2}$ is infinitely divisible. $\square$

We observe

$$\frac{\sin(\pi r)}{\pi} \int_0^\infty \frac{b}{a\lambda+b} \cdot \frac{d\lambda}{\lambda^{1-r}} = (b/a)^r$$

due to (3). After substitution of (5) into (4) we change the order of two integrals, and then we set $t = 1/\gamma$. In this way it is not so difficult to get

$$\left(\frac{1+bx^2}{1+ax^2}\right)^r = (b/a)^r + \int_{-\infty}^\infty \left(\frac{\sin(\pi r)}{\pi} \int_{1/\sqrt{a}}^{1/\sqrt{b}} e^{-t|y|} \left(\frac{1-bt^2}{at^2-1}\right)^r dt\right) e^{ixy}\, dy.$$

This expression will not be used in sequel, and details are left to the reader.

Theorem 2. *Let $f(z)$ be an entire function taking real values for the reals (the restriction to* **R** *is denoted by $f(x)$). We assume:*

(i) $f(0) > 0$ *and* $f'(0) = 0$;
(ii) *all the zeros of $f(z)$ are pure imaginary;*
(iii) *the order ρ of $f(z)$ is less than 2, i.e.,*

$$\rho = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r} < 2 \quad \textit{with } M(r) = \max\bigl\{\bigl|f(z)\bigr|;\ |z| = r\bigr\}.$$

Under these circumstances the (real) functions $1/f(x)$ and $f(\nu x)/f(x)$ ($\nu \in [0, 1]$) are infinitely divisible.

Proof. Firstly we collect all the zeros in the upper half-plane (repeated according to multiplicity). By the assumption (ii) they are of the form $\{i\alpha_n\}_{n=1,2,\ldots}$ with $0 < \alpha_1 \leqslant \alpha_2 \leqslant \alpha_3 \leqslant \cdots$. Then (thanks to the Schwarz reflection principle) all the zeros are given by $\{\pm i\alpha_n\}_{n=1,2,\ldots}$. Let p be the smallest integer satisfying

$$\sum_{n=1}^{\infty} \frac{1}{\alpha_n^{p+1}} < \infty.$$

The basic property $p \leqslant \rho$ can be deduced from the Poisson–Jensen formula on distribution of zeros, and this exponent p is called the rank of $f(z)$ in [6]. The Hadamard factorization theorem enables us to factorize $f(z)$ in the following way:

$$f(z) = e^{P(z)} \prod_{n=1}^{\infty} \left(\left(1 - \frac{z}{i\alpha_n}\right) \exp\left(\frac{z}{i\alpha_n}\right)\right) \prod_{n=1}^{\infty} \left(\left(1 - \frac{z}{-i\alpha_n}\right) \exp\left(\frac{z}{-i\alpha_n}\right)\right)$$

(when $p = 1$) or

$$f(z) = e^{P(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{i\alpha_n}\right) \prod_{n=1}^{\infty} \left(1 - \frac{z}{-i\alpha_n}\right)$$

(when $p = 0$), where the infinite product in the right-hand side is uniformly convergent on compact sets in the complex plane. Here, $P(z)$ is a polynomial of degree q and we have $\max(p, q) \leqslant \rho$. (In [6] $\max(p, q)$ is called the genus of $f(z)$.) The requirement (iii) forces $q = 0$ or $q = 1$, and we have $e^{P(z)} = f(0)e^{az}$ with some a. Observe that (when $p = 1$) the above two exponential factors cancel out, Therefore, $f(z)$ is of the form

$$f(z) = f(0)e^{az} \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{\alpha_n^2}\right).$$

Logarithmic differentiation yields

$$\frac{f'(z)}{f(z)} = a + 2z \sum_{n=0}^{\infty} \frac{\alpha_n^{-2}}{1 + z^2/\alpha_n^2},$$

and hence we must have $a = f'(0)/f(0) = 0$. This computation looks somewhat formal, but it is not. In fact, for each fixed $r > 0$ one can choose an integer n_0 large enough satisfying $|z^2/\alpha_n^2| \leqslant 1/2$ for $n \geqslant n_0$ and $|z| \leqslant r$ (due to $\alpha_n \nearrow \infty$). We thus have $|1 + z^2/\alpha_n^2| \geqslant 1/2$ and estimate

$$\sum_{n=n_0}^{\infty} \left| \frac{\alpha_n^{-2}}{1 + z^2/\alpha_n^2} \right| \leqslant 2 \sum_{n=n_0}^{\infty} \frac{1}{\alpha_n^2} < \infty.$$

From the arguments so far we have the factorization

$$\frac{f(z)}{f(0)} = \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{\alpha_n^2} \right),$$

showing

$$\frac{f(\nu x)}{f(x)} = \lim_{m \to \infty} \prod_{n=1}^{m} \left(\frac{1 + \nu^2 x^2/\alpha_n^2}{1 + x^2/\alpha_n^2} \right).$$

Products of infinitely divisible functions are obviously infinitely divisible. Therefore, the above finite product $\prod_{n=1}^{m}$ is infinitely divisible (for each m) thanks to Lemma 1 and so is the limit $f(\nu x)/f(x)$. $\square$

In Appendix B we will present a general result (Proposition B.1) on infinitely divisible matrices based on the Hadamard factorization, which is motivated by arguments in [1, Section 2.3]. The reasoning in the preceding proof obviously works in the following situation as well:

Corollary 3. *Let $f(z), g(z)$ be functions satisfying the conditions in Theorem* 2 *with the zeros $\{i\alpha_n\}_{n=1,2,\ldots}$, $\{i\beta_n\}_{n=1,2,\ldots}$, respectively, in the upper half-plane* (*satisfying* $0 < \alpha_1 \leqslant \alpha_2 \leqslant \cdots$ *and* $0 < \beta_1 \leqslant \beta_2 \leqslant \cdots$ *with multiplicities included as before*)*. If $\alpha_n \leqslant \beta_n$* ($n = 1, 2, \ldots$)*, then the ratio $g(x)/f(x)$ is an infinitely divisible function.*

A few remarks are in order.

Remark 4.

(i) The functions

$$\cosh z, \quad \sinh z/z \quad \text{and} \quad \cosh z + s \quad \big(\text{with } s \in (-1, 1]\big)$$

are typical examples satisfying all the requirements in the theorem. Other typical examples will be also pointed out in Section 5. The well-known formulas

$$\sinh z = z \prod_{n=1}^{\infty}\left(1 + \frac{z^2}{n^2\pi^2}\right) \quad \text{and} \quad \cosh z = \prod_{n=1}^{\infty}\left(1 + \frac{z^2}{((2n-1)\pi/2)^2}\right)$$

are the Hadamard factorization for $\sinh z/z$ and $\cosh z$ (while that for the third function is to be worked out in the next section). Anyway, the theorem guarantees the infinite divisibility of the following functions:

$$\frac{\cosh(\nu x)}{\cosh x}, \quad \frac{\sinh(\nu x)}{\sinh x}, \quad \frac{x}{\sinh x}, \quad \frac{1}{\cosh x + s}, \quad \frac{\cosh(\nu x) + s}{\cosh x + s}$$

with $\nu \in [0, 1]$ and $s \in (-1, 1]$. In our previous studies on operator means (see [5,12–14,17] for instance) positive definiteness of relevant functions played an essential role. Factorization technique was also used in [5] to establish positive definiteness.

(ii) In [4, Theorem 3] the infinite divisibility of the function

$$\frac{x\cosh(\nu x)}{\sinh x} \quad \bigl(\nu \in [0, 1/2]\bigr)$$

is proved, which can be also easily seen from Corollary 3. In fact, with $f(z) = \sinh z/z$ and $g(z) = \cosh(\nu z)$ we have

$$\alpha_n = n\pi \quad (n = 1, 2, \ldots),$$
$$\beta_n = \frac{1}{\nu}\cdot\frac{(2n-1)\pi}{2} \quad (n = 1, 2, \ldots),$$

and observe $\alpha_n \leqslant \beta_n$ $(n = 1, 2, \ldots)$ as long as $0 < \nu \leqslant 1/2$. In the recent article [8] it is shown that the function $x\cosh(\nu x)/\sinh x$ with $\nu > 1/2$ is not positive definite.

(iii) The function $\tanh x/x$ is infinitely divisible. More generally so are the functions

$$\frac{\sinh x}{x\,(\cosh x + s)} \quad (-1 < s \leqslant 1).$$

Indeed, with $g(z) = \sinh z/z$ and $f(z) = \cosh z + s$ we can use Corollary 3 (see the first part of Section 4 for the zeros of the latter). A different proof for this fact is presented in the forthcoming book [2, Chapter 5].

Remark 5. A probability measure μ is said to be infinitely divisible if for each $m \in \mathbf{N}$ it can be written as the m-fold convolution product $\mu_m * \mu_m * \cdots * \mu_m$ with some probability measure μ_m. The probability distribution $\frac{a}{\pi}\cdot\frac{1}{a^2+(x-m)^2}$ (with parameters $-\infty < m < \infty$ and $a > 0$) is known as the Cauchy distribution and is a typical infinitely divisible distribution. Lemma 1 is of course closely related to this fact. An infinitely divisible probability measure plays an important role in the study of Lévy processes (see [20,21] for instance), which the author is unfortunately not

so familiar with. It is known that μ is infinitely divisible if and only if the Fourier transform $\hat{\mu}$ (called a "characteristic function") admits a Lévy–Khintchine representation, i.e.,

$$\hat{\mu}(t) = \exp\left(-\frac{at^2}{2} + i\gamma t + \int_{-\infty}^{\infty} \left(e^{its} - 1 - its\,\chi_{[-1,1]}(s)\right) d\nu(s)\right)$$

with $a \geqslant 0$, $\gamma \in \mathbf{R}$ and a measure ν satisfying $\nu(\{0\}) = 0$ and $\int_{-\infty}^{\infty} \min(s^2, 1)\, d\nu(s) < \infty$ (see [21, Section 2.8] for details). However, it is practically impossible to check this criterion in our setting. Another related and useful notion is self-decomposability for probability measures (see [21, Chapter 3]): Namely, if a probability measure μ is self-decomposable (i.e., $\hat{\mu}(t)/\hat{\mu}(b^{-1}t)$ is positive definite for each $b > 1$), then it is infinitely divisible. This fact can be also used to see the infinite divisibility for some of the functions in Remark 4(i).

Remark 6. The function $\exp(-ax^2)$ (with $a > 0$) is obviously infinitely divisible, but $\exp(az^2)$ is of order 2 so that this situation is not covered in Theorem 2. Let us assume that an entire function $f(z)$ in Theorem 2 (i.e., $f(z)$ satisfies (i), (ii) and $f(\mathbf{R}) \subseteq \mathbf{R}$) is of order 2. As in the proof of Theorem 2, the zeros of $f(z)$ are of the form $\{\pm i\alpha_n\}_{n=1,2,\ldots}$ with $0 < \alpha_1 \leqslant \alpha_2 \leqslant \cdots$. Let us further assume that the rank p of $f(z)$ is 0 or 1, i.e.,

$$s = \sum_{n=1}^{\infty} \frac{1}{\alpha_n^2} < \infty.$$

Then, the Hadamard factorization theorem shows

$$f(z) = f(0) e^{az+bz^2} \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{\alpha_n^2}\right)$$

for some constants a, b (after canceling exponential factors when $p = 1$ as in the proof of Theorem 2) and with the logarithmic derivative

$$\frac{f'(z)}{f(z)} = a + 2bz + 2z \sum_{n=0}^{\infty} \frac{\alpha_n^{-2}}{1 + z^2/\alpha_n^2}.$$

Hence we get $a = 0$ again from the assumption $f'(0) = 0$. By differentiating the both sides, we observe

$$\frac{f''(z)f(z) - f'(z)^2}{f(z)^2} = 2b + 2\sum_{n=1}^{\infty} \frac{\alpha_n^{-2}}{1 + z^2/\alpha_n^2} - 4z^2 \sum_{n=1}^{\infty} \frac{\alpha_n^{-4}}{(1 + z^2/\alpha_n^2)^2},$$

showing $\frac{f''(0)}{f(0)} = 2(b + s)$ thanks to $f'(0) = 0$. Therefore, with the additional requirement

$$\frac{f''(0)}{f(0)} \geqslant 2s$$

the positivity of b is guaranteed and consequently the same conclusion as Theorem 2 is available (for $\rho = 2$ and $p \neq 2$).

4. Infinite divisibility for $(\cosh(\nu x) + s')/(\cosh x + s)$

We set

$$f_{\nu,s,s'}(x) = \frac{\cosh(\nu x) + s'}{\cosh x + s} \quad \bigl(\text{for } s, s' \in (-1, 1] \text{ and } \nu \in [0, 1]\bigr).$$

We have already known the infinite divisibility of $f_{\nu,s,s}(x)$ (see Remark 4(i)). In this section we will determine when $f_{\nu,s,s'}(x)$ is infinitely divisible. As mentioned in Section 1 this kind of information will be quite useful for investigation on generalized Heinz-type inequalities.

For $\theta \in [0, \pi)$ we set

$$g(z) = \cosh z + \cos\theta,$$

which is an entire function of order 1 due to the obvious estimate $|\cosh z| \leqslant e^{|z|}$. We will explicitly write down the Hadamard factorization for $g(z)$. (The Hadamard factorization for $\cosh z + s$ with $s \geqslant 1$ will be worked out in Appendix A, see Proposition A.1.) Let us begin with the zeros of $g(z)$. We observe

$$g(z) = 0 \quad \Longleftrightarrow \quad \cosh z = -\cos\theta = \cos(\pi - \theta)$$

so that $z = i(\pi - \theta), i(\pi + \theta)$ are zeros. They are simple zeros for $\theta \in (0, \pi)$ while $i\pi$ is a double zero for $\theta = 0$. All the zeros are obviously

$$z = i(\pi - \theta + 2n\pi), \quad i(\pi + \theta + 2n\pi) \quad (n \in \mathbf{Z}),$$

or equivalently,

$$z = \pm i(\pi - \theta + 2n\pi), \quad \pm i(\pi + \theta + 2n\pi) \quad (n = 0, 1, 2, \ldots).$$

We observe

$$\sum_{n=0}^{\infty} \frac{1}{\pi - \theta + 2n\pi} + \sum_{n=0}^{\infty} \frac{1}{\pi + \theta + 2n\pi} = \infty,$$

$$\sum_{n=0}^{\infty} \frac{1}{(\pi - \theta + 2n\pi)^{1+\varepsilon}} + \sum_{n=0}^{\infty} \frac{1}{(\pi + \theta + 2n\pi)^{1+\varepsilon}} < \infty \quad (\text{for } \varepsilon > 0),$$

showing that the exponent p (in the proof of Theorem 2) is 1. Therefore, the Hadamard factorization theorem asserts

$$g(z) = g(0)e^{az} \prod_{n=0}^{\infty} \left(\left(1 - \frac{z}{i(\pi - \theta + 2n\pi)} \right) e^{-iz/(\pi - \theta + 2n\pi)} \right)$$

$$\times \prod_{n=0}^{\infty}\left(\left(1-\frac{z}{-i(\pi-\theta+2n\pi)}\right)e^{iz/(\pi-\theta+2n\pi)}\right)$$
$$\times \prod_{n=0}^{\infty}\left(\left(1-\frac{z}{i(\pi+\theta+2n\pi)}\right)e^{-iz/(\pi+\theta+2n\pi)}\right)$$
$$\times \prod_{n=0}^{\infty}\left(\left(1-\frac{z}{-i(\pi+\theta+2n\pi)}\right)e^{iz/(\pi+\theta+2n\pi)}\right)$$

with some constant a. We can obviously rearrange involved products into the following form:

$$g(z)=g(0)e^{az}\prod_{n=0}^{\infty}\left(1+\frac{z^2}{(\pi-\theta+2n\pi)^2}\right)\cdot\prod_{n=0}^{\infty}\left(1+\frac{z^2}{(\pi+\theta+2n\pi)^2}\right).$$

We note

$$g(0)=1+\cos\theta \quad \text{and} \quad a=\frac{g'(0)}{g(0)}=0$$

(as in the proof of Theorem 2), and hence we have shown

Proposition 7. *For $\theta\in[0,\pi)$ we have the factorization*

$$\frac{\cosh z+\cos\theta}{1+\cos\theta}=\prod_{n=0}^{\infty}\left(1+\frac{z^2}{(\pi-\theta+2n\pi)^2}\right)\cdot\prod_{n=0}^{\infty}\left(1+\frac{z^2}{(\pi+\theta+2n\pi)^2}\right).$$

One should be also able to derive this factorization formula from that for $\cosh z$ (in Remark 4(i)) and the identity

$$\cosh z+\cos\theta=\cosh z+\cosh(i\theta)=2\cosh\bigl((z+i\theta)/2\bigr)\cosh\bigl((z-i\theta)/2\bigr).$$

However, the direct argument presented so far seems easier. The concrete factorization formula for $\cosh z+s$ (i.e., information on zeros) and Lemma 1 are main ingredients in the next lemma.

Lemma 8. *We assume $s,s'\in(-1,1]$ and $\nu\in[0,1]$. The function $f_{\nu,s,s'}(x)$ is infinitely divisible when the following two inequalities are satisfied*:

$$\nu\leqslant\frac{\pi-\cos^{-1}s'}{\pi-\cos^{-1}s} \quad \textit{and} \quad \nu\leqslant\frac{\pi+\cos^{-1}s'}{\pi+\cos^{-1}s}.$$

Proof. Proposition 7 shows

$$\frac{\cosh(\nu z)+s'}{1+s'}=\prod_{n=0}^{\infty}\left(1+\frac{\nu^2z^2}{(\pi-\theta'+2n\pi)^2}\right)\cdot\prod_{n=0}^{\infty}\left(1+\frac{\nu^2z^2}{(\pi+\theta'+2n\pi)^2}\right),$$
$$\frac{\cosh z+s}{1+s}=\prod_{n=0}^{\infty}\left(1+\frac{z^2}{(\pi-\theta+2n\pi)^2}\right)\cdot\prod_{n=0}^{\infty}\left(1+\frac{z^2}{(\pi+\theta+2n\pi)^2}\right)$$

with $\theta = \cos^{-1} s$ and $\theta' = \cos^{-1} s'$. Thus, thanks to Corollary 3 the ratio is infinitely divisible when

$$(\pi - \theta + 2n\pi)^{-2} \geqslant \nu^2(\pi - \theta' + 2n\pi)^{-2} \quad \text{and} \quad (\pi + \theta + 2n\pi)^{-2} \geqslant \nu^2(\pi + \theta' + 2n\pi)^{-2}$$

for each $n = 0, 1, 2, \ldots$, or equivalently,

$$\pi - \theta' + 2n\pi \geqslant \nu(\pi - \theta + 2n\pi) \quad \text{and} \quad \pi + \theta' + 2n\pi \geqslant \nu(\pi + \theta + 2n\pi)$$

for each $n = 0, 1, 2, \ldots$. We observe that as soon as these inequalities for $n = 0$ hold true then so do all the others, meaning that this condition is the same as what is stated in the lemma. □

Let us consider the following two cases:

Case $s' \geqslant s$. We have

$$\frac{\pi - \cos^{-1} s'}{\pi - \cos^{-1} s} \geqslant 1 \geqslant \nu, \tag{6}$$

i.e., the first inequality in Lemma 8 is always satisfied.

Case $s' \leqslant s$. We have

$$\frac{\pi + \cos^{-1} s'}{\pi + \cos^{-1} s} \geqslant 1 \geqslant \nu, \tag{7}$$

i.e., the second inequality in Lemma 8 is always satisfied.

The second inequality in Lemma 8 does not necessarily hold true when $s' \geqslant s$ so that we cannot use the lemma in this circumstance. Instead the following power series expansion is in rescue:

$$(1 - x)^{-r} = \sum_{n=0}^{\infty} a_n x^n \quad \text{for } r \in (0, 1) \text{ and } |x| < 1 \tag{8}$$

with the coefficients

$$a_n = \frac{r(r+1)(r+2)\cdots(r+(n-1))}{n!} \ (\geqslant 0) \quad \text{for } n = 1, 2, \ldots$$

and $a_0 = 1$.

Lemma 9. *Assume $s, s' \in (-1, 1]$. If $s' \geqslant s$, then the function $f_{\nu,s,s'}(x)$ is infinitely divisible for each $\nu \in [0, 1]$.*

Proof. We note

$$\begin{aligned} f_{\nu,s,s'}(x) &= \frac{\cosh(\nu x)+s'}{\cosh x + s' - (s'-s)} \\ &= \frac{\cosh(\nu x)+s'}{\cosh x + s'} \cdot \left(1 - \frac{s'-s}{\cosh x + s'}\right)^{-1} \end{aligned}$$

with $0 \leqslant s' - s < \cosh x + s'$, and hence (8) yields

$$\bigl(f_{\nu,s,s'}(x)\bigr)^r = \left(\frac{\cosh(\nu x)+s'}{\cosh x + s'}\right)^r \cdot \sum_{n=0}^{\infty} \frac{(s'-s)^n a_n}{(\cosh x + s')^n}.$$

The desired infinite divisibility follows from this expression. Indeed, the first factor in the above right-hand side is positive definite thanks to Remark 4(i) while $(\cosh x + s')^{-1}$ as well as its powers are also positive definite. □

The above "power series trick" was quite useful in our previous work [4]. It will be also repeatedly used in Section 5.

Information on Fourier transforms for relevant functions is indispensable for the proof of the next lemma (Lemma 10). Here we record required formulas on Fourier transforms. They can be found in [10] (and detailed computations are presented in [18] for instance).

(i) We have

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\cosh x + 1} e^{ixy}\, dx = \frac{y}{\sinh(\pi y)}.$$

Moreover, for $\nu \in [0,1)$ we have

$$\frac{1}{2\pi} \int_{\infty}^{\infty} \frac{\cosh(\nu x)}{\cosh x + 1} e^{ixy}\, dx = \frac{y \sinh(\pi y)\cos(\pi\nu) + \nu\cosh(\pi y)\sin(\pi\nu)}{\sinh^2(\pi y) + \sin^2(\pi\nu)}.$$

(ii) For $s \in (-1,1)$ we have

$$\frac{\sqrt{1-s^2}}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ixy}}{\cosh x + s}\, dx = \frac{\sinh(\theta y)}{\sinh(\pi y)}$$

with $\theta = \cos^{-1} s$.

(iii) For $s \in (-1,1)$ and $\nu \in [0,1)$ we have

$$\frac{\sqrt{1-s^2}}{2\pi} \int_{-\infty}^{\infty} \frac{\cosh(\nu x)}{\cosh x + s} e^{ixy}\, dx$$

$$= \frac{\cos((\pi-\theta)\nu)\sinh(\pi y)\sinh(\theta y)+\sin(\pi\nu)\sin(\theta\nu)\cosh((\pi-\theta)y)}{\sinh^2(\pi y)+\sin^2(\pi\nu)}$$

with θ given in (ii). The numerator here is equal to

$$\begin{aligned}
&\big(\cos(\pi\nu)\cos(\theta\nu)+\sin(\pi\nu)\sin(\theta\nu)\big)\sinh(\pi y)\sinh(\theta y)\\
&\quad+\sin(\pi\nu)\sin(\theta\nu)\big(\cosh(\pi y)\cosh(\theta y)-\sinh(\pi y)\sinh(\theta y)\big)\\
&=\cosh(\pi y)\cosh(\theta y)\sinh(\pi y)\sinh(\theta y)+\sin(\pi\nu)\sin(\theta\nu)\cosh(\pi y)\cosh(\theta y),
\end{aligned}$$

and we easily observe that it can be also written as

$$\frac{1}{2}\Big(\cos\big((\pi-\theta)\nu\big)\cosh\big((\pi+\theta)y\big)-\cos\big((\pi+\theta)\nu\big)\cosh\big((\pi-\theta)y\big)\Big)$$

(by just expanding everything as above). This last expression actually appears in [10, 3.983, formula 6, p. 506].

It is possible to get the formulas in (i) from those in (iii) by letting $s \nearrow 1$, which is certainly legitimate thanks to the dominated convergence theorem.

Lemma 10. *We assume $s, s' \in (-1, 1]$ and $\nu \in [0, 1]$. If the function $f_{\nu,s,s'}(x)$ is positive definite, then we must have $\nu \leqslant \frac{\pi-\cos^{-1}s'}{\pi-\cos^{-1}s}$.*

Proof. For $\nu = 1$ we note

$$\mathcal{F} f_{1,s,s'} = \mathcal{F}\left(1+\frac{s'-s}{\cosh x+s}\right) = \delta_0 + (s'-s)\,\mathcal{F}\left(\frac{1}{\cosh x+s}\right)$$

with the delta function δ_0 together with the Fourier transform given by either (the first part in) (i) or (ii), showing that the function is positive definite if and only if $s' \geqslant s$. Thus, in the rest we may and do assume $\nu \in [0, 1)$.

Case $s = 1$. Thanks to (i) we compute

$$\begin{aligned}
&\frac{1}{2\pi}(\mathcal{F} f_{\nu,1,s'})(y)\\
&\quad=\frac{y\sinh(\pi y)\cos(\pi\nu)+\nu\cosh(\pi y)\sin(\pi\nu)}{\sinh^2(\pi y)+\sin^2(\pi\nu)}+\frac{s'y}{\sinh(\pi y)}\\
&\quad=\frac{1}{\sinh^2(\pi y)+\sin^2(\pi\nu)}\cdot\bigg[\big(y\sinh(\pi y)\cos(\pi\nu)+\nu\cosh(\pi y)\sin(\pi\nu)\big)\\
&\qquad+\frac{s'y}{\sinh(\pi y)}\cdot\big(\sinh^2(\pi y)+\sin^2(\pi\nu)\big)\bigg]\\
&\quad=\frac{1}{\sinh^2(\pi y)+\cos^2(\pi\nu)}\cdot\bigg[\big(s'+\cos(\pi\nu)\big)y\sinh(\pi y)
\end{aligned}$$

$$+ s' \sin^2(\pi \nu) \cdot \frac{y}{\sinh(\pi y)} + \nu \sin(\pi \nu) \cosh(\pi y)\Bigg].$$

We notice that the inside of the big bracket is asymptotically equal to

$$\big(s' + \cos(\pi \nu)\big) \cdot \frac{|y| e^{\pi |y|}}{2} + \nu \sin(\pi \nu) \cdot \frac{e^{\pi |y|}}{2} \sim \big(s' + \cos(\pi \nu)\big) \cdot \frac{|y| e^{\pi |y|}}{2}$$

as $y \to \pm\infty$. Thus, if the function $f_{\nu,1,s'}(x)$ is positive definite, then the above Fourier transform is non-negative thanks to Bochner's theorem and we must have

$$s' + \cos(\pi \nu) \geqslant 0 \quad \big(\Longleftrightarrow \quad \pi \nu \leqslant \cos^{-1}(-s') = \pi - \cos^{-1} s'\big).$$

Case $s \in (-1, 1)$. Based on (ii) and (iii) (with $\theta = \cos^{-1} s$) we compute

$$\begin{aligned}
&\frac{\sqrt{1-s^2}}{2\pi} (\mathcal{F} f_{\nu,s,s'})(y) \\
&\quad = \frac{\cos((\pi - \theta)\nu) \sinh(\pi y) \sinh(\theta y) + \sin(\pi \nu) \sin(\theta \nu) \cosh((\pi - \theta) y)}{\sinh^2(\pi y) + \sin^2(\pi \nu)} \\
&\qquad + \frac{s' \sinh(\theta y)}{\sinh(\pi y)} \\
&\quad = \frac{\sinh(\theta y)}{\sinh(\pi y)(\sinh^2(\pi y) + \sin^2(\pi \nu))} \cdot \Bigg[\big(s' + \cos\big((\pi - \theta)\nu\big)\big) \sinh^2(\pi y) \\
&\qquad + s' \sin^2(\pi \nu) + \sin(\pi \nu) \sin(\theta \nu) \cdot \frac{\sinh(\pi y) \cosh((\pi - \theta) y)}{\sinh(\theta y)}\Bigg].
\end{aligned}$$

We notice that the inside of the big bracket is asymptotically equal to

$$\begin{aligned}
&\big(s' + \cos\big((\pi - \theta)\nu\big)\big) \cdot \frac{e^{2\pi |y|}}{4} + \sin(\pi \nu) \sin(\theta \nu) \cdot \frac{e^{2(\pi - \theta)|y|}}{2} \\
&\quad \sim \big(s' + \cos\big((\pi - \theta)\nu\big)\big) \cdot \frac{e^{2\pi |y|}}{4}
\end{aligned}$$

as $y \to \pm\infty$. Thus, by the same reasoning as in the previous case, the positive definiteness of $f_{\nu,s,s'}(x)$ forces

$$s' + \cos\big((\pi - \theta)\nu\big) \geqslant 0 \quad \big(\Longleftrightarrow \quad (\pi - \theta)\nu \leqslant \cos^{-1}(-s') = \pi - \cos^{-1} s'\big). \qquad \square$$

Theorem 11. *For the function*

$$\frac{\cosh(\nu x) + s'}{\cosh x + s}$$

with $s, s' \in (-1, 1]$ *and* $\nu \in [0, 1]$ *the following three conditions are equivalent*:

(i) *the function is infinitely divisible*;

(ii) *the function is positive definite*;
(iii) *the inequality*

$$\nu \leqslant \frac{\pi - \cos^{-1} s'}{\pi - \cos^{-1} s} \tag{9}$$

is satisfied.

Proof. The implication (i) ⇒ (ii) is trivial while (ii) ⇒ (iii) is exactly Lemma 10. Hence, it remains to show (iii) ⇒ (i).

Case $s' \geqslant s$. The inequality (9) is always satisfied (see (6)), and the function is indeed infinitely divisible by Lemma 9.

Case $s' \leqslant s$. The second inequality in Lemma 8 comes free (see (7)). Therefore, if (9) (i.e., the first inequality in the lemma) is satisfied, then the desired infinite divisibility is guaranteed. □

The condition (9) means that the infinite divisibility and the positive definiteness are completely governed by the location of just the "first roots" of the entire functions $\cosh(\nu z) + s'$ and $\cosh z + s$ on the imaginary axis (see the proof of Lemma 8). We note that only asymptotic behaviors of relevant Fourier transforms were needed in the preceding arguments. However, the converse of Lemma 10 can be actually proved with a little bit more effort (see [18] for details), giving rise to a direct proof for the equivalence between (ii) and (iii) in the theorem.

Remark 12. In the extreme case $\nu = 1$ the condition (9) in the theorem means $s' \geqslant s$ while this means $s' \geqslant -\sqrt{\frac{1-s}{2}}$ for $\nu = 1/2$. In fact, when $s' \in [0, 1]$, we have $\pi - \cos^{-1} s' \in [\pi/2, \pi]$ so that we always have

$$\frac{\pi - \cos^{-1} s'}{\pi - \cos^{-1} s} \geqslant \frac{\pi}{2} \cdot \frac{1}{\pi - \cos^{-1} s} \geqslant \frac{1}{2}.$$

On the other hand, when $s' \in (-1, 0]$, we have $2\cos^{-1} s' = 2\pi - \cos^{-1}(2s'^2 - 1)$ and notice

$$\begin{aligned} &\bigl(\cos^{-1}(-s) =\bigr)\ \pi - \cos^{-1} s \leqslant 2\bigl(\pi - \cos^{-1} s'\bigr)\ \bigl(= \cos^{-1}\bigl(2s'^2 - 1\bigr)\bigr) \\ &\iff \quad -s \geqslant 2s'^2 - 1 \quad \iff \quad 1 - s \geqslant 2s'^2. \end{aligned}$$

The function $\sin x/x$ is positive definite due to $2\sin x/x = \int_{-1}^{1} e^{ixy}\,dy$, and so is the square

$$\frac{\sin^2 x}{x^2} = \frac{1}{4} \int_{-\infty}^{\infty} (\chi_{[-1,1]} * \chi_{[-1,1]})(y) e^{ixy}\,dy.$$

It is known that an infinitely divisible function has no real zeros (see [20, Theorem 5.3.1, p. 108] or [21, Lemma 7.5]), and hence $\sin^2 x/x^2$ cannot be infinitely divisible. It is also known that a (non-constant) function having the Fourier transform with bounded support cannot be infinitely divisible (see [21, Corollary 24.4]). On the other hand, (non-negative) positive definite functions

appearing naturally in study of operator means (see [5,12–14,17] for instance) do not have these properties. In our previous work [4] and Theorem 11 (see also Remark 4) we have actually observed that many of them are automatically infinitely divisible. It is worthwhile to investigate how general this phenomenon is.

5. Miscellaneous examples

In this section we will study other typical examples of infinitely divisible functions. We begin by recalling

$$\begin{aligned}\cosh z+\cosh(\alpha z)&=2\cosh\big((1+\alpha)z/2\big)\cosh\big((1-\alpha)z/2\big),\\ \cosh z-\cosh(\alpha z)&=2\sinh\big((1+\alpha)z/2\big)\sinh\big((1-\alpha)z/2\big),\\ \sinh z+\sinh(\beta z)&=2\sinh\big((1+\beta)z/2\big)\cosh\big((1-\beta)z/2\big).\end{aligned}$$

These identities make sure that the three functions $\cosh z+\cosh(\alpha z)$, $(\cosh z-\cosh(\alpha z))/z^2$, $(\sinh z+\sinh(\beta z))/z$ (for $\alpha\in[0,1]$ and $\beta\in(-1,1]$) have zeros (only) on the imaginary axis. Therefore, by Theorem 2 (or just by combining the above three formulas and what was stated in Remark 4(i)), we conclude the following: For $\alpha\in[0,1]$ and $\beta\in(-1,1]$ the functions

$$\frac{1}{\cosh z+\cosh(\alpha z)},\qquad \frac{z^2}{\cosh z-\cosh(\alpha z)},\qquad \frac{z}{\sinh z+\sinh(\beta z)}$$

are infinitely divisible, and so are the functions

$$\frac{\cosh(\nu x)+\cosh(\nu\alpha x)}{\cosh x+\cosh(\alpha x)},\qquad \frac{\cosh(\nu x)-\cosh(\nu\alpha x)}{\cosh x-\cosh(\alpha x)},\qquad \frac{\sinh(\nu x)+\sinh(\nu\beta x)}{\sinh x+\sinh(\beta x)}$$

for each $\nu\in[0,1]$.

Here we will mainly deal with the entire function $\cosh z+s\cosh(\alpha z)$ with $\alpha\in[0,1]$ and $s\in(-1,1]$ (and related ones). To know location of zeros, we make use of the (well-known) factorization

$$\cosh(nz)=P_n(\cosh z)\quad (n=1,2,\ldots) \tag{10}$$

with the polynomial

$$P_n(x)=2^{n-1}\prod_{k=1}^{n}\left(x-\cos\left(\frac{(2k-1)\pi}{2n}\right)\right).$$

Indeed, let us recall

$$\begin{aligned}X^{2n}+1&=\prod_{k=1}^{2n}\left(X-\exp\left(\frac{(2k-1)\pi i}{2n}\right)\right)\\ &=\prod_{k=1}^{n}\left(X-\exp\left(\frac{(2k-1)\pi i}{2n}\right)\right)\cdot\prod_{k=1}^{n}\left(X-\exp\left(\frac{(2(2n+1-k)-1)\pi i}{2n}\right)\right).\end{aligned}$$

Since

$$\exp\left(\frac{(2(2n+1-k)-1)\pi i}{2n}\right)=\exp\left(2\pi i-\frac{(2k-1)\pi i}{2n}\right)=\overline{\exp\left(\frac{(2k-1)\pi i}{2n}\right)},$$

the above factorization gives rise to

$$X^{2n}+1=\prod_{k=1}^{n}\left(X^2-2X\cos\left(\frac{(2k-1)\pi i}{2n}\right)+1\right)$$

(see [10, 1.396, formula 4, p. 40] for instance). Therefore, dividing the both sides by X^n and then substituting $X=\exp z$, we conclude

$$2\cosh(nz)=\prod_{k=1}^{n}\left(2\cosh z-2\cos\left(\frac{(2k-1)\pi}{2n}\right)\right),$$

which is exactly (10).

Lemma 13. *We assume $n, m \in \mathbf{N}$ and $n > m$. The equation $P_n(x)+sP_m(x)=0$ has n real roots for each $s \in (-1, 1)$, and moreover all of them fall into the open interval $(-1, 1)$.*

Proof. We have to check behavior of $P_n(x)$ on the interval $[-1, 1]$. We note

$$P_n(1)=P_n(\cosh 0)=\cosh(n0)=1,$$
$$P_n(-1)=P_n\big(\cosh(i\pi)\big)=\cosh(in\pi)=\cos(n\pi)=(-1)^n.$$

Therefore, the graph of $P_n(x)$ starts from the point $(-1, P_n(-1)=(-1)^n)$, cuts the x-axis n-times (at $\cos((2k-1)\pi/2n)$, $k=1,2,\ldots,n$) and ends at the point $(1, P_n(1)=1)$. Note that local minima or maxima occur $n-1$ times somewhere in the open interval $(-1, 1)$. We claim that all of these local extrema have modulus 1. Indeed, from (10) we get

$$P_n'(\cosh x)\sinh x=n\sinh(nx).$$

Squaring the both sides, we observe

$$P_n'(\cosh x)^2\big(\cosh^2 x-1\big)=n^2\big(\cosh^2(nx)-1\big)=n^2\big(P_n(\cosh x)^2-1\big).$$

This means

$$P_n'(x)^2\big(x^2-1\big)=n^2\big(P_n(x)^2-1\big),$$

showing

$$P_n(x)'=0 \quad\Longrightarrow\quad P_n(x)=\pm 1.$$

The discussion so far (with m instead) also shows $|sP_m(x)|\leqslant s<1$ on $[-1, 1]$, and the assertion is now evident. $\square$

The next result is a typical example where Theorem 2 is useful even when the exact location of zeros is unknown.

Theorem 14. *For each* $\alpha \in [0,1]$ *and* $-1 < s \leqslant 1$ *the function* $\cosh z + s\cosh(\alpha z)$ *fulfills all the requirements in Theorem* 2*. In particular, all of the functions*

$$\frac{1}{\cosh x + s\cosh(\alpha x)}, \qquad \frac{\cosh(\nu x) + s\cosh(\nu\alpha x)}{\cosh x + s\cosh(\alpha x)} \quad \bigl(\nu \in [0,1]\bigr)$$

are infinitely divisible.

Proof. We may and do assume $s \in (-1,1)$ (by the discussion in the first paragraph of the section). The only thing that we have to worry about is the requirement (ii) (on the zeros) in Theorem 2. At first we assume that $\alpha = m/n$ is rational (with $m < n$). Lemma 13 says that the polynomial $P_n(x) + sP_m(x)$ (of degree n) is of the form $2^{n-1}\prod_{i=1}^{n}(x - s_i)$ with $s_i \in (-1,1)$, $i = 1,2,\ldots,n$. Therefore, we have the factorization

$$\begin{aligned}\cosh z + s\cosh(mz/n) &= P_n\bigl(\cosh(z/n)\bigr) + sP_m\bigl(\cosh(z/n)\bigr)\\ &= 2^{n-1}\prod_{i=1}^{n}\bigl(\cosh(z/n) - s_i\bigr).\end{aligned}$$

Each factor $\cosh(z/n) - s_i$ admitting only pure imaginary zeros, so does the product $\cosh z + s\cosh(mz/n)$.

For a general $\alpha \in [0,1]$ one chooses a sequence $\{r_i\}_{i=1,2,\ldots}$ of rationals tending to α. Then, the sequence $\{\cosh z + s\cosh(r_i z)\}_{i=1,2,\ldots}$ of entire functions converges uniformly to $\cosh z + s\cosh(\alpha z)$ on each compact set so that the limit function $\cosh z + s\cosh(\alpha z)$ admits only pure imaginary zeros thanks to the first half of the proof and Hurwitz's theorem (see [6, p. 152] for instance). $\square$

The author is unable to determine what happens for $s > 1$. On the other hand, in [3, Theorem 1.2] it was shown that the matrices

$$\left[\frac{1}{\lambda_i^3 + \lambda_j^3 + s(\lambda_i^2\lambda_j + \lambda_i\lambda_j^2)}\right]_{i,j=1,2,\ldots,n} \qquad (\text{for each } s > -1)$$

are always positive for each $\lambda_1, \lambda_2, \ldots, \lambda_n > 0$ (and for each $n \in \mathbf{N}$), i.e., the function

$$\frac{1}{\cosh t + s\cosh(t/3)}$$

is positive definite for each $s > -1$ (by computations analogous to (1)). This positive definiteness remains valid for $\cosh(t/2)$ instead of $\cosh(t/3)$ [18].

Formulas for the Fourier transforms of functions of the form $(\cosh x + s)^{-1}$ were needed in Section 4 (especially for the proof of Lemma 10). One standard way to derive them is the so-called method of residues. Note that this method cannot be used for evaluating the integral

$$\int_{-\infty}^{\infty} \frac{e^{ixy}\,dx}{\cosh x + s\cosh(\alpha x)}$$

(for general α). Indeed, one cannot write down zeros for $\cosh z + s\cosh(\alpha z)$ explicitly so that residues of the integrand (as a complex function) are not computable. Therefore, it seems impossible (at least to the author) to confirm even positive definiteness of the functions in question via Fourier transform approach. Anyway, results so far can be combined with the power series expansion trick (used in the proof of Lemma 9) to show the next result.

Proposition 15. *We assume* $\alpha, \nu \in [0, 1]$ *and* $s, s' \in (-1, 1]$.

(i) *The function* $\frac{\cosh(\nu x)}{\cosh x + s\cosh(\alpha x)}$ *is infinitely divisible as long as* $\nu \leqslant \frac{1+\alpha}{2}$.
(ii) *The function* $\frac{\cosh(\nu x) + s'\cosh(\nu\alpha x)}{\cosh x + s\cosh(\alpha x)}$ *is infinitely divisible as long as* $s \leqslant s'$.

Proof. To see (i) for $s = 1$, we note

$$\frac{\cosh(\nu x)}{\cosh x + \cosh(\alpha x)} = \frac{\cosh(\nu x)}{2\cosh((1+\alpha)x/2)\cosh((1-\alpha)x/2)},$$

which is infinitely divisible due to the assumption $\nu \leqslant (1+\alpha)/2$ (see Remark 4(i)). We then assume $s \in (-1, 1)$ and compute

$$\begin{aligned}\frac{\cosh(\nu x)}{\cosh x + s\cosh(\alpha x)} &= \frac{\cosh(\nu x)}{\cosh x + \cosh(\alpha x) - (1-s)\cosh(\alpha x)} \\ &= \frac{\cosh(\nu x)}{\cosh x + \cosh(\alpha x)} \cdot \left(1 - \frac{(1-s)\cosh(\alpha x)}{\cosh x + \cosh(\alpha x)}\right)^{-1}\end{aligned}$$

with $0 \leqslant (1-s)\cosh(\alpha x) < \cosh x + \cosh(\alpha x)$. Thus, by making use of the power series expansion (8) we get

$$\left(\frac{\cosh(\nu x)}{\cosh x + s\cosh(\alpha x)}\right)^r = \left(\frac{\cosh(\nu x)}{\cosh x + \cosh(\alpha x)}\right)^r \cdot \sum_{n=0}^{\infty} \frac{a_n(1-s)^n\cosh^n(\alpha x)}{(\cosh x + \cosh(\alpha x))^n}$$

for each $r \in (0, 1)$. The first rth power in the right-hand side is positive definite by the first part of the proof while the obvious fact $\alpha \leqslant (1+\alpha)/2$ guarantees the positive definiteness (indeed the infinite divisibility) of each $\cosh^n(\alpha x)/(\cosh x + \cosh(\alpha x))^n$, showing (i).

To prove (ii), we need to observe

$$\frac{\cosh(\nu x) + s'\cosh(\nu\alpha x)}{\cosh x + s\cosh(\alpha x)}$$

$$= \frac{\cosh(\nu x) + s' \cosh(\nu\alpha x)}{\cosh x + s' \cosh(\alpha x)} \cdot \left(1 - \frac{(s' - s)\cosh(\alpha x)}{\cosh x + s' \cosh(\alpha x)}\right)^{-1}.$$

Then, the result follows from Theorem 14 and (i) together with the usual trick based on (8) repeatedly used so far. $\square$

The condition $\nu \leqslant \frac{1+\alpha}{2}$ in (i) does not involve the parameter s, and is not an optimal one. For instance, for $s \in (-1, 0]$ the function in question is infinitely divisible always (i.e., for each $\nu \in [0, 1]$) thanks to (ii). However, some kind of restriction is unavoidable. Indeed, Theorem 11 says that for $\alpha = 0$ and $s = 1$ it is infinitely divisible if and only if $\nu \leqslant 1/2$.

Proposition 16. *We assume* $\alpha \in [0, 1]$. *The functions*

$$\frac{x}{\sinh x + s \sinh(\alpha x)}, \qquad \frac{\sinh(\nu x)}{\sinh x + s \sinh(\alpha x)} \quad \bigl(0 < \nu \leqslant (1+\alpha)/2\bigr)$$

are infinitely divisible.

Proof. We note

$$\frac{\sinh(\nu x)}{\sinh x + \sinh(\alpha x)} = \frac{\sinh(\nu x)}{2\sinh((1+\alpha)x/2)\cosh((1-\alpha)x/2)},$$

$$\frac{\sinh(\nu x)}{\sinh x + s \sinh(\alpha x)} = \frac{\sinh(\nu x)}{\sinh x + \sinh(\alpha x) - (1-s)\sinh(\alpha x)}$$

$$= \frac{\sinh(\nu x)}{\sinh x + \sinh(\alpha x)} \cdot \left(1 - \frac{(1-s)\sinh(\alpha x)}{\sinh x + \sinh(\alpha x)}\right)^{-1}.$$

These together with arguments as in the proof of Proposition 15(i) yield the infinite divisibility of the second function. The infinite divisibility of the first can be obtained by similar arguments or from the obvious fact

$$\frac{x}{\sinh x + \sinh(\alpha x)} = \lim_{\nu \searrow 0} \left(\frac{1}{\nu} \cdot \frac{\sinh(\nu x)}{\sinh x + \sinh(\alpha x)}\right). \qquad \square$$

Appendix A

We will explicitly write down the Hadamard factorization for the function $\cosh z + s$ with $s \geqslant 1$. After some trials one is convinced that the following parameterization is more convenient:

$$f(z) = \cosh(\pi z) + \cosh(\pi a) \quad (\text{with } a \in \mathbf{R}).$$

It is elementary to see that the zeros of $f(z)$ are

$$z = \pm a + i + 2ni, \quad n \in \mathbf{Z},$$

or equivalently,

$$\begin{cases} a + (2n+1)\,i \\ a - (2n+1)\,i \end{cases} \quad (\text{for } n = 0, 1, \ldots) \quad \text{and} \quad \begin{cases} -a + (2n+1)\,i \\ -a - (2n+1)\,i \end{cases} \quad (\text{for } n = 0, 1, \ldots).$$

They are all simple zeros (unless $a = 0$), and the rank of $f(z)$ is obviously 1. We note

$$\frac{1}{a \pm (2n+1)\,i} = \frac{a \mp (2n+1)\,i}{a^2 + (2n+1)^2} \quad \text{and} \quad \frac{1}{-a \pm (2n+1)\,i} = \frac{-a \mp (2n+1)\,i}{a^2 + (2n+1)^2}.$$

Based on these facts we easily compute

$$\begin{aligned}
&\left(1 - \frac{z}{a + (2n+1)i}\right) e^{z/(a+(2n+1)i)} \cdot \left(1 - \frac{z}{a - (2n+1)i}\right) e^{z/(a-(2n+1)i)} \\
&\quad = \left(1 + \frac{z^2 - 2az}{a^2 + (2n+1)^2}\right) e^{2az/(a^2+(2n+1)^2)}, \\
&\left(1 - \frac{z}{-a + (2n+1)i}\right) e^{z/(-a+(2n+1)i)} \cdot \left(1 - \frac{z}{-a - (2n+1)i}\right) e^{z/(-a-(2n+1)i)} \\
&\quad = \left(1 + \frac{z^2 + 2az}{a^2 + (2n+1)^2}\right) e^{-2az/(a^2+(2n+1)^2)}.
\end{aligned}$$

Thus, the Hadamard factorization is

$$\begin{aligned}
f(z) = f(0)e^{bz} &\prod_{n=0}^{\infty} \left(\left(1 + \frac{z^2 - 2az}{a^2 + (2n+1)^2}\right) e^{2az/(a^2+(2n+1)^2)} \right) \\
&\times \prod_{n=0}^{\infty} \left(\left(1 + \frac{z^2 + 2az}{a^2 + (2n+1)^2}\right) e^{-2az/(a^2+(2n+1)^2)} \right)
\end{aligned}$$

with some b. But, we get $b = 0$ again due to $f'(0) = 0$. Therefore, by canceling two exponential factors, we have shown the following factorization:

Proposition A.1. *For a real we have*

$$\begin{aligned}
&\frac{\cosh(\pi z) + \cosh(\pi a)}{1 + \cosh(\pi a)} = \prod_{n=0}^{\infty} \left(1 + \frac{z^2 + 2az}{a^2 + (2n+1)^2}\right) \cdot \prod_{n=0}^{\infty} \left(1 + \frac{z^2 - 2az}{a^2 + (2n+1)^2}\right) \\
&\quad \left(= \prod_{n=0}^{\infty} \left(1 + \frac{z^4 + 2(2n+1+a)(2n+1-a)z^2}{(a^2 + (2n+1)^2)^2}\right) \right).
\end{aligned}$$

The appearance of $\pm 2az$ makes it impossible to use Lemma 1 (unless $a = 0$). In fact, the function in question is not positive definite (as was pointed out in [5]).

Appendix B

The Pascal matrix

$$\left[\frac{(i+j)!}{i!\,j!} \right]_{i,j=0,1,\ldots,n-1}$$

and its generalizations $P = [p_{ij}]_{i,j=1,2,\dots,n}$ with entries

$$p_{ij} = \frac{\Gamma(\lambda_i + \lambda_j + 1)}{\Gamma(\lambda_i + 1)\Gamma(\lambda_j + 1)} \quad \text{(with arbitrary positive reals } \lambda_1, \lambda_2, \dots, \lambda_n\text{)}$$

were studied in [1], and based on the classical Gauss formula

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\,n^z}{z(z+1)\cdots(z+n)} \quad (z \neq 0, -1, -2, \dots)$$

the infinite divisibility of P was proved, i.e., $P^{\circ r} = [p_{ij}^r] \geqslant 0$ for each $r \in (0,1)$ (see Section 2.1). The reasoning in [1] can be formulated in the following general form, and $1/\Gamma(z+1)$ is indeed a typical example satisfying all the requirements below:

Proposition B.1. *Let $f(z)$ be an entire function taking real values for the reals. We assume* (i) $f(0) > 0$, (ii) *all the zeros of $f(z)$ are real and negative, and* (iii) $\rho < 2$. *In this case, we have $f(x) > 0$ for $x \geqslant 0$, and for each $\lambda_1, \lambda_2, \dots, \lambda_n > 0$ the matrix*

$$\left[\frac{f(\lambda_i)f(\lambda_j)}{f(\lambda_i + \lambda_j)}\right]_{i,j=1,2,\dots,n}$$

is infinitely divisible.

Proof. Let $\{-\alpha_n\}_{n=1,2,\dots}$ be the zeros of $f(z)$ with $0 < \alpha_1 \leqslant \alpha_2 \leqslant \cdots$ (repeated according to multiplicity). The rank p of $f(z)$ is 0 or 1. For instance, when $p = 1$, the Hadamard factorization is

$$f(z) = f(0)e^{az} \prod_{n=1}^{\infty} \left(\left(1 + \frac{z}{\alpha_n}\right)e^{-z/\alpha_n}\right)$$

(with $a = f'(0)/f(0)$), showing $f(x) > 0$ for $x \geqslant 0$.

Since the maps $z \to e^{az}$ and $z \to e^{-z/\alpha_n}$ are multiplicative (and infinite divisibility is preserved by taking products), it suffices to show the infinite divisibility of the matrix P with entries

$$p_{ij} = \frac{(1 + \lambda_i/\alpha_n)(1 + \lambda_j/\alpha_n)}{1 + (\lambda_i + \lambda_j)/\alpha_n}.$$

Note that P is congruent to the matrix with entries

$$\frac{1}{1 + (\lambda_i + \lambda_j)/\alpha_n} = \left(\frac{\alpha_n}{(\lambda_i + \alpha_n/2) + (\lambda_j + \alpha_n/2)}\right)$$

so that the desired result follows from the infinite divisibility of the Cauchy matrix $[(\lambda_i + \lambda_j)^{-1}]$ (see Section 1). $\square$

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Existence and uniqueness of martingale solutions for SDEs with rough or degenerate coefficients

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Abstract

In this paper we extend recent results on the existence and uniqueness of solutions of ODEs with non-smooth vector fields to the case of martingale solutions, in the Stroock–Varadhan sense, of SDEs with non-smooth coefficients. In the first part we develop a general theory, which roughly speaking allows to deduce existence, uniqueness and stability of martingale solutions for $\mathcal{L}^d$-almost every initial condition x whenever existence and uniqueness is known at the PDE level in the L^∞-setting (and, conversely, if existence and uniqueness of martingale solutions is known for $\mathcal{L}^d$-a.e. initial condition, then existence and uniqueness for the PDE holds). In the second part of the paper we consider situations where, on the one hand, no pointwise uniqueness result for the martingale problem is known and, on the other hand, well-posedness for the Fokker–Planck equation can be proved. Thus, the theory developed in the first part of the paper is applicable. In particular, we will study the Fokker–Planck equation in two somehow extreme situations: in the first one, assuming uniform ellipticity of the diffusion coefficients and Lipschitz regularity in time, we are able to prove existence and uniqueness in the L^2-setting; in the second one we consider an additive noise and, assuming the drift b to have BV regularity and allowing the diffusion matrix a to be degenerate (also identically 0), we prove existence and uniqueness in the L^∞-setting. Therefore, in these two situations, our theory yields existence, uniqueness and stability results for martingale solutions.

Keywords: Martingale solutions; Existence and uniqueness almost everywhere; Fokker–Planck equation; Absolutely continuous solutions

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1. Introduction and preliminary results

Recent research activity has been devoted to study transport equations with rough coefficients, showing that a well-posedness result for the transport equation in a certain subclass of functions allows to prove existence and uniqueness of a flow for the associated ODE. The first result in this direction is due to DiPerna and P.-L. Lions [10], where the authors study the connection between the transport equation and the associated ODE $\dot\gamma = b(t,\gamma)$, showing that existence and uniqueness for the transport equation is equivalent to a sort of well-posedness of the ODE which says, roughly speaking, that the ODE has a unique solution for $\mathcal{L}^d$-almost every initial condition (here and in the sequel, $\mathcal{L}^d$ denotes the Lebesgue measure in $\mathbb{R}^d$). In that paper they also show that the transport equation $\partial_t u + \sum_i b_i \partial_i u = c$ is well-posed in L^∞ if $b = (b_1, \dots, b_n)$ is Sobolev and satisfies suitable global conditions (including L^∞-bounds on the spatial divergence), which yields the well-posedness of the ODE.

In [1] (see also [2]), using a slightly different philosophy, Ambrosio studied the connection between the continuity equations $\partial_t u + \sum_i \partial_i(b_i u) = c$ and the ODE $\dot\gamma = b(t,\gamma)$. This different approach allows him to develop the general theory of the so-called Regular Lagrangian Flows (see [2, Remark 31] for a detailed comparison with the DiPerna–Lions axiomatization), which relates existence and uniqueness for the continuity equation with well-posedness of the ODE, without assuming any regularity on the vector field b. Indeed, since the transport equation is in a conservative form, it has a meaning in the sense of distributions even when b is only L^∞_{loc} and u is L^1_{loc}. Thus, a general theory is developed in [1] under very general hypotheses, showing as in [10] that existence and uniqueness for the continuity equation is equivalent to a sort of well-posedness of the ODE. After having proved this, in [1] the well-posedness of the continuity equations in L^∞ is proved in the case of vector fields with BV regularity whose distributional divergence belongs to L^∞ (for other similar results on the well-posedness of the transport/continuity equation, see also [6,7,11,13,17]).

Our aim is to develop a stochastic counterpart of this theory: in our setting the continuity equation becomes the Fokker–Planck equation, while the ODE becomes an SDE.

Let us consider the following SDE

$$\begin{cases} dX = b(t,X)\,dt + \sigma(t,X)\,dB(t), \\ X(0) = x, \end{cases} \tag{1}$$

where $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0,T] \times \mathbb{R}^d \to \mathcal{L}(\mathbb{R}^r, \mathbb{R}^d)$ are bounded (here $\mathcal{L}(\mathbb{R}^r, \mathbb{R}^d)$ denotes the vector space of linear maps from $\mathbb{R}^r$ to $\mathbb{R}^d$) and B is an r-dimensional Brownian motion on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We want to study the existence and uniqueness of martingale solutions for this equation. Let us define $a(t,x) := \sigma(t,x)\sigma^*(t,x)$ (that is $a_{ij} := \sum_k \sigma_{ik}\sigma_{jk}$). We consider the so called Fokker–Planck equation

$$\begin{cases} \partial_t \mu_t + \sum_i \partial_i(b_i \mu_t) - \frac{1}{2} \sum_{ij} \partial_{ij}(a_{ij}\mu_t) = 0 & \text{in } [0,T] \times \mathbb{R}^d, \\ \mu_0 = \bar\mu & \text{in } \mathbb{R}^d. \end{cases} \tag{2}$$

We recall that, for a (possibly signed) measure $\mu = \mu(t,x) = \mu_t(x)$, being a solution of (2) simply means that

$$\frac{d}{dt}\int_{\mathbb{R}^d}\varphi(x)\,d\mu_t(x)$$
$$=\int_{\mathbb{R}^d}\left[\sum_i b_i(t,x)\partial_i\varphi(x)+\frac{1}{2}\sum_{ij}a_{ij}(t,x)\partial_{ij}\varphi(x)\right]d\mu_t(x)\quad \forall\varphi\in C_c^\infty\big(\mathbb{R}^d\big) \tag{3}$$

in the distributional sense on $[0,T]$, and the initial condition means that μ_t w^*-converges to $\bar{\mu}$ (i.e. converges in the duality with $C_c(\mathbb{R}^d)$) as $t\to 0$. We observe that, since Eq. (2) is in divergence form, it makes sense without any regularity assumption on a and b, provided that

$$\int_0^T\int_A\big(\big|b(t,x)\big|+\big|a(t,x)\big|\big)\,d|\mu_t|(x)\,dt<+\infty\quad \forall A\Subset\mathbb{R}^d$$

(here and in the sequel, $|\mu_t|$ denotes the total variation of μ_t). Since b and a will always be assumed to be bounded, in the definition of measure-valued solution of the PDE we assume that

$$\int_0^T|\mu_t|(A)\,dt<+\infty\quad \forall A\Subset\mathbb{R}^d, \tag{4}$$

so that (2) surely makes sense. However, if μ_t is singular with respect to the Lebesgue measure $\mathcal{L}^d$, then the products $b(t,\cdot)\mu_t$ and $a(t,\cdot)\mu_t$ are sensitive to modification of $b(t,\cdot)$ and $a(t,\cdot)$ in $\mathcal{L}^d$-negligible sets. Since in the case of singular measures the coefficients a and b will be assumed to be continuous, while in the case of coefficients in L^∞ the measures will be assumed to be absolutely continuous, (2) will always make sense.

Recall also that it is not restrictive to consider only solutions $t\mapsto\mu_t$ of the Fokker–Planck equation that are w^*-continuous on $[0,T]$, i.e. continuous in the duality with $C_c(\mathbb{R}^d)$ (see Lemma 2.1). Thus, we can assume that μ_t is defined for all t and even at the endpoints of $[0,T]$.

For simplicity of notation, we define

$$L_t:=\sum_i b_i(t,\cdot)\partial_i+\frac{1}{2}\sum_{ij}a_{ij}(t,\cdot)\partial_{ij}.$$

In this way the PDE can be written as

$$\partial_t\mu_t=L_t^*\mu_t\quad \text{in } [0,T]\times\mathbb{R}^d,$$

where L_t^* denotes the (formal) adjoint of L_t in $L^2(\mathbb{R}^d)$. Using Itô's formula it is simple to check that, if $X(t,x,\omega)\in L^2(\Omega,C([0,T],\mathbb{R}^d))$ is a family of solutions of (1), measurable in (t,x,ω), then the measure μ_t defined by

$$\int_{\mathbb{R}^d}f(x)\,d\mu_t(x):=\int_{\mathbb{R}^d}\mathbb{E}\big[f\big(X(t,x,\omega)\big)\big]\,d\bar{\mu}(x)\quad \forall f\in C_c\big(\mathbb{R}^d\big)$$

is a solution of (2) with $\mu_0=\bar{\mu}$ (see also Lemma 2.4).

We define $\Gamma_T := C([0,T],\mathbb{R}^d)$, and $e_t : \Gamma_T \to \mathbb{R}^d$, $e_t(\gamma) := \gamma(t)$. Let us recall the Stroock–Varadhan definition of martingale solutions.

Definition 1.1. A measure $\nu_{x,s}$ on Γ_T is a martingale solution of (1) starting from x at time s if:

(i) $\nu_{x,s}(\{\gamma \mid \gamma(s) = x\}) = 1$;
(ii) for any $\varphi \in C_c^\infty(\mathbb{R}^d)$, the stochastic process on Γ_T

$$\varphi\big(\gamma(t)\big) - \int_s^t L_u\varphi\big(\gamma(u)\big)\,du$$

is a $\nu_{x,s}$-martingale after time s with respect to the canonical filtration.

We will say that the martingale problem is well-posed if, for any $(s,x) \in \mathbb{R}^d$, we have existence and uniqueness of martingale solutions.

In the sequel, we will deal with families $\{\nu_x\}_{x\in\mathbb{R}^d}$ of probability measures that are measurable with respect to x according to the following standard definition.

Definition 1.2. We say that a family of probability measures on a probability space $(\Omega, \mathcal{A})$ $\{\nu_x\}_{x\in\mathbb{R}^d}$ is measurable if, for any $A \in \mathcal{A}$, the real-valued map $x \mapsto \nu_x(A)$ is measurable.

1.1. Plan of the paper

1.1.1. The theory of Stochastic Lagrangian Flows

In the first part of the paper, we develop a general theory (independent of specific regularity or ellipticity assumptions), which roughly speaking allows to deduce existence, uniqueness and stability of martingale solutions for $\mathcal{L}^d$-almost every initial condition x whenever existence and uniqueness is known at the PDE level in the L^∞-setting (and, conversely, if existence and uniqueness of martingale solutions is known for $\mathcal{L}^d$-a.e. initial condition, then existence and uniqueness for the PDE in the L^∞-setting holds).

More precisely, in Section 2 we study how uniqueness of the SDE is related to that of the PDE. In Section 2.1 we prove a representation formula for solutions of the PDE, which shows that they can always be seen as a superposition of solutions of the SDE also when standard existence results for martingale solutions of SDE do not apply. In particular, assuming only the boundedness of the coefficients, we will show that, whenever we have existence of a solution of the PDE starting from μ_0, there exists at least one martingale solution of the SDE for μ_0-a.e. initial condition x.

In Section 3 we introduce the main object of our study, what we call Stochastic Lagrangian Flow. In Section 3.1 we state and prove our main result regarding the existence and uniqueness of Stochastic Lagrangian Flows, showing that these flows exist and are unique whenever the PDE is well-posed in the L^∞-setting. We also prove a stability result, and we show that Stochastic Lagrangian Flows satisfy the Chapman–Kolmogorov equation. Moreover, in Section 3.2 we investigate the relation between our result and its deterministic counterpart and, applying our stability result, we deduce a vanishing viscosity theorem for Ambrosio's Regular Lagrangian Flows.

1.1.2. The Fokker–Planck equation

In the second part of the paper we study by purely PDE methods the well-posedness of the Fokker–Planck equation in two extreme (with respect to the regularity imposed in time, or in space) situations: in the first one, assuming uniform ellipticity of the coefficients and Lipschitz regularity in time, we are able to prove existence and uniqueness in the L^2-settings assuming no regularity in space, but only suitable divergence bounds (see Theorem 4.3). This result, together with Proposition 4.4, directly implies the following theorem (here and in the sequel, $\mathcal{S}_+(\mathbb{R}^d)$ denotes the set of symmetric and non-negative definite $d \times d$ matrices).

Theorem 1.3. *Let us assume that* $a : [0,T] \times \mathbb{R}^d \to \mathcal{S}_+(\mathbb{R}^d)$ *and* $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ *are bounded functions such that*:

(i) $\sum_j \partial_j a_{ij} \in L^\infty([0,T] \times \mathbb{R}^d)$ *for* $i = 1, \dots, d$;
(ii) $\partial_t a_{ij} \in L^\infty([0,T] \times \mathbb{R}^d)$ *for* $i, j = 1, \dots, d$;
(iii) $(\sum_i \partial_i b_i - \frac{1}{2}\sum_{ij} \partial_{ij} a_{ij})^- \in L^\infty([0,T] \times \mathbb{R}^d)$;
(iv) $\langle \xi, a(t,x)\xi \rangle \geqslant \alpha |\xi|^2$ $\forall (t,x) \in [0,T] \times \mathbb{R}^d$, *for a certain* $\alpha > 0$;
(v) $\frac{a}{1+|x|^2} \in L^2([0,T] \times \mathbb{R}^d)$, $\frac{b}{1+|x|} \in L^2([0,T] \times \mathbb{R}^d)$.

Then there exists a unique solution of (2) *in* $\mathscr{L}_+$, *where*

$$\mathscr{L}_+ := \bigl\{u \in L^\infty\bigl([0,T], L^1_+\bigl(\mathbb{R}^d\bigr)\bigr) \cap L^\infty\bigl([0,T], L^\infty_+\bigl(\mathbb{R}^d\bigr)\bigr) \bigm| u \in C\bigl([0,T], w^*\text{-}L^\infty\bigl(\mathbb{R}^d\bigr)\bigr)\bigr\},$$

and L^1_+ *and* L^∞_+ *denote the convex subsets of* L^1 *and* L^∞ *consisting of non-negative functions.*

In the second case, a does not depend on the space variables, but it can be degenerate and it is allowed to depend on t even in a measurable way. Since a can also be identically 0, we need to assume BV regularity on the vector field b, and so we can prove:

Theorem 1.4. *Let us assume that* $a : [0,T] \to \mathcal{S}_+(\mathbb{R}^d)$ *and* $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ *are bounded functions such that*:

(i) $b \in L^1([0,T], BV_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^d))$, $\sum_i \partial_i b_i \in L^1_{\mathrm{loc}}([0,T] \times \mathbb{R}^d)$;
(ii) $(\sum_i \partial_i b_i)^- \in L^1([0,T], L^\infty(\mathbb{R}^d))$.

Then there exists a unique solution of (2) *in* $\mathscr{L}_+$.

This theorem is a direct consequence of Theorem 4.12. Other existence and uniqueness results for the Fokker–Planck equation, which are in some sense intermediate with respect the two extreme ones stated above, have been proved in a recent paper of LeBris and P.-L. Lions [14]. As in our case, in that paper the authors are interested in the well-posedness of the Fokker–Planck equation as a tool to deduce existence and uniqueness results at the SDE level (see also [15]). In particular, in [14, Section 4] the authors give a list of interesting situations in the modelization of polymeric fluids when SDEs with irregular drift b and dispersion matrix σ arise (see also [12] and the references therein for other existence and uniqueness results for non-smooth SDEs).

1.1.3. Conclusions and appendix

In Section 5 we apply the theory developed in Section 3.1 to obtain, in the cases considered above, the generic well-posedness of the associated SDE.

Finally, in Appendix A we generalize an important uniqueness result of Stroock and Varadhan (see Theorem 2.2 and the remarks at the end of Theorem 5.4).

2. SDE–PDE uniqueness

In this section we study the main relations between the SDE and the PDE. The main result is a general representation formula for solutions of the PDE (Theorem 2.6) which allows to relate uniqueness of the SDE to that of the PDE (Lemma 2.3).

As we already said in the introduction, here and in the sequel b and a are always assumed to be bounded. Let us recall the following result on the time regularity of $t \mapsto \mu_t$ (see for example [2, Remark 3] or [4, Lemma 8.1.2]):

Lemma 2.1. *Up to modification of μ_t in a negligible set of times, $t \mapsto \mu_t$ is w^*-continuous on $[0,T]$. Moreover, if $|\mu_t|(\mathbb{R}^d) \leqslant C$ for any $t \in [0,T]$, then $t \mapsto \mu_t$ is narrowly continuous.*

We also recall the following important theorem of Stroock and Varadhan (for a proof, see [18, Theorem 6.2.3]).

Theorem 2.2. *Assume that for any $(s,x) \in [0,T] \times \mathbb{R}^d$, for any $\nu_{x,s}$ and $\tilde{\nu}_{x,s}$ martingale solutions of* (1) *starting from x at time s, one has*

$$(e_t)_\# \nu_{x,s} = (e_t)_\# \tilde{\nu}_{x,s} \quad \forall t \in [s,T].$$

Then the martingale solution of (1) *starting from any $(s,x) \in [0,T] \times \mathbb{R}^d$ is unique.*

We start studying how the uniqueness of (1) is related to that of (2).

Lemma 2.3. *Let $A \subset \mathbb{R}^d$ be a Borel set. The following two properties are equivalent*:

(a) *Time-marginals of martingale solutions of the SDE are unique for any $x \in A$.*
(b) *Finite non-negative measure-valued solutions of the PDE are unique for any non-negative Radon measure μ_0 concentrated in A.*

Proof. (b) $\Rightarrow$ (a). Let us choose $\mu_0 = \delta_x$, with $x \in A$. Then, if ν_x and $\tilde{\nu}_x$ are two martingale solutions of the SDE, we get that $\mu_t := (e_t)_\# \nu_x$ and $\tilde{\mu}_t := (e_t)_\# \tilde{\nu}_x$ are two solutions of the PDE with $\mu_0 = \delta_x$ (see Lemma 2.4). This implies that $\mu_t = \tilde{\mu}_t$, that is

$$\langle \mu_t, \varphi \rangle = \int_{\Gamma_T} \varphi\big(\gamma(t)\big)\, d\nu_x(\gamma) = \int_{\Gamma_T} \varphi\big(\gamma(t)\big)\, d\tilde{\nu}_x(\gamma) = \langle \tilde{\mu}_t, \varphi \rangle \quad \forall \varphi \in C_c^\infty\big(\mathbb{R}^d\big),$$

that is $(e_t)_\# \nu_x = (e_t)_\# \tilde{\nu}_x$ (observe in particular that, if $A = \mathbb{R}^d$ and we have uniqueness for the PDE for any initial time $s \geqslant 0$, by Theorem 2.2 we get that $\nu_x = \tilde{\nu}_x$ for any $x \in \mathbb{R}^d$).

(a) $\Rightarrow$ (b). This implication follows by Theorem 2.6, which provides, for every finite non-negative measure-valued solutions of the PDE, the representation

$$\int_{\mathbb{R}^d} \varphi\, d\mu_t = \int_{\mathbb{R}^d \times \Gamma_T} \varphi\big(\gamma(t)\big)\, d\nu_x(\gamma)\, d\mu_0(x), \tag{5}$$

where, for μ_0-a.e. x, ν_x is a martingale solution of SDE starting from x (at time 0). Therefore, by the uniqueness of $(e_t)_\# \nu_x$, we obtain that solutions of the PDE are unique. $\square$

We now prove that, if ν_x is a martingale solution of the SDE starting from x (at time 0) for μ_0-a.e. x, the right-hand side of (5) always defines a non-negative solution of the PDE. We recall that a locally finite measure is a possibly signed measure with locally finite total variation.

Lemma 2.4. *Let μ_0 be a locally finite measure on $\mathbb{R}^d$, and let $\{\nu_x\}_{x \in \mathbb{R}^d}$ be a measurable family of probability measures on Γ_T such that ν_x is a martingale solution of the SDE starting from x (at time* 0) *for $|\mu_0|$-a.e. x. Define on Γ_T the measure $\nu := \int_{\mathbb{R}^d} \nu_x\, d\mu_0(x)$, and assume that*

$$\int_0^T \int_{\mathbb{R}^d \times \Gamma_T} \chi_{B_R}\big(\gamma(t)\big)\, d\nu_x(\gamma)\, d|\mu_0|(x)\, dt < +\infty \quad \forall R > 0 \tag{6}$$

(*this property is trivially true if, for example, $|\mu_0|(\mathbb{R}^d) < +\infty$*). *Then the measure μ_t^ν on $\mathbb{R}^d$ defined by*

$$\big\langle \mu_t^\nu, \varphi \big\rangle := \big\langle (e_t)_\# \nu, \varphi \big\rangle = \int_{\mathbb{R}^d \times \Gamma_T} \varphi\big(\gamma(t)\big)\, d\nu_x(\gamma)\, d\mu_0(x) \quad \forall \varphi \in C_c^\infty\big(\mathbb{R}^d\big)$$

is a solution of the PDE.

Proof. Let us first show that the map $t \mapsto \langle \mu_t^\nu, \varphi \rangle$ is absolutely continuous for any $\varphi \in C_c^\infty(\mathbb{R}^d)$. We recall that a real-valued map $t \mapsto f(t)$ is said absolutely continuous if, for any $\varepsilon > 0$ there exists $\delta > 0$ such that, given any family of disjoint intervals $(s_k, t_k) \subset [0, T]$, the following implication holds:

$$\sum_k |t_k - s_k| \leqslant \delta \quad \Rightarrow \quad \sum_k \big| f(t_k) - f(s_k) \big| \leqslant \varepsilon.$$

Take $R > 0$ such that $\operatorname{supp}(\varphi) \subset B_R$, and let $I = \bigcup_{k=1}^n (s_k, t_k)$ be a subset of $[0, T]$ with (s_k, t_k) disjoint and such that $|t_k - s_k| \leqslant 1$. For μ_0-a.e. x, by the definition of martingale solution we have

$$\int_{\Gamma_T} \varphi\big(\gamma(t_k)\big)\, d\nu_x(\gamma) - \int_{\Gamma_T} \varphi\big(\gamma(s_k)\big)\, d\nu_x(\gamma)$$

$$= \int_{s_k}^{t_k} \int_{\Gamma_T} L_t\varphi\big(\gamma(t)\big)\, d\nu_x(\gamma)\, dt$$

$$= \int_{s_k}^{t_k} \int_{\Gamma_T} \sum_i b_i\big(t,\gamma(t)\big)\partial_i\varphi\big(\gamma(t)\big)\, d\nu_x(\gamma)\, dt + \frac{1}{2}\int_{s_k}^{t_k} \int_{\Gamma_T} \sum_{ij} a_{ij}\big(t,\gamma(t)\big)\partial_{ij}\varphi\big(\gamma(t)\big)\, d\nu_x(\gamma)\, dt$$

and so, integrating with respect to μ_0, we obtain

$$\big|\langle \mu^\nu_{t_k}, \varphi\rangle - \langle \mu^\nu_{s_k}, \varphi\rangle\big| \leqslant \|\varphi\|_{C^2}\bigg[\|b\|_\infty + \frac{1}{2}\|a\|_\infty\bigg] \int_{s_k}^{t_k} \int_{\mathbb{R}^d\times\Gamma_T} \chi_{B_R}\big(\gamma(t)\big)\, d\nu_x(\gamma)\, d|\mu_0|(x)\, dt.$$

Thus

$$\sum_{k=1}^n \big|\langle \mu^\nu_{t_k}, \varphi\rangle - \langle \mu^\nu_{s_k}, \varphi\rangle\big| \leqslant \|\varphi\|_{C^2}\bigg[\|b\|_\infty + \frac{1}{2}\|a\|_\infty\bigg] \sum_{k=1}^n \int_{s_k}^{t_k} \int_{\mathbb{R}^d\times\Gamma_T} \chi_{B_R}\big(\gamma(t)\big)\, d\nu_x(\gamma)\, d|\mu_0|(x)\, dt,$$

which shows that the map $t \mapsto \langle \mu^\nu_t, \varphi\rangle$ is absolutely continuous thanks to (6) and the absolute continuity property of the integral. So, in order to conclude that μ^ν_t solves the PDE, it suffices to compute the time derivative of $t \mapsto \langle \mu^\nu_t, \varphi\rangle$, and, by the computation we made above, one simply gets

$$\frac{d}{dt}\langle \mu^\nu_t, \varphi\rangle = \int_{\mathbb{R}^d} \frac{d}{dt}\bigg(\int_{\Gamma_T} \varphi\big(\gamma(t)\big)\, d\nu_x(\gamma)\bigg)\, d\mu_0(x)$$
$$= \int_{\mathbb{R}^d} \int_{\Gamma_T} L_t\varphi\big(\gamma(t)\big)\, d\nu_x(\gamma)\, d\mu_0(x) = \langle \mu^\nu_t, L_t\varphi\rangle. \qquad \square$$

Remark 2.5. We observe that, by the definition of μ^ν_t, the following implications hold:

(i) $\mu_0 \geqslant 0 \Rightarrow \forall t \geqslant 0,\ \mu^\nu_t \geqslant 0$ and $\mu^\nu_t(\mathbb{R}^d) = \mu_0(\mathbb{R}^d)$ (the total mass can also be infinite);
(ii) μ_0 signed $\Rightarrow \forall t \geqslant 0,\ |\mu^\nu_t|(\mathbb{R}^d) \leqslant |\mu_0|(\mathbb{R}^d)$ (the total variation can also be infinite).

2.1. A representation formula for solutions of the PDE

We denote by $\mathcal{M}_+(\mathbb{R}^d)$ the set of non-negative finite measures on $\mathbb{R}^d$.

Theorem 2.6. *Let μ_t be a solution of the PDE such that $\mu_t \in \mathcal{M}_+(\mathbb{R}^d)$ for any $t \in [0,T]$, with $\mu_t(\mathbb{R}^d) \leqslant C$ for any $t \in [0,T]$. Then there exists a measurable family of probability measures $\{\nu_x\}_{x\in\mathbb{R}^d}$ such that ν_x is a martingale solution of* (1) *starting from x* (*at time* 0) *for μ_0-a.e. x, and the following representation formula holds*:

$$\int_{\mathbb{R}^d} \varphi\, d\mu_t = \int_{\mathbb{R}^d\times\Gamma_T} \varphi\big(\gamma(t)\big)\, d\nu_x(\gamma)\, d\mu_0(x). \tag{7}$$

By this theorem it follows that, whenever we have existence of a solution of the PDE starting from μ_0, there exists a martingale solution of the SDE for μ_0-a.e. initial condition x.

Proof. Up to a renormalization of μ_0, we can assume that $\mu_0(\mathbb{R}^d) = 1$.

Step 1 (*Smoothing*). Let $\rho : \mathbb{R}^d \to (0, +\infty)$ be a convolution kernel such that $|D^k\rho(x)| \leqslant C_k|\rho(x)|$ for any $k \geqslant 1$ ($\rho(x) = Ce^{-\sqrt{1+|x|^2}}$, for instance). We consider the measures $\mu_t^\varepsilon := \mu_t * \rho_\varepsilon$. They are smooth solutions of the PDE

$$\partial_t \mu_t^\varepsilon + \sum_i \partial_i \big(b_i^\varepsilon \mu_t^\varepsilon\big) - \frac{1}{2}\sum_{ij} \partial_{ij}\big(a_{ij}^\varepsilon \mu_t^\varepsilon\big) = 0, \tag{8}$$

where

$$b_t^\varepsilon = b^\varepsilon(t,\cdot) := \frac{(b(t,\cdot)\mu_t) * \rho_\varepsilon}{\mu_t^\varepsilon}, \qquad a_t^\varepsilon = a^\varepsilon(t,\cdot) := \frac{(a(t,\cdot)\mu_t) * \rho_\varepsilon}{\mu_t^\varepsilon}.$$

Then it is immediate to see that

$$\big\|b_t^\varepsilon\big\|_\infty \leqslant \|b_t\|_\infty, \qquad \big\|a_t^\varepsilon\big\|_\infty \leqslant \|a_t\|_\infty. \tag{9}$$

Since $|D^k\rho(x)| \leqslant C_k|\rho(x)|$, it is simple to check that b^ε and a^ε are smooth and bounded together with all their spatial derivatives. By [18, Corollary 6.3.3], the martingale problem for a^ε and b^ε is well-posed (see Definition 1.1) and the family $\{\nu_x^\varepsilon\}_{x\in\mathbb{R}^d}$ of martingale solutions (starting at time 0) is measurable (see Definition 1.2). By (9) we can apply Lemma 2.4, which tells us that $\tilde{\mu}_t^\varepsilon := (e_t)_\# \int_{\mathbb{R}^d} \nu_x^\varepsilon \, d\mu_0^\varepsilon(x)$ is a finite measure which solves the smoothed PDE (8) with initial datum μ_0^ε. Then, since the solution of (8) is unique (Proposition 4.1), we obtain $\tilde{\mu}_t^\varepsilon = \mu_t^\varepsilon$, that is

$$\int_{\mathbb{R}^d} \varphi \, d\mu_t^\varepsilon = \int_{\mathbb{R}^d \times \Gamma_T} \varphi\big(\gamma(t)\big)\, d\nu_x^\varepsilon(\gamma)\, d\mu_0^\varepsilon(x). \tag{10}$$

Step 2 (*Tightness*). It is clear that the measures $\mu_0^\varepsilon = \mu_0 * \rho_\varepsilon$ are tight. So, if we define $\nu^\varepsilon := \int_{\mathbb{R}^d} \nu_x^\varepsilon \, d\mu_0^\varepsilon$, we have

$$\lim_{R\to\infty} \sup_{0<\varepsilon<1} \nu^\varepsilon\big(\big\{\big|\gamma(0)\big| > R\big\}\big) = 0.$$

For any $\varphi \in C_c^\infty(\mathbb{R}^d)$, let us define $A_\varphi := \|\varphi\|_{C^2}[\|b\|_\infty + \frac{1}{2}\|a\|_\infty]$. Since for every $\varphi \in C_c^\infty(\mathbb{R}^d)$ and any $0 < \varepsilon < 1$

$$\varphi\big(\gamma(t)\big) - \int_0^t \bigg(\sum_i b_i^\varepsilon\big(u,\gamma(u)\big)\partial_i\varphi\big(\gamma(u)\big) + \frac{1}{2}\sum_{ij} a_{ij}^\varepsilon\big(u,\gamma(u)\big)\partial_{ij}\varphi\big(\gamma(u)\big)\bigg)\, du$$

is a ν^ε-martingale with respect to the canonical filtration, by (9) we obtain that $\varphi(\gamma(t)) + A_\varphi t$ is a ν^ε-submartingale with respect to the canonical filtration. Thus [18, Theorem 1.4.6] can be applied, and the tightness of ν^ε follows.

Let ν be any limit point of ν^ε, and consider the disintegration of ν with respect to $\mu_0 = (e_0)_\# \nu$, i.e. $\nu = \int_{\mathbb{R}^d} \nu_x \, d\mu_0(x)$. Passing to the limit in (10), we get

$$\int_{\mathbb{R}^d} \varphi \, d\mu_t(x) = \int_{\mathbb{R}^d \times \Gamma_T} \varphi\big(\gamma(t)\big) \, d\nu_x(\gamma) \, d\mu_0(x).$$

Step 3 (ν_x *is a martingale solution of the SDE for* μ_0*-a.e.* x). Let $\varepsilon_n \to 0$ be a sequence such that ν is the weak limit of ν^{ε_n}. Let us fix a continuous function $f : \mathbb{R}^d \to \mathbb{R}$ with $0 \leqslant f \leqslant 1$, $s \in [0, T]$, and an $\mathcal{F}_s$-measurable continuous function $\Phi^s : \Gamma_T \to \mathbb{R}$ with $0 \leqslant \Phi^s \leqslant 1$, where $(\mathcal{F}_s)_{0 \leqslant s \leqslant T}$ denotes the canonical filtration on Γ_T. We define

$$L_t^n := \sum_i b_i^{\varepsilon_n}(t, \cdot)\partial_i + \frac{1}{2} \sum_{ij} a_{ij}^{\varepsilon_n}(t, \cdot)\partial_{ij}.$$

Since each $\nu_x^{\varepsilon_n}$ is a martingale solution, we know that for any $t \in [s, T]$ and for any $\varphi \in C_c^\infty(\mathbb{R}^d)$

$$\begin{aligned}
&\int_{\mathbb{R}^d \times \Gamma_T} \left[\varphi\big(\gamma(t)\big) - \int_0^t L_u^n \varphi\big(\gamma(u)\big) \, du \right] \Phi^s(\gamma) \, d\nu_x^{\varepsilon_n}(\gamma) f(x) \, d\mu_0^{\varepsilon_n}(x) \\
&\quad = \int_{\mathbb{R}^d \times \Gamma_T} \left[\varphi\big(\gamma(s)\big) - \int_0^s L_u^n \varphi\big(\gamma(u)\big) \, du \right] \Phi^s(\gamma) \, d\nu_x^{\varepsilon_n}(\gamma) f(x) \, d\mu_0^{\varepsilon_n}(x)
\end{aligned}$$

(see Definition 1.1), or equivalently

$$\int_{\mathbb{R}^d \times \Gamma_T} \left[\varphi\big(\gamma(t)\big) - \varphi\big(\gamma(s)\big) - \int_s^t L_u^n \varphi\big(\gamma(u)\big) \, du \right] \Phi^s(\gamma) \, d\nu_x^{\varepsilon_n}(\gamma) f(x) \, d\mu_0^{\varepsilon_n}(x) = 0.$$

Let us take $\tilde{b} : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\tilde{a} : [0, T] \times \mathbb{R}^d \to \mathcal{S}_+(\mathbb{R}^d)$ bounded and continuous, and define

$$\begin{aligned}
\tilde{L}_t &:= \sum_i \tilde{b}_i(t, \cdot)\partial_i + \frac{1}{2} \sum_{ij} \tilde{a}_{ij}(t, \cdot)\partial_{ij}, \\
\tilde{L}_t^n &:= \sum_i \tilde{b}_i^{\varepsilon_n}(t, \cdot)\partial_i + \frac{1}{2} \sum_{ij} \tilde{a}_{ij}^{\varepsilon_n}(t, \cdot)\partial_{ij},
\end{aligned}$$

where $\tilde{b}_i^{\varepsilon_n}$ and $\tilde{a}_{ij}^{\varepsilon_n}$ are defined analogously to $b_i^{\varepsilon_n}$ and $a_{ij}^{\varepsilon_n}$. Thus we can write

$$\int_{\mathbb{R}^d \times \Gamma_T} \left[\varphi\big(\gamma(t)\big) - \varphi\big(\gamma(s)\big) - \int_s^t \tilde{L}_u^n \varphi\big(\gamma(u)\big) \, du \right] \Phi^s(\gamma) \, d\nu_x^{\varepsilon_n}(\gamma) f(x) \, d\mu_0^{\varepsilon_n}(x)$$

$$= \int_{\mathbb{R}^d \times \Gamma_T} \left[\int_s^t \left(L_u^n - \tilde{L}_u^n\right)\varphi\left(\gamma(u)\right) du \right] \Phi^s(\gamma)\, d\nu_x^{\varepsilon_n}(\gamma)\, f(x)\, d\mu_0^{\varepsilon_n}(x).$$

Then, recalling that $0 \leqslant f \leqslant 1$ and $0 \leqslant \Phi^s \leqslant 1$, we get

$$\begin{aligned}
&\left| \int_{\mathbb{R}^d \times \Gamma_T} \left[\varphi\left(\gamma(t)\right) - \varphi\left(\gamma(s)\right) - \int_s^t \tilde{L}_u^n \varphi\left(\gamma(u)\right) du \right] \Phi^s(\gamma)\, d\nu_x^{\varepsilon_n}(\gamma)\, f(x)\, d\mu_0^{\varepsilon_n}(x) \right| \\
&\quad \leqslant \int_{\mathbb{R}^d \times \Gamma_T} \left[\int_s^t \left| \left(L_u^n - \tilde{L}_u^n\right)\varphi\left(\gamma(u)\right) \right| du \right] \Phi^s(\gamma)\, d\nu_x^{\varepsilon_n}(\gamma)\, f(x)\, d\mu_0^{\varepsilon_n}(x) \\
&\quad \leqslant \int_{\mathbb{R}^d \times \Gamma_T} \left[\int_s^t \left| \left(L_u^n - \tilde{L}_u^n\right)\varphi\left(\gamma(u)\right) \right| du \right] d\nu_x^{\varepsilon_n}(\gamma)\, d\mu_0^{\varepsilon_n}(x) \\
&\quad = \int_s^t \int_{\mathbb{R}^d} \left| \left(L_u^n - \tilde{L}_u^n\right)\varphi(x) \right| d\mu_u^{\varepsilon_n}(x)\, du \\
&\quad \leqslant \sum_i \int_s^t \int_{\mathbb{R}^d} \left| \left(\frac{(b_i(u,\cdot)\mu_u) * \rho_{\varepsilon_n}}{\mu_u^{\varepsilon_n}} - \frac{(\tilde{b}_i(u,\cdot)\mu_u) * \rho_{\varepsilon_n}}{\mu_u^{\varepsilon_n}} \right) \partial_i \varphi \right| (x)\, d\mu_u^{\varepsilon_n}(x)\, du \\
&\qquad + \frac{1}{2} \sum_{ij} \int_s^t \int_{\mathbb{R}^d} \left| \left(\frac{(a_{ij}(u,\cdot)\mu_u) * \rho_{\varepsilon_n}}{\mu_u^{\varepsilon_n}} - \frac{(\tilde{a}_{ij}(u,\cdot)\mu_u) * \rho_{\varepsilon_n}}{\mu_u^{\varepsilon_n}} \right) \partial_{ij} \varphi \right| (x)\, d\mu_u^{\varepsilon_n}(x)\, du \\
&\quad \leqslant \sum_i \int_s^t \int_{\mathbb{R}^d} \left| b_i(u,\cdot) - \tilde{b}_i(u,\cdot) \right| (x) \partial_i \varphi * \rho_{\varepsilon_n}(x)\, d\mu_u(x)\, du \\
&\qquad + \frac{1}{2} \sum_{ij} \int_s^t \int_{\mathbb{R}^d} \left| a_{ij}(u,\cdot) - \tilde{a}_{ij}(u,\cdot) \right| (x) \partial_{ij} \varphi * \rho_{\varepsilon_n}(x)\, d\mu_u(x)\, du.
\end{aligned}$$

Since $\tilde{a}$ and $\tilde{b}$ are continuous, $\tilde{a}^{\varepsilon_n}$ and $\tilde{b}^{\varepsilon_n}$ converge to $\tilde{a}$ and $\tilde{b}$ locally uniformly. So we can pass to the limit in the above equation as $n \to \infty$, obtaining

$$\begin{aligned}
&\left| \int_{\mathbb{R}^d \times \Gamma_T} \left[\varphi\left(\gamma(t)\right) - \varphi\left(\gamma(s)\right) - \int_s^t \tilde{L}_u \varphi\left(\gamma(u)\right) du \right] \Phi^s(\gamma)\, d\nu_x(\gamma)\, f(x)\, d\mu_0(x) \right| \\
&\quad \leqslant \sum_i \int_s^t \int_{\mathbb{R}^d} \left| b_i(u,x) - \tilde{b}_i(u,x) \right| \partial_i \varphi(x)\, d\mu_u(x)\, du
\end{aligned}$$

$$+\frac{1}{2}\sum_{ij}\int_s^t\int_{\mathbb{R}^d}\big|a_{ij}(u,x)-\tilde{a}_{ij}(u,x)\big|\partial_{ij}\varphi(x)\,d\mu_u(x)\,du.$$

Choosing two sequences of continuous functions $(\tilde{b}^k)_{k\in\mathbb{N}}$ and $(\tilde{a}^k)_{k\in\mathbb{N}}$ converging respectively to b and a in $L^1([0,T]\times\mathbb{R}^d,\eta)$, with $\eta := \int_0^T \mu_t\,dt$, we finally obtain

$$\int_{\mathbb{R}^d\times\Gamma_T}\left[\varphi\big(\gamma(t)\big)-\varphi\big(\gamma(s)\big)-\int_s^t L_u\varphi\big(\gamma(u)\big)\,du\right]\Phi^s(\gamma)\,d\nu_x(\gamma)\,f(x)\,d\mu_0(x)=0,$$

that is

$$\int_{\mathbb{R}^d\times\Gamma_T}\left[\varphi\big(\gamma(t)\big)-\int_0^t L_u\varphi\big(\gamma(u)\big)\,du\right]\Phi^s(\gamma)\,d\nu_x(\gamma)\,f(x)\,d\mu_0(x)$$

$$=\int_{\mathbb{R}^d\times\Gamma_T}\left[\varphi\big(\gamma(s)\big)-\int_0^s L_u\varphi\big(\gamma(u)\big)\,du\right]\Phi^s(\gamma)\,d\nu_x(\gamma)\,f(x)\,d\mu_0(x).$$

By the arbitrariness of f we get that, for any $0\leqslant s\leqslant t\leqslant T$, and for any $\mathcal{F}_s$-measurable function Φ^s, we have

$$\int_{\Gamma_T}\left[\varphi\big(\gamma(t)\big)-\int_0^t L_u\varphi\big(\gamma(u)\big)\,du\right]\Phi^s(\gamma)\,d\nu_x(\gamma)$$

$$=\int_{\Gamma_T}\left[\varphi\big(\gamma(s)\big)-\int_0^s L_u\varphi\big(\gamma(u)\big)\,du\right]\Phi^s(\gamma)\,d\nu_x(\gamma)\quad\text{for } \mu_0\text{-a.e. } x.$$

Letting Φ^s vary in a dense countable subset of $\mathcal{F}_s$-measurable functions, by approximations we deduce that, for any $0\leqslant s\leqslant t\leqslant T$, for μ_0-a.e. x,

$$\int_{\Gamma_T}\left[\varphi\big(\gamma(t)\big)-\int_0^t L_u\varphi\big(\gamma(u)\big)\,du\right]\Phi^s(\gamma)\,d\nu_x(\gamma)$$

$$=\int_{\Gamma_T}\left[\varphi\big(\gamma(s)\big)-\int_0^s L_u\varphi\big(\gamma(u)\big)\,du\right]\Phi^s(\gamma)\,d\nu_x(\gamma)$$

for any $\mathcal{F}_s$-measurable function Φ^s (here the μ_0-a.e. depends on s and t but not on Φ^s). Taking now $s,t\in[0,T]\cap\mathbb{Q}$, we deduce that, for μ_0-a.e. x,

$$\int_{\Gamma_T}\left[\varphi\bigl(\gamma(t)\bigr)-\int_0^t L_u\varphi\bigl(\gamma(u)\bigr)\,du\right]\Phi^s(\gamma)\,d\nu_x(\gamma)$$
$$=\int_{\Gamma_T}\left[\varphi\bigl(\gamma(s)\bigr)-\int_0^s L_u\varphi\bigl(\gamma(u)\bigr)\,du\right]\Phi^s(\gamma)\,d\nu_x(\gamma)$$

for any $s,t\in[0,T]\cap\mathbb{Q}$, for any $\mathcal{F}_s$-measurable function Φ^s. By the continuity of the above equality with respect to both s and t, and the continuity in time of the filtration $\mathcal{F}_s$, we conclude that ν_x is a martingale solution for μ_0-a.e. x. $\square$

Remark 2.7. We observe that by (7) it follows that

$$\mu_t\bigl(\mathbb{R}^d\bigr)\leqslant C\quad\forall t\quad\Rightarrow\quad\mu_t\bigl(\mathbb{R}^d\bigr)=\mu_0\bigl(\mathbb{R}^d\bigr)$$

(this result can also be proved more directly using as test functions in (2) a suitable sequence $(\varphi_n)_{n\in\mathbb{N}}\subset C_c^\infty(\mathbb{R}^d)$, with $0\leqslant\varphi_n\leqslant 1$ and $\varphi_n\nearrow 1$, and, even in the case when the measures μ_t are signed, under the assumption $|\mu_t|(\mathbb{R}^d)\leqslant C$ one obtains the constancy of the map $t\mapsto\mu_t(\mathbb{R}^d)$).

3. Stochastic Lagrangian Flows

In this section we want to prove an existence and uniqueness result for martingale solutions which satisfy certain properties, in the spirit of the Regular Lagrangian Flows (RLF) introduced in [1].

Definition 3.1. Given a measure $\mu_0=\rho_0\mathcal{L}^d\in\mathcal{M}_+(\mathbb{R}^d)$, with $\rho_0\in L^\infty(\mathbb{R}^d)$, we say that a measurable family of probability measures $\{\nu_x\}_{x\in\mathbb{R}^d}$ on Γ_T is a μ_0-*Stochastic Lagrangian Flow* (μ_0-SLF) (starting at time 0), if:

(i) for μ_0-a.e. x, ν_x is a martingale solution of the SDE starting from x (at time 0);
(ii) for any $t\in[0,T]$

$$\mu_t:=(e_t)_\#\left(\int\nu_x\,d\mu_0(x)\right)\ll\mathcal{L}^d,$$

and, denoting $\mu_t=\rho_t\mathcal{L}^d$, we have $\rho_t\in L^\infty(\mathbb{R}^d)$ uniformly in t.

More generally, one can analogously define a μ_0-SLF starting at time s with $s\in(0,T)$ requiring that ν_x is a martingale solution of the SDE starting from x at time s.

Remark 3.2. If $\{\nu_x\}_{x\in\mathbb{R}^d}$ is a μ_0-SLF, then it is also a μ_0'-SLF for any $\mu_0'\in\mathcal{M}_+(\mathbb{R}^d)$ with $\mu_0'\leqslant C\mu_0$. Indeed, this easily follows by the inequality

$$0\leqslant(e_t)_\#\int_{\mathbb{R}^d}\tilde{\nu}_x\,d\mu_0'(x)\leqslant C(e_t)_\#\int_{\mathbb{R}^d}\tilde{\nu}_x\,d\mu_0(x).$$

3.1. Existence, uniqueness and stability of SLF

We denote by L^1_+ and L^∞_+ the convex subsets of L^1 and L^∞ consisting of non-negative functions, and, following [1], we define

$$\mathscr{L} := \left\{u \in L^\infty\left([0,T], L^1\left(\mathbb{R}^d\right)\right) \cap L^\infty\left([0,T], L^\infty\left(\mathbb{R}^d\right)\right) \;\middle|\; u \in C\left([0,T], w^*\text{-}L^\infty\left(\mathbb{R}^d\right)\right)\right\},$$

and

$$\mathscr{L}_+ := \left\{u \in L^\infty\left([0,T], L^1_+\left(\mathbb{R}^d\right)\right) \cap L^\infty\left([0,T], L^\infty_+\left(\mathbb{R}^d\right)\right) \;\middle|\; u \in C\left([0,T], w^*\text{-}L^\infty\left(\mathbb{R}^d\right)\right)\right\}.$$

Under an existence and uniqueness result for the PDE in the class $\mathscr{L}_+$, we prove existence and uniqueness of SLF.

Theorem 3.3 *(Existence of SLF starting from a fixed measure). Let us suppose that, for some initial datum $\mu_0 = \rho_0 \mathcal{L}^d \in \mathcal{M}_+(\mathbb{R}^d)$, with $\rho_0 \in L^\infty(\mathbb{R}^d)$, there exists a solution of the PDE in $\mathscr{L}_+$. Then there exists a μ_0-SLF.*

Proof. It suffices to apply Theorem 2.6 to the solution of the PDE in $\mathscr{L}_+$. $\square$

Let us assume now that forward uniqueness for the PDE holds in the class $\mathscr{L}_+$ for any initial time, that is, for any $s \in [0,T]$, for any $\rho_s \in L^1_+(\mathbb{R}^d) \cap L^\infty_+(\mathbb{R}^d)$, if we denote by $\rho_t \mathcal{L}^d$ and $\tilde{\rho}_t \mathcal{L}^d$ two solutions of the PDE in the class $\mathscr{L}_+$ starting from $\rho_s \mathcal{L}^d$ at time s, then

$$\rho_t = \tilde{\rho}_t \quad \text{for any } t \in [s,T].$$

Before stating and proving our main theorem, we first introduce some notation that will be used also in Appendix A.

Let $\mathcal{B}$ be the Borel σ-algebra on $\Gamma_T = C([0,T], \mathbb{R}^d)$, and define the filtrations $\mathcal{F}_t := \sigma[e_s \mid 0 \leqslant s \leqslant t]$ and $\mathcal{F}^t := \sigma[e_s \mid t \leqslant s \leqslant T]$. Set $\mathcal{P}(\Gamma_T)$ the set of probability measures on Γ_T. Now, given $\nu \in \mathcal{P}(\Gamma_T)$, we denote by

$$\Gamma_T \ni \gamma \mapsto \nu^\gamma_{\mathcal{F}_t} \in \mathcal{P}(\Gamma_T)$$

a regular conditional probability distribution of ν given $\mathcal{F}_t$, that is a family of probability measures on $(\Gamma_T, \mathcal{B})$ indexed by γ such that:

- for each $B \in \mathcal{B}$, $\gamma \mapsto \nu^\gamma_{\mathcal{F}_t}(B)$ is $\mathcal{F}_t$-measurable;
- $$\nu(A \cap B) = \int_A \nu^\gamma_{\mathcal{F}_t}(B)\, d\nu(\gamma) \quad \forall A \in \mathcal{F}_t,\ \forall B \in \mathcal{B}. \tag{11}$$

Since Γ_T is a Polish space and every σ-algebra $\mathcal{F}_t$ is finitely generated, such a function exists and is unique, up to ν-null sets. In particular, up to changing this function in a ν-null set, the following fact holds:

$$\nu^\gamma_{\mathcal{F}_t}\left(\left\{\tilde{\gamma} \mid \tilde{\gamma}(s) = \gamma(s)\ \forall s \in [0,t]\right\}\right) = 1 \quad \forall \gamma \in \Gamma_T. \tag{12}$$

Finally, given $0 \leqslant t_1 \leqslant \cdots \leqslant t_n \leqslant T$, we set $M^{t_1,\dots,t_n} := \sigma[e_{t_1},\dots,e_{t_n}]$, and one can analogously define $\nu^\gamma_{M^{t_1,\dots,t_n}}$. For $\nu^\gamma_{M^{t_1,\dots,t_n}}$ an analogous of (12) holds:

$$\nu^\gamma_{M^{t_1,\dots,t_n}}\big(\big\{\tilde\gamma \bigm| \tilde\gamma(t_i) = \gamma(t_i)\ \forall i = 1,\dots,n\big\}\big) = 1 \quad \forall \gamma \in \Gamma_T. \tag{13}$$

If $\gamma(t_i) = x_i$ for $i = 1,\dots,n$, then we will also use the notation $\nu^\gamma_{M^{t_1,\dots,t_n}} = \nu^{x_1,\dots,x_n}_{M^{t_1,\dots,t_n}}$.

By (11) one can check that $\int_{\Gamma_T} \nu^{\tilde\gamma}_{\mathcal{F}_{t_n}}\, d\nu^\gamma_{M^{t_1,\dots,t_n}}(\tilde\gamma)$ is a regular conditional probability distribution of ν given $M^{t_1,\dots,t_n}$, which implies by uniqueness that

$$\nu^\gamma_{M^{t_1,\dots,t_n}} = \int_{\Gamma_T} \nu^{\tilde\gamma}_{\mathcal{F}_{t_n}}\, d\nu^\gamma_{M^{t_1,\dots,t_n}}(\tilde\gamma) \quad \text{for } \nu\text{-a.e. } \gamma. \tag{14}$$

Theorem 3.4 *(Uniqueness of SLF starting from a fixed measure). Let us assume that forward uniqueness for the PDE holds in the class $\mathscr{L}_+$ for any initial time. Then, for any $\mu_0 = \rho_0\mathcal{L}^d \in \mathcal{M}_+(\mathbb{R}^d)$, with $\rho_0 \in L^\infty(\mathbb{R}^d)$, the μ_0-SLF is uniquely determined μ_0-a.e. (in the sense that, if $\{\nu_x\}$ and $\{\tilde\nu_x\}$ are two μ_0-SLF, then $\nu_x = \tilde\nu_x$ for μ_0-a.e. x).*

Proof. Let $\{\nu_x\}$ and $\{\tilde\nu_x\}$ be two μ_0-SLF. Take now a function $\psi \in C_c(\mathbb{R}^d)$, with $\psi \geqslant 0$. By Remark 3.2, $\{\nu_x\}$ and $\{\tilde\nu_x\}$ are two $\psi\mu_0$-SLF. Thus, by Lemma 2.4 and the uniqueness of the PDE in $\mathscr{L}_+$, for any $\varphi \in C_c(\mathbb{R}^d)$ we have

$$\begin{aligned}&\int_{\mathbb{R}^d\times\Gamma_T} \varphi\big(e_t(\gamma)\big)\, d\nu_x(\gamma)\psi(x)\, d\mu_0(x)\\ &\quad = \int_{\mathbb{R}^d\times\Gamma_T} \varphi\big(e_t(\gamma)\big)\, d\tilde\nu_x(\gamma)\psi(x)\, d\mu_0(x) \quad \forall t \in [0,T].\end{aligned} \tag{15}$$

This clearly implies that, for any $t \in [0,T]$,

$$(e_t)_\#\nu_x = (e_t)_\#\tilde\nu_x \quad \text{for } \mu_0\text{-a.e. } x.$$

We now want to use an analogous argument to deduce that, for any $0 < t_1 < t_2 < \cdots < t_n \leqslant T$,

$$(e_{t_1},\dots,e_{t_n})_\#\nu_x = (e_{t_1},\dots,e_{t_n})_\#\tilde\nu_x \quad \text{for } \mu_0\text{-a.e. } x. \tag{16}$$

The idea is that, given a measure $\tilde\mu_s = \tilde\rho_s\mathcal{L}^d \in \mathcal{M}_+(\mathbb{R}^d)$, with $\tilde\rho_s \in L^\infty$, once we have a $\tilde\mu_s$-SLF starting at time s we can multiply $\tilde\mu_s$ by a function $\psi_s \in C_c(\mathbb{R}^d)$ with $\psi_s \geqslant 0$, and by Remark 3.2 our $\tilde\mu_s$-SLF is also a $\psi_s\tilde\mu_s$-SLF starting at time s. Using this argument n times at different times and the time marginals uniqueness, we will obtain (16).

Fix $0 < t_1 < \cdots < t_n \leqslant T$. Take $\psi_0 \geqslant 0$ with $\psi_0 \in C_c(\mathbb{R}^d)$ and $\int_{\mathbb{R}^d} \psi_0\, d\mu_0 = 1$, and denote by $\mu^{\psi_0}_{t_1}$ the value at time t_1 of the (unique) solution in $\mathscr{L}_+$ of the PDE starting from $\psi_0\mu_0$ (which is induced both by $\{\nu_x\}$ and $\{\tilde\nu_x\}$ by uniqueness, see Eq. (15)). Let $\{\nu_{x,t_1}\}_{x\in\mathbb{R}^d}$ and $\{\tilde\nu_{x,t_1}\}_{x\in\mathbb{R}^d}$ be the families of probability measures on Γ_T given by the disintegration of

$$\nu^{\psi_0} := \int_{\mathbb{R}^d} \nu_x\psi_0(x)\, d\mu_0(x) \quad \text{and} \quad \tilde\nu^{\psi_0} := \int_{\mathbb{R}^d} \tilde\nu_x\psi_0(x)\, d\mu_0(x)$$

with respect to $\mu_{t_1}^{\psi_0} = (e_{t_1})_\# \nu^{\psi_0} = (e_{t_1})_\# \tilde{\nu}^{\psi_0}$, that is

$$\nu^{\psi_0} = \int_{\mathbb{R}^d} \nu_{x,t_1}\, d\mu_{t_1}^{\psi_0}(x), \qquad \tilde{\nu}^{\psi_0} = \int_{\mathbb{R}^d} \tilde{\nu}_{x,t_1}\, d\mu_{t_1}^{\psi_0}(x). \tag{17}$$

It is easily seen that $\{\nu_{x,t_1}\}$ and $\{\tilde{\nu}_{x,t_1}\}$ are regular conditional probability distributions, given $M^{t_1} = \sigma[e_{t_1}]$, of ν^{ψ_0} and $\tilde{\nu}^{\psi_0}$, respectively (that is, with the notation introduced before, $\nu_{x,t_1} = (\nu^{\psi_0})^x_{M_{t_1}}$ and $\tilde{\nu}_{x,t_1} = (\tilde{\nu}^{\psi_0})^x_{M_{t_1}}$). Thus, looking at $\{\nu_{x,t_1}\}$ and $\{\tilde{\nu}_{x,t_1}\}$ as their restriction to $C([t_1,T],\mathbb{R}^d)$, $\{\nu_{x,t_1}\}$ and $\{\tilde{\nu}_{x,t_1}\}$ are $\mu_{t_1}^{\psi_0}$-SLF starting at time t_1. Indeed, by the stability of martingale solutions with respect to regular conditional probability (see [18, Chapter 6]), $\{\nu_{x,t_1}\}$ and $\{\tilde{\nu}_{x,t_1}\}$ are martingale solutions of the SDE starting from x at time t_1 for $\mu_{t_1}^{\psi_0}$-a.e. x (see also the remarks at the end of the proof of Proposition A.1), while (ii) of Definition 3.1 is trivially true since $\{\nu_x\}$ and $\{\tilde{\nu}_x\}$ are $\psi_0\mu_0$-SLF. As before, since $\{\nu_{x,t_1}\}$ and $\{\tilde{\nu}_{x,t_1}\}$ are also $\psi_1\mu_{t_1}^{\psi_0}$-SLF for any $\psi_1 \in C_c(\mathbb{R}^d)$ with $\psi_1 \geqslant 0$, using again the uniqueness of the PDE in $\mathscr{L}_+$ we get

$$\int_{\mathbb{R}^d\times\Gamma_T} \varphi\big(e_{t_2}(\gamma)\big)\, d\nu_{x,t_1}(\gamma)\psi_1(x)\, d\mu_{t_1}^{\psi_0}(x) = \int_{\mathbb{R}^d\times\Gamma_T} \varphi\big(e_{t_2}(\gamma)\big)\, d\tilde{\nu}_{x,t_1}(\gamma)\psi_1(x)\, d\mu_{t_1}^{\psi_0}(x)$$

for any $\varphi \in C_c(\mathbb{R}^d)$, which can also be written as

$$\begin{aligned}
&\int_{\mathbb{R}^d\times\Gamma_T} \varphi\big(e_{t_2}(\gamma)\big)\psi_1\big(e_{t_1}(\gamma)\big)\, d\nu_{x,t_1}(\gamma)\, d\mu_{t_1}^{\psi_0}(x) \\
&\quad = \int_{\mathbb{R}^d\times\Gamma_T} \varphi\big(e_{t_2}(\gamma)\big)\psi_1\big(e_{t_1}(\gamma)\big)\, d\tilde{\nu}_{x,t_1}(\gamma)\, d\mu_{t_1}^{\psi_0}(x).
\end{aligned} \tag{18}$$

Recalling that by (17)

$$\int_{\mathbb{R}^d} \nu_{x,t_1}\, d\mu_{t_1}^{\psi_0}(x) = \int_{\mathbb{R}^d} \nu_x\psi_0(x)\, d\mu_0(x), \qquad \int_{\mathbb{R}^d} \tilde{\nu}_{x,t_1}\, d\mu_{t_1}^{\psi_0}(x) = \int_{\mathbb{R}^d} \tilde{\nu}_x\psi_0(x)\, d\mu_0(x),$$

by (18) we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d\times\Gamma_T} \varphi\big(e_{t_2}(\gamma)\big)\psi_1\big(e_{t_1}(\gamma)\big)\, d\nu_x(\gamma)\psi_0(x)\, d\mu_0(x) \\
&\quad = \int_{\mathbb{R}^d\times\Gamma_T} \varphi\big(e_{t_2}(\gamma)\big)\psi_1\big(e_{t_1}(\gamma)\big)\, d\tilde{\nu}_x(\gamma)\psi_0(x)\, d\mu_0(x)
\end{aligned}$$

for any non-negative $\psi_0, \psi_1, \varphi \in C_c(\mathbb{R}^d)$ (the constraint $\int_{\mathbb{R}^d}\psi_0\, d\mu_0 = 1$ can be easily removed multiplying the above equality by a positive constant). Iterating this argument, we finally get

$$\int\limits_{\mathbb{R}^d \times \Gamma_T} \psi_n\big(e_{t_n}(\gamma)\big) \cdots \psi_1\big(e_{t_1}(\gamma)\big)\, d\nu_x(\gamma)\psi_0(x)\, d\mu_0(x)$$
$$= \int\limits_{\mathbb{R}^d \times \Gamma_T} \psi_n\big(e_{t_n}(\gamma)\big) \cdots \psi_1\big(e_{t_1}(\gamma)\big)\, d\tilde{\nu}_x(\gamma)\psi_0(x)\, d\mu_0(x),$$

for any non-negative $\psi_0, \ldots, \psi_n \in C_c(\mathbb{R}^d)$, and thus (16) follows.

Considering now only rational times, we get that there exists a subset $A \subset \mathbb{R}^d$, with $\mu_0(A^c) = 0$, such that, for any $x \in A$,

$$(e_{t_1}, \ldots, e_{t_n})_\# \nu_x = (e_{t_1}, \ldots, e_{t_n})_\# \tilde{\nu}_x \quad \text{for any } t_1, \ldots, t_n \in [0, T] \cap \mathbb{Q}.$$

By continuity, this implies that, for any $x \in A$, $\nu_x = \tilde{\nu}_x$, as wanted. □

Remark 3.5. Suppose that forward uniqueness for the PDE holds in the class $\mathscr{L}_+$, and take $\mu_0 = \rho_0 \mathcal{L}^d$ and $\tilde{\mu}_0 = \tilde{\rho}_0 \mathcal{L}^d$, with $\rho_0, \tilde{\rho}_0 \in L^1_+(\mathbb{R}^d) \cap L^\infty_+(\mathbb{R}^d)$. If $\{\nu_x\}$ is a μ_0-SLF and $\{\tilde{\nu}_x\}$ is a $\tilde{\mu}_0$-SLF, then

$$\nu_x = \tilde{\nu}_x \quad \text{for } \mu_0 \wedge \tilde{\mu}_0\text{-a.e. } x.$$

In fact, by Remark 3.2 $\{\nu_x\}$ and $\{\tilde{\nu}_x\}$ are both $\mu_0 \wedge \tilde{\mu}_0$-SLF, and thus we conclude by the uniqueness result proved above.

By Theorems 3.3 and 3.4, and by the remark above, we obtain the following:

Corollary 3.6 *(Existence and uniqueness of SLF). Let us assume that we have forward existence and uniqueness for the PDE in $\mathscr{L}_+$. Then there exists a measurable selection of martingale solution $\{\nu_x\}_{x\in\mathbb{R}^d}$ which is a μ_0-SLF for any $\mu_0 = \rho_0\mathcal{L}^d$ with $\rho_0 \in L^1_+(\mathbb{R}^d) \cap L^\infty_+(\mathbb{R}^d)$, and if $\{\tilde{\nu}_x\}_{x\in\mathbb{R}^d}$ is a $\tilde{\mu}_0$-SLF for a fixed $\tilde{\mu}_0 = \tilde{\rho}_0\mathcal{L}^d$ with $\tilde{\rho}_0 \in L^1_+(\mathbb{R}^d) \cap L^\infty_+(\mathbb{R}^d)$, then $\nu_x = \tilde{\nu}_x$ for $\mathcal{L}^d$-a.e. $x \in \operatorname{supp}(\tilde{\mu}_0)$.*

Proof. It suffices to consider a SLF starting from a Gaussian measure (which exists by Theorem 3.3), and to apply Remark 3.5. □

By now, the above selection of martingale solutions $\{\nu_x\}$, which is uniquely determined $\mathcal{L}^d$-a.e., will be called the SLF (starting at time 0 and relative to (b, a)).

We finally prove a stability result for SLF.

Theorem 3.7 *(Stability of SLF starting from a fixed measure). Let us suppose that $b^n, b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $a^n, a : [0, T] \times \mathbb{R}^d \to \mathcal{S}_+(\mathbb{R}^d)$ are uniformly bounded functions, and that we have forward existence and uniqueness for the PDE in $\mathscr{L}_+$ with coefficients (b, a). Let $\mu_0 = \rho_0\mathcal{L}^d \in \mathcal{M}_+(\mathbb{R}^d)$, with $\rho_0 \in L^\infty(\mathbb{R}^d)$, and let $\{\nu^n_x\}_{x\in\mathbb{R}^d}$ and $\{\nu_x\}_{x\in\mathbb{R}^d}$ be μ_0-SLF for (b^n, a^n) and (b, a), respectively. Define $\nu^n := \int_{\mathbb{R}^d} \nu^n_x\, d\mu_0(x)$, $\nu := \int_{\mathbb{R}^d} \nu_x\, d\mu_0(x)$. Assume that:*

(i) *$(b^n, a^n) \to (b, a)$ in $L^1_{\mathrm{loc}}([0, T] \times \mathbb{R}^d)$;*

(ii) *setting* $\rho_t^n \mathcal{L}^d = \mu_t^n := (e_t)_\# \nu^n$, *for any* $t \in [0, T]$

$$\left\|\rho_t^n\right\|_{L^\infty(\mathbb{R}^d)} \leqslant C \quad \text{for a certain constant } C = C(T).$$

Then $\nu^n \rightharpoonup^* \nu$ *in* $\mathcal{M}(\Gamma_T)$.

Proof. Since (b^n, a^n) are uniformly bounded in L^∞, as in Step 2 of the proof of Theorem 2.6 one proves that the sequence of probability measures (ν^n) on $\mathbb{R}^d \times \Gamma_T$ is tight. In order to conclude, we must show that any limit point of (ν^n) is ν.

Let $\tilde{\nu}$ be any limit point of (ν^n). We claim that $\tilde{\nu}$ is concentrated on martingale solutions of the SDE with coefficients (b, a). Indeed, let us define $\tilde{\mu}_t := (e_t)_\# \tilde{\nu}$. Since $\mu_t^n \to \tilde{\mu}_t$ narrowly and ρ_t^n are non-negative functions bounded in $L^\infty(\mathbb{R}^d)$, we get $\tilde{\mu}_t = \rho_t \mathcal{L}^d$ for a certain non-negative function $\rho_t \in L^\infty(\mathbb{R}^d)$. We now observe that the argument used in Step 3 of the proof of Theorem 2.6 was using only the property that, for any $\varphi \in C_c^\infty(\mathbb{R}^d)$,

$$\limsup_{n\to+\infty} \sum_i \int_s^t \int_{\mathbb{R}^d} \left|\left(b_i^n(u,x) - \tilde{b}_i(u,x)\right)\partial_i \varphi(x)\right| \rho_u^n(x)\,dx\,du$$

$$\leqslant \sum_i \int_s^t \int_{\mathbb{R}^d} \left|\left(b_i(u,x) - \tilde{b}_i(u,x)\right)\partial_i \varphi(x)\right| \rho_u(x)\,dx\,du,$$

$$\limsup_{n\to+\infty} \sum_{ij} \int_s^t \int_{\mathbb{R}^d} \left|\left(a_{ij}^n(u,x) - \tilde{a}_{ij}(u,x)\right)\partial_{ij} \varphi(x)\right| \rho_u^n(x)\,dx\,du$$

$$\leqslant \sum_{ij} \int_s^t \int_{\mathbb{R}^d} \left|\left(a_{ij}(u,x) - \tilde{a}_{ij}(u,x)\right)\partial_{ij} \varphi(x)\right| \rho_u(x)\,dx\,du$$

for any $\tilde{b} : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\tilde{a} : [0,T] \times \mathbb{R}^d \to \mathcal{S}_+(\mathbb{R}^d)$ bounded and continuous. This property simply follows by (i) and the w^*-convergence of ρ_t^n to ρ_t in $L^\infty([0,T] \times \mathbb{R}^d)$.

Since $t \mapsto \rho_t \mathcal{L}^d$ is w^*-continuous in the sense of measures, the w^*-continuity of $t \mapsto \rho_t$ in $L^\infty(\mathbb{R}^d)$ follows. Thus, if we write $\tilde{\nu} := \int_{\mathbb{R}^d} \tilde{\nu}_x \, d\mu_0(x)$ (considering the disintegration of $\tilde{\nu}$ with respect to $\mu_0 = (e_0)_\# \tilde{\nu}$), we have proved that $\{\tilde{\nu}_x\}$ is a μ_0-SLF for (b, a). Therefore, by Theorem 3.4, we conclude that $\nu = \tilde{\nu}$. $\square$

We remark that the theory just developed could be generalized to more general situations. Indeed the key property of the convex class $\mathscr{L}_+$ is the following monotonicity property:

$$0 \leqslant \tilde{\mu}_t \leqslant \mu_t \in \mathscr{L}_+ \quad \Rightarrow \quad \tilde{\mu}_t \in \mathscr{L}_+$$

(see also [2, Section 3]).

3.2. SLF versus RLF

We remark that, in the special case $a = 0$, our SLF coincides with a sort of superposition of the RLF introduced in [1]:

Lemma 3.8. *Let us assume $a = 0$. Then $\nu_{x,s}$ is a martingale solution of the SDE (which, in this case, is just an ODE) starting from x at time s if and only if it is concentrated on integral curves of the ODE, that is, for $\nu_{x,s}$-a.e. γ,*

$$\gamma(t) - \gamma(s) = \int_s^t b\big(\tau, \gamma(\tau)\big)\, d\tau \quad \forall t \in [s, T].$$

Proof. It is clear from the definition of martingale solution that, if $\nu_{x,s}$ is concentrated on integral curves on the ODE, then it is a martingale solution. Let us prove the converse implication. By the definition of martingale solution and the fact that $a = 0$, it is a known fact that

$$M_t := \gamma(t) - \gamma(s) - \int_s^t b\big(\tau, \gamma(\tau)\big)\, d\tau, \quad t \in [s, T],$$

is a $\nu_{x,s}$-martingale with zero quadratic variation. This implies that also M_t^2 is a martingale, and since $M_s = 0$ we get

$$0 = \mathbb{E}^{\nu_{x,s}}\big[M_t^2\big] = \int_{\Gamma_T} \left(\gamma(t) - \gamma(s) - \int_s^t b\big(\tau, \gamma(\tau)\big)\, d\tau\right)^2 d\nu_{x,s}(\gamma) \quad \forall t \in [s, T],$$

which gives the thesis. □

Thus, in the case $a = 0$, a martingale solution of the SDE starting from x is simply a measure on Γ_T concentrated on integral curves of b. By the results in [1] we know that, if we have forward uniqueness for the PDE in $\mathscr{L}_+$, then any measure ν on Γ_T concentrated on integral curves of b such that its time marginals induces a solution of the PDE in $\mathscr{L}_+$ is concentrated on a graph, i.e. there exists a function $x \mapsto X(\cdot, x) \in \Gamma_T$ such that

$$\nu = X(\cdot, x)_\# \mu_0, \quad \text{with } \mu_0 := (e_0)_\# \nu$$

(see for instance [3, Theorem 18]). Then, if we assume forward uniqueness for the PDE in $\mathscr{L}_+$, our SLF coincides exactly with the RLF in [1]. Applying the stability result proved in the above paragraph, we obtain that, as the noise tends to 0, our SLF converges to the RLF associated to the ODE $\dot{\gamma} = b(\gamma)$. So we have a vanishing viscosity result for RLF.

Corollary 3.9. *Let us suppose that $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ is uniformly bounded, and that we have forward existence and uniqueness for the PDE in $\mathscr{L}_+$ with coefficients $(b, 0)$. Let $\{\nu_x^\varepsilon\}_{x \in \mathbb{R}^d}$ and $\{\nu_x\}_{x \in \mathbb{R}^d}$ be the SLF relative to $(b, \varepsilon I)$ and $(b, 0)$, respectively (existence and uniqueness of martingale solutions for the SDE with coefficients $(b, \varepsilon I)$, together with the measurability*

of the family $\{\nu_x^\varepsilon\}_{x\in\mathbb{R}^d}$*, follows by* [18, *Theorem* 7.2.1]*). Let* $\mu_0 = \rho_0\mathcal{L}^d \in \mathcal{M}_+(\mathbb{R}^d)$*, with* $\rho_0 \in L^\infty(\mathbb{R}^d)$*, and define* $\nu^\varepsilon := \int_{\mathbb{R}^d} \nu_x^\varepsilon \, d\mu_0(x)$*,* $\nu := \int_{\mathbb{R}^d} \nu_x \, d\mu_0(x)$*.*

Set $\rho_t^\varepsilon \mathcal{L}^d = \mu_t^\varepsilon := (e_t)_\# \nu^\varepsilon$*, and assume that for any* $t \in [0, T]$

$$\left\| \rho_t^\varepsilon \right\|_{L^\infty(\mathbb{R}^d)} \leqslant C \quad \textit{for a certain constant } C = C(T).$$

Then $\nu^\varepsilon \rightharpoonup^* \nu$ *in* $\mathcal{M}(\Gamma_T)$*.*

In [1], the uniqueness of RLF implies the semigroup law (see [1,2] for more details). In our case, by the uniqueness of SLF, we have as a consequence that the Chapman–Kolmogorov equation holds:

Proposition 3.10. *For any* $s \geqslant 0$*, let* $\{\nu_{x,s}\}_{x\in\mathbb{R}^d}$ *denotes the unique SLF starting at time* s*. Let us denote by* $\nu_{s,x}(t, dy)$ *the probability measure on* $\mathbb{R}^d$ *given by* $\nu_{s,x}(t, \cdot) := (e_t)_\# \nu_{s,x}$*. Then, for any* $0 \leqslant s < t < u \leqslant T$*,*

$$\int_{\mathbb{R}^d} \nu_{t,y}(u, \cdot)\nu_{s,x}(t, dy) = \nu_{s,x}(u, \cdot) \quad \textit{for } \mathcal{L}^d\textit{-a.e. } x.$$

Proof. Let us define

$$\tilde{\nu}_{s,x} := \begin{cases} \nu_{s,x} & \text{on } C([s,t], \mathbb{R}^d), \\ \int_{\mathbb{R}^d} \nu_{t,y}\nu_{s,x}(t, dy) & \text{on } C([t,T], \mathbb{R}^d). \end{cases}$$

This gives a family of martingale solution starting from x at time s (see [18]), and, using that $\{\nu_{x,s}\}$ and $\{\nu_{x,t}\}$ are SLF starting at time s and t, respectively, it is simple to check that $\{\tilde{\nu}_{s,x}\}_{x\in\mathbb{R}^d}$ is a SLF starting at time s. Thus, by Theorem 3.4, we have the thesis. □

4. Fokker–Planck equation

We now want to study the Fokker–Planck equation

$$\partial_t \mu_t + \sum_i \partial_i(b_i \mu_t) - \frac{1}{2}\sum_{ij} \partial_{ij}(a_{ij}\mu_t) = 0 \quad \text{in } [0, T] \times \mathbb{R}^d, \tag{19}$$

where $a = (a_{ij})$ is symmetric and non-negative definite (that is, $a : [0, T] \times \mathbb{R}^d \to \mathcal{S}_+(\mathbb{R}^d)$).

4.1. Existence and uniqueness of measure-valued solutions

Proposition 4.1. *Let us assume that* $a : [0, T] \times \mathbb{R}^d \to \mathcal{S}_+(\mathbb{R}^d)$ *and* $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ *are bounded functions, having two bounded continuous spatial derivatives. Then, for any finite measure* μ_0 *there exists a unique finite measure-valued solution of* (19) *starting from* μ_0 *such that* $|\mu_t|(\mathbb{R}^d) \leqslant C$ *for any* $t \in [0, T]$*.*

Proof. *Existence*. Let $\{\nu_x\}_{x\in\mathbb{R}^d}$ be the measurable family of martingale solutions of the SDE

$$\begin{cases} dX = b(t, X)\,dt + \sqrt{a(t, X)}\,dB(t), \\ X(0) = x \end{cases}$$

(which exists and is unique by [18, Corollary 6.3.3]). Then, by Lemma 2.4 and Remark 2.5, the measure $\mu_t := (e_t)_\# \int_{\mathbb{R}^d} \nu_x \, d\mu_0(x)$ solves (19) and $|\mu_t|(\mathbb{R}^d) \leqslant |\mu_0|(\mathbb{R}^d)$.

Uniqueness. By linearity, it suffices to prove that, if $\mu_0 = 0$, then $\mu_t = 0$ for all $t \in [0, T]$. Fix $\psi \in C_c^\infty(\mathbb{R}^d)$, $\bar{t} \in [0, T]$, and let $f(t, x)$ be the (unique) solution of

$$\begin{cases} \partial_t f + \sum_i b_i \partial_i f + \frac{1}{2} \sum_{ij} a_{ij} \partial_{ij} f = 0 & \text{in } [0, \bar{t}] \times \mathbb{R}^d, \\ f(\bar{t}) = \psi & \text{on } \mathbb{R}^d \end{cases}$$

(which exists and is unique by [18, Theorem 3.2.6]). By [18, Theorems 3.1.1 and 3.2.4], we know that $f \in C_b^{1,2}$, i.e. it is uniformly bounded with one bounded continuous time derivative and two bounded continuous spatial derivatives. Since μ_t is a finite measure by assumption, and $t \mapsto \mu_t$ is narrowly continuous (Lemma 2.1), we can use $f(t, \cdot)$ as test functions in (3), and we get

$$\begin{aligned} &\frac{d}{dt} \int_{\mathbb{R}^d} f(t, x)\, d\mu_t(x) \\ &\quad = \int_{\mathbb{R}^d} \left[\partial_t f(t, x) + \sum_i b_i(t, x) \partial_i f(t, x) + \frac{1}{2} \sum_{ij} a_{ij}(t, x) \partial_{ij} f(t, x) \right] d\mu_t(x) = 0 \end{aligned}$$

(the above computation is admissible since $f \in C_b^{1,2}$). This implies in particular that

$$0 = \int_{\mathbb{R}^d} f(0, x)\, d\mu_0(x) = \int_{\mathbb{R}^d} f(\bar{t}, x)\, d\mu_{\bar{t}}(x) = \int_{\mathbb{R}^d} \psi(x)\, d\mu_{\bar{t}}(x).$$

By the arbitrariness of ψ and $\bar{t}$ we obtain $\mu_t = 0$ for all $t \in [0, T]$. □

We remark that, in the uniformly parabolic case, the above proof still works under weaker regularity assumptions. Indeed, in that case, one has existence of a measurable family of martingale solutions of the SDE and of a solution $f \in C_b^{1,2}([0, \bar{t}] \times \mathbb{R}^d)$ of the adjoint equation if a and b are just Hölder continuous (see [18, Theorem 3.2.1]). So we get:

Proposition 4.2. *Let us assume that* $a : [0, T] \times \mathbb{R}^d \to \mathcal{S}_+(\mathbb{R}^d)$ *and* $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ *are bounded functions such that*:

(i) $\langle \xi, a(t, x)\xi \rangle \geqslant \alpha |\xi|^2$ $\forall (t, x) \in [0, T] \times \mathbb{R}^d$, *for a certain* $\alpha > 0$;
(ii) $|b(t, x) - b(s, y)| + \|a(t, x) - a(s, y)\| \leqslant C(|x - y|^\delta + |t - s|^\delta)$ $\forall (t, x), (s, y) \in [0, T] \times \mathbb{R}^d$, *for some* $\delta \in (0, 1]$, $C \geqslant 0$.

Then, for any finite measure μ_0 *there exists a unique finite measure-valued solution of* (19) *starting from* μ_0.

4.2. Existence and uniqueness of absolutely continuous solutions in the uniformly parabolic case

We are now interested in absolutely continuous solutions of (2). Therefore, we consider the following equation

$$\begin{cases} \partial_t u + \sum_i \partial_i(b_i u) - \frac{1}{2}\sum_{ij}\partial_{ij}(a_{ij}u) = 0 & \text{in } [0,T]\times\mathbb{R}^d, \\ u(0) = u_0, \end{cases} \tag{20}$$

which must be understood in the distributional sense on $[0,T]\times\mathbb{R}^d$. We now first prove an existence and uniqueness result in the L^2-setting under a regularity assumption on the divergence of a, which enables us to write (20) in a variational form, and thus to apply classical existence results (the uniqueness part in L^2 is much more involved). After, we will give a maximum principle result.

Let us make the following assumptions on the coefficients:

$$\begin{aligned} &\sum_j \partial_j a_{ij} \in L^\infty\big([0,T]\times\mathbb{R}^d\big) \quad \text{for } i=1,\dots,d, \\ &\Bigg(\sum_i \partial_i b_i - \frac{1}{2}\sum_{ij}\partial_{ij}a_{ij}\Bigg)^- \in L^\infty\big([0,T]\times\mathbb{R}^d\big), \\ &\big\langle \xi, a(t,x)\xi\big\rangle \geqslant \alpha|\xi|^2 \quad \forall (t,x)\in[0,T]\times\mathbb{R}^d, \text{ for a certain } \alpha>0. \end{aligned} \tag{21}$$

Theorem 4.3. *Let us assume that $a:[0,T]\times\mathbb{R}^d\to\mathcal{S}_+(\mathbb{R}^d)$ and $b:[0,T]\times\mathbb{R}^d\to\mathbb{R}^d$ are bounded functions such that* (21) *is fulfilled. Then, for any $u_0\in L^2(\mathbb{R}^d)$,* (20) *has a unique solution $u\in Y$, where*

$$Y := \big\{u\in L^2\big([0,T],H^1\big(\mathbb{R}^d\big)\big) \bigm| \partial_t u \in L^2\big([0,T],H^{-1}\big(\mathbb{R}^d\big)\big)\big\}.$$

If moreover $\partial_t a_{ij}\in L^\infty([0,T]\times\mathbb{R}^d)$ for $i,j=1,\dots,d$, then existence and uniqueness holds in $L^2([0,T]\times\mathbb{R}^d)$, and so in particular any solution $u\in L^2([0,T]\times\mathbb{R}^d)$ of (20) *belongs to Y.*

The proof the above theorem is quite standard, except for the uniqueness result in the large space L^2, which is indeed quite technical and involved. The motivation for this more general result is that $L^1_+(\mathbb{R}^d)\cap L^\infty_+(\mathbb{R}^d)\subset L^2(\mathbb{R}^d)$, and $L^1_+(R^d)\cap L^\infty_+(R^d)$ is the space where we need well-posedness of the PDE if we want to apply the theory on martingale solutions developed in the last section (see Theorems 1.3 and 5.1).

We now give some properties of the family of solutions of (20):

Proposition 4.4. *We assume that $a:[0,T]\times\mathbb{R}^d\to\mathcal{S}_+(\mathbb{R}^d)$ and $b:[0,T]\times\mathbb{R}^d\to\mathbb{R}^d$ are bounded functions, and that* (21) *is fulfilled. Then the solution $u\in Y$ provided by Theorem* 4.3 *satisfies*:

(a) $u_0\geqslant 0 \Rightarrow u\geqslant 0$;

(b) $u_0 \in L^\infty(\mathbb{R}^d) \Rightarrow u \in L^\infty([0,T] \times \mathbb{R}^d)$ *and we have*

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} \leqslant \|u_0\|_{L^\infty(\mathbb{R}^d)} e^{t\|(\sum_i \partial_i b_i - \frac{1}{2}\sum_{ij}\partial_{ij}a_{ij})^-\|_\infty};$$

(c) *if moreover*

$$\frac{a}{1+|x|^2} \in L^2([0,T] \times \mathbb{R}^d), \qquad \frac{b}{1+|x|} \in L^2([0,T] \times \mathbb{R}^d),$$

then $u_0 \in L^1 \Rightarrow \|u(t)\|_{L^1(\mathbb{R}^d)} \leqslant \|u_0\|_{L^1(\mathbb{R}^d)}\ \forall t \in [0,T]$.

We observe that, by the above results together with Proposition 4.2, we obtain:

Corollary 4.5. *Let us assume that* $a : [0,T] \times \mathbb{R}^d \to \mathcal{S}(\mathbb{R}^d)$ *and* $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ *are bounded functions such that*:

(i) $$\langle \xi, a(t,x)\xi \rangle \geqslant \alpha|\xi|^2 \quad \forall (t,x) \in [0,T] \times \mathbb{R}^d,\ \text{for a certain } \alpha > 0;$$

(ii) $$\begin{gathered}|b(t,x) - b(s,y)| + \|a(t,x) - a(s,y)\| \leqslant C(|x-y|^\gamma + |t-s|^\gamma)\\ \forall (t,x), (s,y) \in [0,T] \times \mathbb{R}^d,\ \text{for some } \gamma \in (0,1],\ C \geqslant 0;\end{gathered}$$

(iii) $$\begin{gathered}\sum_j \partial_j a_{ij} \in L^\infty([0,T] \times \mathbb{R}^d) \quad \text{for } i = 1,\dots,d,\\ \Bigl(\sum_i \partial_i b_i - \frac{1}{2}\sum_{ij}\partial_{ij}a_{ij}\Bigr)^- \in L^\infty([0,T] \times \mathbb{R}^d);\end{gathered}$$

(iv) $$\frac{a}{1+|x|^2} \in L^2([0,T] \times \mathbb{R}^d), \quad \frac{b}{1+|x|} \in L^2([0,T] \times \mathbb{R}^d).$$

Then, for any $\mu_0 \in \mathcal{M}_+(\mathbb{R}^d)$ *there exists a unique finite measure-valued solution* $\mu_t \in \mathcal{M}_+(\mathbb{R}^d)$ *of* (2) *starting from* μ_0. *Moreover, if such that* $\mu_0 = \rho_0\mathcal{L}^d$ *with* $\rho_0 \in L^2(\mathbb{R}^d)$, *then* $\mu_t \ll \mathcal{L}^d$ *for all* $t \in [0,T]$.

Proof. Existence and uniqueness of finite measure-valued solutions follows by Proposition 4.2. So the only thing to prove is that, if $\rho_0 \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ is non-negative, then $\mu_t \in \mathcal{M}_+(\mathbb{R}^d)$ and $\mu_t \ll \mathcal{L}^d$ for all $t \in [0,T]$. This simply follows by the fact that the solution $u \in Y$ provided by Theorem 4.3 belongs to $L^1_+(\mathbb{R}^d)$ by Proposition 4.4, and thus coincides with μ_t by uniqueness in the set of finite measure-valued solutions. □

In order to prove the results stated before, we need the following theorem of J.-L. Lions (see [16]).

Theorem 4.6. *Let H be an Hilbert space, provided with a norm* $|\cdot|$, *and inner product* $(\cdot,\cdot)$. *Let* $\Phi \subset H$ *be a subspace endowed with a prehilbertian norm* $\|\cdot\|$, *such that the injection* $\Phi \hookrightarrow H$ *is continuous. We consider a bilinear form* $B : H \times \Phi \to \mathbb{R}$ *such that*:

- *$H \ni u \mapsto B(u,\varphi)$ is continuous on H for any fixed $\varphi \in \Phi$;*
- *there exists $\alpha > 0$ such that $B(\varphi,\varphi) \geqslant \alpha\|\varphi\|^2$ for any $\varphi \in \Phi$.*

Then, for any linear continuous form L on Φ there exists $v \in H$ such that

$$B(v,\varphi) = L(\varphi) \quad \forall \varphi \in \Phi.$$

Proof of Theorem 4.3. We will first prove existence and uniqueness of a solution in the space Y. Once this will be done, we will show that, if u is a weak solution of (20) belonging to $L^2([0,T]\times\mathbb{R}^d)$ and $\partial_t a_{ij} \in L^\infty([0,T]\times\mathbb{R}^d)$ for $i,j = 1,\dots,d$, then u belongs to Y, and so it coincides with the unique solution provided before.

The change of unknown

$$v(t,x) = e^{-\lambda t}u(t,x)$$

leads to the equation

$$\begin{cases} \partial_t v + \sum_i \partial_i(\tilde{b}_i v) - \frac{1}{2}\sum_{ij}\partial_i(a_{ij}\partial_j v) + \lambda v = 0 & \text{in } [0,T]\times\mathbb{R}^d, \\ v_0 = u_0, \end{cases} \tag{22}$$

where $\tilde{b}_i := b_i - \frac{1}{2}\sum_j \partial_j a_{ij} \in L^\infty([0,T]\times\mathbb{R}^d)$. Assuming that λ satisfies $\lambda > \frac{1}{2}\|(\sum_i \partial_i \tilde{b}_i)^-\|_\infty$, we will prove existence and uniqueness for u.

Step 1 (*Existence in Y*). We want to apply Theorem 4.6.

Let us take $H := L^2([0,T], H^1(\mathbb{R}^d))$, $\Phi := \{\varphi \in C^\infty([0,T]\times\mathbb{R}^d) \mid \operatorname{supp}\varphi \Subset [0,T)\times\mathbb{R}^d\}$. Φ is endowed with the norm

$$\|\varphi\|_\Phi^2 := \|\varphi\|_H^2 + \frac{1}{2}\int_{\mathbb{R}^d} \left|\varphi(0,x)\right|^2 dx.$$

The bilinear form B and the linear form L are defined as

$$B(u,\varphi) := \int_0^T \int_{\mathbb{R}^d} \left[u\left(-\partial_t\varphi - \sum_i \tilde{b}_i\partial_i\varphi + \lambda\varphi\right) + \frac{1}{2}\sum_{ij} a_{ij}\partial_j u \partial_i\varphi\right] dx\,dt,$$

$$L(\varphi) := \int_{\mathbb{R}^d} u_0(x)\varphi(0,x)\,dx.$$

Thanks to these definitions and our assumptions, Lions' theorem applies, and we find a distributional solution v of (22). In particular,

$$\partial_t v = -\sum_i \partial_i(\tilde{b}_i v) + \frac{1}{2}\sum_{ij}\partial_i(a_{ij}\partial_j v) - \lambda v \in H^* = L^2\big([0,T], H^{-1}\big(\mathbb{R}^d\big)\big),$$

and thus $v \in Y$. In order to give a meaning to the initial condition and to show the uniqueness, we recall that for functions in Y there exists a well-defined notion of trace at 0 in $L^2(\mathbb{R}^d)$, and the following Gauss–Green formula holds:

$$\int_0^T \int_{\mathbb{R}^d} \partial_t u\tilde{u} + \partial_t \tilde{u} u \, dx \, dt = \int_{\mathbb{R}^d} u(T,x)\tilde{u}(T,x)\, dx - \int_{\mathbb{R}^d} u(0,x)\tilde{u}(0,x)\, dx \quad \forall u, \tilde{u} \in Y \tag{23}$$

(both facts follow by a standard approximation with smooth functions and by the fact that, if u is smooth and compactly supported in $[0,T) \times \mathbb{R}^d$, $\int_{\mathbb{R}^d} u^2(0,x)\, dx \leqslant 2\|\partial_t u\|_{H^*}\|u\|_H$). Thus, by (22) and (23), we obtain that v satisfies

$$\int_{\mathbb{R}^d} \big(v(0,x) - u_0(x)\big)\varphi(0,x)\, dx = 0 \quad \forall \varphi \in \Phi,$$

and therefore the initial condition is satisfied in $L^2(\mathbb{R}^d)$.

Step 2 (*Uniqueness in Y*). For the uniqueness, if $v \in Y$ is a solution of (22) with $u_0 = 0$, again by (23) we get

$$\begin{aligned}
0 &= \int_0^T \int_{\mathbb{R}^d} \left(\partial_t v + \sum_i \partial_i(\tilde{b}_i v) - \frac{1}{2}\sum_{ij} \partial_i(a_{ij}\partial_j v) + \lambda v\right) v \, dx \, dt \\
&= \frac{1}{2}\int_0^T \int_{\mathbb{R}^d} \left[\frac{d}{dt} v^2 - \sum_i \tilde{b}_i \partial_i\big(v^2\big) + \sum_{ij} a_{ij}\partial_i v \partial_j v + 2\lambda v^2\right] dx \\
&\geqslant \frac{1}{2}\int_{\mathbb{R}^d} v^2(T,x)\, dx + \left(\lambda - \frac{1}{2}\left\|\left(\sum_i \partial_i \tilde{b}_i\right)^-\right\|_\infty\right) \int_0^T \int_{\mathbb{R}^d} v^2 \, dx \, dt \\
&\geqslant \left(\lambda - \frac{1}{2}\left\|\left(\sum_i \partial_i \tilde{b}_i\right)^-\right\|_\infty\right) \int_0^T \int_{\mathbb{R}^d} v^2 \, dx \, dt.
\end{aligned}$$

Since $\lambda > \frac{1}{2}\|(\sum_i \partial_i \tilde{b}_i)^-\|_\infty$, we get $v = 0$.

Remark 4.7. We observe that the above proof still works for the PDE

$$\begin{cases} \partial_t u + \sum_i \partial_i(b_i u) - \frac{1}{2}\sum_{ij} \partial_{ij}(a_{ij}u) = U & \text{in } [0,T] \times \mathbb{R}^d, \\ u(0) = u_0, \end{cases}$$

with $U \in H^* = L^2([0,T], H^{-1}(\mathbb{R}^d))$. Indeed, it suffices to define L as

$$L(\varphi) := \langle U, \varphi\rangle_{H^*,H} + \int_{\mathbb{R}^d} u_0(x)\varphi(x)\, dx,$$

and all the rest of the proof works without any changes.

Thanks to this remark, we can now prove uniqueness in the larger space $L^2([0,T]\times\mathbb{R}^d)$ under the assumption $\partial_t a_{ij} \in L^\infty([0,T]\times\mathbb{R}^d)$ for $i,j=1,\dots,d$.

Step 3 (*Uniqueness in L^2*). If $u\in L^2([0,T]\times\mathbb{R}^d)$ is a (distributional) solution of (19), then

$$\partial_t u - \frac{1}{2}\sum_{ij}\partial_i(a_{ij}\partial_j u) = -\sum_i \partial_i(\tilde{b}_i u) \in L^2\big([0,T], H^{-1}\big(\mathbb{R}^d\big)\big).$$

By Remark 4.7, there exists $\tilde{u}\in Y$ solution of the above equation, with the same initial condition. Let us define $w := u-\tilde{u}\in L^2([0,T]\times\mathbb{R}^d)$. Then w is a distributional solution of

$$\begin{cases} \partial_t w - A(\partial_x)w := \partial_t w - \frac{1}{2}\sum_{ij}\partial_i(a_{ij}\partial_j w) = 0 & \text{in } [0,T]\times\mathbb{R}^d,\\ w(0)=0. \end{cases}$$

In order to conclude the proof, it suffices to prove that $w=0$.

Step 3.1 (*Regularization*). Let us consider the PDE

$$w_\varepsilon - \varepsilon A(\partial_x)w_\varepsilon = w \quad \text{in } [0,T]\times\mathbb{R}^d \tag{24}$$

(this is an elliptic problem degenerate in the time variable). Applying Theorem 4.6, with $H = \Phi := L^2([0,T], H^1(\mathbb{R}^d))$,

$$B(u,\varphi) := \int_0^T\int_{\mathbb{R}^d}\bigg(u\varphi + \frac{\varepsilon}{2}\sum_{ij}a_{ij}\partial_j u\partial_i\varphi\bigg)\,dx\,dt,$$

$$L(\varphi) := \int_0^T\int_{\mathbb{R}^d} w\varphi\,dx\,dt,$$

we find a unique solution w_ε of (24) in $L^2([0,T],H^1(\mathbb{R}^d))$, that is $w_\varepsilon = (I-\varepsilon A(\partial_x))^{-1}w$, with $(I-\varepsilon A(\partial_x)) : L^2([0,T],H^1(\mathbb{R}^d)) \to L^2([0,T],H^{-1}(\mathbb{R}^d))$ isomorphism. Now we want to find the equation solved by w_ε. We observe that, since $(I-\varepsilon A(\partial_x))^{-1}$ commutes with $A(\partial_x)$ and $\partial_t w = A(\partial_x)w$, the parabolic equation solved by w_ε formally looks

$$\partial_t w_\varepsilon - A(\partial_x)w_\varepsilon = \big[\partial_t, \big(I-\varepsilon A(\partial_x)\big)^{-1}\big]w.$$

Formally computing the commutator between ∂_t and $(I-\varepsilon A(\partial_x))^{-1}$, one obtains

$$\partial_t w_\varepsilon - A(\partial_x)w_\varepsilon = \varepsilon\big(I-\varepsilon A(\partial_x)\big)^{-1}\sum_{ij}\partial_j\big(\partial_t a_{ij}\partial_i w^\varepsilon\big) \tag{25}$$

in the distributional sense (see (27) below). Let us assume for a moment that (25) has been rigorously justified, and let us see how we can conclude.

Step 3.2 (*Gronwall argument*). By (25) it follows that $\partial_t w_\varepsilon \in L^2([0,T], H^{-1}(\mathbb{R}^d))$. Thus, recalling that $w_\varepsilon \in L^2([0,T], H^1(\mathbb{R}^d))$, we can multiply (25) by w_ε and integrate on $\mathbb{R}^d$, obtaining

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} |w_\varepsilon|^2\,dx + \alpha\int_{\mathbb{R}^d} |\nabla_x w_\varepsilon|^2\,dx \leqslant -\varepsilon\int_{\mathbb{R}^d}\sum_{ij}(\partial_t a_{ij})\partial_i w_\varepsilon \partial_j\big(\big(I-\varepsilon A(\partial_x)\big)^{-1} w_\varepsilon\big)\,dx.$$

We observe that $w_\varepsilon(t) \to 0$ in L^2 as $t \searrow 0$. Indeed, since $w_\varepsilon \in Y$ there is a well-defined notion of trace at 0 in L^2 (see (23)), and it is not difficult to see that this trace is 0 since $w(0) = 0$ in the sense of distributions. Thus, integrating in time the above inequality, we get

$$\begin{aligned}&\big\|w_\varepsilon(t)\big\|^2_{L^2(\mathbb{R}^d)} + 2\alpha\|\nabla_x w_\varepsilon\|^2_{L^2([0,T]\times\mathbb{R}^d)}\\ &\quad\leqslant 2C\varepsilon\|\nabla_x w_\varepsilon\|_{L^2([0,T]\times\mathbb{R}^d)}\big\|\nabla_x\big(\big(I-\varepsilon A(\partial_x)\big)^{-1}w_\varepsilon\big)\big\|_{L^2([0,T]\times\mathbb{R}^d)} \quad \forall t\in[0,T].\end{aligned} \tag{26}$$

Let us consider, for a general $v \in L^2$, the function $v_\varepsilon := (I-\varepsilon A(\partial_x))^{-1}v$. Multiplying the identity $v_\varepsilon - \varepsilon A(\partial_x)v_\varepsilon = v$ by v_ε and integrating on $[0,T]\times\mathbb{R}^d$, we get

$$\|v_\varepsilon\|^2_{L^2} + \alpha\varepsilon\|\nabla_x v_\varepsilon\|^2_{L^2} \leqslant \|v_\varepsilon\|_{L^2}\|v\|_{L^2},$$

which implies $\|v_\varepsilon\|_{L^2} \leqslant \|v\|_{L^2}$, and therefore $\alpha\varepsilon\|\nabla_x v_\varepsilon\|^2_{L^2} \leqslant \|v\|^2_{L^2}$. Applying this last inequality with $v = w_\varepsilon$, we obtain

$$\big\|\nabla_x\big(\big(I-\varepsilon A(\partial_x)\big)^{-1}w_\varepsilon\big)\big\|_{L^2([0,T]\times\mathbb{R}^d)} \leqslant \frac{1}{\sqrt{\alpha\varepsilon}}\|w_\varepsilon\|_{L^2([0,T]\times\mathbb{R}^d)}.$$

Substituting the above inequality in (26), we have

$$\begin{aligned}\big\|w_\varepsilon(t)\big\|^2_{L^2(\mathbb{R}^d)} + 2\alpha\|\nabla_x w_\varepsilon\|^2_{L^2([0,T]\times\mathbb{R}^d)} &\leqslant 2C\sqrt{\frac{\varepsilon}{\alpha}}\|\nabla_x w_\varepsilon\|_{L^2([0,T]\times\mathbb{R}^d)}\|w_\varepsilon\|_{L^2([0,T]\times\mathbb{R}^d)}\\ &\leqslant C\sqrt{\frac{\varepsilon}{\alpha}}\|\nabla_x w_\varepsilon\|^2_{L^2([0,T]\times\mathbb{R}^d)} + C\sqrt{\frac{\varepsilon}{\alpha}}\|w_\varepsilon\|^2_{L^2([0,T]\times\mathbb{R}^d)},\end{aligned}$$

which implies, for ε small enough (say $\varepsilon \leqslant 4\frac{\alpha^3}{C^2}$),

$$\big\|w_\varepsilon(t)\big\|^2_{L^2(\mathbb{R}^d)} \leqslant C\sqrt{\frac{\varepsilon}{\alpha}}\|w_\varepsilon\|^2_{L^2([0,T]\times\mathbb{R}^d)} \quad \forall t\in[0,T].$$

By Gronwall inequality $w_\varepsilon = 0$, and thus by (24) $w = 0$.

Step 3.3 (*Rigorous justification of (25)*). In order to conclude the proof of the theorem, we only need to rigorously justify (25).

Let $(a^n_{ij})_{n\in\mathbb{N}}$ be a sequence of smooth functions bounded in L^∞, such that $\langle a^n\xi,\xi\rangle \geqslant \frac{\alpha}{2}|\xi|^2$, $\sum_j \partial_j a^n_{ij}$ and $\partial_t a^n_{ij}$ are uniformly bounded, and $a^n_{ij} \to a_{ij}$, $\sum_j \partial_j a^n_{ij} \to \sum_j \partial_j a_{ij}$, $\partial_t a^n_{ij} \to \partial_t a_{ij}$ a.e. We now compute $[\partial_t, (I-\varepsilon A^n(\partial_x))^{-1}]$, where $A^n(\partial_x) := \sum_{ij}\partial_i(a^n_{ij}\partial_j\cdot)$:

$$\begin{aligned}\big[\partial_t, \big(I-\varepsilon A^n(\partial_x)\big)^{-1}\big] &= \Big[\partial_t, \sum_{k\geqslant 0}\varepsilon^k A^n(\partial_x)^k\Big] = \sum_{n\geqslant 0}\varepsilon^k\big[\partial_t, A^n(\partial_x)^k\big]\\ &= \varepsilon\sum_{k=0}^{\infty}\sum_{i=0}^{k-1}\big(\varepsilon A^n(\partial_x)\big)^i\big[\partial_t, A^n(\partial_x)\big]\big(\varepsilon A^n(\partial_x)\big)^{k-i-1}\\ &= \varepsilon\sum_{i=0}^{\infty}\big(\varepsilon A^n(\partial_x)\big)^i\big[\partial_t, A^n(\partial_x)\big]\sum_{k>i}\big(\varepsilon A^n(\partial_x)\big)^{k-i-1}\\ &= \varepsilon\big(I-\varepsilon A^n(\partial_x)\big)^{-1}\big[\partial_t, A^n(\partial_x)\big]\big(I-\varepsilon A^n(\partial_x)\big)^{-1},\end{aligned}\tag{27}$$

where at the second equality we used the algebraic identity $[A, B^k] = \sum_{i=0}^{k-1} B^i[A,B]B^{k-i-1}$. Thus, for any $\varphi, \psi \in C_c^\infty([0,T]\times\mathbb{R}^d)$, we have

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^d}\psi\,\partial_t\big(\big(I-\varepsilon A^n(\partial_x)\big)^{-1}\varphi\big)\,dx\,dt\\ &\quad=\int_0^T\int_{\mathbb{R}^d}\psi\big[\big(I-\varepsilon A^n(\partial_x)\big)^{-1}\partial_t\varphi\big]\,dx\,dt\\ &\qquad+\varepsilon\int_0^T\int_{\mathbb{R}^d}\psi\big[\big(I-\varepsilon A^n(\partial_x)\big)^{-1}\big[\partial_t, A^n(\partial_x)\big]\big(I-\varepsilon A^n(\partial_x)\big)^{-1}\varphi\big]\,dx\,dt.\end{aligned}\tag{28}$$

We now want to pass to the limit in the above identity as $n\to\infty$. Since $(I-\varepsilon A^n(\partial_x))^{-1}$ is selfadjoint in $L^2([0,T]\times\mathbb{R}^d)$ and it commutes with $A^n(\partial_x)$, we get

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^d}\psi\big[\big(I-\varepsilon A^n(\partial_x)\big)^{-1}\big[\partial_t, A^n(\partial_x)\big]\big(I-\varepsilon A^n(\partial_x)\big)^{-1}\varphi\big]\,dx\,dt\\ &\quad=\int_0^T\int_{\mathbb{R}^d}\big[\big(I-\varepsilon A^n(\partial_x)\big)^{-1}\psi\big]\big[\big[\partial_t, A^n(\partial_x)\big]\big(I-\varepsilon A^n(\partial_x)\big)^{-1}\varphi\big]\,dx\,dt\\ &\quad=-\int_0^T\int_{\mathbb{R}^d}\big[\partial_t\big(\big(I-\varepsilon A^n(\partial_x)\big)^{-1}\psi\big)\big]\big[\big(I-\varepsilon A^n(\partial_x)\big)^{-1}A^n(\partial_x)\varphi\big]\,dx\,dt\\ &\qquad-\int_0^T\int_{\mathbb{R}^d}\big[\big(I-\varepsilon A^n(\partial_x)\big)^{-1}A^n(\partial_x)\psi\big]\big[\partial_t\big(\big(I-\varepsilon A^n(\partial_x)^{-1}\big)\varphi\big)\big]\,dx\,dt.\end{aligned}$$

By (27) we have

$$\begin{aligned}\partial_t\big(\big(I-\varepsilon A^n(\partial_x)\big)^{-1}\varphi\big)&=\big(I-\varepsilon A^n(\partial_x)\big)^{-1}\partial_t\varphi\\&\quad+\varepsilon\big(I-\varepsilon A^n(\partial_x)\big)^{-1}\big[\partial_t,A^n(\partial_x)\big]\big(I-\varepsilon A^n(\partial_x)\big)^{-1}\varphi,\end{aligned}$$

and, observing that $[\partial_t, A^n(\partial_x)] = \sum_{ij}\partial_i(\partial_t a^n_{ij}\partial_j\cdot)$, we deduce that the right-hand side is uniformly bounded in $L^2([0,T],H^1(\mathbb{R}^d))$. In the same way one obtains

$$\begin{aligned}\partial_t\big(\big(I-\varepsilon A^n(\partial_x)\big)^{-1}A^n(\partial_x)\varphi\big)&=\big(I-\varepsilon A^n(\partial_x)\big)^{-1}\partial_t\big(A^n(\partial_x)\varphi\big)\\&\quad+\varepsilon\big(I-\varepsilon A^n(\partial_x)\big)^{-1}\big[\partial_t,A^n(\partial_x)\big]\big(I-\varepsilon A^n(\partial_x)\big)^{-1}A^n(\partial_x)\varphi\\&=\big(I-\varepsilon A^n(\partial_x)\big)^{-1}\big[\partial_t,A^n(\partial_x)\big]\varphi\\&\quad+\big(I-\varepsilon A^n(\partial_x)\big)^{-1}A^n(\partial_x)\partial_t\varphi\\&\quad+\varepsilon\big(I-\varepsilon A^n(\partial_x)\big)^{-1}\big[\partial_t,A^n(\partial_x)\big]\big(I-\varepsilon A^n(\partial_x)\big)^{-1}A^n(\partial_x)\varphi,\end{aligned}$$

and, as above, the right-hand side is uniformly bounded in $L^2([0,T],H^1(\mathbb{R}^d))$. Thus $\partial_t(I-\varepsilon A^n(\partial_x))^{-1}\varphi$ is uniformly bounded in $L^2([0,T],H^1(\mathbb{R}^d))\subset L^2([0,T]\times\mathbb{R}^d)$ (the same obviously holds for ψ in place of φ), while $(I-\varepsilon A^n(\partial_x))^{-1}A^n(\partial_x)\varphi$ is uniformly bounded in $H^1([0,T]\times\mathbb{R}^d)$ (again the same fact holds for ψ in place of φ). Therefore, since $H^1_{\mathrm{loc}}([0,T]\times\mathbb{R}^d)\hookrightarrow L^2_{\mathrm{loc}}([0,T]\times\mathbb{R}^d)$ compactly, all we have to check is that

$$\partial_t\big(\big(I-\varepsilon A^n(\partial_x)\big)^{-1}\varphi\big)\to\partial_t\big(\big(I-\varepsilon A(\partial_x)\big)^{-1}\varphi\big)$$

and

$$\big(I-\varepsilon A^n(\partial_x)\big)^{-1}A^n(\partial_x)\varphi\to\big(I-\varepsilon A(\partial_x)\big)^{-1}A(\partial_x)\varphi$$

in the sense of distribution (indeed, by what we have shown above, $\partial_t((I-\varepsilon A^n(\partial_x))^{-1}\varphi)$ will converge weakly in L^2 while $(I-\varepsilon A^n(\partial_x))^{-1}A^n(\partial_x)\varphi$ will converge strongly in L^2_{loc}, and therefore it is not difficult to see that the product converges to the product of the limits). We observe that, since the solution of

$$\varphi_\varepsilon-\varepsilon A(\partial_x)\varphi_\varepsilon=\varphi\quad\text{in }[0,T]\times\mathbb{R}^d\tag{29}$$

belonging to $L^2([0,T],H^1(\mathbb{R}^d))$ is unique, and any limit point of $(I-\varepsilon A^n(\partial_x))^{-1}\varphi$ belongs to $L^2([0,T],H^1(\mathbb{R}^d))$ and is a distributional solution of (29), one obtains that

$$\big(I-\varepsilon A^n(\partial_x)\big)^{-1}\varphi\to\big(I-\varepsilon A(\partial_x)\big)^{-1}\varphi$$

in the distributional sense, which implies the convergence of $\partial_t(I-\varepsilon A^n(\partial_x))^{-1}\varphi$ to $\partial_t(I-\varepsilon A(\partial_x))^{-1}\varphi$. Regarding $(I-\varepsilon A^n(\partial_x))^{-1}A^n(\partial_x)\varphi$, let us take $\chi\in C^\infty_{\mathrm c}([0,T]\times\mathbb{R}^d)$. Then we consider

$$\int_0^T\int_{\mathbb{R}^d}A^n(\partial_x)\varphi\big[\big(I-\varepsilon A^n(\partial_x)\big)^{-1}\chi\big]\,dx\,dt=-\int_0^T\int_{\mathbb{R}^d}\sum_{ij}a^n_{ij}\partial_j\varphi\big(\partial_i\big(I-\varepsilon A^n(\partial_x)\big)^{-1}\chi\big)\,dx\,dt.$$

Recalling that $(I-\varepsilon A^n(\partial_x))^{-1}\chi$ is uniformly bounded in $L^2([0,T],H^1(\mathbb{R}^d))$, we get that $\partial_j(I-\varepsilon A^n(\partial_x))^{-1}\chi$ converges to $\partial_j(I-\varepsilon A(\partial_x))^{-1}\chi$ weakly in $L^2([0,T]\times\mathbb{R}^d)$ while $a^n_{ij}\to a_{ij}$ a.e., and so the convergence of $(I-\varepsilon A^n(\partial_x))^{-1}A^n(\partial_x)\varphi$ to $(I-\varepsilon A(\partial_x))^{-1}A(\partial_x)\varphi$ follows.

Thus we are able to pass to the limit in (28), and we get

$$\partial_t\big(\big(I-\varepsilon A(\partial_x)\big)^{-1}\varphi\big)\in L^2\big([0,T],H^1\big(\mathbb{R}^d\big)\big)$$

and

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^d}\psi\,\partial_t\big(\big(I-\varepsilon A(\partial_x)\big)^{-1}\varphi\big)\,dx\,dt\\
&\quad=\int_0^T\int_{\mathbb{R}^d}\psi\big[\big(I-\varepsilon A(\partial_x)\big)^{-1}\partial_t\varphi\big]\,dx\,dt\\
&\qquad+\varepsilon\int_0^T\int_{\mathbb{R}^d}\psi\big[\big(I-\varepsilon A(\partial_x)\big)^{-1}\big[\partial_t,A(\partial_x)\big]\big(I-\varepsilon A(\partial_x)\big)^{-1}\varphi\big]\,dx\,dt.
\end{aligned}$$

Observing that $(I-\varepsilon A(\partial_x))^{-1}$ is selfadjoint in $L^2([0,T]\times\mathbb{R}^d)$ (for instance, this can be easily proved by approximation), we have that the second integral in the right-hand side can be written as

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^d}\psi\big[\big(I-\varepsilon A(\partial_x)\big)^{-1}\big[\partial_t,A(\partial_x)\big]\big(I-\varepsilon A(\partial_x)\big)^{-1}\varphi\big]\,dx\,dt\\
&\quad=\int_0^T\int_{\mathbb{R}^d}\big[\big(I-\varepsilon A(\partial_x)\big)^{-1}\psi\big]\big[\big[\partial_t,A(\partial_x)\big]\big(\big(I-\varepsilon A(\partial_x)\big)^{-1}\varphi\big)\big]\,dx\,dt.
\end{aligned}$$

Using now that $[\partial_t,A(\partial_x)]=\sum_{ij}\partial_i(\partial_t a_{ij}\partial_j\cdot)$ in the sense of distributions, it can be easily proved by approximation that the right-hand side above coincides with

$$-\int_0^T\int_{\mathbb{R}^d}\sum_{ij}(\partial_t a_{ij})\partial_i\big(\big(I-\varepsilon A(\partial_x)\big)^{-1}\psi\big)\partial_j\big(\big(I-\varepsilon A(\partial_x)\big)^{-1}\varphi\big)\,dx\,dt.$$

Therefore we finally obtain

$$\int_0^T\int_{\mathbb{R}^d}\psi\,\partial_t\big(\big(I-\varepsilon A(\partial_x)\big)^{-1}\varphi\big)\,dx\,dt$$

$$= \int_0^T \int_{\mathbb{R}^d} \psi\big[\big(I - \varepsilon A(\partial_x)\big)^{-1} \partial_t \varphi\big]\,dx\,dt$$
$$- \varepsilon \int_0^T \int_{\mathbb{R}^d} \sum_{ij} (\partial_t a_{ij}) \partial_i \big(\big(I - \varepsilon A(\partial_x)\big)^{-1} \psi\big) \partial_j \big(\big(I - \varepsilon A(\partial_x)\big)^{-1} \varphi\big)\,dx\,dt. \tag{30}$$

By what we have proved above, it follows that

$$\begin{aligned} &\partial_t \big(\big(I - \varepsilon A(\partial_x)\big)^{-1} \varphi\big) \in L^2\big([0,T], H^1\big(\mathbb{R}^d\big)\big), \\ &A(\partial_x)\big(\big(I - \varepsilon A(\partial_x)\big)^{-1} \varphi\big) = \big(I - \varepsilon A(\partial_x)\big)^{-1} A(\partial_x)\varphi \in L^2\big([0,T], H^1\big(\mathbb{R}^d\big)\big). \end{aligned} \tag{31}$$

This implies that (30) holds also for $\psi \in L^2([0,T] \times \mathbb{R}^d)$, and that $(I - \varepsilon A(\partial_x))^{-1}\varphi$ is an admissible test function in the equation $\partial_t w - A(\partial_x) w = 0$. By these two facts we obtain

$$\begin{aligned} 0 &= \int_0^T \int_{\mathbb{R}^d} w\big[\big(\partial_t + A(\partial_x)\big)\big(I - \varepsilon A(\partial_x)\big)^{-1} \varphi\big]\,dx\,dt \\ &= \int_0^T \int_{\mathbb{R}^d} w\big[\big(I - \varepsilon A(\partial_x)\big)^{-1} \big(\partial_t + A(\partial_x)\big)\varphi\big]\,dx\,dt \\ &\quad - \varepsilon \int_0^T \int_{\mathbb{R}^d} \sum_{ij} (\partial_t a_{ij}) \partial_i \big(\big(I - \varepsilon A(\partial_x)\big)^{-1} w\big) \partial_j \big(\big(I - \varepsilon A(\partial_x)\big)^{-1} \varphi\big)\,dx\,dt \\ &= \int_0^T \int_{\mathbb{R}^d} w_\varepsilon \big[\big(\partial_t + A(\partial_x)\big)\varphi\big]\,dx\,dt - \varepsilon \int_0^T \int_{\mathbb{R}^d} \sum_{ij} (\partial_t a_{ij}) \partial_i w_\varepsilon \partial_j \big(\big(I - \varepsilon A(\partial_x)\big)^{-1} \varphi\big)\,dx\,dt, \end{aligned}$$

which exactly means that

$$\partial_t w_\varepsilon - A(\partial_x) w_\varepsilon = \varepsilon \big(I - \varepsilon A(\partial_x)\big)^{-1} \sum_{ij} \partial_j \big(\partial_t a_{ij} \partial_i w^\varepsilon\big)$$

in the distributional sense. □

Proof of Proposition 4.4. (a) Arguing as in the first part of the proof of Theorem 4.3, with the same notation we have

$$0 = \int_0^T \int_{\mathbb{R}^d} \bigg(\partial_t v + \sum_i \partial_i (\tilde{b}_i v) - \frac{1}{2} \sum_{ij} \partial_i (a_{ij} \partial_j v) + \lambda v\bigg) v^- \,dx\,dt$$

$$= \frac{1}{2}\int_0^T\int_{\mathbb{R}^d}\left[-\frac{d}{dt}(v^-)^2 - \sum_i \tilde{b}_i\partial_i\big((v^-)^2\big) - \sum_{ij} a_{ij}\partial_i v^-\partial_j v^- - 2\lambda(v^-)^2\right]dx$$

$$\leqslant -\frac{1}{2}\int_{\mathbb{R}^d}(v^-)^2(T,x)\,dx - \left(\lambda - \frac{1}{2}\left\|\left(\sum_i \partial_i\tilde{b}_i\right)^-\right\|_\infty\right)\int_0^T\int_{\mathbb{R}^d}(v^-)^2\,dx\,dt$$

$$\leqslant -\left(\lambda - \frac{1}{2}\left\|\left(\sum_i \partial_i\tilde{b}_i\right)^-\right\|_\infty\right)\int_0^T\int_{\mathbb{R}^d}(v^-)^2\,dx\,dt,$$

and then $v^- = 0$.

(b) It suffices to observe that the above argument works for every $v \in Y$ such that $v(0) \geqslant 0$ and

$$\partial_t v + \sum_i \partial_i(\tilde{b}_i v) - \frac{1}{2}\sum_{ij}\partial_i(a_{ij}\partial_j v) \geqslant 0.$$

Applying this remark to the function $v := \|u_0\|_{L^\infty(\mathbb{R}^d)} - ue^{-\lambda t}$ with $\lambda > \|(\sum_i \partial_i\tilde{b}_i)^-\|_\infty$, and then letting $\lambda \to \|(\sum_i \partial_i\tilde{b}_i)^-\|_\infty$, the thesis follows.

(c) The argument we use here is reminiscent of the one that we will use in the next subsection for renormalized solutions. Indeed, in order to prove the thesis, we will implicitly prove that, if $u \in L^2([0,T], H^1(\mathbb{R}^d))$ is a solution of (20), then it is also a renormalized solution (see Definition 4.9).

Let us define

$$\beta_\varepsilon(s) := \left(\sqrt{s^2+\varepsilon^2} - \varepsilon\right) \in C^2(\mathbb{R}).$$

Notice that β_ε is convex and

$$\beta_\varepsilon(s) \to |s| \quad \text{as } \varepsilon \to 0, \qquad \beta_\varepsilon(s) - s\beta_\varepsilon'(s) \in [-\varepsilon, 0].$$

Moreover, since $\beta_\varepsilon', \beta_\varepsilon'' \in W^{1,\infty}(\mathbb{R})$, it is easily seen that

$$u \in L^2\big([0,T], H^1\big(\mathbb{R}^d\big)\big) \quad \Rightarrow \quad \beta_\varepsilon(u), \beta_\varepsilon'(u) \in L^2\big([0,T], H^1\big(\mathbb{R}^d\big)\big).$$

Fix now a non-negative cut-off function $\varphi \in C_c^\infty(\mathbb{R}^d)$ with $\mathrm{supp}(\varphi) \subset B_2(0)$, and $\varphi = 1$ in $B_1(0)$, and consider the functions $\varphi_R(x) := \varphi(x/R)$ for $R \geqslant 1$.

Thus, since $\beta_\varepsilon'' \geqslant 0$ and a_{ij} is positive definite, recalling that $\tilde{b}_i = b_i - \frac{1}{2}\sum_j \partial_j a_{ij}$, for any $t \in [0,T]$ we have

$$0 = \int_0^t\int_{\mathbb{R}^d}\left(\partial_t u + \sum_i \partial_i(\tilde{b}_i u) - \frac{1}{2}\sum_{ij}\partial_i(a_{ij}\partial_j u)\right)\beta_\varepsilon'(u)\varphi_R\,dx\,ds$$

$$
\begin{aligned}
&= \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} \biggl(\frac{d}{dt}\bigl(\varphi_R \beta_\varepsilon(u)\bigr) - 2\sum_i \tilde{b}_i \partial_i \bigl(u\beta_\varepsilon'(u)\varphi_R\bigr) + 2\sum_i \tilde{b}_i \partial_i \bigl(\beta_\varepsilon(u)\bigr)\varphi_R \\
&\quad + \sum_{ij} a_{ij}\partial_i u \partial_j u \beta_\varepsilon''(u)\varphi_R + \sum_{ij} a_{ij}\partial_i\bigl(\beta_\varepsilon(u)\bigr)\partial_j\varphi_R \biggr)\,dx\,ds \\
&\geqslant \frac{1}{2}\int_{\mathbb{R}^d} \varphi_R\beta_\varepsilon\bigl(u(t)\bigr)\,dx - \frac{1}{2}\int_{\mathbb{R}^d} \varphi_R\beta_\varepsilon\bigl(u(0)\bigr)\,dx \\
&\quad - \int_0^t \int_{\mathbb{R}^d} \sum_i \tilde{b}_i \bigl(\partial_i\bigl(\bigl(u\beta_\varepsilon'(u) - \beta_\varepsilon(u)\bigr)\varphi_R\bigr) + \beta_\varepsilon(u)\partial_i\varphi_R\bigr)\,dx\,ds \\
&\quad - \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} \sum_{ij} \bigl((\partial_j a_{ij})\partial_i\varphi_R + a_{ij}\partial_{ij}\varphi_R\bigr)\beta_\varepsilon(u)\,dx\,ds \\
&\geqslant \frac{1}{2}\int_{\mathbb{R}^d} \varphi_R\beta_\varepsilon\bigl(u(t)\bigr)\,dx - \frac{1}{2}\int_{\mathbb{R}^d} \varphi_R\beta_\varepsilon\bigl(u(0)\bigr)\,dx - \int_0^t \int_{\mathbb{R}^d} \biggl(\sum_i \partial_i\tilde{b}_i\biggr)^- \bigl(u\beta_\varepsilon'(u) - \beta_\varepsilon(u)\bigr)\varphi_R\,dx\,ds \\
&\quad - \int_0^t \int_{\mathbb{R}^d} \biggl(\sum_i b_i\partial_i\varphi_R + \frac{1}{2}\sum_{ij} a_{ij}\partial_{ij}\varphi_R\biggr)\beta_\varepsilon(u)\,dx\,ds.
\end{aligned}
$$

Observing that $|\beta_\varepsilon(u)| \leqslant |u|$, and using Hölder inequality and the inequalities

$$
\frac{1}{R}\chi_{\{R\leqslant|x|\leqslant 2R\}} \leqslant \frac{3}{1+|x|}\chi_{\{|x|\geqslant R\}}, \qquad \frac{1}{R^2}\chi_{\{R\leqslant|x|\leqslant 2R\}} \leqslant \frac{5}{1+|x|^2}\chi_{\{|x|\geqslant R\}}, \tag{32}
$$

we get

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \varphi_R\beta_\varepsilon\bigl(u(t)\bigr)\,dx \\
&\quad \leqslant \int_{\mathbb{R}^d} \varphi_R\beta_\varepsilon\bigl(u(0)\bigr)\,dx + 2\varepsilon\int_0^t \int_{|x|\leqslant 2R} \biggl(\sum_i \partial_i\tilde{b}_i\biggr)^-\,dx\,ds \\
&\qquad + \|\varphi\|_{C^2}\biggl(6\biggl\|\frac{b}{1+|x|}\biggr\|_{L^2([0,T]\times\{|x|\geqslant R\})} + 5\biggl\|\frac{a}{1+|x|^2}\biggr\|_{L^2([0,T]\times\{|x|\geqslant R\})}\biggr)\|u\|_{L^2([0,t]\times\mathbb{R}^d)}.
\end{aligned}
$$

Letting first $\varepsilon \to 0$ and then $R \to \infty$, we obtain

$$
\bigl\|u(t)\bigr\|_{L^1(\mathbb{R}^d)} \leqslant \bigl\|u(0)\bigr\|_{L^1(\mathbb{R}^d)} \quad \forall t \in [0,T]. \qquad \square
$$

4.3. Existence and uniqueness in the degenerate parabolic case

We now want to drop the uniform ellipticity assumption on a. In this case, to prove existence and uniqueness in $\mathscr{L}_+$, we will need to assume a independent of the space variables.

4.3.1. Uniqueness in $\mathscr{L}$

The uniqueness result is a consequence of the following comparison principle in $\mathscr{L}$ (recall that the comparison principle in said to hold if the inequality between two solutions at time 0 is preserved at later times).

Theorem 4.8 *(Comparison principle in $\mathscr{L}$). Let us assume that $a : [0,T] \to \mathcal{S}_+(\mathbb{R}^d)$ and $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ are such that*:

(i) $b \in L^1([0,T], BV_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{R}^d))$, $\sum_i \partial_i b_i \in L^1_{\mathrm{loc}}([0,T]\times\mathbb{R}^d)$;
(ii) $a \in L^\infty([0,T], \mathcal{S}_+(\mathbb{R}^d))$.

Then (19) *satisfies the comparison principle in $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. In particular solutions of the PDE in $\mathscr{L}$, if they exist, are unique.*

Since we do not assume any ellipticity of the PDE, in order to prove the above result we use the technique of renormalized solutions, which was first introduced in the study of the Boltzmann equation by DiPerna and P.-L. Lions [8,9], and then applied in the context of transport equations by many authors (see for example [1,5–7,10]).

Definition 4.9. Let $a : [0,T] \times \mathbb{R}^d \to \mathcal{S}_+(\mathbb{R}^d)$, $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ be such that:

(i) $b, \sum_i \partial_i b_i \in L^1_{\mathrm{loc}}([0,T]\times\mathbb{R}^d)$;
(ii) $a, \sum_j \partial_j a_{ij}, \sum_{ij} \partial_{ij} a_{ij} \in L^1_{\mathrm{loc}}([0,T]\times\mathbb{R}^d)$.

Let $u \in L^\infty_{\mathrm{loc}}([0,T]\times\mathbb{R}^d)$ and assume that

$$c := \partial_t u + \sum_i b_i \partial_i u - \frac{1}{2}\sum_{ij} a_{ij}\partial_{ij} u \in L^1_{\mathrm{loc}}\big([0,T]\times\mathbb{R}^d\big). \tag{33}$$

We say that u is a renormalized solution of (33) if, for any convex function $\beta : \mathbb{R} \to \mathbb{R}$ of class C^2, we have

$$\partial_t \beta(u) + \sum_i b_i \partial_i \beta(u) - \frac{1}{2}\sum_{ij} a_{ij}\partial_{ij}\beta(u) \leqslant c\beta'(u).$$

Equivalently the definition could be given in a partially conservative form:

$$\partial_t \beta(u) + \sum_i \partial_i\big(b_i\beta(u)\big) - \frac{1}{2}\sum_{ij} a_{ij}\partial_{ij}\beta(u) \leqslant c\beta'(u) + \left(\sum_i \partial_i b_i\right)\beta(u).$$

Recalling that a is non-negative definite and β is convex, it is simple to check that, if everything is smooth so that one can apply the standard chain rule, every solution of (33) is a renormalized solution. Indeed, in that case, one gets

$$\partial_t\beta(u)+\sum_i b_i\partial_i\beta(u)-\frac{1}{2}\sum_{ij}a_{ij}\partial_{ij}\beta(u)=c\beta'(u)-\frac{1}{2}\beta''(u)\sum_{ij}a_{ij}\partial_i u\partial_j u\leqslant c\beta'(u).$$

In our case, a solution of the Fokker–Planck equation is renormalized if

$$\partial_t\beta(u)+\sum_i\Bigl(b_i-\sum_j\partial_j a_{ij}\Bigr)\partial_i\beta(u)-\frac{1}{2}\sum_{ij}a_{ij}\partial_{ij}\beta(u)\leqslant\Bigl(\frac{1}{2}\sum_{ij}\partial_{ij}a_{ij}-\sum_i\partial_i b_i\Bigr)u\beta'(u),$$

or equivalently, writing everything in the partially conservative form,

$$\begin{aligned}&\partial_t\beta(u)+\sum_i\partial_i\Bigl(\Bigl(b_i-\sum_j\partial_j a_{ij}\Bigr)\beta(u)\Bigr)-\frac{1}{2}\sum_{ij}a_{ij}\partial_{ij}\beta(u)\\&\quad\leqslant\Bigl(\frac{1}{2}\sum_{ij}\partial_{ij}a_{ij}-\sum_i\partial_i b_i\Bigr)u\beta'(u)+\sum_i\partial_i\Bigl(b_i-\sum_j\partial_j a_{ij}\Bigr)\beta(u)\\&\quad=\Bigl(\sum_i\partial_i b_i-\frac{1}{2}\sum_{ij}\partial_{ij}a_{ij}\Bigr)\bigl(\beta(u)-u\beta'(u)\bigr)-\frac{1}{2}\Bigl(\sum_{ij}\partial_{ij}a_{ij}\Bigr)\beta(u).\end{aligned}$$

Now, since

$$\begin{aligned}\sum_{ij}a_{ij}\partial_{ij}\beta(u)&=\sum_{ij}\partial_j\bigl(a_{ij}\partial_i\beta(u)\bigr)-\sum_{ij}\partial_j a_{ij}\partial_i\beta(u)\\&=\sum_{ij}\partial_{ij}\bigl(a_{ij}\beta(u)\bigr)-2\sum_{ij}\partial_i\bigl((\partial_j a_{ij})\beta(u)\bigr)+\Bigl(\sum_{ij}\partial_{ij}a_{ij}\Bigr)\beta(u),\end{aligned}$$

the above expression can be simplified, and we obtain that a solution of the Fokker–Planck equation is renormalized if and only if

$$\begin{aligned}&\partial_t\beta(u)+\sum_i\partial_i\bigl(b_i\beta(u)\bigr)-\frac{1}{2}\sum_{ij}\partial_{ij}\bigl(a_{ij}\beta(u)\bigr)\\&\quad\leqslant\Bigl(\sum_i\partial_i b_i-\frac{1}{2}\sum_{ij}\partial_{ij}a_{ij}\Bigr)\bigl(\beta(u)-u\beta'(u)\bigr).\end{aligned}\tag{34}$$

It is not difficult to prove the following lemma.

Lemma 4.10. *Assume that there exist $p,q\in[1,\infty]$ such that*

$$\frac{a}{1+|x|^2}\in L^1\bigl([0,T],L^p\bigl(\mathbb{R}^d\bigr)\bigr),\qquad\frac{b}{1+|x|}\in L^1\bigl([0,T],L^q\bigl(\mathbb{R}^d\bigr)\bigr),$$

and that

$$\left(\sum_i \partial_i b_i - \frac{1}{2}\sum_{ij}\partial_{ij}a_{ij}\right)^- \in L^1_{\mathrm{loc}}\big([0,T]\times\mathbb{R}^d\big).$$

Setting $a, b = 0$ for $t < 0$, assume moreover that any solution $u \in \mathscr{L}$ of the Fokker–Planck equation in $(-\infty, T)\times\mathbb{R}^d$ is renormalized. Then the comparison principle holds in $\mathscr{L}$.

Proof. By the linearity of the equation, it suffices to prove that

$$u_0 \leqslant 0 \quad\Rightarrow\quad u(t)\leqslant 0 \quad \forall t\in[0,T].$$

Fix a non-negative cut-off function $\varphi \in C^\infty_{\mathrm{c}}(\mathbb{R}^d)$ with $\mathrm{supp}(\varphi)\subset B_2(0)$, and $\varphi = 1$ in $B_1(0)$, and take as renormalization function

$$\beta_\varepsilon(s) := \frac{1}{2}\big(\sqrt{s^2+\varepsilon^2}+s-\varepsilon\big)\in C^2(\mathbb{R}).$$

Notice that β_ε is convex and

$$\beta_\varepsilon(s)\to s^+ \quad \text{as } \varepsilon\to 0, \qquad \beta_\varepsilon(s) - s\beta_\varepsilon'(s)\in[-\varepsilon,0].$$

By (34), we know that

$$\partial_t\beta_\varepsilon(u)+\sum_i\partial_i\big(b_i\beta_\varepsilon(u)\big)-\frac{1}{2}\sum_{ij}\partial_{ij}\big(a_{ij}\beta_\varepsilon(u)\big)\leqslant\left(\sum_i\partial_i b_i-\frac{1}{2}\sum_{ij}\partial_{ij}a_{ij}\right)\big(\beta_\varepsilon(u)-u\beta_\varepsilon'(u)\big)$$

in the sense of distributions in $(-\infty,T)\times\mathbb{R}^d$. Using as test function $\varphi_R(x):=\varphi(\frac{x}{R})$ for $R\geqslant 1$, we get

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d}\varphi_R\beta_\varepsilon(u)\,dx &\leqslant \int_{\mathbb{R}^d}\left(\sum_i b_i(t)\partial_i\varphi_R+\frac{1}{2}\sum_{ij}a_{ij}(t)\partial_{ij}\varphi_R\right)\beta_\varepsilon(u)\,dx\\ &\quad+\int_{\mathbb{R}^d}\varphi_R\left(\sum_i\partial_i b_i(t)-\frac{1}{2}\sum_{ij}\partial_{ij}a_{ij}(t)\right)\big(\beta_\varepsilon(u)-u\beta_\varepsilon'(u)\big)\,dx.\end{aligned}$$

Observing that $|\beta_\varepsilon(u)|\leqslant|u|$, by Hölder inequality and the inequalities (32) we can bound the first integral in the right-hand side, uniformly with respect to ε, with

$$\begin{aligned}&\|\varphi\|_{C^2}\int_{\{|x|\geqslant R\}}\left(3\frac{|b(t,x)|}{1+|x|}+\frac{5}{2}\frac{|a(t,x)|}{(1+|x|^2)}\right)\big|u(t,x)\big|\,dx\\ &\quad\leqslant\|\varphi\|_{C^2}\left(3\left\|\frac{b(t)}{1+|x|}\right\|_{L^p(\{|x|\geqslant R\})}\big\|u(t)\big\|_{L^{p'}(\mathbb{R}^d)}+\frac{5}{2}\left\|\frac{a(t)}{1+|x|^2}\right\|_{L^q(\{|x|\geqslant R\})}\big\|u(t)\big\|_{L^{q'}(\mathbb{R}^d)}\right)\end{aligned}$$

(recall that $u \in \mathscr{L}$, and thus $u \in L^\infty([0,T], L^r(\mathbb{R}^d))$ for any $r \in [1,\infty]$), while the second integral is bounded by

$$\varepsilon \int\limits_{\{|x| \leqslant 2R\}} \left(\sum_i \partial_i b_i - \frac{1}{2} \sum_{ij} \partial_{ij} a_{ij} \right)^- dx.$$

Letting first $\varepsilon \to 0$ and then $R \to \infty$, we get

$$\frac{d}{dt} \int\limits_{\mathbb{R}^d} u^+ \, dx \leqslant 0$$

in the sense of distribution in $(-\infty, T)$. Since the function vanishes for negative times, we conclude $u^+ = 0$. $\square$

Now Theorem 4.8 is a direct consequence of the following proposition.

Proposition 4.11. *Let us assume that* $a : [0,T] \to \mathcal{S}_+(\mathbb{R}^d)$ *and* $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ *are such that*:

(i) $b \in L^1([0,T], BV_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^d))$, $\sum_i \partial_i b_i \in L^1_{\mathrm{loc}}([0,T] \times \mathbb{R}^d)$;
(ii) $a \in L^\infty([0,T], \mathcal{S}_+(\mathbb{R}^d))$.

Then any distributional solution $u \in L^\infty_{\mathrm{loc}}([0,T] \times \mathbb{R}^d)$ *of* (33) *is renormalized.*

Proof. We take η, a smooth convolution kernel in $\mathbb{R}^d$, and we mollify the equation with respect to the spatial variable obtaining

$$\partial_t u^\varepsilon + \sum_i b_i \partial_i u^\varepsilon - \frac{1}{2} \sum_{ij} a_{ij} \partial_{ij} u^\varepsilon = c * \eta_\varepsilon - r^\varepsilon, \tag{35}$$

where

$$r^\varepsilon := \sum_i (b_i \partial_i u) * \eta_\varepsilon - \sum_i b_i \partial_i (u * \eta_\varepsilon), \quad u^\varepsilon := u * \eta_\varepsilon.$$

By the smoothness of u^ε with respect to x, by (35) we have that $\partial_t u^\varepsilon \in L^1_{\mathrm{loc}}$. Thus by the standard chain rule in Sobolev spaces we get that u^ε is a renormalized solution, that is

$$\partial_t \beta(u^\varepsilon) + \sum_i b_i \partial_i \beta(u^\varepsilon) - \frac{1}{2} \sum_{ij} a_{ij} \partial_{ij} \beta(u^\varepsilon) \leqslant (c * \eta_\varepsilon - r^\varepsilon) \beta'(u^\varepsilon)$$

for any $\beta \in C^2(\mathbb{R})$ convex. Passing to the limit in the distributional sense as $\varepsilon \to 0$ in the above identity, the convergence of all the terms is trivial except for $r^\varepsilon \beta'(u^\varepsilon)$.

Let σ_η be any weak limit point of $r^\varepsilon \beta'(u^\varepsilon)$ in the sense of measures (such a cluster point exists since $r^\varepsilon \beta'(u^\varepsilon)$ is bounded in L^1_{loc}). Thus we get

$$\partial_t \beta(u) + \sum_i b_i \partial_i \beta(u) - \frac{1}{2} \sum_{ij} a_{ij} \partial_{ij} \beta(u) - c\beta'(u) \leqslant -\sigma_\eta \leqslant |\sigma_\eta|.$$

Since the left-hand side is independent of η, in order to conclude the proof it suffices to prove that $\bigwedge_\eta |\sigma_\eta| = 0$, where η varies in a dense countable set of convolution kernels. This fact is implicitly proved in [2, Theorem 34], see in particular Step 3 therein. □

4.3.2. Existence in $\mathscr{L}_+$

We can now prove an existence and uniqueness result in the class $\mathscr{L}_+$.

Theorem 4.12. *Let us assume that $a : [0,T] \times \mathbb{R}^d \to \mathcal{S}(\mathbb{R}^d)$ and $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ are bounded functions such that*

$$\left(\sum_i \partial_i b_i - \frac{1}{2} \sum_{ij} \partial_{ij} a_{ij} \right)^- \in L^1\big([0,T], L^\infty\big(\mathbb{R}^d\big)\big).$$

Then, for any $\mu_0 = \rho_0 \mathcal{L}^d \in \mathcal{M}_+(\mathbb{R}^d)$, with $\rho_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, there exists a solution of (2) *in $\mathscr{L}_+$. If moreover $b \in L^1([0,T], BV_{\mathrm{loc}}(\mathbb{R}^d))$, $\sum_i \partial_i b_i \in L^1_{\mathrm{loc}}([0,T] \times \mathbb{R}^d)$, and a is independent of x, then this solution turns out to be unique.*

Proof. *Existence*. It suffices to approximate the coefficients a and b locally uniformly with smooth uniformly bounded coefficients a^n and b^n such that $(\sum_i \partial_i b_i^n - \frac{1}{2} \sum_{ij} \partial_{ij} a_{ij}^n)^-$ is uniformly bounded in $L^1([0,T], L^\infty(\mathbb{R}^d))$. Indeed, if we now consider the approximate solutions $\mu_t^n = \rho_t^n \mathcal{L}^d \in \mathcal{M}_+(\mathbb{R}^d)$, we know that

$$\partial_t \rho_t^n + \sum_i \partial_i \big(b_i^n \rho_t^n\big) - \frac{1}{2} \sum_{ij} \partial_{ij} \big(a_{ij}^n \rho_t^n\big) = 0,$$

that is

$$\partial_t \rho_t^n - \frac{1}{2} a_{ij}^n \partial_{ij} \rho_t^n + \sum_i \left(b_i^n - \sum_j \partial_j a_{ij}^n \right) \partial_i \rho_t^n + \left(\sum_i \partial_i b_i^n - \frac{1}{2} \sum_{ij} \partial_{ij} a_{ij}^n \right) \rho_t^n = 0.$$

Using the Feynman–Kac's formula, we obtain the bound

$$\big\| \rho_t^n \big\|_{L^\infty(\mathbb{R}^d)} \leqslant \| \rho_0 \|_{L^\infty(\mathbb{R}^d)} e^{\int_0^t \| (\sum_i \partial_i b_i^n(s) - \frac{1}{2} \sum_{ij} \partial_{ij} a_{ij}^n(s))^- \|_{L^\infty(\mathbb{R}^d)}\, ds}.$$

So we see that the approximate solutions are non-negative and uniformly bounded in $L^1 \cap L^\infty$ (the bound in L^1 follows by the constancy of the map $t \mapsto \|\rho_t^n\|_{L^1}$ (observe that $\rho_t^n \geqslant 0$ and recall Remark 2.7)). Therefore, any weak limit is a solution of the PDE in $\mathscr{L}_+$.

Uniqueness. It follows by Theorem 4.8. □

5. Conclusions

Let us now combine the results proved in Sections 2 and 4 in order to get existence and uniqueness of SLF. The first theorem follows directly by Corollary 3.6 and Theorem 1.3, while the second is a consequence of Corollary 3.6 and Theorem 1.4.

Theorem 5.1. *Let us assume that* $a : [0,T] \times \mathbb{R}^d \to \mathcal{S}_+(\mathbb{R}^d)$ *and* $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ *are bounded functions such that*:

(i) $\sum_j \partial_j a_{ij} \in L^\infty([0,T] \times \mathbb{R}^d)$ *for* $i = 1, \dots, d$;
(ii) $\partial_t a_{ij} \in L^\infty([0,T] \times \mathbb{R}^d)$ *for* $i, j = 1, \dots, d$;
(iii) $(\sum_i \partial_i b_i - \frac{1}{2} \sum_{ij} \partial_{ij} a_{ij})^- \in L^\infty([0,T] \times \mathbb{R}^d)$;
(iv) $\langle \xi, a(t,x)\xi \rangle \geqslant \alpha |\xi|^2 \ \forall (t,x) \in [0,T] \times \mathbb{R}^d$, *for a certain* $\alpha > 0$;
(v) $\frac{a}{1+|x|^2} \in L^2([0,T] \times \mathbb{R}^d)$, $\frac{b}{1+|x|} \in L^2([0,T] \times \mathbb{R}^d)$.

Then there exists a unique SLF (*in the sense of Corollary* 3.6).

If moreover $(b^n, a^n) \to (b, a)$ *in* $L^1_{\mathrm{loc}}([0,T] \times \mathbb{R}^d)$ *and* $(\sum_i \partial_i b_i^n - \frac{1}{2} \sum_{ij} \partial_{ij} a_{ij}^n)^-$ *are uniformly bounded in* $L^1([0,T], L^\infty(\mathbb{R}^d))$, *then the Feynman–Kac formula implies* (ii) *of Theorem* 3.7 (*see the proof of Theorem* 4.12). *Thus we have stability of SLF.*

Theorem 5.2. *Let us assume that* $a : [0,T] \to \mathcal{S}(\mathbb{R}^d)$ *and* $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ *are bounded functions such that*:

(i) $b \in L^1([0,T], BV_{\mathrm{loc}}(\mathbb{R}^d))$, $\sum_i \partial_i b_i \in L^1_{\mathrm{loc}}([0,T] \times \mathbb{R}^d)$;
(ii) $(\sum_i \partial_i b_i)^- \in L^1([0,T], L^\infty(\mathbb{R}^d))$.

Then there exists a unique SLF (*in the sense of Corollary* 3.6).

If moreover $(b^n, a^n) \to (b, a)$ *in* $L^1_{\mathrm{loc}}([0,T] \times \mathbb{R}^d)$ *and* $(\sum_i \partial_i b_i^n - \frac{1}{2} \sum_{ij} \partial_{ij} a_{ij}^n)^-$ *are uniformly bounded in* $L^1([0,T], L^\infty(\mathbb{R}^d))$, *then the Feynman–Kac formula implies* (ii) *of Theorem* 3.7 (*see the proof of Theorem* 4.12). *Thus we have stability of SLF.*

In particular, by Corollary 3.9 and the Feynman–Kac formula (see the proof of Theorem 4.12), the following vanishing viscosity result for RLF holds:

Theorem 5.3. *Let us assume that* $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ *is bounded and*:

(i) $b \in L^1([0,T], BV_{\mathrm{loc}}(\mathbb{R}^d))$, $\sum_i \partial_i b_i \in L^1_{\mathrm{loc}}([0,T] \times \mathbb{R}^d)$;
(ii) $(\sum_i \partial_i b_i)^- \in L^1([0,T], L^\infty(\mathbb{R}^d))$.

Let $\{\nu_x^\varepsilon\}_{x \in \mathbb{R}^d}$ *be the unique SLF relative to* $(b, \varepsilon I)$, *with* $\varepsilon > 0$, *and* $\{\nu_x\}_{x \in \mathbb{R}^d}$ *be the RLF relative to* $(b, 0)$ (*which is uniquely determined* $\mathcal{L}^d$*-a.e. by the results in* [1]). *Then, as* $\varepsilon \to 0$,

$$\int_{\mathbb{R}^d} \nu_x^\varepsilon f(x)\,dx \rightharpoonup^* \int_{\mathbb{R}^d} \nu_x f(x)\,dx \quad \text{in } \mathcal{M}(\Gamma_T) \text{ for any } f \in C_c\big(\mathbb{R}^d\big).$$

We finally combine an important uniqueness result of Stroock and Varadhan (see Theorem 2.2) with the well-posedness results on Fokker–Planck of the previous section. By Theorem 2.2, Lemma 2.3 applied with $A = \mathbb{R}^d$ and Corollary 4.5, we have:

Theorem 5.4. *Let us assume that* $a : [0,T] \times \mathbb{R}^d \to \mathcal{S}_+(\mathbb{R}^d)$ *and* $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ *are bounded functions such that*:

(i) $\langle \xi, a(t,x)\xi \rangle \geqslant \alpha|\xi|^2$ $\forall (t,x) \in [0,T] \times \mathbb{R}^d$, *for a certain* $\alpha > 0$;
(ii) $|b(t,x) - b(s,y)| + \|a(t,x) - a(s,y)\| \leqslant C(|x-y|^\gamma + |t-s|^\gamma)$ $\forall (t,x), (s,y) \in [0,T] \times \mathbb{R}^d$, *for some* $\gamma \in (0,1]$, $C \geqslant 0$;
(iii) $\sum_j \partial_j a_{ij} \in L^\infty([0,T] \times \mathbb{R}^d)$ *for* $i = 1, \dots, d$, $(\sum_i \partial_i b_i - \frac{1}{2}\sum_{ij} \partial_{ij} a_{ij})^- \in L^\infty([0,T] \times \mathbb{R}^d)$;
(iv) $\frac{a}{1+|x|^2} \in L^2([0,T] \times \mathbb{R}^d)$, $\frac{b}{1+|x|} \in L^2([0,T] \times \mathbb{R}^d)$.

Then, there exists a unique martingale solution starting from x (*at time* 0) *for any* $x \in \mathbb{R}^d$.

We remark that this result is not interesting by itself, since it can be proved that the martingale problem starting from any $x \in \mathbb{R}^d$ at any initial time $s \in [0,T]$ is well-posed also under weaker regularity assumptions (see [18, Chapters 6 and 7]). We stated it just because we believe that it is an interesting example of how existence and uniqueness at the PDE level can be combined with a refined analysis at the level of the uniqueness of martingale solutions. It is indeed in this spirit that we generalize Theorem 2.2 in Appendix A, hoping that it could be useful for further analogous applications.

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Appendix A. A generalized uniqueness result for martingale solutions

Here we generalize Theorem 2.2, using the notation introduced in Section 3.1.

Proposition A.1. *For any* $(s,x) \in [0,T] \times \mathbb{R}^d$, *let* $C_{x,s}$ *be a subset of martingale solutions of the SDE starting from* x *at time* s, *and let us make the following assumptions*: *there exists a measure* $\mu_0 \in \mathcal{M}_+(\mathbb{R}^d)$ *such that*:

(i) $\forall s \in [0,T]$, $C_{x,s}$ *is convex for* μ_0*-a.e.* x;
(ii) $\forall s \in [0,T]$, $\forall t \in [s,T]$,

$$\text{for } \mu_0\text{-a.e. } x, \quad (e_t)_\# \nu^1_{x,s} = (e_t)_\# \nu^2_{x,s} \quad \forall \nu^1_{x,s}, \nu^2_{x,s} \in C_{x,s};$$

(iii) *for* μ_0*-a.e.* x, *for any* $\nu_x \in C_x := C_{x,0}$, *for* ν_x*-a.e.* γ,

$$\forall t \in [0,T], \quad \nu^{i,\gamma}_{x,\mathcal{F}_t} := \left(\nu^i_x\right)^\gamma_{\mathcal{F}_t} \in C_{\gamma(t),t},$$

where, with the above notation, we mean that the restriction of $\nu^{i,\gamma}_{x,\mathcal{F}_t}$ *to* $\Gamma^t_T := C([t,T],\mathbb{R}^d)$ *is a martingale solution starting from* $\gamma(t)$ *at time* t;

(iv) *the solution of* (2) *starting from* μ_0 *given by* $\mu_t := (e_t)_\# \int_{\mathbb{R}^d} \nu^1_x \, d\mu_0(x)$ *for a measurable selections* $\{\nu_x\}_{x\in\mathbb{R}^d}$ *with* $\nu_x \in C_x$ (*observe that* μ_t *does not depends on the choice of* $\nu_x \in C_x$ *by* (ii)), *satisfies* $\mu_t \ll \mu_0$ *for any* $t \in [0,T]$.

Then, given two measurable families of probability measures $\{\nu^1_x\}_{x\in\mathbb{R}^d}$ *and* $\{\nu^2_x\}_{x\in\mathbb{R}^d}$ *with* $\nu^1_x, \nu^2_x \in C_x$, $\nu^1_x = \nu^2_x$ *for* μ_0*-a.e.* x. *In particular, by standard measurable selection theorems* (*see for instance* [18, *Chapter* 12]), C_x *is a singleton for* μ_0*-a.e.* x.

Proof. Let $\{\nu^1_x\}_{x\in\mathbb{R}^d}$ and $\{\nu^2_x\}_{x\in\mathbb{R}^d}$ be two measurable families of probability measures with $\nu^1_x, \nu^2_x \in C_x$, and fix $0 < t_1 < \cdots < t_n \leqslant T$.

Claim. *For* μ_0*-a.e.* x, *for* ν^i_x*-a.e.* γ $(i = 1,2)$,

$$\nu^{i,\tilde\gamma}_{x,\mathcal{F}_{t_n}} \in C_{\tilde\gamma(t_n),t_n} \quad \textit{for } \nu^{i,\gamma}_{x,M^{t_1,\dots,t_n}}\textit{-a.e. } \tilde\gamma$$

where $\nu^{i,\gamma}_{x,M^{t_1,\dots,t_n}} := (\nu^i_x)^\gamma_{M^{t_1,\dots,t_n}}$.

This claim follows observing that, by assumption (iii), for μ_0-a.e. x there exists a subset $\Gamma_x \subset \Gamma_T$ such that $\nu^i_x(\Gamma_x) = 1$ and $\nu^{i,\gamma}_{x,\mathcal{F}_{t_n}} \in C_{\gamma(t_n),t_n}$ for any $\gamma \in \Gamma_x$. Thus, by (11) applied with $\nu := \nu^i_x$, $A := \Gamma_T$, $B := \Gamma_x$, and with $M^{t_1,\dots,t_n}$ in place of $\mathcal{F}_{t_n}$, one obtains

$$0 = \nu^i_x\big(\Gamma^c_x\big) = \int_{\Gamma_T} \nu^{i,\gamma}_{x,M^{t_1,\dots,t_n}}\big(\Gamma^c_x\big)\, d\nu^i_x(\gamma),$$

that is,

$$\nu^{i,\gamma}_{x,M^{t_1,\dots,t_n}}(\Gamma_x) = 1 \quad \text{for } \nu^i_x\text{-a.e. } \gamma.$$

This, together with assumption (iii), implies the claim.

By (13), $\nu^{i,\gamma}_{x,M^{t_1,\dots,t_n}}$ is concentrated on the set $\{\tilde\gamma \mid \tilde\gamma(t_n) = \gamma(t_n)\}$, and so, by the claim above, we get

$$\nu^{i,\tilde\gamma}_{x,\mathcal{F}_{t_n}} \in C_{\gamma(t_n),t_n} \quad \text{for } \nu^{i,\gamma}_{x,M^{t_1,\dots,t_n}}\text{-a.e. } \tilde\gamma.$$

Let $A \subset \mathbb{R}^d$ be such that $\mu_0(A^c) = 0$ and assumption (i) is true for any $x \in A$. By assumption (iv), we have $\mu_{t_n}(A^c) = 0 = \int_{\mathbb{R}^d\times\Gamma_T} 1_{A^c}(\gamma(t_n))\, d\nu^i_x(\gamma)\, d\mu_0(x)$, that is

$$\text{for } \mu_0\text{-a.e. } x, \quad \gamma(t_n) \in A \quad \text{for } \nu^i_x\text{-a.e } \gamma. \tag{A.1}$$

Thus, for μ_0-a.e. x, $C_{\gamma(t_n),t_n}$ is convex for ν^i_x-a.e γ, and so, by (14) applied with ν^i_x, we obtain that

$$\text{for } \mu_0\text{-a.e. } x, \quad \nu^{i,\gamma}_{x,M^{t_1,\dots,t_n}} \in C_{\gamma(t_n),t_n} \quad \text{for } \nu^i_x\text{-a.e. } \gamma \tag{A.2}$$

(where, with the above notation, we again mean that the restriction of $\nu^{i,\gamma}_{x,M^{t_1,\dots,t_n}}$ to $\Gamma_T^{t_n}$ is a martingale solution starting from $\gamma(t_n)$ at time t_n). We now want to prove that, for all $n \geqslant 1$, $0 < t_1 < \dots < t_n \leqslant T$, we have that, for μ_0-a.e. x,

$$\int_{\Gamma_T} f_1\big(e_{t_1}(\gamma)\big)\dots f_n\big(e_{t_n}(\gamma)\big)\,d\nu^1_x(\gamma) = \int_{\Gamma_T} f_1\big(e_{t_1}(\gamma)\big)\dots f_n\big(e_{t_n}(\gamma)\big)\,d\nu^2_x(\gamma) \tag{A.3}$$

for any $f_i \in C_c(\mathbb{R}^d)$. We observe that (A.3) is true for $n = 1$ by assumption (ii). We want to prove it for any n by induction. Let us assume (A.3) true for $n-1$, and let us prove it for n. We want to show that

$$\int_{\Gamma_T} f_1\big(e_{t_1}(\gamma)\big)\dots f_n\big(e_{t_n}(\gamma)\big)\,d\nu^1_x(\gamma) = \int_{\Gamma_T} f_1\big(e_{t_1}(\gamma)\big)\dots f_n\big(e_{t_n}(\gamma)\big)\,d\nu^2_x(\gamma),$$

which can be written also as

$$\mathbb{E}^{\nu^1_x}\big[f_1(e_{t_1})\dots f_n(e_{t_n})\big] = \mathbb{E}^{\nu^2_x}\big[f_1(e_{t_1})\dots f_n(e_{t_n})\big],$$

where $\mathbb{E}^\nu := \int_{\Gamma_T} d\nu$. Now we observe that, for $i = 1, 2$,

$$\begin{aligned}
\mathbb{E}^{\nu^i_x}\big[f_1(e_{t_1})\dots f_n(e_{t_n})\big] &= \mathbb{E}^{\nu^i_x}\big[\mathbb{E}^{\nu^i_x}\big[f_1(e_{t_1})\dots f_n(e_{t_n}) \bigm| M^{t_1,\dots,t_{n-1}}\big]\big] \\
&= \mathbb{E}^{\nu^i_x}\big[f_1(e_{t_1})\dots f_{n-1}(e_{t_{n-1}})\mathbb{E}^{\nu^i_x}\big[f_n(e_{t_n}) \mid M^{t_1,\dots,t_{n-1}}\big]\big] \\
&= \mathbb{E}^{\nu^i_x}\big[f_1(e_{t_1})\dots f_{n-1}(e_{t_{n-1}})\psi^i_x(e_{t_1},\dots,e_{t_{n-1}})\big],
\end{aligned}$$

where $\psi^i_x(e_{t_1},\dots,e_{t_{n-1}}) := \mathbb{E}^{\nu^i_x}[f_n(e_{t_n}) \mid M^{t_1,\dots,t_{n-1}}]$. Let $\phi \in C_c(\mathbb{R}^d)$, and let us prove that

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mathbb{E}^{\nu^1_x}\big[f_1(e_{t_1})\dots f_{n-1}(e_{t_{n-1}})\psi^1_x(e_{t_1},\dots,e_{t_{n-1}})\big]\phi(x)\,d\mu_0(x) \\
&\quad = \int_{\mathbb{R}^d} \mathbb{E}^{\nu^2_x}\big[f_1(e_{t_1})\dots f_{n-1}(e_{t_{n-1}})\psi^2_x(e_{t_1},\dots,e_{t_{n-1}})\big]\phi(x)\,d\mu_0(x).
\end{aligned} \tag{A.4}$$

Let $B \subset \mathbb{R}^d$ be such that $\mu_0(B^c) = 0$ and assumption (ii′) is true for any $x \in B$. By assumption (iv), we also have $\mu_{t_{n-1}}(B^c) = 0 = \int_{\mathbb{R}^d\times\Gamma_T} 1_{B^c}(e_{t_{n-1}}(\gamma))\,d\nu^i_x(\gamma)\,d\mu_0(x)$, that is

$$\text{for } \mu_0\text{-a.e. } x, \quad \gamma(t_{n-1}) \in B \quad \text{for } \nu^i_x\text{-a.e. } \gamma. \tag{A.5}$$

Let us consider $\nu^{i,\gamma}_{x,M^{t_1,\dots,t_{n-1}}}$. By (A.2),

$$\text{for } \mu_0\text{-a.e. } x, \quad \nu^{i,\gamma}_{x,M^{t_1,\dots,t_{n-1}}} \in C_{\gamma(t_{n-1}),t_{n-1}} \quad \text{for } \nu^i_x\text{-a.e. } \gamma,$$

and, combining this with (A.5), we obtain

$$\text{for } \mu_0\text{-a.e. } x, \quad \nu^{i,\gamma}_{x,M^{t_1,\dots,t_{n-1}}} \in C_{\gamma(t_{n-1}),t_{n-1}} \quad \text{and} \quad \gamma(t_{n-1}) \in B \quad \text{for } \nu^i_x\text{-a.e. } \gamma.$$

By assumption (ii) applied with $t = t_n$, this implies that

$$\text{for } \mu_0\text{-a.e. } x, \quad (e_{t_n})_\# \nu^{1,\gamma}_{x,M^{t_1,\dots,t_{n-1}}} = (e_{t_n})_\# \nu^{2,\gamma}_{x,M^{t_1,\dots,t_{n-1}}} \quad \text{for } \nu^i_x\text{-a.e. } \gamma,$$

which give us that

$$\text{for } \mu_0\text{-a.e. } x, \quad \psi^1_x(e_{t_1},\dots,e_{t_{n-1}}) = \psi^2_x(e_{t_1},\dots,e_{t_{n-1}}) \quad \text{for } \nu^i_x\text{-a.e. } \gamma. \tag{A.6}$$

Thus we get

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mathbb{E}^{\nu^1_x}\big[f_1(e_{t_1})\dots f_{n-1}(e_{t_{n-1}})\psi^1_x(e_{t_1},\dots,e_{t_{n-1}})\big]\phi(x)\,d\mu_0(x) \\
&\quad= \int_{\mathbb{R}^d} \mathbb{E}^{\nu^2_x}\big[f_1(e_{t_1})\dots f_{n-1}(e_{t_{n-1}})\psi^1_x(e_{t_1},\dots,e_{t_{n-1}})\big]\phi(x)\,d\mu_0(x) \\
&\quad\overset{(A.6)}{=} \int_{\mathbb{R}^d} \mathbb{E}^{\nu^2_x}\big[f_1(e_{t_1})\dots f_{n-1}(e_{t_{n-1}})\psi^2_x(e_{t_1},\dots,e_{t_{n-1}})\big]\phi(x)\,d\mu_0(x),
\end{aligned}$$

where the first equality in the above equation follows by the inductive hypothesis. Now, by (A.4) and the arbitrariness of ϕ and of f_j, with $j = 1,\dots,n$, we obtain that, for all $n \geqslant 1$, $0 < t_1 < \dots < t_n \leqslant T$, we have

$$\text{for } \mu_0\text{-a.e. } x, \quad (e_{t_1},\dots,e_{t_n})_\#\nu_x = (e_{t_1},\dots,e_{t_n})_\#\tilde{\nu}_x \quad \forall t_1,\dots,t_n \in [0,T].$$

Considering only rational times, we get that there exists a subset $D \subset \mathbb{R}^d$, with $\mu_0(D^c) = 0$, such that, for any $x \in D$,

$$(e_{t_1},\dots,e_{t_n})_\#\nu_x = (e_{t_1},\dots,e_{t_n})_\#\tilde{\nu}_x \quad \text{for any } t_1,\dots,t_n \in [0,T] \cap \mathbb{Q}.$$

By continuity, this implies that, for any $x \in D$, $\nu_x = \tilde{\nu}_x$, as wanted. □

The above result apply, for example, in the case when $C_{x,s}$ denotes the set of all martingale solutions starting from x. In particular, we remark that, by the above proof, one obtains the well-known fact that, if ν_x is a martingale solution starting from x (at time 0), then, for any $0 \leqslant t_1 \leqslant \dots \leqslant t_n \leqslant T$, $\nu^\gamma_{x,M^{t_1,\dots,t_n}}$ is a martingale solution starting from $\gamma(t_n)$ at time t_n. More generally, since martingale solutions are closed by convex combination, is μ is a probability measure on $\mathbb{R}^d$, the average $\int_{\mathbb{R}^d} \nu^\gamma_{x,M^{t_1,\dots,t_n}}\,d\mu(x)$ is a martingale solution starting from $\gamma(t_n)$ at time t_n.

Observe that assumption (iv) in the above theorem was necessary only to deduce, from a μ_0-a.e. assumption, a μ_t-a.e. property. Thus, the above proof give us the following result:

Proposition A.2. *For any $(s,x) \in [0,T] \times \mathbb{R}^d$, let $C_{x,s}$ be a convex subset of martingale solutions of the SDE starting from x at time s, and let us make the following assumption: there exists a measure $\mu_0 \in \mathcal{M}_+(\mathbb{R}^d)$ such that:*

(i) $\forall t \in [0,T]$, *for μ_0-a.e. x,*

$$(e_t)_\# \nu_x^1 = (e_t)_\# \nu_x^2 \quad \forall \nu_x^1, \nu_x^2 \in C_x := C_{x,0}.$$

If (i) *holds, we can define $\mu_t := (e_t)_\# \int_{\mathbb{R}^d} \nu_x \, d\mu_0(x)$ for a measurable selections $\{\nu_x\}_{x \in \mathbb{R}^d}$ with $\nu_x \in C_x$, and this definition does not depends on the choice of $\nu_x \in C_x$. We now assume that:*

(i′) $\forall s \in [0,T]$, $\forall t \in [s,T]$, *for μ_s-a.e. x,*

$$(e_t)_\# \nu_{x,s}^1 = (e_t)_\# \nu_{x,s}^2 \quad \forall \nu_{x,s}^1, \nu_{x,s}^2 \in C_{x,s};$$

(ii) $\forall s \in [0,T]$, *$C_{x,s}$ is convex for μ_s-a.e. x;*
(iii) *for μ_0-a.e. x, for any $\nu_x \in C_x$, for ν_x-a.e. γ,*

$$\forall t \in [0,T], \quad \nu_{x,\mathcal{F}_t}^{i,\gamma} := \left(\nu_x^i\right)_{\mathcal{F}_t}^{\gamma} \in C_{\gamma(t),t},$$

where, with the above notation, we mean that the restriction of $\nu_{x,\mathcal{F}_t}^{i,\gamma}$ to Γ_T^t is a martingale solution starting from $\gamma(t)$ at time t.

Then, given two measurable families of probability measures $\{\nu_x^1\}_{x \in \mathbb{R}^d}$ and $\{\nu_x^2\}_{x \in \mathbb{R}^d}$ with $\nu_x^1, \nu_x^2 \in C_x$, $\nu_x^1 = \nu_x^2$ for μ_0-a.e. x. In particular, by standard measurable selection theorems (*see for instance* [18, *Chapter* 12]), *C_x is a singleton for μ_0-a.e. x.*

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Contractions with rank one defect operators and truncated CMV matrices

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Dedicated to the memory of Moshe Livšic, an outstanding human being and great mathematician

Abstract

The main issue we address in the present paper are the new models for completely nonunitary contractions with rank one defect operators acting on some Hilbert space of dimension $N \leqslant \infty$. These models complement nicely the well-known models of Livšic and Sz.-Nagy–Foias. We show that each such operator acting on some finite-dimensional (respectively, separable infinite-dimensional Hilbert space) is unitarily equivalent to some finite (respectively semi-infinite) truncated CMV matrix obtained from the "full" CMV matrix by deleting the first row and the first column, and acting in $\mathbb{C}^N$ (respectively $\ell^2(\mathbb{N})$). This result can be viewed as a nonunitary version of the famous characterization of unitary operators with a simple spectrum due to Cantero, Moral and Velázquez, as well as an analog for contraction operators of the result from [Yu. Arlinskiĭ, E. Tsekanovskiĭ, Non-self-adjoint Jacobi matrices with a rank-one imaginary part, J. Funct. Anal. 241 (2006) 383–438] concerning dissipative non-self-adjoint operators with a rank one imaginary part. It is shown that another functional model for contractions with rank one defect operators takes the form of the compression $f(\zeta) \to P_{\mathcal{K}}(\zeta f(\zeta))$ on the Hilbert space $L^2(\mathbb{T}, d\mu)$ with a probability measure μ onto the subspace $\mathcal{K} = L^2(\mathbb{T}, d\mu) \ominus \mathbb{C}$. The relationship between characteristic functions of sub-matrices of the truncated CMV matrix with rank one defect operators and the corresponding Schur iterates is established. We develop direct and inverse spectral analysis for finite and semi-infinite truncated CMV matrices. In particular, we study the problem of reconstruction of such matrices from their spectrum or the mixed

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spectral data involving Schur parameters. It is pointed out that if the mixed spectral data contains zero eigenvalue, then no solution, unique solution or infinitely many solutions may occur in the inverse problem for truncated CMV matrices. The uniqueness theorem for recovered truncated CMV matrix from the given mixed spectral data is established. In this part the paper is closely related to the results of Hochstadt and Gesztesy–Simon obtained for finite self-adjoint Jacobi matrices.

Keywords: Contractions; CMV matrices; Truncated CMV matrices; Functional models

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Contents

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1. Introduction

It is well known [2] that every self-adjoint or unitary operator with a simple spectrum acting on some separable Hilbert space is unitarily equivalent to the operator of multiplication by the independent variable on the Hilbert space $L^2(\mathbb{R}, d\mu)$ or $L^2(\mathbb{T}, d\mu)$, respectively, where $d\mu$ is a probability measure on the real line $\mathbb{R}$ or on the unit circle $\mathbb{T} = \{\zeta \in \mathbb{C}\colon |\zeta| = 1\}$. The matrix representation of self-adjoint operators with simple spectrum was established for the first time by Stone [1]. He proved that every self-adjoint operator with a simple spectrum is unitarily equivalent to a certain Jacobi (tri-diagonal) matrix of the form

$$J = \begin{pmatrix} b_1 & a_1 & 0 & 0 & 0 & \cdot & \cdot \\ a_1 & b_2 & a_2 & 0 & 0 & \cdot & \cdot \\ 0 & a_2 & b_3 & a_3 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \tag{1.1}$$

where $a_k > 0$, and b_k are real numbers for all $k \in \mathbb{N}$. The non-self-adjoint version of the Stone theorem has been recently obtained in [4] for dissipative non-self-adjoint operators with rank one imaginary part. It turned out that the matrix representation of such operators is a non-self-adjoint Jacobi matrix of the form (1.1) with only nonreal first entry b_1 satisfying $\operatorname{Im} b_1 > 0$.

The problem of the canonical matrix representation of a unitary operator with a simple spectrum has been recently solved by M. Cantero, L. Moral and L. Velázquez in [13]. They introduced and studied five-diagonal unitary matrices of the form

$$\mathcal{C} = \mathcal{C}\big(\{\alpha_n\}\big) = \begin{pmatrix} \bar{\alpha}_0 & \bar{\alpha}_1\rho_0 & \rho_1\rho_0 & 0 & 0 & \dots \\ \rho_0 & -\bar{\alpha}_1\alpha_0 & -\rho_1\alpha_0 & 0 & 0 & \dots \\ 0 & \bar{\alpha}_2\rho_1 & -\bar{\alpha}_2\alpha_1 & \bar{\alpha}_3\rho_2 & \rho_3\rho_2 & \dots \\ 0 & \rho_2\rho_1 & -\rho_2\alpha_1 & -\bar{\alpha}_3\alpha_2 & -\rho_3\alpha_2 & \dots \\ 0 & 0 & 0 & \bar{\alpha}_4\rho_3 & -\bar{\alpha}_4\alpha_3 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}. \tag{1.2}$$

Such matrix appears as a matrix representation of the unitary operator $(Uf)(\zeta) = \zeta f(\zeta)$ in $L_2(\mathbb{T}, d\mu)$ with respect to the orthonormal system $\{\chi_n\}$ obtained by orthonormalization of the sequence

$$\big\{1, \zeta, \zeta^{-1}, \zeta^2, \zeta^{-2}, \dots\big\}.$$

The so called Schur parameters or Verblunsky coefficients $\{\alpha_n\}$, $|\alpha_n| < 1$, arise in the Szegő recurrence formula

$$\zeta \Phi_n(\zeta) = \Phi_{n+1}(\zeta) + \bar{\alpha}_n \zeta^n \overline{\Phi_n(1/\bar{\zeta})}, \quad n = 0, 1, \dots$$

for monic orthogonal with respect to $d\mu$ polynomials $\{\Phi_n\}$, and $\rho_n := \sqrt{1 - |\alpha_n|^2}$. The matrices $\mathcal{C}(\{\alpha_n\})$ are called the *CMV matrices*. The spectral analysis of unitary CMV matrices has recently attracted much attention, and we refer on this matter to the papers [13,14,23,24,39–41].

As pointed out by Simon in a recent paper [41], the actual history of CMV matrices is more involved as it started in 1991 with Bunse-Gerstner and Elsner [12], and then with Watkins in 1993 [44], before Cantero, Moral, and Velázquez (CMV) re-discovered them in 2003. In a context different from orthogonal polynomials on the unit circle, Bourget, Howland, and Joye [8] introduced a set of doubly infinite matrices with three sets of parameters which for special choices of the parameters reduces to two-sided CMV matrices on $\ell^2(\mathbb{Z})$.

The spectral theory of non-self-adjoint and nonunitary operators and their models is based on the concept of *characteristic function* of the corresponding operator or the operator colligation [6,10,11,29–36,42].

In this paper we employ the Sz.-Nagy–Foias theory [42] and the Brodskiĭ–Livšic unitary colligations approach [10] to the spectral analysis of contractions acting on Hilbert spaces. The corresponding characteristic function belongs to the Schur class of operator-valued functions holomorphic in the open unit disk $\mathbb{D}$. By Sz.-Nagy–Foias theorem [42, Proposition VI.2.1] each completely nonunitary contraction T with rank one defect operators $D_T = (I - T^*T)^{1/2}$ and $D_{T^*} = (I - TT^*)^{1/2}$ (shortly, with rank one *defects*) is unitarily equivalent to the operator (functional model) of the form

$$\begin{aligned}\mathfrak{H}_\Theta &= \big(H^2 \oplus \operatorname{clos} \Delta L^2(\mathbb{T})\big) \ominus \big\{\Theta u \oplus \Delta u\colon u \in H^2\big\} \\ &= \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \colon f \in H^2,\ g \in \operatorname{clos} \Delta L^2(\mathbb{T}),\ P_{H^2}(\overline{\Theta} f + \Delta g) = 0 \right\},\end{aligned}$$

$$\mathfrak{T}_\Theta \begin{pmatrix} f \\ g \end{pmatrix} = P_{\mathfrak{H}_\Theta} \zeta \begin{pmatrix} f \\ g \end{pmatrix}, \qquad \mathfrak{T}^*_\Theta \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \bar{\zeta}(f - f(0)) \\ \bar{\zeta} g \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} \in \mathfrak{H}_\Theta,$$

where H^2 is the Hardy space,

$$\Theta = \Theta_T(z) = \big(-T + z D_{T^*}(I - zT^*)^{-1} D_T\big) \upharpoonright \mathfrak{D}_T$$

is the characteristic function of T, $\Delta^2 = 1 - |\Theta|^2$, P_{H^2} is the orthogonal projection onto H^2 in $L^2(\mathbb{T})$, and $P_{\mathfrak{H}_\Theta}$ is the orthogonal projection onto the model space $\mathfrak{H}_\Theta$.

We obtain a new functional model that complements the above mentioned Sz.-Nagy–Foias functional model, and show that every completely nonunitary contraction T with rank one defects is unitarily equivalent to the compression $f(\zeta) \to P_{\mathcal{K}}(\zeta f(\zeta))$ on the Hilbert space $L^2(\mathbb{T}, d\mu)$ with a probability measure μ onto subspace $\mathcal{K} = L^2(\mathbb{T}, d\mu) \ominus \mathbb{C}$.

We study the so called *truncated* CMV matrix $\mathcal{T}$ obtained from the "full" CMV matrix $\mathcal{C} = \mathcal{C}(\{\alpha_n\})$ (1.2) by deleting the first row and the first column:

$$\mathcal{T} = \mathcal{T}\big(\{\alpha_n\}\big) = \begin{pmatrix} -\bar{\alpha}_1 \alpha_0 & -\rho_1 \alpha_0 & 0 & 0 & \dots \\ \bar{\alpha}_2 \rho_1 & -\bar{\alpha}_2 \alpha_1 & \bar{\alpha}_3 \rho_2 & \rho_3 \rho_2 & \dots \\ \rho_2 \rho_1 & -\rho_2 \alpha_1 & -\bar{\alpha}_3 \alpha_2 & -\rho_3 \alpha_2 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

In the semi-infinite case $\mathcal{T}$ takes on the block-matrix form (see Section 4.3)

$$\mathcal{T} = \begin{pmatrix} \mathcal{B}_1 & \mathcal{C}_1 & 0 & 0 & 0 & \cdot & \cdot \\ \mathcal{A}_1 & \mathcal{B}_2 & \mathcal{C}_2 & 0 & 0 & \cdot & \cdot \\ 0 & \mathcal{A}_2 & \mathcal{B}_3 & \mathcal{C}_3 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

It turned out that the truncated CMV matrix $\mathcal{T}(\{\alpha_n\})$ is a contraction with rank one defects, and the Sz.-Nagy–Foias characteristic function agrees with the Schur function which has $\{\alpha\}$ as its Schur parameters. Moreover, we show that the sub-matrix $\mathcal{T}^{(k)}(\{\alpha_n\})$ obtained from $\mathcal{T}(\{\alpha_n\})$ by deleting the first k rows and columns is also a contraction with rank one defects, and its characteristic function agrees with the well-known kth Schur iterate

$$f_k(z) = \frac{f_{k-1}(z) - \alpha_{k-1}}{z(1 - \bar{\alpha}_{k-1} f_{k-1}(z))}, \qquad f_0(z) = f(z).$$

This relation is an analog of the corresponding relation between the m-function of a Jacobi matrix and the m-function of its sub-matrix (cf. [22]).

Our main result states that an arbitrary completely nonunitary contraction T with rank one defects is unitarily equivalent to any operator from the one-parameter family $\mathcal{T}(\{e^{it}\alpha_n\})$, where $\{\alpha_n\}$ are the Schur parameters of the Sz.-Nagy–Foias characteristic function of T. We develop di-

rect and inverse spectral analysis for finite and semi-infinite truncated CMV matrices. It is shown that given an arbitrary set of N not necessarily distinct numbers from $\mathbb{D}$ there is a one-parameter family of unitarily equivalent $N \times N$ truncated CMV matrices having those numbers as the eigenvalues counting algebraic multiplicity. We prove the uniqueness of $N \times N$ truncated CMV matrix $\mathcal{T}$ with given not necessarily distinct eigenvalues $z_1, \ldots, z_r$ and given first $N - r + 1$ Schur parameters $\alpha_0(\mathcal{T}), \ldots, \alpha_{N-r}(\mathcal{T})$. This result on inverse spectral analysis of finite truncated CMV matrices is an analog of the Hochstadt [26] *and Gesztesy–Simon* [22] *uniqueness theorems for finite self-adjoint Jacobi matrices as well as for established in* [4] *uniqueness theorem for finite non-self-adjoint Jacobi matrices with rank one imaginary part. We obtain the existence of $N \times N$ truncated CMV matrix $\mathcal{T}$ when its eigenvalues $z_1, \ldots, z_m$ and the last Schur parameters $\alpha_m(\mathcal{T}), \ldots, \alpha_N(\mathcal{T})$ are known.*

Here is a summary of the rest of the paper. In Sections 2 and 3 we discuss some basics from the Sz.-Nagy–Foias theory and the unitary colligations with the focus upon the characteristic function and its properties. Section 4 provides a brief overview of the theory of orthogonal polynomials on the unit circle and CMV matrices. The main results concerning truncated CMV matrices and the models of completely nonunitary contractions with rank one defects are presented in Sections 5 and 6. The final Section 7 deals with the inverse spectral analysis for truncated CMV matrices.

2. Contractions, unitary colligations, and their characteristic functions

2.1. Contractions and the Sz.-Nagy–Foias characteristic functions

Let H be a separable Hilbert space with the inner product $(\cdot,\cdot)$. A bounded linear operator T in H is called a *contraction* if $\|T\| \leqslant 1$ (for the basic properties of contractions see [42, Chapter I]). If T is a contraction then the operators

$$D_T := (I - T^*T)^{1/2}, \qquad D_{T^*} := (I - TT^*)^{1/2}$$

are called the *defect operators* of T or, shortly, *defects*, and the subspaces $\mathfrak{D}_T = \overline{\operatorname{ran}} D_T$, $\mathfrak{D}_{T^*} = \overline{\operatorname{ran}} D_{T^*}$ the *defect subspaces* of T. The dimensions $\dim \mathfrak{D}_T$, $\dim \mathfrak{D}_{T^*}$ are known as the *defect numbers* of T. Given a pair of numbers $n, n^* = 0, 1, \ldots, \infty$ it is easy to construct a contraction with $n = \dim \mathfrak{D}_T$, $n^* = \dim \mathfrak{D}_{T^*}$. Each contraction T acting on a finite-dimensional Hilbert space has equal defect numbers: $n = n^*$.

The defect operators satisfy the following intertwining relations:

$$TD_T = D_{T^*}T, \qquad T^*D_{T^*} = D_T T^*, \tag{2.1}$$

and the block-operators

$$\begin{pmatrix} -T^* & D_T \\ D_{T^*} & T \end{pmatrix} : \begin{pmatrix} \mathfrak{D}_{T^*} \\ H \end{pmatrix} \to \begin{pmatrix} \mathfrak{D}_T \\ H \end{pmatrix}, \qquad \begin{pmatrix} -T & D_{T^*} \\ D_T & T^* \end{pmatrix} : \begin{pmatrix} \mathfrak{D}_T \\ H \end{pmatrix} \to \begin{pmatrix} \mathfrak{D}_{T^*} \\ H \end{pmatrix}$$

are unitary operators in the corresponding orthogonal sums of the spaces. It follows from (2.1) that $T\mathfrak{D}_T \subset \mathfrak{D}_{T^*}$, $T^*\mathfrak{D}_{T^*} \subset \mathfrak{D}_T$, and $T(\ker D_T) = \ker D_{T^*}$, $T^*(\ker D_{T^*}) = \ker D_T$. Moreover, $T \restriction \ker D_T$ and $T^* \restriction \ker D_{T^*}$ are isometric operators. It follows that T is a *quasi-unitary extension* [29] of the isometric operator $V = T \restriction \ker D_T$ (for the definition see Section 6.2).

A contraction T is called *completely nonunitary* if there is no nontrivial reducing subspace of T, on which T generates a unitary operator. One of the fundamental results of the contractions theory [42, Theorem I.3.2] reads that, given a contraction T in H, there is a canonical orthogonal decomposition

$$H = H_0 \oplus H_1, \qquad T = T_0 \oplus T_1, \quad T_j = T \restriction H_j,\ j = 0, 1,$$

where H_0 and H_1 reduce T, T_0 is a completely nonunitary contraction, and T_1 is a unitary operator. Moreover,

$$H_1 = \Bigl(\bigcap_{n \geqslant 1} \ker D_{T^n}\Bigr) \cap \Bigl(\bigcap_{n \geqslant 1} \ker D_{T^{*n}}\Bigr),$$

so,

$$\begin{aligned} &T \text{ is completely nonunitary} \\ \Leftrightarrow\quad &\Bigl(\bigcap_{n \geqslant 1} \ker D_{T^n}\Bigr) \cap \Bigl(\bigcap_{n \geqslant 1} \ker D_{T^{*n}}\Bigr) = \{0\}. \end{aligned} \tag{2.2}$$

Clearly,

$$\begin{aligned} &\bigcap_{n \geqslant 1} \ker D_{T^n} = H \ominus \overline{\operatorname{span}}\bigl\{T^{*n} D_T H,\ n = 0, 1, \ldots\bigr\}, \\ &\bigcap_{n \geqslant 1} \ker D_{T^{*n}} = H \ominus \overline{\operatorname{span}}\bigl\{T^{n} D_{T^*} H,\ n = 0, 1, \ldots\bigr\}. \end{aligned} \tag{2.3}$$

Let V be an isometry in H. A subspace Ω in H is called wandering for V if $V^p\Omega \perp V^q\Omega$ for all $p, q \in \mathbb{Z}_+$, $p \neq q$. Since V is an isometry, the latter is equivalent to $V^n\Omega \perp \Omega$ for all $n \in \mathbb{N}$. If $H = \bigoplus_{n=0}^{\infty} V^n\Omega$, then V is called a unilateral shift and Ω is called the generating subspace. The dimension of Ω is called the multiplicity of the unilateral shift V. It is well known [42, Theorem I.1.1] that V is a unilateral shift if and only if $\bigcap_{n=0}^{\infty} V^n H = \{0\}$. Clearly, if an isometry V is the unilateral shift in H, then $\Omega = H \ominus VH$ is the generating subspace for V.

Given a contraction T in H and a subspace $\mathfrak{H} \subset H$, the unilateral shift $V : \mathfrak{H} \to \mathfrak{H}$ is said to be *contained in* T, if $\mathfrak{H}$ is invariant for T, and $T \restriction \mathfrak{H} = V$ [15]. The subspaces $\bigcap_{n \geqslant 1} \ker D_{T^n}$ and $\bigcap_{n \geqslant 1} \ker D_{T^{*n}}$ are invariant for T and T^*, respectively, and the operators $V_T := T \restriction \bigcap_{n \geqslant 1} \ker D_{T^n}$ and $V_{T^*} := T^* \restriction \bigcap_{n \geqslant 1} \ker D_{T^{*n}}$ are unilateral shifts. Moreover, V_T and V_{T^*} are the maximal unilateral shifts contained in T and T^*. The multiplicities of the shifts V_T and V_{T^*} do not exceed the defect numbers $\dim \mathfrak{D}_{T^*}$ and $\dim \mathfrak{D}_T$, respectively [17]. If T is a completely nonunitary contraction with rank one defects, then (see [15], [17, Theorem 1.7])

$$\begin{aligned} &T \text{ does not contain the unilateral shift} \\ \Leftrightarrow\quad &T^* \text{ does not contain the unilateral shift} \\ \Leftrightarrow\quad &\bigcap_{n \geqslant 1} \ker D_{T^n} = \{0\} \quad \Leftrightarrow \quad \bigcap_{n \geqslant 1} \ker D_{T^{*n}} = \{0\}. \end{aligned} \tag{2.4}$$

The function (see [42, Chapter VI])

$$\Theta_T(z) = \bigl(-T + zD_{T^*}(I - zT^*)^{-1}D_T\bigr)\restriction\mathfrak{D}_T$$

is known as the *characteristic function* of the Sz.-Nagy–Foias type of a contraction T. This function belongs to the *Schur class* $\mathbf{S}(\mathfrak{D}_T, \mathfrak{D}_{T^*})$ of $\mathcal{L}(\mathfrak{D}_T, \mathfrak{D}_{T^*})$-valued holomorphic in the unit disk $\mathbb{D}$ operator-functions, i.e., $\|\Theta_T(z)\| \leqslant 1$ for $z \in \mathbb{D}$. Moreover, the function Θ_T satisfies the condition $\|\Theta_T(0)f\| < \|f\|$ for all $f \in \mathfrak{D}_T \setminus \{0\}$. The characteristic functions of T and T^* are connected by the relation

$$\Theta_{T^*}(z) = \Theta_T^*(\bar{z}), \quad z \in \mathbb{D}.$$

Two operator-valued functions $\Theta_1 \in \mathbf{S}(\mathfrak{M}_1, \mathfrak{N}_1)$ and $\Theta_2 \in \mathbf{S}(\mathfrak{M}_2, \mathfrak{N}_2)$ are said to *agree* if there are two unitary operators $V : \mathfrak{N}_1 \to \mathfrak{N}_2$ and $W : \mathfrak{M}_2 \to \mathfrak{M}_1$ such that

$$V\Theta_1(z)W = \Theta_2(z), \quad z \in \mathbb{D}.$$

It is well known [42, Theorem VI.3.4], that two completely nonunitary contractions T_1 and T_2 are unitarily equivalent if and only if their characteristic functions Θ_{T_1} and Θ_{T_2} agree.

Every operator-valued function Θ from the Schur class $\mathbf{S}(\mathfrak{M}, \mathfrak{N})$ has almost everywhere nontangential strong limit values $\Theta(\zeta)$, $\zeta \in \mathbb{T}$. A function $\Theta \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$ is called *inner* if $\Theta^*(\zeta)\Theta(\zeta) = I_{\mathfrak{M}}$ for a.e. $\zeta \in \mathbb{T}$, and *co-inner* if $\Theta(\zeta)\Theta^*(\zeta) = I_{\mathfrak{N}}$ for a.e. $\zeta \in \mathbb{T}$. A function $\Theta \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$ is called *bi-inner*, if it is both inner and co-inner. A contraction T on a Hilbert space $\mathfrak{H}$ belongs to the classes $C_{0\cdot}$ ($C_{\cdot 0}$), if

$$\text{s-}\lim_{n\to\infty} T^n = 0 \qquad \bigl(\text{s-}\lim_{n\to\infty} T^{*n} = 0\bigr),$$

respectively. By definition $C_{00} := C_{0\cdot} \cap C_{\cdot 0}$. The completely nonunitary part of a contraction T belongs to the class $C_{\cdot 0}$, $C_{0\cdot}$, or C_{00} if and only if its characteristic function $\Theta_T(z)$ is inner, co-inner, or bi-inner, respectively (cf. [42, Section VI.2]).

In the following statement [42, Theorem VI.4.1] the spectrum of completely nonunitary contractions is described.

Theorem 2.1. *Let T be a completely nonunitary contraction on H. Denote by S_T the set of points $z \in \mathbb{D}$ for which the operator $\Theta_T(z)$ is not boundedly invertible, together with those $z \in \mathbb{T}$ not lying on any of the open arcs of $\mathbb{T}$ on which Θ_T is a unitary operator valued analytic function. Furthermore, denote by S_T^0 the set of points $z \in \mathbb{D}$ for which $\Theta_T(z)$ is not invertible at all. Then the spectrum $\sigma(T)$ of T agrees with S_T, and the point spectrum $\sigma_p(T)$ with S_T^0.*

If T is a completely nonunitary contraction with rank one defects, and if z_0 is an eigenvalue of T, then the geometric multiplicity of z_0 is one, the algebraic multiplicity is finite, and the characteristic function Θ_T admits the following factorization:

$$\Theta_T(z) = c\prod \frac{\bar{z}_k}{z_k}\frac{z_k - z}{1 - \bar{z}_k z}\exp\left(-\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\,d\mu(t)\right)$$

$$\times \exp\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\ln k(t)\,dt\right),$$

where $|c|=1$, $k(t)\geqslant 0$, $\ln k(t)\in L_1[0,2\pi]$, μ is a finite nonnegative measure singular with respect to the Lebesgue measure, and $\{z_k\}$ are the eigenvalues of T. In addition, if $\dim H = N<\infty$, and T is a completely nonunitary contraction in H with rank one defects, then its characteristic function is the finite Blaschke product of order N of the form

$$b(z)=e^{i\varphi}\prod_{k=1}^{m}\left(\frac{z-z_k}{1-\bar{z}_k z}\right)^{l_k},$$

where $z_1,\dots,z_m$ are distinct eigenvalues of T with the algebraic multiplicities $l_1,\dots,l_m$, respectively, $l_1+\dots+l_m=N$, and $\varphi\in[0,2\pi)$. Hence, a finite-dimensional completely nonunitary contraction T with rank one defects belongs to the class C_{00}, and $\lim_{n\to\infty}\|T^n\|=0$. It is easily seen from Theorem 2.1 that the point spectrum of a contraction T with rank one defects agrees with $\mathbb{D}$ if and only if $\Theta_T\equiv 0$.

2.2. *Unitary colligations and their characteristic functions*

Every contraction T acting on Hilbert space H can be included into the *unitary operator colligation* [11][1]

$$\Delta=\left\{\begin{pmatrix} S & G\\ F & T\end{pmatrix};\ \mathfrak{M},\mathfrak{N},H\right\},$$

where $\mathfrak{M}$ and $\mathfrak{N}$ are separable Hilbert spaces, and

$$U=\begin{pmatrix} S & G\\ F & T\end{pmatrix}:\begin{pmatrix}\mathfrak{M}\\ H\end{pmatrix}\to\begin{pmatrix}\mathfrak{N}\\ H\end{pmatrix}$$

is a unitary operator. T is called the *basic operator* of the unitary colligation Δ. The spaces $\mathfrak{M}$ and $\mathfrak{N}$ are called the *left outer space* and *right outer space*, respectively. The unitarity of U means

$$U^*U=\begin{pmatrix} I_{\mathfrak{M}} & 0\\ 0 & I_H\end{pmatrix},\qquad UU^*=\begin{pmatrix} I_{\mathfrak{N}} & 0\\ 0 & I_H\end{pmatrix}$$

or equivalently,

$$\begin{aligned} &T^*T+G^*G=I_H, &&F^*F+S^*S=I_{\mathfrak{M}}, &&T^*F+G^*S=0,\\ &TT^*+FF^*=I_H, &&GG^*+SS^*=I_{\mathfrak{N}}, &&TG^*+FS^*=0. \end{aligned}\tag{2.5}$$

[1] Also known as the *conservative system* [5].

The colligation

$$\Delta_0 = \left\{ \begin{pmatrix} -T^* & D_T \\ D_{T^*} & T \end{pmatrix};\ \mathfrak{D}_T, \mathfrak{D}_{T^*}, H \right\} \tag{2.6}$$

provides an example of the unitary colligation with given basic operator T.

Let $\Delta = \{\left(\begin{smallmatrix} S & G \\ F & T \end{smallmatrix}\right);\ \mathfrak{M}, \mathfrak{N}, H\}$ be a unitary colligation. Define the following subspaces in H:

$$\begin{aligned} H^{(c)} &= \overline{\operatorname{span}}\{T^n F\mathfrak{M},\ n = 0, 1, \ldots\}, \\ H^{(o)} &= \overline{\operatorname{span}}\{T^{*n} G^*\mathfrak{N},\ n = 0, 1, \ldots\}. \end{aligned} \tag{2.7}$$

The subspaces $H^{(c)}$ and $H^{(o)}$ are called the *controllable* and the *observable* subspaces, respectively. Let

$$\left(H^{(c)}\right)^{\perp} := H \ominus H^{(c)}, \qquad \left(H^{(o)}\right)^{\perp} := H \ominus H^{(o)}. \tag{2.8}$$

A unitary colligation Δ is called *prime* if $\overline{H^{(c)} + H^{(o)}} = H$. Clearly, the latter condition is equivalent to

$$\left(H^{(c)}\right)^{\perp} \cap \left(H^{(o)}\right)^{\perp} = \{0\}.$$

From (2.5) and (2.8) we get

$$\begin{aligned} \left(H^{(c)}\right)^{\perp} &= \bigcap_{n\geqslant 0} \ker\left(F^* T^{*n}\right) = \bigcap_{n\geqslant 0} \ker\left(D_{T^*} T^{*n}\right) = \bigcap_{n\geqslant 1} \ker(D_{T^{*n}}), \\ \left(H^{(o)}\right)^{\perp} &= \bigcap_{n\geqslant 0} \ker\left(G T^{n}\right) = \bigcap_{n\geqslant 0} \ker\left(D_T T^{n}\right) = \bigcap_{n\geqslant 1} \ker(D_{T^{n}}). \end{aligned} \tag{2.9}$$

It follows now from (2.2) that *the unitary colligation*

$$\Delta = \left\{ \begin{pmatrix} S & G \\ F & T \end{pmatrix};\ \mathfrak{M}, \mathfrak{N}, H \right\}$$

is prime if and only if T is a completely nonunitary operator.

Given a unitary colligation

$$\Delta = \left\{ \begin{pmatrix} S & G \\ F & T \end{pmatrix};\ \mathfrak{M}, \mathfrak{N}, H \right\},$$

its characteristic function[2] [11, Section 3] is defined by

$$\Theta_\Delta(z) = S + zG(I_H - zT)^{-1}F, \quad z \in \mathbb{D}.$$

[2] *The transfer function* of the system [5].

This function belongs to the Schur class $\mathbf{S}(\mathfrak{M},\mathfrak{N})$ of $\mathcal{L}(\mathfrak{M},\mathfrak{N})$-valued holomorphic in the unit disk $\mathbb{D}$ operator-functions. In particular, the characteristic function of the unitary colligation Δ_0 (2.6)

$$\Theta_0(z) = \left(-T^* + zD_T(I - zT)^{-1}D_{T^*}\right)\restriction\mathfrak{D}_{T^*}$$

is in fact the Sz.-Nagy–Foias characteristic function of the operator T^*.

Two prime unitary colligations

$$\Delta_1 = \left\{\begin{pmatrix} S & G_1 \\ F_1 & T_1 \end{pmatrix};\ \mathfrak{M},\mathfrak{N},H_1\right\} \quad \text{and} \quad \Delta_2 = \left\{\begin{pmatrix} S & G_2 \\ F_2 & T_2 \end{pmatrix};\ \mathfrak{M},\mathfrak{N},H_2\right\}$$

which have equal characteristic functions are unitarily equivalent in the following sense [11, Theorem 3.2]: there exists a unitary operator $V : H_1 \to H_2$ such that

$$\begin{aligned} &VT_1 = T_2V, \qquad VF_1 = F_2, \qquad G_2V = G_1 \\ \Leftrightarrow \quad &\begin{pmatrix} I_{\mathfrak{N}} & 0 \\ 0 & V \end{pmatrix}\begin{pmatrix} S & G_1 \\ F_1 & T_1 \end{pmatrix} = \begin{pmatrix} S & G_2 \\ F_2 & T_2 \end{pmatrix}\begin{pmatrix} I_{\mathfrak{M}} & 0 \\ 0 & V \end{pmatrix}. \end{aligned}$$

Besides, given $\Theta \in \mathbf{S}(\mathfrak{M},\mathfrak{N})$, there exists a prime unitary colligation

$$\Delta = \left\{\begin{pmatrix} S & G \\ F & T \end{pmatrix};\ \mathfrak{M},\mathfrak{N},H\right\}$$

such that $\Theta_\Delta = \Theta$ in $\mathbb{D}$ [11, Theorem 5.1].

Later on in Section 3 we will need the following result.

Theorem 2.2. *Let T be a contraction with finite defect numbers acting on Hilbert space H. Suppose that $\mathfrak{M}$ and $\mathfrak{N}$ are two given Hilbert spaces such that* $\dim\mathfrak{N} = \dim\mathfrak{D}_T$ *and* $\dim\mathfrak{M} = \dim\mathfrak{D}_{T^*}$. *Then all unitary colligations with the basic operator T and outer subspaces $\mathfrak{M}$ and $\mathfrak{N}$ take the form*

$$\Delta = \left\{\begin{pmatrix} -KT^*M & KD_T \\ D_{T^*}M & T \end{pmatrix};\ \mathfrak{M},\mathfrak{N},H\right\}, \tag{2.10}$$

where $K : \mathfrak{D}_T \to \mathfrak{N}$ and $M : \mathfrak{M} \to \mathfrak{D}_{T^*}$ are unitary operators. The characteristic function of Δ is

$$\Theta_\Delta(z) = K\Theta_{T^*}(z)M, \quad z \in \mathbb{D},$$

i.e., Θ_Δ agrees with the characteristic function Θ_{T^*} of T^*.

Proof. Let $\Delta = \{\left(\begin{smallmatrix} S & G \\ F & T \end{smallmatrix}\right);\ \mathfrak{M},\mathfrak{N},H\}$ be a unitary colligation. From the relation $G^*G + T^*T = I_H$ it follows that

$$\|Gf\|^2 = \|D_Tf\|^2, \quad f \in H.$$

Hence, the operator $K : \mathfrak{D}_T \to \mathfrak{N}$ defined by

$$KD_Tf = Gf, \quad f \in H,$$

is isometric, and $\operatorname{ran} K = \mathfrak{N}$. Similarly, the relation $FF^* + TT^* = I_H$ yields that the operator $N : \mathfrak{D}_{T^*} \to \mathfrak{M}$ given by the relation

$$ND_{T^*} f = F^* f, \quad f \in H,$$

is isometric, and $\operatorname{ran} N = \mathfrak{M}$. So $M = N^* : \mathfrak{M} \to \mathfrak{D}_{T^*}$ is unitary, and $F = D_{T^*} M$.

From the relation $T^*F + G^*S = 0$ we get $T^* D_{T^*} M + D_T K^* S = 0$. Hence by (2.1) $T^*M + K^*S = 0$. As $\operatorname{ran} M = \mathfrak{D}_{T^*}$, $\operatorname{ran} K^* = \mathfrak{D}_T$, and $T\mathfrak{D}_{T^*} \subset \mathfrak{D}_T$, we have

$$S = -KT^*M.$$

Observe also that

$$\begin{aligned}
TG^* + FS^* &= TD_T K^* - D_{T^*} MM^* TK^* = 0, \\
SS^* + GG^* &= KT^*MM^*TK^* + KD_T^2 K^* \\
&= K(T^*T + I - T^*T)K^* = I_{\mathfrak{N}}, \\
S^*S + F^*F &= M^*TK^*KT^*M + M^*D_{T^*}M \\
&= M^*(TT^* + I - TT^*)M = I_{\mathfrak{M}}.
\end{aligned}$$

Thus, all conditions (2.5) are satisfied, i.e., the colligation Δ is of the form (2.10).

Conversely, if $\dim \mathfrak{N} = \dim \mathfrak{D}_T < \infty$, $\dim \mathfrak{M} = \dim \mathfrak{D}_{T^*} < \infty$, and $K : \mathfrak{D}_T \to \mathfrak{N}$ and $M : \mathfrak{M} \to \mathfrak{D}_{T^*}$ are unitary operators, then one can easily see that

$$U = \begin{pmatrix} -KT^*M & KD_T \\ D_{T^*}M & T \end{pmatrix} : \begin{pmatrix} \mathfrak{M} \\ H \end{pmatrix} \to \begin{pmatrix} \mathfrak{N} \\ H \end{pmatrix}$$

is a unitary operator, i.e., the relations (2.5) are satisfied. It follows that

$$\Delta = \left\{ \begin{pmatrix} S & G \\ F & T \end{pmatrix}; \ \mathfrak{M}, \mathfrak{N}, H \right\}$$

is a unitary colligation, where $G = KD_T$, $F = D_{T^*}M$, $S = -KT^*M$.

For the characteristic function Θ_Δ we obtain for all $z \in \mathbb{D}$

$$\begin{aligned}
\Theta_\Delta(z) &= S + zG(I - zT)^{-1} F \\
&= -KT^*M + zKD_T(I - zT)^{-1} D_{T^*} M = K\Theta_{T^*}(z) M. \qquad \square
\end{aligned}$$

Corollary 2.3. *Let T be a contraction with finite defect numbers,* $\dim \mathfrak{N} = \dim \mathfrak{D}_T$, $\dim \mathfrak{M} = \dim \mathfrak{D}_{T^*}$*, and let*

$$\Delta = \left\{ \begin{pmatrix} S & G \\ F & T \end{pmatrix}; \ \mathfrak{M}, \mathfrak{N}, H \right\}$$

be a unitary colligation. Then all other unitary colligations with the basic operator T and outer subspaces $\mathfrak{M}$ and $\mathfrak{N}$ take the form

$$\widetilde{\Delta} = \left\{ \begin{pmatrix} C_1 S C_2 & C_1 G \\ F C_2 & T \end{pmatrix};\ \mathfrak{M}, \mathfrak{N}, H \right\},$$

where C_1 and C_2 are unitary operators in $\mathfrak{N}$ and $\mathfrak{M}$, respectively.

Proof. By Theorem 2.2 we have

$$G = K D_T, \qquad F = D_{T^*} M, \qquad S = -K T^* M,$$

where $K : \mathfrak{D}_T \to \mathfrak{M}$ and $M : \mathfrak{N} \to \mathfrak{D}_{T^*}$ are unitary operators. If $\widetilde{\Delta} = \{ \left(\begin{smallmatrix} \widetilde{S} & \widetilde{G} \\ \widetilde{F} & T \end{smallmatrix}\right);\ \mathfrak{M}, \mathfrak{N}, H \}$ is some other unitary colligation then $\widetilde{G} = \widetilde{K} D_T$, $\widetilde{F} = D_{T^*} \widetilde{M}$, $\widetilde{S} = -\widetilde{K} T^* \widetilde{M}$, where $\widetilde{K} : \mathfrak{D}_T \to \mathfrak{M}$ and $\widetilde{M} : \mathfrak{N} \to \mathfrak{D}_{T^*}$ are unitary operators. Let $C_1 := \widetilde{K} K^{-1}$, $C_2 := M^{-1} \widetilde{M}$. Then C_1 and C_2 are unitary operators in $\mathfrak{N}$ and $\mathfrak{M}$, respectively, and

$$\widetilde{G} = C_1 G, \qquad \widetilde{F} = F C_2, \qquad \widetilde{S} = C_1 S C_2,$$

as needed. □

3. Completely nonunitary contractions with rank one defects and the corresponding unitary colligations

Theorem 3.1. *Each contraction T with rank one defects on the Hilbert space H can be included into the unitary colligation*

$$\Delta = \left\{ \begin{pmatrix} S & G \\ F & T \end{pmatrix};\ \mathbb{C}, \mathbb{C}, H \right\}.$$

Let $\vec{1} = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \in \mathbb{C} \oplus H$, and let the subspaces $(H^{(c)})^\perp$ and $(H^{(o)})^\perp$ in H be defined by (2.8). *Then*

$$\begin{aligned} \left(H^{(c)}\right)^\perp &= (\mathbb{C} \oplus H) \ominus \overline{\operatorname{span}}\left\{U^n \vec{1};\ n = 0, 1, \ldots\right\}, \\ \left(H^{(o)}\right)^\perp &= (\mathbb{C} \oplus H) \ominus \overline{\operatorname{span}}\left\{U^{*n} \vec{1};\ n = 0, 1, \ldots\right\}, \end{aligned} \tag{3.1}$$

and so the following conditions are equivalent:

(i) *the unitary colligation $\Delta = \{ \left(\begin{smallmatrix} S & G \\ F & T \end{smallmatrix}\right);\ \mathbb{C}, \mathbb{C}, H \}$ is prime*;
(ii) *T is a completely nonunitary contraction*;
(iii) *$\vec{1}$ is the cyclic vector for U: $\overline{\operatorname{span}}\{U^n \vec{1},\ n \in \mathbb{Z}\} = \mathbb{C} \oplus H$.*

All other unitary colligations with the basic operator T and the outer spaces $\mathbb{C}$ are of the form

$$\widetilde{\Delta} = \left\{ \begin{pmatrix} c_1 c_2 S & c_1 G \\ c_2 F & T \end{pmatrix};\ \mathbb{C}, \mathbb{C}, H \right\}, \tag{3.2}$$

where $|c_1| = |c_2| = 1$.

Proof. Since $\dim \mathfrak{D}_T = \dim \mathfrak{D}_{T^*} = 1$, by Theorem 2.2 we can choose the unitary colligation $\Delta = \{\left(\begin{smallmatrix} S & G \\ F & T \end{smallmatrix}\right);\ \mathbb{C}, \mathbb{C}, H\}$ of the form (2.10), i.e., $S = -KT^*M$, $G = KD_T$, $F = D_{T^*}M$, and $K : \operatorname{ran} D_T \to \mathbb{C}$, $M : \mathbb{C} \to \operatorname{ran} D_{T^*}$ are isometric operators. So $U = \left(\begin{smallmatrix} S & G \\ F & T \end{smallmatrix}\right) : \left(\begin{smallmatrix} \mathbb{C} \\ H \end{smallmatrix}\right) \to \left(\begin{smallmatrix} \mathbb{C} \\ H \end{smallmatrix}\right)$ is the unitary operator.

To prove (3.1), suppose that the vector $\vec{h} = \left(\begin{smallmatrix} z \\ h \end{smallmatrix}\right) \in \mathbb{C} \oplus H$ is orthogonal to the subspace $\overline{\operatorname{span}}\{U^n \vec{1},\ n = 0, 1, \ldots\}$. Then $U^{*n}\vec{h} \perp \vec{1}$, $n = 0, 1, \ldots,$ so $z = 0$ and $\vec{h} = \left(\begin{smallmatrix} 0 \\ h \end{smallmatrix}\right)$. By using $U^* = \left(\begin{smallmatrix} S^* & F^* \\ G^* & T^* \end{smallmatrix}\right)$, we get consequently

$$F^*h = 0, \quad F^*T^*h = 0, \quad F^*T^{*2}h = 0, \ \ldots, \ F^*T^{*k}h = 0, \ldots.$$

It follows from (2.9) that $h \in (H^{(c)})^\perp$. Conversely, if $h \in (H^{(c)})^\perp$ then $h \perp \overline{\operatorname{span}}\{U^n\vec{1},\ n = 0, 1, \ldots\}$. Similarly, $(H^{(o)})^\perp = (\mathbb{C} \oplus H) \ominus (\overline{\operatorname{span}}\{U^{*n}\vec{1},\ n = 0, 1, \ldots\})$, as needed.

We arrive at the following conclusion:

$$\begin{aligned}
&\vec{1} \text{ is a cyclic vector for } U \\
&\Leftrightarrow \quad \big(H^{(c)}\big)^\perp \cap \big(H^{(o)}\big)^\perp = \{0\} \\
&\Leftrightarrow \quad \text{the unitary colligation } \Delta = \left\{\begin{pmatrix} S & G \\ F & T \end{pmatrix}; \mathbb{C}, \mathbb{C}, H\right\} \text{ is prime} \\
&\Leftrightarrow \quad \text{the operator } T \text{ is completely nonunitary.}
\end{aligned}$$

By Corollary 2.3 all other unitary colligations with the basic operator T and the outer subspace $\mathbb{C}$ are given by (3.2) with $|c_1| = |c_2| = 1$. □

Remark 3.2. In terms of the Naimark dilations of a probability operator-valued measure on the unit circle, the main result of Theorem 3.1 is proved in [16, Theorem 1.20].

Let us give more precise expressions for the operators F, G, and S. Let $\widehat{\varphi}_1 \in \mathfrak{D}_T$, $\widehat{\varphi}_2 \in \mathfrak{D}_{T^*}$. Put

$$\varphi_1 = \frac{\widehat{\varphi}_1}{\|\widehat{\varphi}_1\|}, \qquad \varphi_2 = \frac{\widehat{\varphi}_2}{\|\widehat{\varphi}_2\|}.$$

Then

$$\begin{aligned}
Kh &= b_1(h, \varphi_1), \quad h \in \operatorname{ran} D_T, \\
M^*g &= b_2(g, \varphi_2), \quad g \in \operatorname{ran} D_{T^*},
\end{aligned}$$

where $|b_1| = |b_2| = 1$. Observe that $T\varphi_1 = -\alpha_0\varphi_2$ and $T^*\varphi_2 = -\bar{\alpha}_0\varphi_1$, where α_0 is a complex number from $\mathbb{D}$. It follows that

$$D_T^2\varphi_1 = \big(1 - |\alpha_0|^2\big)\varphi_1, \qquad D_{T^*}^2\varphi_2 = \big(1 - |\alpha_0|^2\big)\varphi_2.$$

Let $\rho_0 = \sqrt{1 - |\alpha_0|^2}$. Since $\dim(\operatorname{ran} D_T^2) = \dim(\operatorname{ran} D_{T^*}^2) = 1$, the number ρ_0 is a unique positive eigenvalue of $D_T(D_{T^*})$. Next,

$$Gh = b_1(D_T h, \varphi_1) = b_1(h, D_T\varphi_1) = b_1\rho_0(h, \varphi_1),$$
$$F^*h = b_2(D_{T^*}h, \varphi_2) = b_2(h, D_{T^*}\varphi_2) = b_2\rho_0(h, \varphi_2), \quad h \in H.$$

Hence $F1 = \rho_0\bar{b}_2\varphi_2$. Since $S = -KT^*M$, we get

$$S1 = -b_1\bar{b}_2(T^*\varphi_2, \varphi_1) = b_1\bar{b}_2\bar{\alpha}_0.$$

In the case $\dim H = N < \infty$ the operator T can be given by the $N \times N$ matrix with respect to some orthonormal basis and we can choose $\widehat{\varphi_1}$ (respectively, $\widehat{\varphi_2}$), as one of the nonzero columns of the matrix $I - T^*T$ $(I - TT^*)$. In addition,

$$\operatorname{Trace}(I - T^*T) = \operatorname{Trace}(I - TT^*) = \rho_0^2.$$

Thus, if

$$\varphi_2 = \begin{pmatrix} \varphi_2^{(1)} \\ \varphi_2^{(2)} \\ \vdots \\ \varphi_2^{(N)} \end{pmatrix},$$

then the column F takes the form

$$F = \bar{b}_2\rho_0 \begin{pmatrix} \varphi_2^{(1)} \\ \varphi_2^{(2)} \\ \vdots \\ \varphi_2^{(N)} \end{pmatrix}.$$

If

$$\varphi_1 = \begin{pmatrix} \varphi_1^{(1)} \\ \varphi_1^{(2)} \\ \vdots \\ \varphi_1^{(N)} \end{pmatrix},$$

then the row G takes the form $G = b_1\rho_0(\bar{\varphi}_1^{(1)}\ \bar{\varphi}_1^{(2)}\ \dots\ \bar{\varphi}_1^{(N)})$. Finally, the number S is given by $-b_1\bar{b}_2(T^*\varphi_2, \varphi_1)$.

If $\dim H = N$ and T is a completely nonunitary contraction with rank one defects, then Θ_Δ is a finite Blaschke product

$$\Theta_\Delta(z) = e^{i\varphi} \prod_{k=1}^{N} \frac{z - \bar{z}_k}{1 - z_k z},$$

where the numbers $z_1, \dots, z_N$ are the eigenvalues of T. Since all other unitary colligations are of the form (3.2), for the characteristic function $\Theta_{\widetilde{\Delta}}(z)$ we get $\Theta_{\widetilde{\Delta}}(z) = c_1c_2\Theta_\Delta(z) = e^{it}\Theta_\Delta(z)$, $z \in \mathbb{D}$, and $t \in [0, 2\pi)$.

Let U be a unitary operator with a cyclic vector e, acting on the Hilbert space H. The spectral measure μ associated with U and e provides the relation

$$\big(F(U)e,e\big)=\int\limits_{\mathbb{T}}F(\zeta)\,d\mu(\zeta),$$

which is the Spectral Theorem for unitaries. For instance,

$$F(z)=\big((U+zI)(U-zI)^{-1}e,e\big)=\int\limits_{\mathbb{T}}\frac{\zeta+z}{\zeta-z}\,d\mu(\zeta),\quad z\in\mathbb{D},\tag{3.3}$$

is the Carathéodory function (4.11), i.e., F is holomorphic in the unit disc $\mathbb{D}$, $\operatorname{Re}F>0$ in $\mathbb{D}$, and $F(0)=1$.

Theorem 3.3. *Let T be a completely nonunitary contraction with rank one defects, $\Delta=\{\left(\begin{smallmatrix}S&G\\F&T\end{smallmatrix}\right);\ \mathbb{C},\mathbb{C},H\}$ be the prime unitary colligation, and Θ_Δ be its characteristic function. Put*

$$F(z)=\big((U+zI)(U-zI)^{-1}\vec{1},\vec{1}\big),\quad z\in\mathbb{D},\tag{3.4}$$

where $U=\left(\begin{smallmatrix}S&G\\F&T\end{smallmatrix}\right):\left(\begin{smallmatrix}\mathbb{C}\\H\end{smallmatrix}\right)\to\left(\begin{smallmatrix}\mathbb{C}\\H\end{smallmatrix}\right)$. Then

$$\overline{\Theta_\Delta(\bar z)}=\frac{1}{z}\,\frac{F(z)-1}{F(z)+1},\qquad F(z)=\frac{1+z\overline{\Theta_\Delta(\bar z)}}{1-z\overline{\Theta_\Delta(\bar z)}},\quad z\in\mathbb{D}.\tag{3.5}$$

Proof. We use the well-known Schur–Frobenius formula for the inverse of block operators (see, e.g., [18, Section 0.2], [19, p. 57]). Let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be two Hilbert spaces, and Φ an operator in $\mathfrak{H}_1\oplus\mathfrak{H}_2$ given by the block operator matrix

$$\Phi=\begin{pmatrix}A&B\\C&D\end{pmatrix}:\begin{pmatrix}\mathfrak{H}_1\\\mathfrak{H}_2\end{pmatrix}\to\begin{pmatrix}\mathfrak{H}_1\\\mathfrak{H}_2\end{pmatrix}.$$

Suppose that $D^{-1}\in\mathcal{L}(\mathfrak{H}_2)$ and $(A-BD^{-1}C)^{-1}\in\mathcal{L}(\mathfrak{H}_1)$. Then $\Phi^{-1}\in\mathcal{L}(\mathfrak{H}_1\oplus\mathfrak{H}_2,\mathfrak{H}_1\oplus\mathfrak{H}_2)$ and

$$\Phi^{-1}=\begin{pmatrix}K^{-1}&-K^{-1}BD^{-1}\\-D^{-1}CK^{-1}&D^{-1}+D^{-1}CK^{-1}BD^{-1}\end{pmatrix},$$

where $K=A-BD^{-1}C$.

Applying this formula for

$$\Phi=I-zU=\begin{pmatrix}1-zS&-zG\\-zF&I-zT\end{pmatrix}:\begin{pmatrix}\mathbb{C}\\H\end{pmatrix}\to\begin{pmatrix}\mathbb{C}\\H\end{pmatrix},\quad z\in\mathbb{D},$$

we get $K=1-zS-z^2G(I-zT)^{-1}F=1-z\Theta_\Delta(z)$. Therefore

$$\big((I-zU)^{-1}\vec{1},\vec{1}\big)=\frac{1}{1-z\Theta_\Delta(z)},\quad z\in\mathbb{D}.$$

Let

$$\Psi(z) = \big((I + zU)(I - zU)^{-1}\vec{1}, \vec{1}\big), \quad z \in \mathbb{D}.$$

Clearly, the equality $F(z) = \overline{\Psi(\bar{z})}$ holds, which yields (3.5). $\square$

Remark 3.4. Relations (3.5) is proved in [16, Theorem 1.20, Comments 2.8]. Our proof is different.

4. OPUC and CMV matrices

4.1. Basics of OPUC

It is well recognized now that the theory of orthogonal polynomials on the real line plays an important role in the spectral theory of self-adjoint operators (and close to such operators) acting on Hilbert spaces. Likewise, the theory of orthogonal polynomials on the unit circle (OPUC) appears in the same fashion in the study of unitary operators and close to such operators. Here we recall some rudiments and advances of the OPUC theory.

If μ is a nontrivial probability measure on $\mathbb{T}$ (that is, not supported on a finite set), the monic orthogonal polynomials $\Phi_n(z, \mu)$ (or Φ_n if μ is understood) are uniquely determined by

$$\Phi_n(z) = \prod_{j=1}^{n} (z - z_{n,j}), \qquad \int_{\mathbb{T}} \zeta^{-j} \Phi_n(\zeta)\, d\mu = 0, \quad j = 0, 1, \ldots, n-1, \tag{4.1}$$

so on the Hilbert space $L^2(\mathbb{T}, d\mu)$, $\langle \Phi_n, \Phi_m \rangle = 0$, $n \neq m$. We also consider the orthonormal polynomials ϕ_n of the form $\phi_n = \Phi_n/\|\Phi_n\|$.

In case when μ is supported on a finite set, that is,

$$\mu = \sum_{k=1}^{N} \mu_k \delta(\zeta_k), \quad \zeta_k \in \mathbb{T}, \tag{4.2}$$

a finite number of orthogonal polynomials $\{\Phi_k\}_{k=0}^{N-1}$ can be defined in the same manner.

Clearly, (4.1) and the fact that the space of polynomials of degree at most n has dimension $n+1$ imply

$$\deg(P) = n, \quad P \perp \zeta^j,\ j = 0, 1, \ldots, n-1 \quad \Rightarrow \quad P = c\Phi_n. \tag{4.3}$$

On $L^2(\mathbb{T}, d\mu)$ the anti-unitary map $f^*(\zeta) := \zeta^n \overline{f(\zeta)}$ (which depends on n) is naturally defined. The set of polynomials of degree at most n is left invariant:

$$P(z) = \sum_{j=0}^{n} p_j z^j \quad \Rightarrow \quad P^*(z) = \sum_{j=0}^{n} \bar{p}_{n-j} z^j. \tag{4.4}$$

(4.3) now implies

$$\deg(P) \leqslant n, \quad P \perp \zeta^j, \quad j = 1, \ldots, n \quad \Rightarrow \quad P = c\Phi_n^*. \tag{4.5}$$

A key feature of the unit circle is that the multiplication $Uf = zf$ in $L^2(\mathbb{T}, d\mu)$ is a unitary operator. So the difference $\Phi_{n+1}(z) - z\Phi_n(z)$ is of degree n and orthogonal to z^j for $j = 1, 2, \dots, n$, and by (4.5)

$$\Phi_{n+1}(z) = z\Phi_n(z) - \bar{\alpha}_n(\mu)\Phi_n^*(z) \tag{4.6}$$

with some complex numbers $\alpha_n(\mu)$, called the *Verblunsky coefficients* [40]. (4.6) is known as the *Szegő recurrences* after its first occurrence in the celebrated book [43] of G. Szegő. (4.6) at $z = 0$ imply

$$\alpha_n(\mu) = \alpha_n = -\overline{\Phi_{n+1}(0)}. \tag{4.7}$$

It is known that for nontrivial measures $|\alpha_n| < 1$ for all $n = 0, 1, 2, \dots,$ and for trivial measures (4.2) one has a finite set of Verblunsky coefficients $\{\alpha_n\}_{n=0}^{N-1}$ with $|\alpha_n| < 1$, $n = 0, 1, \dots, N-2$, and $|\alpha_{N-1}| = 1$. Since it arises often, define

$$\rho_j := \sqrt{1 - |\alpha_j|^2}, \quad 0 < \rho_j \leqslant 1, \quad |\alpha_j|^2 + \rho_j^2 = 1. \tag{4.8}$$

The inverse Szegő recurrences are also of interest (cf. [40, Theorem 1.5.4]):

$$z\Phi_n(z) = \rho_n^{-2}\big(\Phi_{n+1}(z) + \bar{\alpha}_n\Phi_{n+1}^*(z)\big). \tag{4.9}$$

The norm of the polynomials Φ_n in $L^2(\mathbb{T}, d\mu)$ can be computed by

$$\|\Phi_n\| = \prod_{j=0}^{n-1}\rho_j, \quad n = 1, 2, \dots.$$

Let $\mathbb{D}^\infty$ be the set of complex sequences $\{\alpha_j\}_{j=0}^\infty$ with $|\alpha_j| < 1$. The map $\mathcal{S}$, from $\mu \to \{\alpha_j(\mu)\}_{j=0}^\infty$, is a well-defined map from the set $\mathcal{P}$ of nontrivial probability measures on $\mathbb{T}$ to $\mathbb{D}^\infty$. It was S. Verblunsky who proved that $\mathcal{S}$ is a bijection. As a matter of fact, $\mathcal{S}$ is a homeomorphism, provided $\mathcal{P}$ is equipped with the weak*-topology, and $\mathbb{D}^\infty$ with the topology of component convergence. Moreover, it follows directly from (4.6) that for two measures μ_1 and μ_2

$$\begin{aligned}&\alpha_j(\mu_1) = \alpha_j(\mu_2), \quad j = 0, 1, \dots, n-1\\ &\quad\Rightarrow \quad \Phi_j(z, \mu_1) = \Phi_j(z, \mu_2), \quad j = 0, 1, \dots, n.\end{aligned}$$

Conversely, by (4.9)

$$\Phi_n(z, \mu_1) = \Phi_n(z, \mu_2) \quad \Rightarrow \quad \alpha_j(\mu_1) = \alpha_j(\mu_2), \quad j = 0, 1, \dots, n-1.$$

The orthonormal set $\{\phi_n\}_{n\geqslant 0}$ does not necessarily form a basis in $L^2(\mathbb{T}, d\mu)$ (e.g., if $d\mu = dm$ is the normalized Lebesgue measure on $\mathbb{T}$, then $\phi_n = \zeta^n$ and ζ^{-1} is orthogonal to all ϕ_n). A celebrated result of Szegő–Kolmogorov–Krein reads that $\{\phi_n\}$ is a basis in $L^2(\mathbb{T}, d\mu)$ if and only if $\log\mu' \notin L^1(\mathbb{T})$, where μ' is the Radon–Nikodym derivative of μ with respect to dm. In addition, the following result holds true (cf. [40, Theorem 1.5.7]).

Theorem 4.1. *For any nontrivial probability measure μ on the unit circle, the following are equivalent*:

(i) $\lim_{n\to\infty} \|\Phi_n\| = 0$;
(ii) $\sum_{n=0}^{\infty} |\alpha_n|^2 = \infty$;
(iii) *the system $\{\phi_n\}_{n=0}^{\infty}$ is the orthonormal basis in $L^2(\mathbb{T}, d\mu)$.*

Note that if $\sum_{n=0}^{\infty} |\alpha_n|^2 < \infty$ and P is the orthogonal projection in $L^2(\mathbb{T}, d\mu)$ onto $\overline{\operatorname{span}}\{\zeta^n,\ n = 0, 1, \ldots\}$, then (see [39])

$$\|(I - P)\bar{\zeta}\| = \prod_{n=0}^{\infty} \rho_n. \tag{4.10}$$

Let us now turn to the basic properties of zeros $\{z_{n,j}\}_{j=1}^{n}$ of OPUC. It is well known (cf., e.g., [40, Theorem 1.7.1]) that $|z_{n,j}| < 1$ for all n and j. Moreover, a result of Geronimus [40, Theorem 1.7.5] reads that given a monic polynomial P_n of degree n with all its zeros inside $\mathbb{D}$, there is a (nontrivial) probability measure μ on $\mathbb{T}$ such that $P_n = \Phi_n(\mu)$. Actually, there are infinitely many such measures, all of them have the same Verblunsky coefficients up to the order $n-1$, and the same moments up to the order n. Given a monic polynomial P_n with all its zeros inside the disk, let us call a monic polynomial Q_{n+m} an *extension* of P_n, if there is a measure μ such that

$$P_n = \Phi_n(\mu), \qquad Q_{n+m} = \Phi_{n+m}(\mu).$$

To obtain all such extensions one just has to extend a sequence of Verblunsky coefficients $\alpha_0, \ldots, \alpha_{n-1}$, which are completely determined by P_n, by a sequence $\beta_0, \ldots, \beta_{m-1}$ with arbitrary $\beta_j \in \mathbb{D}$ and then apply (4.6).

One of the most recent advances in the study of zeros of OPUC is the theorem of Simon and Totik [40, Theorem 1.7.15], which claims that given a polynomial P_n as above, and an arbitrary set of points $z_1, \ldots, z_m$ in the unit disk, not necessarily distinct, there is an extension Q_{n+m} of P_n such that $Q_{n+m}(z_j) = 0$, $j = 1, 2, \ldots, m$, counting the multiplicity. The latter as usual means that

$$z_k = z_{k+1} = \cdots = z_{k+p} \quad \Rightarrow \quad Q_{n+m}(z_k) = Q'_{n+m}(z_k) = \cdots = Q^{(p)}_{n+m}(z_k) = 0.$$

The uniqueness of such extension is an open problem. A particular case $m = 1$ appeared earlier in [3]. Now $\beta_0 = \alpha_n$ is defined uniquely from (4.6) by

$$0 = Q_{n+1}(z_1) = z_1 P_n(z_1) - \bar{\alpha}_n P_n^*(z_1).$$

This result will play a key role in the inverse problems with mixed data in Section 7.

4.2. Geronimus theory

There is an important analytic aspect of the OPUC theory which was developed by Geronimus [20,21] in 1940s.

Given a probability measure μ on $\mathbb{T}$, define the *Carathéodory function* by

$$F(z)=F(z,\mu):=\int_{\mathbb{T}}\frac{\zeta+z}{\zeta-z}\,d\mu(\zeta)=1+2\sum_{n=1}^{\infty}\beta_n z^n,\quad \beta_n=\int_{\mathbb{T}}\zeta^{-n}\,d\mu \tag{4.11}$$

the moments of μ. F is an analytic function in $\mathbb{D}$ which obeys $\operatorname{Re}F>0$, $F(0)=1$. The *Schur function* is then defined by

$$f(z)=f(z,\mu):=\frac{1}{z}\frac{F(z)-1}{F(z)+1},\quad F(z)=\frac{1+zf(z)}{1-zf(z)}, \tag{4.12}$$

so it is an analytic function in $\mathbb{D}$ with $\sup_{\mathbb{D}}|f(z)|\leqslant 1$. A one-to-one correspondence can be easily set up between the three classes (probability measures, Carathéodory and Schur functions). Under this correspondence μ is trivial, that is, supported on a finite set, if and only if the associate Schur function is a finite Blaschke product. Moreover, this Blaschke product has the order $N-1$ for measures (4.2).

We proceed with the Schur algorithm. Given a Schur function $f=f_0$, which is not a finite Blaschke product, define inductively

$$f_{n+1}(z)=\frac{f_n(z)-\gamma_n}{z(1-\bar{\gamma}_n f_n(z))},\quad \gamma_n=f_n(0). \tag{4.13}$$

It is clear that the sequence $\{f_n\}$ is an *infinite* sequence of Schur functions (called the nth Schur iterates) and neither of its terms is a finite Blaschke product. The numbers $\{\gamma_n\}$ are called the *Schur parameters*:

$$\mathcal{S}f=\{\gamma_0,\gamma_1,\ldots\}.$$

In case when

$$f(z)=e^{i\varphi}\prod_{k=1}^{N}\frac{z-z_k}{1-\bar{z}_k z}$$

is a finite Blaschke product of order N, the Schur algorithm terminates at the Nth step. The sequence of Schur parameters $\{\gamma_k\}_{k=0}^N$ is finite, $|\gamma_k|<1$ for $k=0,1,\ldots,N-1$, and $|\gamma_N|=1$.

If a Schur function f is not a finite Blaschke product, the connection between the non-tangential limit values $f(\zeta)$ and its Schur parameters $\{\gamma_n\}$ is given by the formula

$$\prod_{n=0}^{\infty}\bigl(1-|\gamma_n|^2\bigr)=\exp\Biggl\{\int_{\mathbb{T}}\ln\bigl(1-\bigl|f(\zeta)\bigr|^2\bigr)\,dm\Biggr\} \tag{4.14}$$

(see [9]). It follows that

$$\sum_{n=0}^{\infty}|\gamma_n|^2=\infty\quad\Leftrightarrow\quad \ln\bigl(1-\bigl|f(\zeta)\bigr|^2\bigr)\notin L^1(\mathbb{T}).$$

In addition, if one of the conditions

(1) $\limsup_{n\to\infty} |\gamma_n| = 1$,
(2) $\lim_{n\to\infty} \gamma_n\gamma_{n+m} = 0$ for each $m = 1, 2, \dots$, but $\limsup_{n\to\infty} |\gamma_n| > 0$,

is fulfilled, then f is the inner function (see [27,37]).

Later in Section 7 we will make use of the following fundamental result of Schur [38]: the set of all Schur functions f with prescribed first Schur parameters $\gamma_0, \dots, \gamma_n$ is given by the linear fractional transformation

$$f(z) = \frac{A(z) + zB^*(z)s(z)}{B(z) + zA^*(z)s(z)}, \tag{4.15}$$

where s is an arbitrary Schur function, and A, B are polynomials of degree at most n. Moreover,

$$\mathcal{S}f = \bigl\{\gamma_0, \dots, \gamma_n, \gamma_0(s), \gamma_1(s), \dots\bigr\}.$$

The pair (A, B), known as the Wall pair, is completely determined by $\{\gamma_j\}_{j=0}^n$. Specifically,

$$W(z) := \begin{pmatrix} zB^*(z) & A(z) \\ zA^*(z) & B(z) \end{pmatrix} = Q_{\gamma_0}(z)\, Q_{\gamma_1}(z) \cdots Q_{\gamma_n}(z),$$

where

$$Q_\omega(z) = \frac{1}{\sqrt{1-|\omega|^2}} \begin{pmatrix} z & \omega \\ z\bar{\omega} & 1 \end{pmatrix}, \quad \omega \in \mathbb{D}.$$

By computing determinants, we see that

$$B^*(z)B(z) - A^*(z)A(z) = z^n \prod_{j=0}^n \bigl(1 - |\gamma_j|^2\bigr)^{1/2},$$

so A and B have no common zeros in $\mathbb{C} \setminus \{0\}$. In fact they have no common zeros at all since $B(0) = 1$. It is known also that $B \neq 0$ in $\overline{\mathbb{D}}$, and both AB^{-1} and A^*B^{-1} are Schur functions.

A straightforward computation shows that Q_ω (and hence W) are j-inner matrix functions:

$$W^*(z)jW(z) \geqslant j \quad \text{for } z \in \mathbb{D},$$
$$W^*(z)jW(z) = j \quad \text{for } z \in \mathbb{T}$$

with the signature matrix

$$j = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For further properties of the Wall pairs see [27, Section 4], [40, Chapter 1.3.8].

A curious situation when the Schur parameters for a finite Blaschke product can be computed explicitly was found by Khrushchev [28, formula (1.12)]. Let μ be a nontrivial probability measure (or measure of the form (4.2) with big enough N) with Verblunsky coefficients $\{\alpha_k\}$, and Φ_n be its nth monic orthogonal polynomial. Consider the following Blaschke product of order n:

$$b_0(z) := \frac{\Phi_n(z)}{\Phi_n^*(z)} = \prod_{j=1}^{n} \frac{z - z_{n,j}}{1 - \bar{z}_{n,j} z}, \quad b_0(0) = -\bar{\alpha}_{n-1}.$$

It is a matter of a simple computation based on (4.9) to make sure that

$$b_1(z) = \frac{b_0(z) - b_0(0)}{z(1 - \bar{b}_0(0) b_0(z))} = \frac{\Phi_{n-1}(z)}{\Phi_{n-1}^*(z)}.$$

Hence the Schur parameters of b_0 are of the form

$$\mathcal{S} b_0 = \{-\bar{\alpha}_{n-1}, -\bar{\alpha}_{n-2}, \ldots, -\bar{\alpha}_0, 1\}. \tag{4.16}$$

The fundamental paper of Schur [38] had appeared a few years before Szegő introduced the notion of orthogonal polynomials on the unit circle. Amazingly, neither of them benefited from the ideas of the other. Only 20 years later Geronimus put these ideas together and came up with the following fundamental result (see [20, Theorem IX, p. 111]).

Theorem 4.2. *Let μ be a nontrivial probability measure on $\mathbb{T}$ and f its Schur function with the Schur parameters $\gamma_n(f)$. Then $\gamma_n(f) = \alpha_n(\mu)$. For measures* (4.2) *the latter equality holds for $n = 0, 1, \ldots, N-1$.*

It is clear now why a minus and conjugate is taken in (4.6).

We complete with the result which will be used later on in Section 7.

Theorem 4.3. *Given two sets $\alpha_0, \ldots, \alpha_{n-1}$ and $z_1, \ldots, z_m$ of complex numbers in $\mathbb{D}$, and $\gamma \in \mathbb{T}$, there exists a finite Blaschke product b of order $n + m$ such that*:

(i) $\mathcal{S} b = \{\omega_0, \ldots, \omega_{m-1}, \alpha_0, \ldots, \alpha_{n-1}, \gamma\}$,
(ii) $b(z_j) = 0$, $j = 1, \ldots, m$, *counting multiplicity.*

Proof. Denote $\beta_k := -\gamma \bar{\alpha}_{n-k-1}$, $k = 0, 1, \ldots, n-1$ and construct a system of monic orthogonal polynomials $\{\Phi_k(z, \beta)\}_{k=0}^{n}$ by (4.6). The theorem of Simon–Totik claims that there is a measure μ with

$$\Phi_n(z, \mu) = \Phi_n(z, \beta), \qquad \Phi_{n+m}(z_j, \mu) = 0, \quad j = 1, \ldots, m,$$

counting the multiplicity. The first equality means that $\alpha_k(\mu) = \beta_k$, $k = 1, \ldots, n-1$. Finally, put

$$b(z) := \gamma \frac{\Phi_{n+m}(z, \mu)}{\Phi_{n+m}^*(z, \mu)}.$$

The result now follows from Khrushchev's formula (4.16). $\square$

Note that for $m = 1$ the Blaschke product is uniquely determined.

4.3. CMV matrices

One of the most interesting developments in the OPUC theory in recent years is the discovery by Cantero, Moral, and Velázquez [13,14] of a matrix realization for the operator of multiplication by ζ on $L^2(\mathbb{T}, d\mu)$ which is a unitary matrix of finite band size (i.e., $|\langle \zeta\chi_m, \chi_n\rangle| = 0$ if $|m-n| > k$ for some k); in this case, $k = 2$ to be compared with $k = 1$ for the Jacobi matrices, which correspond to the real line case. The CMV basis (complete, orthonormal system) $\{\chi_n\}$ is obtained by orthonormalizing the sequence $1, \zeta, \zeta^{-1}, \zeta^2, \zeta^{-2}, \dots$, and the matrix, called the *CMV matrix*,

$$\mathcal{C} = \mathcal{C}(\mu) = \|c_{n,m}\|_{m,n=0}^{\infty} = \big\|\langle \zeta\chi_m, \chi_n\rangle\big\|, \quad m, n \in \mathbb{Z}_+$$

is five-diagonal. Remarkably, the χ's can be expressed in terms of ϕ's and ϕ^*'s:

$$\chi_{2n}(z) = z^{-n}\phi_{2n}^*(z), \qquad \chi_{2n+1}(z) = z^{-n}\phi_{2n+1}(z), \quad n \in \mathbb{Z}_+,$$

and the matrix elements in terms of α's and ρ's:

$$\mathcal{C} = \mathcal{C}\big(\{\alpha_n\}\big) = \begin{pmatrix} \bar\alpha_0 & \bar\alpha_1\rho_0 & \rho_1\rho_0 & 0 & 0 & \dots \\ \rho_0 & -\bar\alpha_1\alpha_0 & -\rho_1\alpha_0 & 0 & 0 & \dots \\ 0 & \bar\alpha_2\rho_1 & -\bar\alpha_2\alpha_1 & \bar\alpha_3\rho_2 & \rho_3\rho_2 & \dots \\ 0 & \rho_2\rho_1 & -\rho_2\alpha_1 & -\bar\alpha_3\alpha_2 & -\rho_3\alpha_2 & \dots \\ 0 & 0 & 0 & \bar\alpha_4\rho_3 & -\bar\alpha_4\alpha_3 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \tag{4.17}$$

α's are the Verblunsky coefficients and ρ's are given in (4.8).

It is not hard to write down a general formula for the matrix entries c_{ij} (see [25]). Let $2\epsilon_m := 1 - (-1)^m$, $m \in \mathbb{Z}_+$, and $\epsilon_{-1} = 1$, so $\{\epsilon_m\}_{m\geqslant 0} = \{0, 1, 0, 1, \dots\}$,

$$\epsilon_m + \epsilon_{m+1} = 1, \qquad \epsilon_m\epsilon_{m+1} = 0, \qquad \epsilon_m - \epsilon_{m+1} = (-1)^{m+1}.$$

Then

$$\begin{aligned} c_{mm} &= -\bar\alpha_m\alpha_{m-1}, \\ c_{m+2,m} &= \rho_m\rho_{m+1}\epsilon_m, \\ c_{m,m+2} &= \rho_m\rho_{m+1}\epsilon_{m+1}, \end{aligned} \tag{4.18}$$

and

$$\begin{aligned} c_{m+1,m} &= \bar\alpha_{m+1}\rho_m\epsilon_m - \alpha_{m-1}\rho_m\epsilon_{m+1}, \\ c_{m,m+1} &= \bar\alpha_{m+1}\rho_m\epsilon_{m+1} - \alpha_{m-1}\rho_m\epsilon_m. \end{aligned} \tag{4.19}$$

It is clear (cf. [7, Theorem 1]), that any semi-infinite CMV matrix $\mathcal{C}$ (4.17) can be written in the three-diagonal block-matrix form

$$\mathcal{C}=\begin{pmatrix} \mathcal{B}_0 & \mathcal{C}_0 & 0 & 0 & 0 & \cdot & \cdot \\ \mathcal{A}_0 & \mathcal{B}_1 & \mathcal{C}_1 & 0 & 0 & \cdot & \cdot \\ 0 & \mathcal{A}_1 & \mathcal{B}_2 & \mathcal{C}_2 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \tag{4.20}$$

with

$$\begin{aligned} &\mathcal{B}_0=(\bar{\alpha}_0), \qquad \mathcal{C}_0=(\bar{\alpha}_1\rho_0 \quad \rho_1\rho_0), \qquad \mathcal{A}_0=\begin{pmatrix}\rho_0\\0\end{pmatrix}, \\ &\mathcal{A}_n=\begin{pmatrix}\rho_{2n}\rho_{2n-1} & -\rho_{2n}\alpha_{2n-1}\\ 0 & 0\end{pmatrix}, \qquad \mathcal{B}_n=\begin{pmatrix}-\bar{\alpha}_{2n-1}\alpha_{2n-2} & -\rho_{2n-1}\alpha_{2n-2}\\ \bar{\alpha}_{2n}\rho_{2n-1} & -\bar{\alpha}_{2n}\alpha_{2n-1}\end{pmatrix}, \\ &\mathcal{C}_n=\begin{pmatrix}0 & 0\\ \bar{\alpha}_{2n+1}\rho_{2n} & \rho_{2n+1}\rho_{2n}\end{pmatrix}, \quad n=1,2,\ldots. \end{aligned} \tag{4.21}$$

There is a nice multiplicative structure of the CMV matrices. In the semi-infinite case $\mathcal{C}$ is the product of two matrices: $\mathcal{C}=\mathcal{L}\mathcal{M}$, where

$$\begin{aligned} \mathcal{L}&=\Psi(\alpha_0)\oplus\Psi(\alpha_2)\oplus\cdots\oplus\Psi(\alpha_{2m})\oplus\cdots, \\ \mathcal{M}&=\mathbf{1}_{1\times 1}\oplus\Psi(\alpha_1)\oplus\Psi(\alpha_3)\oplus\cdots\oplus\Psi(\alpha_{2m+1})\oplus\cdots, \end{aligned} \tag{4.22}$$

and $\Psi(\alpha)=\left(\begin{smallmatrix}\bar{\alpha} & \rho\\ \rho & -\alpha\end{smallmatrix}\right)$. The finite $(N+1)\times(N+1)$ CMV matrix $\mathcal{C}$ obeys $\alpha_0,\alpha_1,\ldots,\alpha_{N-1}\in\mathbb{D}$, $|\alpha_N|=1$, and is also the product $\mathcal{C}=\mathcal{L}\mathcal{M}$, where in this case $\Psi(\alpha_N)=(\bar{\alpha}_N)$.

It is just natural to take the ordered set $1,\zeta^{-1},\zeta,\zeta^{-2},\zeta^2,\ldots$ instead of $1,\zeta,\zeta^{-1},\zeta^2,\zeta^{-2},\ldots$, that leads to the alternate CMV basis $\{x_n\}$ and the alternate CMV matrix

$$\widetilde{\mathcal{C}}=\big\|\langle\zeta x_m,x_n\rangle\big\|=\begin{pmatrix} \bar{\alpha}_0 & \rho_0 & 0 & 0 & 0 & \ldots \\ \bar{\alpha}_1\rho_0 & -\bar{\alpha}_1\alpha_0 & \bar{\alpha}_2\rho_1 & \rho_2\rho_1 & 0 & \ldots \\ \rho_1\rho_0 & -\rho_1\alpha_0 & -\bar{\alpha}_2\alpha_1 & -\rho_2\alpha_1 & 0 & \ldots \\ 0 & 0 & \bar{\alpha}_3\rho_2 & -\bar{\alpha}_3\alpha_2 & \bar{\alpha}_4\rho_3 & \ldots \\ 0 & 0 & \rho_3\rho_2 & -\rho_3\alpha_2 & -\bar{\alpha}_4\alpha_3 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}, \tag{4.23}$$

which turns out to be the transpose of $\mathcal{C}$ (see [40, Corollary 4.2.6]). Furthermore, $\mathcal{L}=\mathcal{L}^t$ and $\mathcal{M}=\mathcal{M}^t$ imply $\widetilde{\mathcal{C}}=\mathcal{C}^t=\mathcal{M}\mathcal{L}$.

An important relation between CMV matrices and monic orthogonal polynomials is similar to the well-known property of orthogonal polynomials on the real line

$$\Phi_n(z)=\det\bigl(zI_n-\mathcal{C}^{(n)}\bigr)$$

holds, where $\mathcal{C}^{(n)}$ is the principal $n\times n$ block of $\mathcal{C}$.

One of the most important results of Cantero, Moral, and Velázquez [13] states that each unitary operator U with the simple spectrum (i.e., having a cyclic vector e_1) acting on some infinite-dimensional separable Hilbert space (respectively, finite-dimensional Hilbert space) is

unitarily equivalent to a certain CMV matrix in $\ell^2(\mathbb{Z}_+)$ (respectively, in $\mathbb{C}^n$). The corresponding α's come up as the Verblunsky coefficients of the spectral measure $d\mu$ of U associated with e_1. This is the analog of Stone's self-adjoint cyclic model theorem. To be more precise, let us, following [41], call a *cyclic unitary model* a unitary operator U acting on a separable Hilbert space $\mathcal{H}$ with the distinguished cyclic unit vector v_0. Two cyclic unitary models, $(\mathcal{H}, U, v_0)$ and $(\tilde{\mathcal{H}}, \tilde{U}, \tilde{v}_0)$ are called equivalent if there is a unitary operator W from $\mathcal{H}$ onto $\tilde{\mathcal{H}}$ such that $Wv_0 = \tilde{v}_0$ and $WUW^{-1} = \tilde{U}$. It is clear that $\delta_0 = (1, 0, 0, \ldots)^t$ is cyclic for any CMV matrix $\mathcal{C}$. Moreover, every class of equivalent unitary models contains exactly one CMV model $(\ell^2, \mathcal{C}, \delta_0)$.

5. A model in the space $L^2(\mathbb{T}, d\mu)$ of a completely nonunitary contraction with rank one defects

Theorem 5.1. *Let T be a completely nonunitary contraction with rank one defects. Then there exists a probability measure μ on $\mathbb{T}$ such that T is unitarily equivalent to the following operator*

$$\mathfrak{T}h(\zeta) = P_{\mathfrak{H}}\big(\zeta h(\zeta)\big), \quad h \in \mathfrak{H} := L^2(\mathbb{T}, d\mu) \ominus \mathbb{C}, \tag{5.1}$$

where $P_{\mathfrak{H}}$ is the orthogonal projection in $L^2(\mathbb{T}, d\mu)$ onto $\mathfrak{H}$. The Schur function associated with μ is exactly the characteristic function of T.

Proof. Include T into a prime unitary colligation

$$\Delta = \left\{ \begin{pmatrix} S & G \\ F & T \end{pmatrix};\ \mathbb{C}, \mathbb{C}, H \right\}.$$

The characteristic function Θ_Δ agrees with the characteristic function of T^*. By Theorem 3.1 the vector $\vec{1} = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ is cyclic for the unitary operator $U = \left(\begin{smallmatrix} S & G \\ F & T \end{smallmatrix}\right)$.

Let $E_U(\zeta)$ be the resolution of identity for U. Define $d\mu(\zeta) := (dE_U(\zeta)\vec{1}, \vec{1})$ and put

$$\mathcal{U}f(\zeta) = \zeta f(\zeta)$$

the unitary multiplication operator in $L^2(\mathbb{T}, d\mu)$. By the spectral theorem for unitaries with cyclic vectors (cf. [40, Section 1.4.5]) there exists a unitary operator $W : \mathbb{C} \oplus H \to L^2(\mathbb{T}, d\mu)$ such that

$$U = W^{-1}\mathcal{U}W \quad \text{and} \quad W\vec{1} = 1.$$

It follows that W takes the block-operator form

$$W = \begin{pmatrix} 1 & 0 \\ 0 & V \end{pmatrix} : \begin{pmatrix} \mathbb{C} \\ H \end{pmatrix} \to \begin{pmatrix} \mathbb{C} \\ \mathfrak{H} \end{pmatrix},$$

where $\mathfrak{H} = L^2(\mathbb{T}, d\mu) \ominus \mathbb{C}$, $V : H \to L^2(\mathbb{T}, d\mu) \ominus \mathbb{C}$ is a unitary operator. If $\mathfrak{T}$ is given by (5.1), then

$$\mathfrak{T} := P_{\mathfrak{H}}\mathcal{U}\restriction\mathfrak{H} = VTV^{-1},$$

i.e., T is unitarily equivalent to $\mathfrak{T}$. Clearly, $\mathcal{U}$ has the block form

$$\mathcal{U} = \begin{pmatrix} P_{\mathbb{C}}\mathcal{U}\restriction\mathbb{C} & P_{\mathbb{C}}\mathcal{U}\restriction\mathfrak{H} \\ P_{\mathfrak{H}}\mathcal{U}\restriction\mathfrak{H} & \mathfrak{T} \end{pmatrix},$$

where $P_{\mathbb{C}}$ is the orthogonal projection in $L^2(\mathbb{T}, d\mu)$ onto the subspace $\mathbb{C}$ of the constant functions in $L^2(\mathbb{T}, d\mu)$. The unitary colligation Δ is unitarily equivalent to the unitary colligation

$$\left\{ \begin{pmatrix} P_{\mathbb{C}}\mathcal{U}\restriction\mathbb{C} & P_{\mathbb{C}}\mathcal{U}\restriction\mathfrak{H} \\ P_{\mathfrak{H}}\mathcal{U}\restriction\mathfrak{H} & \mathfrak{T} \end{pmatrix}, \mathbb{C}, \mathbb{C}, \mathfrak{H} \right\}. \tag{5.2}$$

Note that

$$P_{\mathbb{C}}(\mathcal{U}1) = \int_{\mathbb{T}} \zeta\, d\mu, \qquad P_{\mathfrak{H}}(\mathcal{U}\,1) = \zeta - \int_{\mathbb{T}} \zeta\, d\mu, \qquad P_{\mathbb{C}}(\mathcal{U}^*\,1) = \bar{\zeta} - \int_{\mathbb{T}} \bar{\zeta}\, d\mu.$$

Let $F(z) = ((U + zI)(U - zI)^{-1}\vec{1}, \vec{1})$. Then

$$F(z) = \big((\mathcal{U} + zI)(\mathcal{U} - zI)^{-1}1, 1\big) = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, d\mu(\zeta),$$

i.e., F is the Carathéodory function associated with μ. From Theorem 3.3 we conclude

$$\overline{\Theta_\Delta(\bar{z})} = \frac{1}{z} \frac{F(z) - 1}{F(z) + 1},$$

and so by (4.12) $\overline{\Theta_\Delta(\bar{z})}$ agrees with the Schur function associated with μ. $\square$

Let $\{\Phi_n\}$ be the system of monic polynomials orthogonal with respect to μ, and let $\{\alpha_n\}$ be the corresponding Verblunsky coefficients. By Geronimus' theorem $\{\alpha_n\}$ are the Schur parameters of f. Let $\mathfrak{H}^{(c)}$ be the controllable subspace of the unitary colligation (5.2). From (3.1) it follows that

$$\big(\mathfrak{H}^{(c)}\big)^\perp = L^2(\mathbb{T}, d\mu) \ominus \overline{\operatorname{span}}\big\{\zeta^n,\ n = 0, 1, \ldots\big\}.$$

If μ is a nontrivial measure, then in view of (4.10) we obtain

$$\|P_{(\mathfrak{H}^{(c)})^\perp}\bar{\zeta}\| = \prod_{n=0}^{\infty} \big(1 - |\alpha_n|^2\big)^{1/2}.$$

The latter is equivalent to

$$\big\|P_{(\mathfrak{H}^{(c)})^\perp} P_{\mathbb{C}}(\mathcal{U}^*1)\big\| = \prod_{n=0}^{\infty} \big(1 - |\alpha_n|^2\big)^{1/2}.$$

Hence, from (2.10) and (2.7) we have the equivalence

$$\overline{\operatorname{span}}\{\mathfrak{T}^n\mathfrak{D}_{\mathfrak{T}^*},\ n=0,1,\ldots\}=\mathfrak{H} \quad\Leftrightarrow\quad \sum_{n=0}^{\infty}|\alpha_n|^2=\infty. \tag{5.3}$$

Remark 5.2. By the construction of Theorem 5.1, the Schur function f associated with μ is exactly $\overline{\Theta_\Delta(\bar{z})}$. Another (unitary equivalent) models of T are connected with the operators $U_\lambda = \left(\begin{smallmatrix}\bar\lambda S & G\\ \bar\lambda F & T\end{smallmatrix}\right)$, where $|\lambda|=1$. The characteristic function of the unitary colligation

$$\Delta_\lambda=\left\{\begin{pmatrix}\bar\lambda S & G\\ \bar\lambda F & T\end{pmatrix},\mathbb{C},\mathbb{C},H\right\}$$

is $\bar\lambda\Theta_\Delta$. The model operator $\mathfrak{T}_\lambda$ takes the form

$$\mathfrak{H}_\lambda=L^2(\mathbb{T},d\mu_\lambda)\ominus\mathbb{C},\qquad \mathfrak{T}_\lambda h(\zeta)=P_{\mathfrak{H}_\lambda}\big(\zeta h(\zeta)\big),\quad h(\zeta)\in\mathfrak{H}_\lambda.$$

The Schur function f_λ associated with μ_λ is $f_\lambda=\lambda f$. The connection between the Carathéodory functions $F_\lambda(z)=((U_\lambda+zI)(U_\lambda-zI)^{-1}\vec{1},\vec{1})$ and F is given by

$$F_\lambda(z)=\frac{(1-\lambda)+(1+\lambda)F(z)}{(1+\lambda)+(1-\lambda)F(z)}.$$

The measures μ_λ are known as the *Aleksandrov measures* associated with μ [40, Section 1.3.9].

6. Truncated CMV matrices

6.1. Truncated CMV matrix as a model for contractions with rank one defects

Let $\mathcal{C}=\mathcal{C}(\{\alpha_n\})$ be the CMV matrix given by (4.17). Recall that $\mathcal{C}(\{\alpha_n\})$ is the matrix representation of the unitary operator $\mathcal{U}$ of multiplication by ζ in $L^2(\mathbb{T},d\mu)$, where μ is the probability measure with Verblunsky coefficients $\{\alpha_n\}$. By the Geronimus theorem the Schur parameters of the Schur function (4.12) associated with μ are $\{\alpha_n\}$.

The matrix $\mathcal{C}$ determines the unitary operator in the space $\ell^2(\mathbb{Z}_+)$ (respectively in $\mathbb{C}^{N+1}$ in the case of $(N+1)\times(N+1)$ matrix). The vector $\delta_0=(1,0,0,\ldots)^t$ is cyclic for $\mathcal{C}$. Consider the matrix

$$\mathcal{T}=\mathcal{T}\big(\{\alpha_n\}\big)=\begin{pmatrix}-\bar\alpha_1\alpha_0 & -\rho_1\alpha_0 & 0 & 0 & \ldots\\ \bar\alpha_2\rho_1 & -\bar\alpha_2\alpha_1 & \bar\alpha_3\rho_2 & \rho_3\rho_2 & \ldots\\ \rho_2\rho_1 & -\rho_2\alpha_1 & -\bar\alpha_3\alpha_2 & -\rho_3\alpha_2 & \ldots\\ 0 & 0 & \bar\alpha_4\rho_3 & -\bar\alpha_4\alpha_3 & \ldots\\ \ldots & \ldots & \ldots & \ldots & \ldots\end{pmatrix} \tag{6.1}$$

obtained from $\mathcal{C}$ by deleting the first row and the first column. It is clear from (4.20) that a semi-infinite $\mathcal{T}$ takes on the three-diagonal 2×2 block-matrix form

$$\mathcal{T} = \begin{pmatrix} \mathcal{B}_1 & \mathcal{C}_1 & 0 & 0 & 0 & \cdot & \cdot \\ \mathcal{A}_1 & \mathcal{B}_2 & \mathcal{C}_2 & 0 & 0 & \cdot & \cdot \\ 0 & \mathcal{A}_2 & \mathcal{B}_3 & \mathcal{C}_3 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

where $\mathcal{A}_n$, $\mathcal{B}_n$, and $\mathcal{C}_n$ are defined in (4.21). Henceforth $\mathcal{T}$ is called a *truncated* CMV matrix. $\mathcal{T}$ is the matrix of the operator $\mathfrak{T} = P_{\mathfrak{H}}\mathcal{U}\restriction\mathfrak{H}$, where $P_{\mathfrak{H}}$ is the orthogonal projection in $L^2(\mathbb{T}, d\mu)$ onto the subspace $\mathfrak{H} = L^2(\mathbb{T}, d\mu) \ominus \mathbb{C}$.

It is easy to see that given $\mathcal{T}$ (6.1), the values α_n are uniquely determined. Indeed, from $(2,2)$ and $(3,2)$ entries we have by (4.8) $|\alpha_1|^2 = |\bar{\alpha}_2\alpha_1|^2 + \rho_2^2|\alpha_1|^2$, so $|\alpha_1|$ and $\rho_1 > 0$ are known, and we find α_0, α_2 from $(1,2)$ and $(2,1)$ entries of (6.1). From $(2,1)$ and $(2,2)$ entries we get $\rho_2 > 0$, then α_1, α_3, etc. We call $\alpha_n = \alpha_n(\mathcal{T})$ the *parameters* of $\mathcal{T}$ (6.1).

As it was mentioned in Section 4.3, $\mathcal{C} = \mathcal{L}\mathcal{M}$, $\mathcal{L}$ and $\mathcal{M}$ are defined in (4.22). Given a matrix A, we denote by A_{r} (A_{c}) the matrix obtained from A by deleting the first row (column). Clearly, $A_{\mathrm{rc}} = (A_{\mathrm{r}})_{\mathrm{c}}$. So we have $\mathcal{T} = \mathcal{C}_{\mathrm{rc}} = \mathcal{L}_{\mathrm{r}}\mathcal{M}_{\mathrm{c}}$. $\mathcal{M}_{\mathrm{c}}$ is isometric with $\dim\operatorname{ran}(I - \mathcal{M}_{\mathrm{c}}\mathcal{M}_{\mathrm{c}}^*) = 1$, whereas $\mathcal{L}_{\mathrm{r}}$ is coisometric with $\dim\operatorname{ran}(I - \mathcal{L}_{\mathrm{r}}^*\mathcal{L}_{\mathrm{r}}) = 1$.

Let $P_{\delta_0{}^\perp}$ be the orthogonal projection in $\ell^2(\mathbb{Z}_+)$ ($\mathbb{C}^{N+1}$) onto the subspace $\delta_0{}^\perp \cong \ell^2(\mathbb{N})$ ($\mathbb{C}^N$). Then the matrix $\mathcal{T}$ determines on the Hilbert space $\delta_0^\perp$ the operator $\mathcal{T} = P_{\delta_0^\perp}\mathcal{C}\restriction\delta_0{}^\perp$. Let the operators (matrices) $\mathcal{S}:\mathbb{C} \to \mathbb{C}$, $\mathcal{F}:\mathbb{C} \to \delta_0^\perp$ and $\mathcal{G}:\delta_0^\perp \to \mathbb{C}$ be given by

$$\mathcal{S}1 = \bar{\alpha}, \qquad \mathcal{F}1 = \begin{pmatrix} \rho_0 \\ 0 \\ \vdots \\ 0 \\ \vdots \end{pmatrix}, \qquad \mathcal{G}\begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \\ \vdots \end{pmatrix} = \bar{\alpha}_1\rho_0 h_1 + \rho_1\rho_0 h_2.$$

Hence, the matrix $\mathcal{C}$ takes the block form

$$\mathcal{C} = \begin{pmatrix} \mathcal{S} & \mathcal{G} \\ \mathcal{F} & \mathcal{T} \end{pmatrix}.$$

From (2.10) it follows that

$$\left\| \mathcal{G}\begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \\ \vdots \end{pmatrix} \right\|^2 = \left\| D_{\mathcal{T}}\begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \\ \vdots \end{pmatrix} \right\| = \rho_0^2|\bar{\alpha}_1 h_1 + \rho_1 h_2|^2, \quad \mathfrak{D}_{\mathcal{T}} = \bigl\{\lambda(\alpha_1\delta_1 + \rho_1\delta_2),\ \lambda \in \mathbb{C}\bigr\},$$

$$\left\| \mathcal{F}^*\begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \\ \vdots \end{pmatrix} \right\|^2 = \left\| D_{\mathcal{T}^*}\begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \\ \vdots \end{pmatrix} \right\| = \rho_0^2|h_1|^2, \quad \mathfrak{D}_{\mathcal{T}^*} = \{\lambda\delta_1,\ \lambda \in \mathbb{C}\},$$

$$D_{\mathcal{T}}h = \rho_0(h, \alpha_1\delta_1 + \rho_1\delta_2)(\alpha_1\delta_1 + \rho_1\delta_2), \qquad D_{\mathcal{T}^*}h = \rho_0(h, \delta_1)\delta_1, \quad h \in \ell^2(\mathbb{N})\bigl(\mathbb{C}^N\bigr),$$
$$\mathcal{T}(\alpha_1\delta_1 + \rho_1\delta_2) = -\alpha_0\delta_1. \tag{6.2}$$

Since δ_0 is the cyclic vector for $\mathcal{C}$, then by Theorem 3.1 the unitary colligation

$$\Delta_{\mathcal{C}} = \left\{ \begin{pmatrix} \mathcal{S} & \mathcal{G} \\ \mathcal{F} & \mathcal{T} \end{pmatrix};\ \mathbb{C}, \mathbb{C}, \delta_0^{\perp} \right\} \tag{6.3}$$

is prime, and $\mathcal{T}$ is a completely nonunitary operator with rank one defects on the Hilbert spaces $\ell_2(\mathbb{N})$ or $\mathbb{C}^N$.

Let

$$F(z) = \bigl((\mathcal{C} + zI)(\mathcal{C} - zI)^{-1}\delta_0, \delta_0\bigr), \quad f(z) = \frac{1}{z}\frac{F(z) - 1}{F(z) + 1} \tag{6.4}$$

be the Carathéodory and the Schur functions associated with $\mathcal{C}$. By Theorems 2.2 and 3.3, f agrees with the characteristic function of $\mathcal{T}$.

Proposition 6.1.

(1) *For a semi-infinite truncated CMV matrix $\mathcal{T} = \mathcal{T}(\{\alpha_n\})$ the following statements are equivalent*:
  (a) *the matrix $\mathcal{T}$ does not contain a unilateral shift*;
  (b) *the matrix $\mathcal{T}^*$ does not contain a unilateral shift*;
  (c) $\overline{\operatorname{span}}\{\mathcal{T}^n\delta_1,\ n = 0, 1, \ldots\} = \ell^2(\mathbb{N})$;
  (d) $\overline{\operatorname{span}}\{\mathcal{T}^{*n}(\alpha_1\delta_1 + \rho_1\delta_2),\ n = 0, 1, \ldots\} = \ell^2(\mathbb{N})$;
  (e) $\sum_{n=0}^{\infty} |\alpha_n|^2 = \infty$;
  (f) $\ln(1 - |f(e^{it})|^2) \notin L^1[-\pi, \pi]$.

(2) *If $\mathcal{T}$ is a semi-infinite truncated CMV matrix, and one of the conditions*
  (a) $\limsup_{n\to\infty} |\alpha_n| = 1$,
  (b) $\lim_{n\to\infty} \alpha_n\alpha_{n+m} = 0$ *for* $m = 1, 2, \ldots,$ *but*

$$\limsup_{n\to\infty} |\alpha_n| > 0$$

is fulfilled, then

$$\text{s-}\lim_{n\to\infty} \mathcal{T}^n = \text{s-}\lim_{n\to\infty} \mathcal{T}^{*n} = 0.$$

(3) *If $\mathcal{T}$ is a finite truncated CMV matrix, then* $\lim_{n\to\infty} \|\mathcal{T}^n\| = 0$.

Proof. (1) Since $\{\alpha_n\}$ are the Schur parameters of the Schur function f associated with the full CMV matrix $\mathcal{C}(\{\alpha_n\})$, and f agrees with the characteristic function of $\mathcal{T}(\{\alpha_n\})$, the equivalence of the statements (a)–(f) follows from (2.3), (2.4), (2.7), (2.9), (4.14), (6.2), (5.3), and Theorems 3.1 and 4.1.

(2) Each condition (a) or (b) implies f is inner (see Section 4.2). Hence $\mathcal{T}$ belongs to the class C_{00}, i.e., $\text{s-}\lim_{n\to\infty} \mathcal{T}^n = \text{s-}\lim_{n\to\infty} \mathcal{T}^{*n} = 0$.

(3) The function f is a finite Blaschke product and so inner. Since $\mathcal{T}$ is finite-dimensional, we get $\lim_{n\to\infty} \|\mathcal{T}^n\| = 0$. $\square$

Proposition 6.2. *Let $\mathcal{T}(\{\alpha_n\})$, and $\mathcal{T}(\{\beta_n\})$ be truncated CMV matrices. Then $\mathcal{T}(\{\alpha_n\})$ and $\mathcal{T}(\{\beta_n\})$ are unitarily equivalent if and only if $\beta_n = e^{it}\alpha_n$ for all n and $t \in [0, 2\pi)$. Moreover, if $\mathcal{V}$ is the diagonal unitary matrix of the form*

$$\mathcal{V} = \operatorname{diag}\bigl(e^{it}, 1, e^{it}, 1, \dots\bigr), \tag{6.5}$$

then

$$\mathcal{V}\mathcal{T}\bigl(\{\alpha_n\}\bigr)\mathcal{V}^{-1} = \mathcal{T}\bigl(\bigl\{e^{it}\alpha_n\bigr\}\bigr). \tag{6.6}$$

Proof. Consider two CMV matrices $\mathcal{C}(\{\alpha_n\})$ and $\mathcal{C}(\{\beta_n\})$, and associated with them Schur functions f_α and f_β. Since these functions agree with characteristic functions of $\mathcal{T}(\{\alpha_n\})$ and $\mathcal{T}(\{\beta_n\})$, respectively, the operators $\mathcal{T}(\{\alpha_n\})$ and $\mathcal{T}(\{\beta_n\})$ are unitarily equivalent if and only if f_α and f_β differ by a scalar unimodular factor, which in turn yields $\beta_n = e^{it}\alpha_n$ for all n and $t \in [0, 2\pi)$.

Equality (6.6) with $\mathcal{V}$ (6.5) can be verified by the direct calculation based on (4.18), (4.19). So $\mathcal{T}(\{\alpha_n\})$ and $\mathcal{T}(\{e^{it}\alpha_n\})$ are unitarily equivalent. $\square$

Remark 6.3. The similar problem for "full" CMV matrices can be considered as well. Let two CMV matrices $\mathcal{C}(\{\alpha_n\})$ and $\mathcal{C}(\{\beta_n\})$ be unitarily equivalent by a unitary preserving δ_0. Then they are identical (see [41, Theorem 2.3]). In general, two unitaries with simple spectra are unitarily equivalent if and only if their spectral measures are in the same measure class. This is a standard issue in what is called multiplicity theory. So, two CMV matrices are unitarily equivalent if and only if their measures are mutually absolutely continuous. For instance, a CMV matrix is unitarily equivalent to the free one ($\alpha_n \equiv 0$) if and only if the associated measure μ has the property $\mu' > 0$ a.e. and does not have a singular part.

From (6.6) it follows that

$$\mathcal{T}\bigl(\bigl\{e^{it}\alpha_n\bigr\}\bigr) = e^{it\mathcal{A}}\mathcal{T}\bigl(\{\alpha_n\}\bigr)e^{-it\mathcal{A}}, \quad t \in \mathbb{R},$$

where $\mathcal{A}$ is a self-adjoint diagonal matrix $\mathcal{A} = \operatorname{diag}(1, 0, 1, 0 \dots)$. Hence the matrix $\mathcal{T}(\{e^{it}\alpha_n\})$ satisfies the differential equation

$$\frac{d\mathcal{T}(t)}{dt} = i\bigl(\mathcal{A}\mathcal{T}(t) - \mathcal{T}(t)\mathcal{A}\bigr)$$

and $\mathcal{T}(0) = \mathcal{T}(\{\alpha_n\})$.

The next theorem states that truncated CMV matrices are models of completely nonunitary contractions with rank one defects.

Theorem 6.4. *Let T be a completely nonunitary contraction with rank one defects acting on infinite-dimensional separable Hilbert space H (respectively, finite-dimensional Hilbert space). Then T is unitarily equivalent to the operator acting on $\ell^2(\mathbb{N})$ (respectively, on $\mathbb{C}^N$ in the case $\dim H = N$) determined by the truncated CMV matrix $\mathcal{T} = \mathcal{T}(\{\alpha_n\})$, where $\{\alpha_n\}$ are the Schur*

parameters of the characteristic function of T. In particular, every completely nonunitary contraction with rank one defects is a product of co-isometric and isometric operators with rank one defects.

Proof. Include T into a prime unitary colligation

$$\Delta=\left\{\begin{pmatrix} S & G\\ F & T\end{pmatrix};\ \mathbb{C},\mathbb{C},H\right\}.$$

By Theorem 3.1 the vector $\vec{1}=\binom{1}{0}$ is a cyclic for the unitary operator $U=\left(\begin{smallmatrix} S & G\\ F & T\end{smallmatrix}\right)$. From the results of [13,14] (see also [39,40]) there exists a unique CMV matrix $\mathcal{C}$ such that

$$U=W^{-1}\mathcal{C}W,\qquad \delta_0=W\vec{1},$$

where W is a unitary operator from $\mathbb{C}\oplus H$ onto $\ell^2(\mathbb{Z}_+)$ $(\mathbb{C}^{N+1})$, and $\delta_0=(1,0,0,\dots)^t$. It follows that the operator W takes the block-operator form

$$W=\begin{pmatrix} 1 & 0\\ 0 & \mathcal{X}\end{pmatrix}:\begin{pmatrix}\mathbb{C}\\ H\end{pmatrix}\to\begin{pmatrix}\mathbb{C}\\ \delta_0^{\perp}\end{pmatrix},$$

where $\mathcal{X}:H\to\delta_0^{\perp}$ is a unitary operator. Hence $\mathcal{T}=\mathcal{X}T\mathcal{X}^{-1}$, i.e., the operator T is unitarily equivalent to the operator in $l_2(\mathbb{N})$ $(\mathbb{C}^N)$ given by the truncated CMV matrix $\mathcal{T}=\mathcal{T}(\{\alpha_n\})$. From representation (4.11) of $F(z)=((U+zI)(U-zI)^{-1}\vec{1},\vec{1})$ and Theorem 3.3 it follows that $\{\alpha_n\}$ are the Schur parameters of function $\overline{\Theta_\Delta(\bar z)}$ that agrees with the characteristic function of T.

Let $\mathcal{Q}$ be an arbitrary unitary operator in $\delta_0^{\perp}$. Since $\mathcal{T}=\mathcal{L}_{\rm r}\mathcal{M}_{\rm c}$, we get

$$T=\mathcal{X}^{-1}\mathcal{T}\mathcal{X}=\mathcal{X}^{-1}\mathcal{L}_{\rm r}\mathcal{M}_{\rm c}\mathcal{X}=\mathcal{X}^{-1}\mathcal{L}_{\rm r}\mathcal{Q}\mathcal{Q}^{-1}\mathcal{M}_{\rm c}\mathcal{X}=LM,$$

where $M=\mathcal{Q}^{-1}\mathcal{M}_{\rm c}\mathcal{X}$ is an isometric operator with rank one defect, and $L=\mathcal{X}^{-1}\mathcal{L}_{\rm r}\mathcal{Q}$ is a co-isometric operator with rank one defect. □

Note that the unitary colligation (6.3) is unitary equivalent to the unitary colligation (5.2).

6.2. The Livšic theorem for quasi-unitary contractive extensions and the corresponding truncated CMV matrix

Let V be an isometric operator acting on some Hilbert space H with the domain $\operatorname{dom} V$ and the range $\operatorname{ran} V$. The numbers $\dim(H\ominus\operatorname{dom} V)$ and $\dim(H\ominus\operatorname{ran} V)$ are called the defect indices of V. The isometric operator V is called prime if there is no nontrivial subspace on which V is unitary. In [29,30] M. Livšic developed the spectral theory of isometric operators with equal defect indices, and their quasi-unitary extensions. A nonunitary operator S on H is called a quasi-unitary extension of the isometric operator V with the defect indices (n,n), if S agrees with V on $\operatorname{dom} V$ and maps $H\ominus\operatorname{dom} V$ into $H\ominus\operatorname{ran} V$.

Let $\vec{U}$ be the bilateral shift in $\ell^2(\mathbb{Z})$, i.e., $\vec{U}\delta_k=\delta_{k-1}$, $k\in\mathbb{Z}$, where $\{\delta_k,\ k\in\mathbb{Z}\}$ is the canonical orthonormal basis in $\ell^2(\mathbb{Z})$. Define $\vec{V}_0$ by

$$\operatorname{dom}\vec{V}_0=\delta_o^{\perp},\qquad \vec{V}_0=\vec{U}\restriction\operatorname{dom}\vec{V}_0.$$

Then $\operatorname{ran}\vec{V}_0 = \delta_{-1}^{\perp}$. Let the quasi-unitary extension $\vec{S}_0$ of $\vec{V}_0$ be given by $\vec{S}_0\delta_0 = 0$, $\vec{S}_0 \upharpoonright \operatorname{dom}\vec{V}_0 = \vec{V}_0$. Then each point of $\mathbb{D}$ is the eigenvalue of $\vec{S}_0$. So the spectrum of $\vec{S}_0$ agrees with $\overline{\mathbb{D}}$. The following result is essentially due to M. Livšic [29].

Theorem 6.5. *Let S be a quasi-unitary contractive extension of a prime isometric operator V with the defect indices* $(1,1)$. *If the whole open disk $\mathbb{D}$ consists of the point spectrum of S, then V and S are unitarily equivalent to $\vec{V}_0$ and $\vec{S}_0$, respectively.*

Clearly, the rank of the defect operators $(I - \vec{S}_0^*\vec{S}_0)^{1/2}$ and $(I - \vec{S}_0\vec{S}_0^*)^{1/2}$ is equal to one. Since the point spectrum of $\vec{S}_0$ is $\mathbb{D}$, the Sz.-Nagy–Foias characteristic function Θ of $\vec{S}_0$ is identically equal to zero. On the other hand, one can easily show (and it is well known) that a completely nonunitary contraction with rank one defects and zero characteristic function is unitarily equivalent to the operator $S \oplus S^*$, where S is the unilateral shift in $\ell^2(\mathbb{N})$. So the operators $\vec{S}_0$ and $S \oplus S^*$ are unitarily equivalent. Since all Schur parameters of the function $\Theta = 0$ are zeros, the corresponding truncated CMV matrix $\mathcal{T}_0 = \|t_0(i,j)\|$ takes the form

$$\mathcal{T}_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix},$$

i.e., $t_0(2k, 2k+2) = t_0(2k+1, 2k-1) = 1$, $k \geqslant 1$, and the rest $t_0(i,j) = 0$. The matrix $\mathcal{T}_0$ is a submatrix of the free CMV matrix $\mathcal{C}_0$ corresponding to zero Schur parameters. Each point z of $\mathbb{D}$ is the eigenvalue of $\mathcal{T}_0$. The corresponding eigensubspace is

$$\mathfrak{N}_z = \{\lambda(0, 1, 0, z, 0, z^2, 0, z^3, \dots)^t,\ \lambda \in \mathbb{C}\}.$$

Hence, the spectrum of $\mathcal{T}_0$ is the closed unit disk $\overline{\mathbb{D}}$.

Let $\mathcal{V}_0$ be the operator in $\ell^2(\mathbb{N})$

$$\operatorname{dom}\mathcal{V}_0 = \ell^2(\mathbb{N}) \ominus \{c\delta_2\} = \ker D_{\mathcal{T}_0}, \qquad \mathcal{V}_0 = \mathcal{T}_0 \upharpoonright \operatorname{dom}\mathcal{V}_0. \tag{6.7}$$

Then $\operatorname{ran}\mathcal{V}_0 = \ell^2(\mathbb{N}) \ominus \{c\delta_1\} = \ker D_{\mathcal{T}_0^*}$, and $\mathcal{V}_0$ is isometric with the defect indices $(1,1)$. The contraction $\mathcal{T}_0$ is the quasi-unitary extension of $\mathcal{V}_0$ with the zero characteristic function. Therefore, the truncated CMV matrix $\mathcal{T}_0$ is unitarily equivalent to the operator $\vec{S}_0$, and by Livšic theorem [29] the isometric operator $\mathcal{V}_0$ is unitarily equivalent to $\vec{V}_0$.

All other quasi-unitary contractive extensions of $\mathcal{V}_0$ are given by the truncated CMV matrices $\mathcal{T} = \|t(i,j)\|$

$$\mathcal{T} = \begin{pmatrix} 0 & -re^{i\varphi} & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \tag{6.8}$$

i.e., $t(2k, 2k+2) = t(2k+1, 2k-1) = 1$, $k \geqslant 1$, $t(1,2) = -re^{i\varphi}$, $r \in (0,1)$, φ is an arbitrary number from the interval $[0, 2\pi)$, and the rest $t(i,j) = 0$. The characteristic function of $\mathcal{T}$ is the constant function $\Theta = re^{i\varphi}$. The spectrum of each such matrix is the unit circle $\mathbb{T}$. Because $|\Theta^{-1}| = r^{-1}$, each of such matrix is similar to unitary matrix [42, Theorem IX.1.2].

The matrices $\mathcal{T}_0$ and $\mathcal{T}$ contain the shift

$$\operatorname{dom}\mathcal{W} = \overline{\operatorname{span}}\{\delta_1, \delta_3, \ldots, \delta_{2n-1}, \ldots\}, \quad \mathcal{W}\left(\sum_{n=1}^{\infty} h_n\delta_{2n-1}\right) = \sum_{n=1}^{\infty} h_n\delta_{2n+1}.$$

The matrices $\mathcal{T}_0^*$ and $\mathcal{T}^*$ contain the shift

$$\operatorname{dom}\mathcal{W}_* = \overline{\operatorname{span}}\{\delta_2, \delta_4, \ldots, \delta_{2n}, \ldots\}, \quad \mathcal{W}_*\left(\sum_{n=1}^{\infty} h_n\delta_{2n}\right) = \sum_{n=1}^{\infty} h_n\delta_{2n+2}.$$

Let T be a completely nonunitary contraction with rank one defects and the constant characteristic function Θ, $0 < |\Theta(z)| = r < 1$. Then by Theorem 6.4 T is unitarily equivalent to the truncated CMV matrices (6.8).

6.3. *Sub-matrices of truncated CMV matrices and iterates of their Schur functions*

Along with truncated CMV matrices $\mathcal{T}(\{\alpha_n\})$ (6.1), we consider here truncated CMV matrices $\widetilde{\mathcal{T}}(\{\alpha_n\})$ obtained from the alternate CMV matrix $\widetilde{\mathcal{C}}(\{\alpha_n\})$ (4.23) by the same procedure. The matrix $\widetilde{\mathcal{T}}(\{\alpha_n\}$ is the transpose of $\mathcal{T}(\{\alpha_n\})$

$$\widetilde{\mathcal{T}} = \begin{pmatrix} -\bar\alpha_1\alpha_0 & \bar\alpha_2\rho_1 & \rho_2\rho_1 & 0 & \ldots \\ -\rho_1\alpha_0 & -\bar\alpha_2\alpha_1 & -\rho_2\alpha_1 & 0 & \ldots \\ 0 & \bar\alpha_3\rho_2 & -\bar\alpha_3\alpha_2 & \bar\alpha_4\rho_3 & \ldots \\ 0 & \rho_3\rho_2 & -\rho_3\alpha_2 & -\bar\alpha_4\alpha_3 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}, \tag{6.9}$$

and

$$\widetilde{\mathcal{T}}\big(\{\alpha_n\}\big) = \mathcal{T}^t\big(\{\alpha_n\}\big) = (\mathcal{M}_{\mathrm{c}})^t(\mathcal{L}_{\mathrm{r}})^t = \mathcal{M}_{\mathrm{r}}\mathcal{L}_{\mathrm{c}}.$$

As in Section 6.1, it is not hard to show that $\widetilde{\mathcal{T}}(\{\alpha_n\})$ is a completely nonunitary contraction with rank one defects, and its characteristic function $\widetilde{f}$ agrees with the Schur function associated with Verblunsky coefficients (Schur parameters) $\{\alpha_n\}$. Indeed (cf. (6.4))

$$(\widetilde{\mathcal{C}} + zI)(\widetilde{\mathcal{C}} - zI)^{-1} = \big(\mathcal{C}^t + zI\big)\big(\mathcal{C}^t - zI\big)^{-1} = \big((\mathcal{C} + zI)(\mathcal{C} - zI)^{-1}\big)^t,$$

and so $\widetilde{F}(z) := ((\widetilde{\mathcal{C}} + zI)(\widetilde{\mathcal{C}} - zI)^{-1}\delta_0, \delta_0) = F(z)$, $\widetilde{f} = f$, as claimed. So, the matrices $\mathcal{T}(\{\alpha_n\})$ and $\widetilde{\mathcal{T}}(\{\alpha_n\})$ are unitarily equivalent.

Denote by $\mathcal{T}^{(k)}$ ($\widetilde{\mathcal{T}}^{(k)}$) the matrix obtained from $\mathcal{T}$ ($\widetilde{\mathcal{T}}$) by deleting the first k rows and columns. The following result provides the characteristic function of $\mathcal{T}^{(k)}$.

Theorem 6.6. *Let μ be a probability measure on $\mathbb{T}$ with Verblunsky coefficients $\{\alpha_n\}_{n=0}^N$, $N \leqslant \infty$, and let f, $\mathcal{C}(\{\alpha_n\})$, $\widetilde{\mathcal{C}}(\{\alpha_n\})$, $\mathcal{T}(\{\alpha_n\})$, $\widetilde{\mathcal{T}}(\{\alpha_n\})$ be the corresponding Schur function, CMV and truncated CMV matrices, respectively. Then $\mathcal{T}^{(k)}$, $\widetilde{\mathcal{T}}^{(k)}$ are completely nonunitary contractions with rank one defects, and the following relations hold*:

$$\begin{aligned}\mathcal{T}^{(2m-1)}\big(\{\alpha_n\}_{n=0}^N\big) &= \widetilde{\mathcal{T}}\big(\{\alpha_n\}_{n=2m-1}^N\big),\\ \mathcal{T}^{(2m)}\big(\{\alpha_n\}_{n=0}^N\big) &= \mathcal{T}\big(\{\alpha_n\}_{n=2m}^N\big), \quad m = 1, 2, \ldots.\end{aligned}$$

So, the characteristic function of $\mathcal{T}^{(k)}$ agrees with the kth Schur iterate of f.

Proof. The relations

$$\mathcal{T}^{(1)}\big(\{\alpha_n\}_{n=0}^N\big) = \widetilde{\mathcal{T}}\big(\{\alpha_n\}_{n=1}^N\big), \qquad \widetilde{\mathcal{T}}^{(1)}\big(\{\alpha_n\}_{n=1}^N\big) = \mathcal{T}\big(\{\alpha_n\}_{n=2}^N\big)$$

follows directly from (6.1) and (6.9). The rest is a matter of simple induction and the definition of the kth Schur iterates. □

The relation between the characteristic functions of the sub-matrices $\mathcal{T}^{(k)}(\{\alpha\}_{n=0}^N)$ and the kth Schur iterates established in the above theorem is a complete analog of the result concerning the connections between m-functions of a Jacobi matrix and its sub-matrices [22].

Let us now go back to the model of Section 5.

Theorem 6.7. *Let μ be a probability measure on $\mathbb{T}$ with Verblunsky coefficients $\{\alpha_n\}_{n=0}^N$, $N \leqslant \infty$. Consider three subspaces in $L^2(\mathbb{T}, \mu)$:*

$$\begin{aligned}\mathcal{H}_{2m} &:= \operatorname{span}\big\{1, \zeta, \bar\zeta, \zeta^2, \bar\zeta^2, \ldots, \zeta^m, \bar\zeta^m\big\},\\ \mathcal{H}_{2m-1} &:= \operatorname{span}\big\{1, \zeta, \bar\zeta, \zeta^2, \bar\zeta^2, \ldots, \bar\zeta^{m-1}, \zeta^m\big\},\\ \widetilde{\mathcal{H}}_{2m-1} &:= \operatorname{span}\big\{1, \bar\zeta, \zeta, \bar\zeta^2, \zeta^2, \ldots, \zeta^{m-1}, \bar\zeta^m\big\}.\end{aligned}$$

Denote by $\mathfrak{H}_{2m}$ $(\mathfrak{H}_{2m-1}, \widetilde{\mathfrak{H}}_{2m-1})$ their orthogonal complements in $L^2(\mathbb{T}, \mu)$, and by P_{2m} $(P_{2m-1},\ \widetilde{P}_{2m-1})$ the orthogonal projections onto $\mathfrak{H}_{2m}$ $(\mathfrak{H}_{2m-1},\ \widetilde{\mathfrak{H}}_{2m-1})$, respectively. Then the operators

$$\begin{aligned}\mathfrak{T}_k h(\zeta) &= P_k\big(\zeta h(\zeta)\big), \quad h(\zeta) \in \mathfrak{H}_k,\\ \widetilde{\mathfrak{T}}_{2m-1} h(\zeta) &= \widetilde{P}_m\big(\zeta h(\zeta)\big), \quad h(\zeta) \in \widetilde{\mathfrak{H}}_{2m-1},\end{aligned} \tag{6.10}$$

are completely nonunitary contractions with rank one defects. The characteristic function of $\mathfrak{T}_k$ agrees with the kth Schur iterate of the Schur function $f(\mu)$, the characteristic function of $\widetilde{\mathfrak{T}}_{2m-1}$ agrees with $(2m-1)$th Schur iterate of $f(\mu)$. So, the operator $\mathfrak{T}_k$ is unitarily equivalent to the operator

$$h(\zeta) \to P_0^{(k)}\big(\zeta h(\zeta)\big), \quad h(\zeta) \in L^2\big(\mathbb{T}, d\mu\big(\{\alpha_n\}_{n=k}^N\big)\big) \ominus \mathbb{C}, \tag{6.11}$$

where $P_0^{(k)}$ is the orthogonal projection onto $L^2(\mathbb{T}, d\mu(\{\alpha_n\}_{n=k}^N)) \ominus \mathbb{C}$. In addition, $\mathfrak{T}_{2m-1}$ is unitarily equivalent to $\widetilde{\mathfrak{T}}_{2m-1}$.

Proof. Recall that CMV matrices $\mathcal{C}(\{\alpha_n\}$ and $\widetilde{\mathcal{C}}(\{\alpha_n\})$ represent the unitary operator $Uh(\zeta) = \zeta h(\zeta)$ in $L^2(\mathbb{T}, d\mu(\{\alpha_n\}))$ with respect to the complete orthonormal systems $\{\chi_n\}$ and $\{x_n\}$, respectively. Moreover

$$\begin{aligned} \mathcal{H}_{2m} &= \operatorname{span}\{\chi_0, \chi_1, \dots, \chi_{2m}\} = \operatorname{span}\{x_0, x_1, \dots, x_{2m}\}, \\ \mathcal{H}_{2m-1} &= \operatorname{span}\{\chi_0, \chi_1, \dots, \chi_{2m-1}\}, \\ \widetilde{\mathcal{H}}_{2m-1} &= \operatorname{span}\{x_0, x_1, \dots, x_{2m-1}\}. \end{aligned}$$

Since $\mathcal{T}(\{\alpha_n\}_{n=0}^N\})$ $(\widetilde{\mathcal{T}}(\{\alpha_n\}_{n=0}^N\})$ is the matrix of $\mathfrak{T}$ (5.1) with respect to the basis $\{\chi_n\}_{n=1}^N$ $(\{x_n\}_{n=1}^N)$, the operators $\mathfrak{T}_{2m}$, $\mathfrak{T}_{2m-1}$, and $\widetilde{\mathfrak{T}}_{2m-1}$ have the matrices $\mathcal{T}^{(2m)}$, $\mathcal{T}^{(2m-1)}$, and $\widetilde{\mathcal{T}}^{(2m-1)}$, respectively. From Theorem 6.6 it follows that $\mathfrak{T}_k$ are completely nonunitary contractions with rank one defects for all k, and their characteristic functions agree with the kth Schur iterates of f. By Theorems 6.6 and 5.1 the operator $\mathfrak{T}_k$ is unitarily equivalent to the operator given by (6.11). We also have

$$\widetilde{\mathcal{T}}^{(2m-1)}\bigl(\{\alpha_n\}_{n=0}^N\bigr) = \mathcal{T}\bigl(\{\alpha_n\}_{n=2m-1}^N\bigr).$$

Therefore, the characteristic function of $\widetilde{\mathfrak{T}}^{(2m-1)}(\{\alpha_n\}_{n=0}^N)$ agrees with $(2m-1)$th iterate f_{2m-1} of f, and hence the operators $\widetilde{\mathfrak{T}}^{(2m-1)}(\{\alpha_n\}_{n=0}^N)$ and $\mathfrak{T}^{(2m-1)}(\{\alpha_n\}_{n=0}^N)$ are unitarily equivalent. $\square$

We complete the section with the general result from the contractions theory which is proved with the help of the truncated CMV model.

Theorem 6.8. *Let T be a completely nonunitary contraction with rank one defects in a separable Hilbert space H,* $\dim H \geqslant 2$*, and let $P_{\ker D_{T^*}}$, $P_{\ker D_T}$ be the orthogonal projections onto* $\ker D_{T^*}$ *and* $\ker D_T$ *in H, respectively. Then the operators*

$$T_1 := P_{\ker D_{T^*}} T \restriction \ker D_{T^*}, \qquad \widetilde{T}_1 := P_{\ker D_T} T \restriction \ker D_T$$

are unitarily equivalent completely nonunitary contractions with rank one defects, and their characteristic functions agree with the function

$$h_1(z) := \frac{1}{z} \frac{h(z) - h(0)}{1 - \overline{h(0)} h(z)},$$

where h is the characteristic function of T.

Proof. By Theorem 6.4 the operator T is unitarily equivalent to the truncated CMV matrices $\mathcal{T} = \mathcal{T}(\{\alpha_n\}_{n=0}^N)$ and $\widetilde{\mathcal{T}} = \widetilde{\mathcal{T}}(\{\alpha_n\}_{n=0}^N)$, where $\{\alpha_n\}_{n=0}^N$ are the Schur parameters of h, $N \leqslant \infty$. So, there exists a unitary operators V, $\widetilde{V} : \delta_0^\perp \to H$ such that

$$V \mathcal{T} V^{-1} = \widetilde{V} \widetilde{\mathcal{T}} \widetilde{V}^{-1} = T.$$

It follows that

$$V D_{\mathcal{T}^*} V^{-1} = D_{T^*}, \qquad \widetilde{V} D_{\widetilde{\mathcal{T}}} \widetilde{V}^{-1} = D_T,$$

and hence $V \ker D_{\mathcal{T}^*} = \ker D_{T^*}$, $\widetilde{V} \ker D_{\widetilde{\mathcal{T}}} = \ker D_T$. Due to (6.2) we have

$$\mathfrak{D}_{\mathcal{T}^*} = \mathfrak{D}_{\widetilde{\mathcal{T}}} = \operatorname{span}\{\delta_1\}$$

and

$$\mathcal{T}^{(1)} = P_{\ker D_{\mathcal{T}^*}} \mathcal{T} \restriction \ker D_{\mathcal{T}^*}, \qquad \widetilde{\mathcal{T}}^{(1)} = P_{\ker D_{\widetilde{\mathcal{T}}}} \widetilde{\mathcal{T}} \restriction \ker D_{\widetilde{\mathcal{T}}}.$$

Hence

$$V\mathcal{T}^{(1)}V^{-1} = T_1, \qquad \widetilde{V}\widetilde{\mathcal{T}}^{(1)}\widetilde{V}^{-1} = \widetilde{T}_1.$$

Now from Theorem 6.6 it follows that T_1 and $\widetilde{T}_1$ are completely nonunitary contractions with rank one defects, and their characteristic functions agree with the first Schur iterate h_1 of h. Hence T_1 and $\widetilde{T}_1$ are unitarily equivalent. □

7. Inverse spectral problems for finite and semi-infinite truncated CMV matrices

Consider a $N \times N$ truncated CMV matrix

$$\mathcal{T} = \mathcal{T}\big(\{\alpha_n\}\big) = \begin{pmatrix} -\bar{\alpha}_1\alpha_0 & -\rho_1\alpha_0 & 0 & \dots & 0 \\ \bar{\alpha}_2\rho_1 & -\bar{\alpha}_2\alpha_1 & \bar{\alpha}_3\rho_2 & \dots & 0 \\ \rho_2\rho_1 & -\rho_2\alpha_1 & -\bar{\alpha}_3\alpha_2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \bar{\alpha}_N\rho_{N-1} \\ \dots & \dots & \dots & -\rho_{N-1}\alpha_{N-2} & -\bar{\alpha}_N\alpha_{N-1} \end{pmatrix} \tag{7.1}$$

(for even N it looks a bit different). The problem under investigation in the present section is the reconstruction of the matrix $\mathcal{T}$ (7.1) from either the complete set of its eigenvalues or from the mixed spectral data: the part of the spectrum and the part of the parameters $\alpha_n(\mathcal{T})$.

7.1. Existence of a finite truncated CMV matrix with the given spectrum

Theorem 7.1. *Let $z_1, z_2, \dots, z_N$ be not necessarily distinct numbers from the open unit disk. Then there exists a truncated $N \times N$ CMV matrix $\mathcal{T}$ (7.1) which has eigenvalues $z_1, z_2, \dots, z_N$, counting their algebraic multiplicities. Such matrix is determined uniquely up to multiplication of its parameters $\alpha_n(\mathcal{T})$ by the same unimodular factor.*

Proof. Let

$$b(z) = e^{i\varphi} \prod_{k=1}^{N} \frac{z - z_k}{1 - \bar{z}_k z}, \quad z \in \mathbb{D},\ \varphi \in [0, 2\pi). \tag{7.2}$$

We want to show that b is the characteristic function of a truncated CMV matrix $\mathcal{T}$ (7.1). Put

$$F(z) = \frac{1 + zb(z)}{1 - zb(z)},$$

which is a rational function with $N+1$ distinct simple poles lying on $\mathbb{T}$, $\operatorname{Re} F(z) > 0$, $z \in \mathbb{D}$, and $F(0) = 1$. It follows that there exists a probability measure $d\mu$ on the unit circle supported at those poles, so that

$$F(z) = \int\limits_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, d\mu(\zeta).$$

Let $\{\alpha_0, \ldots, \alpha_{N-1}, \alpha_N\}$ be the Schur parameters of b, that is the same as the Verblunsky coefficients of μ. Construct the $(N+1) \times (N+1)$ unitary CMV matrix $\mathcal{C}$ of the form (4.17). Then

$$F(z) = \big((\mathcal{C} + zI)(\mathcal{C} - zI)^{-1}\delta_0, \delta_0\big), \quad |z| < 1,$$

where $\delta_0 = (1, 0, \ldots, 0)^t \in \mathbb{C}^{N+1}$. Let $\mathcal{T}$ be $N \times N$ be truncated CMV matrix of the form (7.1). $\mathcal{C}$ has the block form

$$\mathcal{C} = \begin{pmatrix} \mathcal{S} & \mathcal{G} \\ \mathcal{F} & \mathcal{T} \end{pmatrix},$$

where $\mathcal{S} = \bar\alpha_0$, $\mathcal{G} = (\bar\alpha_1\rho_0, \rho_1\rho_0, 0, \ \ldots, \ 0)$, and

$$\mathcal{F} = \begin{pmatrix} \rho_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since δ_0 is a cyclic vector for $\mathcal{C}$, the unitary colligation

$$\Delta = \left\{ \begin{pmatrix} \mathcal{S} & \mathcal{G} \\ \mathcal{F} & \mathcal{T} \end{pmatrix}, \ \mathbb{C}, \mathbb{C}, \mathbb{C}^N \right\}$$

is prime. Hence $\mathcal{T}$ is a completely nonunitary contraction with rank one defect operators. Let $\Theta_\Delta(z)$ be the characteristic function of Δ. By Theorem 3.3 we have

$$\overline{\Theta_\Delta(\bar z)} = \frac{1}{z}\frac{F(z) - 1}{F(z) + 1}, \qquad \Theta_\Delta(z) = \overline{b(\bar z)}.$$

So $b(z)$ agrees with the characteristic function of $\mathcal{T}$. Therefore $\mathcal{T}$ has eigenvalues $z_1, \ldots, z_N$, counting their algebraic multiplicities [42].

Finally, let $\mathcal{T}(\{\alpha_n\})$ and $\mathcal{T}(\{\beta_n\})$ be two such matrices. Each of them is a completely nonunitary matrix with rank one defects, and their characteristic functions agree with b (7.2). Hence they are unitarily equivalent, and Proposition 6.2 completes the proof. □

Example 7.2. Let T be a completely nonunitary contraction with rank one defects on N-dimensional Hilbert space, and let T have just one eigenvalue $z = 0$ of the algebraic multiplicity N. Then its characteristic function agrees with $f(z) = e^{i\varphi} z^N$. The corresponding Schur

parameters are $\{\underbrace{0,\ldots,0}_{N}, e^{i\varphi}\}$. It follows that $\rho_n = 1$ for $n = 0,\ldots,N-1$. Hence T is unitarily equivalent to the $N \times N$ truncated CMV matrix $\mathcal{T}_N$ (see the expressions for $\mathcal{T}_5$ and $\mathcal{T}_6$):

$$\mathcal{T}_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{i\varphi} \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \qquad \mathcal{T}_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{i\varphi} & 0 \end{pmatrix}.$$

7.2. Uniqueness and reconstruction of a finite truncated CMV matrix from mixed spectral data

It is easily seen from (7.1) that a truncated $N \times N$ CMV matrix $\mathcal{T}$ is completely determined by $N+1$ independent parameters $\alpha_j(\mathcal{T})$, $j = 0,1,\ldots,N$. The problem we discuss here is whether $\mathcal{T}$ can be restored from the part of its spectrum (the eigenvalues $z_1,\ldots,z_m$, of the algebraic multiplicity l_k, $k = 1,\ldots,m$, with $l_1 + \cdots + l_m = r$), and the *first* $N - r + 1$ parameters $\alpha_0(\mathcal{T}),\ldots,\alpha_{N-r}(\mathcal{T})$. As we will see later on, the solution of this problem is unique (if it exists).

We begin with a simple result from complex analysis. We do not know where exactly it appears in the literature, but by all means it is known to experts.

Lemma 7.3. *Let $z_1,\ldots,z_m$ be distinct points in $\mathbb{D}$, $l_1,\ldots,l_m$ positive integers, and $r = l_1 + \cdots + l_m$. Suppose that the Nevanlinna–Pick interpolation problem with multiple nodes*

$$b^{(j)}(z_k) = w_k^{(j)}, \quad j = 0,1,\ldots,l_k-1,\ k = 1,2,\ldots,m \tag{7.3}$$

has two solutions b_1 and b_2, both the Blaschke products of order $\leqslant r-1$. Then $b_1 = b_2$.

Proof. Assume first that $z_k \neq 0$, $w_k^{(0)} \neq 0$, $k = 1,\ldots,m$. Given a Blaschke product s, we see by differentiating the equality $\overline{s(1/\bar{z})} = s^{-1}(z)$ that

$$\overline{s^{(j)}\left(\frac{1}{\bar{z}}\right)} = \frac{P_j(s(z), s'(z),\ldots,s^{(j)}(z))}{s^{2^j}(z)},$$

where P_j is a polynomial of its variables. Hence

$$\overline{s^{(j)}\left(\frac{1}{\bar{z}_k}\right)} = \frac{P_j(s(z_k),\ldots,s^{(j)}(z_k))}{s^{2^j}(z_k)}, \quad k = 1,2,\ldots,m,$$

so we have

$$b_1^{(j)}(z_k) = b_2^{(j)}(z_k), \qquad b_1^{(j)}\left(\frac{1}{\bar{z}_k}\right) = b_2^{(j)}\left(\frac{1}{\bar{z}_k}\right),$$
$$j = 0,1,\ldots,l_k-1,\ k = 1,2,\ldots,m.$$

Then for the difference $u = b_1 - b_2$ the relations

$$u^{(j)}(z_k) = u^{(j)}\left(\frac{1}{\bar{z}_k}\right) = 0, \quad j = 0, 1, \dots, l_k - 1, \ k = 1, 2, \dots, m \tag{7.4}$$

hold. Let now

$$b_l(z) = \frac{p_l(z)}{q_l(z)}, \quad l = 1, 2, \qquad u(z) = \frac{p_1(z)q_2(z) - p_2(z)q_1(z)}{q_1(z)q_2(z)} = \frac{p(z)}{q(z)},$$

where p, q are polynomials of degree $\leqslant 2r - 2$. The Leibniz formula

$$u^{(n)}(z) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\, p^{(k)}(z)\left(\frac{1}{q}\right)^{(n-k)}(z)$$

shows by induction that (7.4) imply

$$p^{(j)}(z_k) = p^{(j)}\left(\frac{1}{\bar{z}_k}\right) = 0, \quad j = 0, 1, \dots, l_k - 1, \ k = 1, 2, \dots, m. \tag{7.5}$$

But $\deg p \leqslant 2r - 2$, and there are $2r$ conditions in (7.5), so $p \equiv 0$, as needed.

Assume next that $z_k \neq 0$, $k = 1, \dots, m$ and some of $w_k^{(0)}$ are zero. Take $\varepsilon \in \mathbb{D}$, $\varepsilon \neq w_k^{(0)}$ and put

$$s_0 := \frac{z - \varepsilon}{1 - \bar{\varepsilon} z}, \qquad \widehat{b_l}(z) := s_0\big(b_l(z)\big), \quad l = 1, 2.$$

Then both $\widehat{b_1}$ and $\widehat{b_2}$ are Blaschke products of order $\leqslant r - 1$ which solve the interpolation problem

$$\widehat{b}_l^{(j)}(z_k) = \widehat{w}_k^{(j)}, \quad j = 0, 1, \dots, l_k - 1, \ k = 1, 2, \dots, m, \ l = 1, 2,$$

where $\widehat{w}_k^{(0)} = s_0(w_k^{(0)}) \neq 0$ and $\widehat{w}_k^{(j)} = (s_0(b_l(z)))^{(j)}|_{z=z_k}$. The above argument applied to $\widehat{b_l}$ gives $\widehat{b_1} = \widehat{b_2} \Rightarrow b_1 = b_2$, as needed.

Finally, assume that $z_1 = 0$. Let $\varepsilon \neq -z_k$ for all k, and put

$$\widetilde{b_l}(z) := b_l\big(s_0(z)\big), \quad l = 1, 2.$$

Then the Blaschke products $\widetilde{b_1}, \widetilde{b_2}$ of order $\leqslant r - 1$ satisfy

$$\widetilde{b}_l^{(j)}(\widetilde{z}_k) = \widetilde{w}_k^{(j)}, \quad j = 0, 1, \dots, l_k - 1, \ k = 1, 2, \dots, m, \ l = 1, 2$$

and $\widetilde{z}_k = (z_k + \varepsilon)(1 + \bar{\varepsilon} z_k)^{-1} \neq 0$. Hence $\widetilde{b_1} = \widetilde{b_2}$, and so $b_1 = b_2$. The proof is complete. □

Theorem 7.4. *Let $z_1, \dots, z_m$ be distinct nonzero points in $\mathbb{D}$, $l_1, \dots, l_m$ be positive integers, and $r = l_1 + \dots + l_m \leqslant N$. Let $\alpha_0, \dots, \alpha_{N-r} \in \mathbb{D}$. If there exists a $N \times N$ truncated CMV matrix $\mathcal{T}$* (7.1) *such that $z_1, \dots, z_m$ are eigenvalues of $\mathcal{T}$ with the algebraic multiplicities $l_1, \dots, l_m$, and $\alpha_j(\mathcal{T}) = \alpha_j$, $j = 0, \dots, N - r$, then this matrix is unique.*

Proof. If the required $\mathcal{T}$ exists then its characteristic function $\Theta_{\mathcal{T}}(z)$ is the Blaschke product of order N and of the form

$$b(z) = e^{it} \prod_{k=1}^{m} \left(\frac{z - z_k}{1 - \bar{z}_k z} \right)^{l_k} \prod_{j=1}^{N-r} \frac{z - v_j}{1 - \bar{v}_j z}, \tag{7.6}$$

with the given first $N - r + 1$ Schur parameters $\alpha_0(b), \ldots, \alpha_{N-r}(b)$. Our goal is to prove the uniqueness of such function b.

According to the result of Schur [38] (see Section 4.2) the set of all Schur functions b with given first $N - r + 1$ Schur parameters is parametrized by

$$b(z) = \frac{A(z) + zB^*(z)s(z)}{B(z) + zA^*(z)s(z)}, \tag{7.7}$$

where $s(z)$ is an arbitrary Schur function, and A, B are polynomials of degree at most $N - r$. Since b is the Blaschke product of order N, it is clear that so is $s(z)$, $\deg s(z) = r - 1$, and

$$\mathcal{S}b = \{\alpha_0, \ldots, \alpha_{N-r}, \alpha_0(s), \ldots, \alpha_{r-1}(s)\}.$$

Let us solve (7.7) for s:

$$s(z) = \frac{A(z) - B(z)b(z)}{-zB^*(z) + zA^*(z)b(z)},$$

so $s(z)$ satisfies the Nevanlinna–Pick interpolation problem (7.3), where $w_k^{(j)}$ are completely determined from the given nonzero z_k's and α_j's. By Lemma 7.3 there is at most one such $s(z)$, and the uniqueness of b is proved. □

Remark 7.5. Suppose that $z_1, \ldots, z_m$ are distinct nonzero points in $\mathbb{D}$, and $l_1 + \cdots + l_m = N$, so the only α_0 is prescribed. It is clear that α_0 is completely determined by the choice of z_j and their multiplicities l_j:

$$b(z) = e^{it} \prod_{k=1}^{m} \left(\frac{z - z_k}{1 - \bar{z}_k z} \right)^{l_k}, \qquad \alpha_0 = b(0) = e^{it} \prod_{j=1}^{m} \left(-z_k^{l_k}\right).$$

So for all other α_0 the inverse problem has no solution.

In the case when one of the eigenvalues is zero, all three possibilities (no solution, unique solution, and infinitely many solutions) may occur for the inverse problem in question. For instance, there is no solution at all as long as $z_1 = 0$, $\alpha_0 \neq 0$. Assume next, that $r = l_1 = 1$, $z_1 = 0$, and the points $\alpha_0, \alpha_1, \ldots, \alpha_{N-1}$ are taken in $\mathbb{D}$, with the only restriction $\alpha_0 = 0$, $\alpha_1 \neq 0$. The Blaschke products b_γ with the Schur parameters $\{\alpha_0, \alpha_1, \ldots, \alpha_{N-1}; \gamma\}$ and arbitrary $\gamma \in \mathbb{T}$ are of the form

$$b_\gamma(z) = e^{it} z \prod_{j=1}^{N-1} \frac{z - v_j}{1 - \bar{v}_j z},$$

and the corresponding $N \times N$ truncated CMV matrices $\mathcal{T}_\gamma$ solve the problem.

Finally, assume that except for the zero eigenvalue of multiplicity k ($z_1 = z_2 = \cdots = z_k = 0$), a few more nonzero (and not necessarily distinct) eigenvalues $\lambda_1, \ldots, \lambda_r$ are given, as well as the points $\alpha_0 = \cdots = \alpha_{k-1} = 0$, $\alpha_k \neq 0, \ldots, \alpha_{N-r}$ in $\mathbb{D}$. If the solution of the corresponding mixed inverse problem $\mathcal{T}$ exists, its characteristic function takes the form

$$b(z) = e^{it} z^k \prod_{j=1}^{r} \frac{z - \lambda_j}{1 - \bar{\lambda}_j z} g(z),$$

where g is the Blaschke product of order $N - k - r$, $g(0) \neq 0$, and the first $N - k - r + 1$ Schur parameters of $h = z^{-k} b$ are given numbers $\alpha_k, \ldots, \alpha_{N-r}$. Clearly, h is exactly the kth Schur iterate of b. If the required truncated CMV matrix $\mathcal{T}$ exists, then by Theorem 6.6 the characteristic function of $\mathcal{T}^{(k)}$ agrees with h. It follows now from Theorem 7.4 that $\mathcal{T}^{(k)}$ is unique, and since $\alpha_j(\mathcal{T}) = 0$, $j = 0, \ldots, k-1$, the matrix $\mathcal{T}$ is unique as well.

The situation changes dramatically if we assume that the *last* parameters of $\mathcal{T}$ (7.1) are known. In this case we can prove the existence, but not the uniqueness of the solution.

Theorem 7.6. *Let $z_1, \ldots, z_m$ and $\alpha_m, \ldots \alpha_{N-1}$ be two collections of arbitrary complex numbers from the open unit disk, and let $\alpha_N \in \mathbb{T}$. Then there exists a $N \times N$ truncated CMV matrix $\mathcal{T}$ of the form* (7.1) *such that*:

(i) *$z_1, \ldots, z_m$ are eigenvalues of $\mathcal{T}$, counting the algebraic multiplicity,*
(ii) *$\alpha_n(\mathcal{T}) = \alpha_n$, $n = m, m+1, \ldots, N$.*

Proof. By Theorem 4.3 there exists a Blaschke product $b(z)$ of order N such that $b(z_k) = 0$, $k = 1, \ldots, m$, with the Schur parameters

$$\alpha_n(b) = \alpha_n, \quad n = m, m+1, \ldots, N.$$

Take now the matrix $\mathcal{T}$ (7.1) with $\alpha_n(\mathcal{T}) = \alpha_n$, $n = 0, 1, \ldots, N$. By Theorem 3.3 the characteristic function of $\mathcal{T}$ agrees with $b(z)$, that completes the proof. □

Theorem 7.6 thereby says that a $N \times N$ truncated CMV matrix $\mathcal{T}$ can be reconstructed from its m eigenvalues and the lower principal block of order $N - m$. The latter is either the truncated CMV matrix $\mathcal{T}(\{\alpha_n\}_{n=m}^{N})$ or its transpose $\widetilde{\mathcal{T}}$.

7.3. Inverse problem for semi-infinite truncated CMV matrix

In this section we consider the criterion when given complex numbers z_n, $n = 1, 2, \ldots,$ from $\mathbb{D}$ are the eigenvalues counting algebraic multiplicity of some semi-infinite truncated CMV matrix.

Proposition 7.7. *Given complex numbers z_n, $n = 1, 2, \ldots$ are eigenvalues counting algebraic multiplicity of some semi-infinite truncated CMV matrix if and only if*

$$\sum_{n=1}^{\infty} \bigl(1 - |z_n|\bigr) < \infty.$$

Proof. The convergence of the sum is equivalent to the convergence of the Blashke product

$$b(z) = \prod_{k=1}^{\infty} \frac{\bar{z}_k}{z_k} \frac{z_k - z}{1 - \bar{z}_k z}.$$

Let $\{\alpha_n\}$ be the Schur parameters of b. The characteristic function of the truncated CMV matrix $\mathcal{T}(\{\alpha_n\})$ agrees with b. Hence the eigenvalues of $\mathcal{T}(\{\alpha_n\})$ are precisely the complex numbers $\{z_n\}$. □

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An L^2 theory for differential forms on path spaces I ☆

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Abstract

An L^2 theory of differential forms is proposed for the Banach manifold of continuous paths on a Riemannian manifold M furnished with its Brownian motion measure. Differentiation must be restricted to certain Hilbert space directions, the H-tangent vectors. To obtain a closed exterior differential operator the relevant spaces of differential forms, the H-forms, are perturbed by the curvature of M. A Hodge decomposition is given for L^2 H-one-forms, and the structure of H-two-forms is described. The dual operator d^* is analysed in terms of a natural connection on the H-tangent spaces. Malliavin calculus is a basic tool.

Keywords: Path space; L^2 cohomology; Hodge decomposition; Malliavin calculus; Banach manifolds; Bismut tangent spaces; Markovian connection; Itô map; Infinite dimensional; Curvature; Exterior products; Differential forms

1. Introduction

Background. We are concerned with the construction of an L^2 Hodge theory on path spaces with respect to a suitable reference measure and a collection of 'admissible' vector fields. Consider the space of continuous paths on a compact Riemannian manifold, over a fixed time interval $[0, T]$. Path spaces are Banach manifolds with the usual concepts of differentiable functions and

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differential forms, for example see Eells [24], Eliasson [25], Lang [54]. They also have a natural measure, their *Brownian motion*, or *Wiener* measure.

From the works of Bismut [10], Léandre [47], Driver [20] and others following pioneering work by L. Gross [44] in the classical Wiener space case, it seems the natural Sobolev differential calculus for functions on path spaces using such measures is of differentiation in directions given by Hilbert spaces of tangent vectors at each point: essentially the tangent vectors of finite energy. These are the so-called Bismut tangent spaces. The integration by parts formula given by Driver [20], and subsequent results suggest that these notions will lead to a satisfactory, and useful, Malliavin type calculus in this context. However the construction of differential form theory using Bismut tangent spaces leads to difficulties even at the level of the definition of exterior derivative. This is because of the lack of integrability of Bismut tangent 'bundle': the Lie bracket of suitable Bismut tangent space valued vector fields does determine a vector field, but in the presence of curvature it no longer takes values in the Bismut tangent spaces. Several ways of getting round this problem have been formulated, and carried out, especially by Léandre [55,56, 58] who gave analytical de Rham groups and showed that they agree with the singular cohomology of the spaces. See also [57]. But we are not aware of any which have led to an L^2 theory with Hodge–Kodaira Laplacian on our path spaces in the presence of curvature. In flat Wiener space the problem does not arise and the L^2 theory was defined and shown to be cohomologically trivial by Shigekawa [69,70]. See also Mitoma [62] and Arai and Mitoma [5]. For abstract Wiener manifolds, a class of infinite dimensional manifolds with an integrable Hilbert bundle of admissible directions, see Piech [65]. For M a compact Lie group with bi-invariant metric the corresponding results were proved by Fang and Franchi [42], but using the Bismut tangent spaces obtained from the flat left invariant connection on M so the problem again is avoided. They also considered loop groups [42]. For work done on 'sub-manifolds' of Wiener space see Airault and van Biesen [4], van Biesen [71] and especially Kusuoka [52,53], Kazumi and Shigekawa [48]. These submanifolds were constructed to replicate loop spaces over Riemannian manifolds, with their natural "Brownian bridge" measures. For a general survey see Léandre [59], and for a more introductory article concentrating on the approach taken here, see [34].

Let M be a compact C^∞ Riemannian manifold. For a fixed positive number T, consider the space $C_{x_0}M$ of continuous paths $\sigma : [0,T] \to M$ starting at a given point x_0 of M, furnished with its natural structure as a C^∞ Banach manifold and Brownian motion measure μ_{x_0}. For smooth differential forms there are the de Rham cohomology groups $H^q_{de\ Rham}(C_{x_0}M)$. C.J. Atkin informs us that the techniques of [7,8] can be extended to show that the de Rham groups would be equal to the singular cohomology groups, even though $C_{x_0}M$ does not admit smooth partitions of unity, and so trivial for $q \geqslant 0$ since based path spaces are contractible. For related work, also see Lempert and Zhang [60] on Dolbeault cohomology of a loop space. Since our primary interest is in the differential analysis associated with the Brownian motion measure μ on $C_{x_0}M$, which could equally well be considered on Hölder paths of any exponent smaller than a half, we could use Hölder rather than continuous paths and it is really only for notational convenience that we do not. In that case we would have smooth partitions of unity, see Bonic, Frampton and Tromba [11]. However contractibility need not imply triviality of the de Rham cohomology groups when some restriction is put on the spaces of forms. For example if $f : \mathbf{R} \to \mathbf{R}$ is given by $f(x) = x$ then df determines a non-trivial class in the first bounded de Rham group of $\mathbf{R}$. If f has value $+1$ for $x > 1$ and -1 for $x < 1$ then df is non-trivial in L^2-cohomology. In finite dimensions the L^2-cohomology of a cover $\tilde{M}$ of a compact manifold M gives important topological invariants of M even when $\tilde{M}$ is contractible, e.g. see Atiyah [6]; note also Bueler and Prokhorenkov [12], Ahmed and Stroock [1], and Gong and Wang [43].

The Bismut tangent spaces H^1_σ are defined by the parallel translation

$$/\!/_t(\sigma): T_{x_0}M \to T_{\sigma(t)}M$$

of the Levi-Civita connection and consist of those $v \in T_\sigma C_{x_0}M$ such that $v_t = /\!/_t(\sigma)h_t$ for $h_\cdot \in L_0^{2,1}([0,T]; T_{x_0}M)$. To have a satisfying L^2 theory of differential forms on $C_{x_0}M$ the obvious choice would be to consider 'H-forms,' i.e. for 1-forms these would be ϕ with $\phi_\sigma \in (H^1_\sigma)^*$, $\sigma \in C_{x_0}M$, and this agrees with the natural H-derivative $d_{\mathcal{H}}f$ for $f: C_{x_0}M \to \mathbf{R}$. For L^2 q-forms the obvious choice would be ϕ with $\phi_\sigma \in \bigwedge^q(H^1_\sigma)^*$, using here the Hilbert space completion for the exterior product. An L^2 de Rham theory would come from the complex of spaces of L^2 sections

$$\cdots \xrightarrow{\bar{d}} L^2\Gamma \bigwedge^q \big(H^1_\sigma\big)^* \xrightarrow{\bar{d}} L^2\Gamma \bigwedge^{q+1}\big(H^1_\sigma\big)^* \xrightarrow{\bar{d}} \cdots \tag{1.1}$$

where $\bar{d}$ would be a closed operator obtained by closure from the usual exterior derivative: for V^j, $j = 1$ to $q+1$, C^1 vector fields, and ϕ a differentiable one-form:

$$\begin{aligned} &d\phi\big(V^1 \wedge \cdots \wedge V^{q+1}\big) \\ &\quad= \frac{1}{q+1}\sum_{i=1}^{q+1}(-1)^{i+1}L_{V^i}\big[\phi\big(V^1 \wedge \cdots \wedge \widehat{V^i} \wedge \cdots \wedge V^{q+1}\big)\big] \\ &\qquad+ \frac{1}{q+1}\sum_{1\leqslant i<j\leqslant q+1}(-1)^{i+j}\phi\big(\big[V^i, V^j\big] \wedge V^1 \wedge \cdots \wedge \widehat{V^i} \wedge \cdots \wedge \widehat{V^j} \wedge \cdots \wedge V^{q+1}\big) \end{aligned} \tag{1.2}$$

where $[V^i, V^j]$ is the Lie bracket and $\widehat{V^j}$ means omission of the vector field V^j.

From this would come the de Rham–Hodge–Kodaira Laplacians $\bar{d}\bar{d}^* + \bar{d}^*\bar{d}$ and an associated Hodge decomposition. However the brackets $[V^i, V^j]$ of sections of $H^1_\cdot$ are not in general sections of $H^1_\cdot$, e.g. see Cruzeiro and Malliavin [18], Driver [21], see also [33], and formula (1.2), below, for d does not make sense for ϕ_σ defined only on $\bigwedge^q H^1_\sigma$, each σ, as mentioned earlier.

Our proposal is to replace the Hilbert spaces $\bigwedge^q H^1_\sigma$ in (1.1) by a family of different Hilbert spaces $\mathcal{H}^q_\sigma$, $q = 2, 3, \ldots$, continuously included in $\bigwedge^q T_\sigma C_{x_0}M$, though keeping the exterior derivative a closure of the classical exterior derivative on smooth cylindrical forms.

In Elworthy and Li [32], for $q = 1, 2$, we identified a class of Hilbert subspaces $\mathcal{H}^q_\sigma$, of the completed exterior powers $\bigwedge^q T_\sigma C_{x_0}$ of the tangent space $T_\sigma C_{x_0}$ to $C_{x_0}M$ at a path σ which could be the basic building blocks of an L^2 de Rham and Hodge theory for $C_{x_0}M$. We described $\mathcal{H}^2_\sigma$ without proof, proved closability of exterior differentiation on corresponding L^2 1-forms, defined a self-adjoint Hodge–Kodaira Laplacian on such L^2 1-forms and established the Hodge decomposition.

The article [33] both discusses some of the constructions here for more general diffusion measures and connections, and relates them to the Bismut type formulae for differential forms on M [31], see also Driver and Thalmaier [22]. In particular it shows that a very natural class of two-vector fields on $C_{x_0}M$ are of the type we consider here (i.e. are sections of $\mathcal{H}^2$).

Main results. Here we give a detailed analysis of $\mathcal{H}^2_\sigma$ and define $\mathcal{H}^q_\sigma$ for $q > 0$. For $q = 1$, as a space $\mathcal{H}^1_\sigma = H^1_\sigma$. For flat manifolds, $\mathcal{H}^q_\sigma = \bigwedge^q H^1_\sigma$ for all q and the standard Hodge decomposition theorem follows. However in general, the spaces $\mathcal{H}^q_\sigma$ we construct are different from

$\bigwedge^q H$, the exterior products of the Bismut tangent bundle. Sections of $\mathcal{H}^q$ are called H-q-vector fields and sections of $(\mathcal{H}^q)^*$ are called H-differential forms of degree q. In fact $\mathcal{H}^2_\sigma$ is a deformation of $\bigwedge^2 H^1_\sigma$ inside $\mathbf{L}_{skew}(\mathcal{H}^1_\sigma, \mathcal{H}^1_\sigma)$ by the curvature of M. As a Hilbert space $\mathcal{H}^2_\sigma$ is defined to be isometric to $\bigwedge^2 H^1_\sigma$ by a map involving the curvature of the so-called damped Markovian connection on the Bismut tangent "bundle." Algebraic operations such as interior products acting on H-two-vectors, and the exterior products of H-one-forms, as well as the derivation property for the exterior derivative are shown to make sense. A Hodge decomposition is given for H-one-forms. In a sequel, Part II, we establish the analogous decomposition for L^2 2-forms, and we show that the spaces $\mathcal{H}^q_\sigma$ defined by suitable Itô maps $\mathcal{I}$ depend only on the Riemannian structure of the base manifold M.

Organisation. The article is organised as follows:

- Section 2. Review of basic results concerning exterior powers of relevant spaces of tangent vectors to $C_{x_0}M$.
- Section 3. Special Itô maps and the definition of $\mathcal{H}^q$.
- Section 4. Characterisation of $\mathcal{H}^1$ and $\mathcal{H}^2$.
- Section 5. H-one-forms: exterior differentiation and Hodge decomposition.
- Section 6. Tensor products as operators: algebraic operations on H-one-forms.
- Section 7. The derivation property of $\bar{d}^1$.
- Section 8. Infinitesimal rotations as divergences.
- Section 9. Differential geometry of the space $\mathcal{H}^2$ of two-vectors.
- Appendix A. Conventions.
- Appendix B. Brackets of vector fields, torsion, and $d\phi(v^1 \wedge v^2)$.

In Section 2 we discuss the various completed tensor products of tangent, and other spaces which we will use. Properties of these relating to tensor products of abstract Wiener spaces are used in order to define our spaces $\mathcal{H}^q$ in Section 3. The aim is to show that these constructions are well behaved and have interesting geometry.

One of the main results, see Section 4, is a characterisation of $\mathcal{H}^2$ as a perturbation of $\bigwedge^2 \mathcal{H}$ by a curvature of the Levi-Civita connection on M. Write $\mathcal{H}_\sigma = \mathcal{H}^1_\sigma$, then

$$\mathcal{H}^2_\sigma = (I + Q_\sigma)\bigwedge^2 \mathcal{H}_\sigma \tag{1.3}$$

for some operator Q_σ on $\bigwedge^2_\epsilon T_\sigma C_{x_0}$. Equivalently

$$u \in \mathcal{H}^2 \quad \text{if and only if} \quad u - \mathbb{R}(u) \in \bigwedge^2 \mathcal{H}$$

where $\mathbb{R}$ is identified in Section 9 as the curvature of the damped Markovian connection on the H-tangent spaces.

In Section 5 we rapidly recall the results concerning closability of our exterior derivative on H-one-forms and the Hodge decomposition for H-one-forms.

The remainder, the main part, of the article is an analysis of the space $\mathcal{H}^2$, its associated H-two-forms, and the adjoint of the exterior derivative, an operator from H-two-forms to H-one-forms, together with the corresponding divergence operator from two-vector fields to vector fields. In Section 6 it is shown that the exterior product of two H-one-forms is naturally an H-two-form, and the interior product of an H-two-form with a H-one-form is a H-one-form. The

operator Q has image in $\mathcal{L}_{skew}(\mathcal{H};\mathcal{H})$, which implies an element of $\mathcal{H}^2_\sigma$ can be considered to be an element of $\mathcal{L}_{skew}(\mathcal{H}_\sigma;\mathcal{H}_\sigma)$, cf. Corollary 6.2, although in general it is not compact and so not in $\bigwedge^2 \mathcal{H}_\sigma$. In Section 7 a corresponding derivation formula for the exterior derivative of H-one-forms, Theorem 7.1, is shown to hold.

In Section 8 it is shown that the elements of the image of suitable smooth sections of $\bigwedge^2 \mathcal{H}$ by Q "have a divergence" in the sense of satisfying an integration by parts formula and a formula is given in 8.1 for $\operatorname{div} Q(V^1 \wedge V^2)$. Vector fields which are not H-vector fields also make their appearance, especially as Lie brackets. The latter involve infinitesimal rotations which "have a divergence," and in their case the divergence is zero. It is natural to ask if they themselves are divergences, in this extended sense, of some two-vector field. In Section 8 this is shown to be true in a wide class of adapted situations on flat Wiener space, Proposition 8.2. This has independent interest, but it is extended, in Theorem 9.3, to show that the torsion of the damped Markovian connection when applied to suitable non-anticipating H-vector fields is the divergence of the perturbing factor in the definition of $\mathcal{H}^2$:

$$\operatorname{div} Q\big(u^1 \wedge u^2\big) = \frac{1}{2}\mathbb{T}\big(u^1, u^2\big). \tag{1.4}$$

Here $\mathbb{T}$ is the torsion of the damped Markovian connection $\breve{\nabla}$. This helps explain the "cancellation" of the bracket occurring with our exterior derivative, and fits in with the result of Cruzeiro and Fang [16], concerning the vanishing of the divergence of such torsions. The damped Markovian connection, introduced by Cruzeiro and Fang [16], plays an important role here, as it did in [35]. As in [35] we introduce it by giving a $C_{id}([0,T];O(n))$-bundle structure to $\mathcal{H}$. This is done in Section 9. Here we also relate the divergence of our H-two-vector fields to the adjoint of the damped Markovian covariant derivative in a non-anticipating situation, Corollary 9.7: For suitable non-anticipating U, V,

$$\breve{\nabla}^*(U \wedge V) = \operatorname{div}(I + Q)(U \wedge V). \tag{1.5}$$

We also describe the curvature of the damped Markovian connection in Section 9D, to establish our claim that $\mathcal{H}^2$ is a perturbation of $\bigwedge^2 \mathcal{H}^1$ using this curvature operator, Theorem 4.3(iii). In Section 9D we essentially show that $\mathbb{D}^{2,1}$ H-two-forms are in the domain of the adjoint of $\bar{d}^{1*}$, extending the result for one-forms proved in [35].

List of symbols

- $C_{x_0}M$ or C_{x_0}—space of continuous paths over M starting from x_0.
- $T_\sigma C_{x_0}M$ or $T_\sigma C_{x_0}$—tangent space at σ to $C_{x_0}M$.
- $\mathcal{H}^1_\sigma$ or $\mathcal{H}_\sigma$—Bismut tangent space, a Hilbert space included in $T_\sigma C_{x_0}$.
- $\mathcal{H}^1$ or $\mathcal{H}$—corresponding Bismut tangent "bundle," $\bigcup \mathcal{H}^1_\sigma$.
- $\mathcal{H}^2$—vector "bundle" with fibres $\mathcal{H}^2_\sigma \subset \bigwedge^2 T_\sigma C_{x_0}M$.
- ΓB—sections of a vector bundle B.
- $L^2\Gamma B$—L^2 sections of a vector bundle B.
- $C_0\mathbf{R}^m$—Wiener space with Wiener measure $\mathbf{P}$, the canonical probability space.
- $L^{2,1}_0(G)$—for G a Hilbert space, this is $\{h:[0,T]\to G$ such that $\int_0^T |\dot{h}_s|^2\,ds < \infty\}$. When $G = \mathbf{R}^m$, this is the Cameron–Martin space, denoted by H.
- $(\xi_t, t \geqslant 0)$—a Brownian stochastic flow of diffeomorphisms of M.

$T\xi_t$—space derivative of ξ_t.
μ—Brownian motion measure, also called Wiener measure, on $C_{x_0}M$.
$\mathcal{I}$—the Itô map induced by $(\xi_t(x_0),\ t \geqslant 0)$, $\mathcal{I}(\omega) := \xi_\cdot(x_0, \omega)$.
$T\mathcal{I}$—H-derivative of the Itô map.
$\mathcal{F}^{x_0}$—the algebra generated by $(\xi_t(x_0), t \geqslant 0)$ on M.
$\overline{f}(\sigma)$—conditional expectation of f given $\mathcal{I} = \sigma$, $\sigma \in C_{x_0}$, e.g. $\overline{T\mathcal{I}}_\sigma$.
$W_t^{(q)}$—Weitzenbock flow of q-vectors, Eq. (4.1).
${W^{(q)}}_t^s$—Weitzenbock flow starting from time s.
W_t—damped parallel translation, $W_t = W_t^{(1)}$.
$\frac{\mathbb{D}^q}{dt}$, $\frac{\mathbb{D}}{dt}$—see Section 4.
$L_2 T_\sigma C_{x_0}$—the space of L^2 tangent vectors at σ, Definition 4.1.
$\mathcal{W}$—isometry between $\mathcal{H}^1$ and $L_2TC_{x_0}$, Eq. (4.3).
$\mathcal{L}(E_1; E_2)$—the space of continuous linear maps between Banach spaces.
$\mathcal{L}_2(H_1; H_2)$—Hilbert–Schmidt maps between Hilbert spaces.
$\mathcal{R}$, $\mathcal{R}^q$, Ric—respectively the curvature operator, the Weitzenbock curvature on q forms, and the Ricci curvature on M.

In general we shall use $|\ |$ to denote norms of finite dimensional spaces. $\|\ \|$ for infinite dimensional spaces, with $\boldsymbol{|}\ \boldsymbol{|}$ for spaces such as $L^2(\Omega; \mathbf{R}^n)$, or $L^2(C_{x_0}M; \mathbf{R})$, where integration over probability spaces are involved.

2. Exterior powers: Notation

For convenience the conventions we use for tensor products, exterior powers, etc. are gathered together as Appendix A. Please note that they differ from those used in our previous articles, such as [32].

A. All linear spaces are over $\mathbf{R}$. We shall deal with tensor products of Hilbert spaces and of Banach spaces of continuous paths. For any linear space E let $\bigotimes_0^q E$ denote the qth algebraic tensor product of E with itself and $\bigwedge_0^q E$ the linear subspace of antisymmetric elements. For infinite dimensional Banach spaces E we will need completions of these spaces, e.g. see Ruston [67] or Cigler, Losert and Michor [14]:

(i) When $E = T_\sigma C_{x_0}$ or $C_0\mathbf{R}^m$ let $\bigotimes^q E$ and $\bigwedge^q E$ denote the completions using the largest cross norm, i.e. the projective tensor products $\|-\|_\pi$. For general Banach spaces E_i, if v is in the algebraic tensor product $E_1 \otimes_0 \cdots \otimes_0 E_q$,

$$\|v\|_\pi = \inf\left\{\sum_{i=1}^n \prod_{k=1}^q \|a_i^k\|,\ \text{where } v = \sum_{i=1}^n \bigotimes_{k=1}^q a_i^k,\ a_i^k \in E_k,\ n < \infty\right\}.$$

(ii) When E is a Hilbert space H, let $\bigotimes^q H$ and $\bigwedge^q H$ denote the standard Hilbert space completions, (so $\bigotimes^2 H$ can be identified with the space of Hilbert–Schmidt operators on H).

(iii) In general let $\bigotimes_\varepsilon^q E$ and $\bigwedge_\varepsilon^q E$ refer to the completions with respect to the smallest reasonable cross norm, i.e. the inductive cross norm,

$$\|w\|_\epsilon = \sup_{\|u_k^*\|_{E^*} \leqslant 1,\, u_k^* \in E^*} \left|\left(u_1^* \otimes \cdots \otimes u_q^*\right)(w)\right|.$$

We shall use the natural inclusion maps as identifications and so consider

$$\bigotimes_0^q E \subset \bigotimes^q E \subset \bigotimes_\varepsilon^q E.$$

Thus a differential q-form ϕ on $C_{x_0}M$ which by definition gives a continuous antisymmetric multilinear map $\phi_\sigma : T_\sigma C_{x_0} \times \cdots \times T_\sigma C_{x_0} \to \mathbf{R}$, Lang [54], can equivalently be defined as a section of the bundle $\mathcal{L}(\bigwedge^q TC_{x_0}; \mathbf{R})$ with fibres the dual spaces $(\bigwedge^q T_\sigma C_{x_0})^*$, $\sigma \in C_{x_0}M$.

B. If $S : E_1 \to E_2$ and $T : F_1 \to F_2$ are two linear maps of linear spaces, there is the induced linear map $S \otimes T : E_1 \otimes_0 F_1 \to E_2 \otimes_0 F_2$. The Banach space constructions are functorial so that if $S, T \in \mathcal{L}(C_0\mathbf{R}^m; T_\sigma C_{x_0})$ then $S \otimes T$ determines a continuous linear map of the completed tensor spaces $\bigotimes^2 C_0\mathbf{R}^m$ to $\bigotimes^2 T_\sigma C_{x_0}M$ and if $S = T$ we have its restriction $\bigwedge^2 S : \bigwedge^2 C_0\mathbf{R}^m \to \bigwedge^2 T_\sigma C_{x_0}M$, Ruston [67, p. 63] and Cigler, Losert and Michor [14]; with the corresponding result for the inductive tensor product, for the Hilbert space case, and for $q > 2$. There is also the estimate on operator norms

$$\|S^1 \otimes \cdots \otimes S^q\| \leqslant \|S^1\| \cdot \cdots \cdot \|S^q\|$$

so that in particular

$$\left\|\bigwedge^q S\right\| \leqslant \|S\|^q$$

in all of these cases, see Ruston [67] and Cigler, Losert and Michor [14].

For example let $H \equiv L_0^{2,1}\mathbf{R}^m$ be the (Cameron–Martin) Hilbert space of functions $h : [0, T] \to \mathbf{R}^m$ of the form $h_t = \int_0^t \dot{h}_s\,ds$ with $\dot{h} \in L^2([0, T]; \mathbf{R}^m)$ and inner product $\langle h^1, h^2\rangle = \int_0^T \langle \dot{h}_s^1, \dot{h}_s^2\rangle_{\mathbf{R}^m}\,ds$. Thus the indefinite integral

$$\int_0^{\cdot} : L^2\big([0, T]; \mathbf{R}^m\big) \to H$$

is an isometry with inverse which we will write as

$$\frac{d}{d\cdot} : H \to L^2\big([0, T]; \mathbf{R}^m\big).$$

From this we obtain the isometry

$$\bigwedge^q \left(\int_0^{\cdot}\right) : \bigwedge^q L^2\big([0, T]; \mathbf{R}^m\big) \to \bigwedge^q H$$

with inverse

$$\bigwedge^q \left(\frac{d}{d\cdot}\right) : \bigwedge^q L_0^{2,1}\mathbf{R}^m \to \bigwedge^q L^2\big([0, T]; \mathbf{R}^m\big).$$

C. We will regularly make use of the well-known isometries

$$\bigotimes_{\varepsilon}^{q} C_0\mathbf{R}^m \xrightarrow{\rho} C_0\big([0,T]^q; \bigotimes^q \mathbf{R}^m\big)$$

where the right-hand side consists of those continuous $\alpha:[0,T]^q \to \bigotimes^q \mathbf{R}^m$ for which $\alpha(t_1,\dots,t_q)=0$ if $t_j=0$ for any j. For example see Cigler, Losert and Michor [14, p. 66]. For $V\in \bigotimes_{\varepsilon}^{q} C_0\mathbf{R}^m$, write

$$V_{t_1,\dots,t_q} := \rho(V)(t_1,\dots,t_q).$$

Let $\mathrm{ev}_t : C_0\mathbf{R}^m \to \mathbf{R}^m$ be the evaluation map at time t, then

$$V_{t_1,\dots,t_q} = (\mathrm{ev}_{t_1}\otimes\cdots\otimes \mathrm{ev}_{t_q})V.$$

Also note that such V lies in $\bigwedge_{\varepsilon}^{q} C_0\mathbf{R}^m$ if and only if $\rho(V):[0,T]^q \to \bigotimes^q \mathbf{R}^m$ anti-commutes with permutations, i.e.

$$V_{t_{\pi(1)},\dots,t_{\pi(q)}} = (-1)^{\pi} S_\pi V_{t_1,\dots,t_q}$$

for any permutation π on $\{1,\dots,q\}$ with S_π the induced action on $\bigotimes^q \mathbf{R}^m$. If so,

$$V_{t,\dots,t} \in \bigwedge^q \mathbf{R}^m$$

and

$$V_{t_1,t_2,\dots,t_2} \in \mathbf{R}^m \otimes \bigwedge^{q-1}\mathbf{R}^m,$$

etc. From this we see that elements of $\bigwedge_{\varepsilon}^{q} C_0\mathbf{R}^m$ and hence those of the smaller spaces $\bigwedge^q C_0\mathbf{R}^m$ are determined by their values on the simplex $0\leqslant t_1\leqslant\cdots\leqslant t_q\leqslant T$.

Similarly, to any $V\in\bigotimes_{\varepsilon}^{q} T_\sigma C_{x_0}$ we have $V_{t_1,\dots,t_q}\in T_{\sigma_{t_1}}M\otimes\cdots\otimes T_{\sigma_{t_q}}M$ corresponding to an isometric isomorphism of $\bigotimes_{\varepsilon}^{q} T_\sigma C_{x_0}$ with the space of continuous maps $V_\cdot$ on $[0,T]^q$ such that

$$\begin{tikzcd} & \bigotimes^q TM \arrow[d, "\pi"] \\ {[0,T]^q} \arrow[ur, "V"] \arrow[r, "\sigma_\cdot\times\cdots\times\sigma_\cdot"] & M\times\cdots\times M \end{tikzcd}$$

commutes and $V_{t_1,\dots,t_q}=0$ when $t_j=0$ for any j.

D. By functorality the inclusion $i: L_0^{2,1}\mathbf{R}^m \to C_0\mathbf{R}^m$ gives rise to a continuous linear inclusion $\bigotimes^q i : \bigotimes^q H \to \bigotimes_{\varepsilon}^{q} C_0\mathbf{R}^m$. From paragraph **B** we see that $V\in \mathrm{Image}\bigotimes^q i$ if and only if

$$V_{t_1,\dots,t_q} = \int_0^{t_1}\int_0^{t_2}\cdots\int_0^{t_q} U_{s_1,\dots,s_q}\,ds_1\dots ds_q, \tag{2.1}$$

$(t_1, \dots, t_q) \in [0, T]^q$, for some $U_\cdot \in L^2([0, T]^q; \bigotimes \mathbf{R}^m)$. Here we use the isometry ρ of $\bigotimes^q L^2([0, T]; \mathbf{R}^m)$ with $L^2([0, T]^q; \bigotimes^q \mathbf{R}^m)$. In fact the $U_\cdot$ in the above formula is just $\rho(\bigotimes^q(\frac{d}{d\cdot})V)$ or equivalently $U_{t_1,\dots,t_q}$ is the weak derivative $\frac{\partial^q}{\partial t_1 \dots \partial t_q} V_{t_1,\dots,t_q}$.

E. Given a bounded linear operator $S: E \to F$ of Banach spaces there is also the functorial construction

$$\big(d\bigotimes^q\big)(S): \bigotimes_0^q E \to \bigotimes_0^q F$$

defined by linearity and

$$\begin{aligned}&\big(\big(d\bigotimes^q\big)(S)\big)\big(e^1 \otimes \dots \otimes e^q\big)\\&\quad = S\big(e^1\big) \otimes e^2 \otimes \dots \otimes e^q + e^1 \otimes S\big(e^2\big) \otimes \dots \otimes e^q + \dots + e^1 \otimes e^2 \otimes \dots \otimes S\big(e^q\big).\end{aligned}$$

This is just a sum of operators described in paragraph **B** and so extends over the relevant completion. The same notation will be kept for these extensions.

Note that if V is in $\bigotimes^q H$ then $((d\bigotimes^q)(\frac{d}{d\cdot}))(V)$ is in $\bigotimes^q L^2([0, T]; \mathbf{R}^m)$ with kernel

$$\left(\big(d\bigotimes^q\big)\left(\frac{d}{d\cdot}\right)\right)(V)_{t_1,\dots,t_q} = \sum_{j=1}^q \frac{\partial}{\partial t_j} V_{t_1,\dots,t_q}. \tag{2.2}$$

The restriction $(d\Lambda^q(S))$ of $(d\bigotimes^q(S))$ to $\bigwedge_0^q E$ has the form

$$\big(d\Lambda^q(S)\big)\big(v^1 \wedge v^2 \wedge \dots \wedge v^q\big) = S\big(v^1\big) \wedge v^2 \wedge \dots \wedge v^q + \dots + v^1 \wedge v^2 \wedge \dots \wedge S\big(v^q\big)$$

and for $q = 2$

$$\big(d\Lambda^2(S)\big)\big(v^1 \wedge v^2\big) = \frac{1}{2}\big\{Sv^1 \otimes v^2 + v^1 \otimes Sv^2 - Sv^2 \otimes v^1 - v^2 \otimes Sv^1\big\}. \tag{2.3}$$

3. Special Itô maps and the definition of $\mathcal{H}_\sigma^q$

A. Take a surjective C^∞ vector bundle morphism, $X: \underline{\mathbf{R}}^m \to TM$, of the trivial $\mathbf{R}^m$ bundle over M onto TM, for some $m \geqslant n = \dim M$. Suppose that X induces the given Riemannian metric on M and let Y be the $\mathbf{R}^m$-valued 1-form such that $Y_x = X(x)^*: T_x M \to \mathbf{R}^m$. For U a vector field and $v \in T_x M$, set

$$\nabla_v U = X(x)\, d\big[y \to Y_y U(y)\big](v), \tag{3.1}$$

as in Elworthy, LeJan and Li [29,30], where it was called LW connection for X. Suppose that the connection ∇ is the Levi-Civita connection. Take (B_t) to be the canonical Brownian motion on $\mathbf{R}^m$ with probability space $C_0\mathbf{R}^m$ and Wiener measure $\mathbb{P}$ and consider the stochastic differential equation on M

$$dx_t = X(x_t) \circ dB_t, \quad 0 \leqslant t \leqslant T. \tag{3.2}$$

Then the solutions are Brownian motions on M. Let μ_{x_0} be the Brownian motion measure on $C_{x_0}M$, the probability distribution of the solution starting from x_0. An example is the gradient system induced from an isometric immersion $\alpha : M \to \mathbf{R}^m$ with $X(x) : \mathbf{R}^m \to T_x M$ defined to be the orthogonal projection for each $x \in M$. Another class of examples arises from symmetric space structures on M, see [30].

For our fixed x_0 in M there is the solution map, or *Itô map*,

$$\mathcal{I} : C_0\mathbf{R}^m \to C_{x_0}M,$$

of (3.2) defined by

$$\mathcal{I}(\omega)_t = x_t(\omega), \quad \omega \in C_0\mathbf{R}^m,$$

where x_t is the solution starting at x_0. Thus $\mathcal{I}_*(\mathbb{P}) = \mu_{x_0}$. This Itô map has an H-derivative in the sense of Malliavin calculus which is a continuous linear map from the Cameron–Martin space $H \equiv L_0^{2,1}\mathbf{R}^m$,

$$T_\omega\mathcal{I} : H \to T_{\mathcal{I}(\omega)}C_{x_0},$$

for almost all $\omega \in C_0\mathbf{R}^m$. Thus for $h \in H$ and $0 \leqslant t \leqslant T$ we have $T\mathcal{I}(h)_t \in T_{x_t}M$, a.s.

B. Let $\{\xi_t\colon 0 \leqslant t \leqslant T\}$ denote the flow of (3.2) so $\mathcal{I}(\omega)_t = x_t(\omega) = \xi_t(x_0, \omega)$. It can be taken to consist of random C^∞ diffeomorphisms $\xi_t : M \to M$ with derivative maps $T\xi_t : TM \to TM$, so that $T_{x_0}\xi_t \in \mathcal{L}(T_{x_0}M; T_{x_t}M)$.

Take $h \in H$. Set $v_t = T\mathcal{I}(h)_t$. Bismut showed that $v_\cdot$ satisfies the covariant equation along the paths of $\{x_t\colon 0 \leqslant t \leqslant T\}$

$$Dv_t = \nabla_{v_t}X \circ dB_t + X(x_t)\dot{h}_t\, dt \tag{3.3}$$

with solution

$$v_t = T\xi_t \int_0^t (T\xi_s)^{-1}\big(X(x_s)\dot{h}_s\big)\, ds. \tag{3.4}$$

Lemma 3.1. *(See [30,38].) There is a canonical decomposition of the noise $\{B_t\colon 0 \leqslant t \leqslant T\}$ given by*

$$dB_t = \tilde{/\!/}_t\, d\tilde{B}_t + \tilde{/\!/}_t\, d\beta_t \tag{3.5}$$

where

(i) *$\{\tilde{B}_t\colon 0 \leqslant t \leqslant T\}$ is a Brownian motion on the orthogonal complement $[\ker X(x_0)]^\perp$ of the kernel of $X(x_0)$ in $\mathbf{R}^m$;*
(ii) *$\{\beta_t\colon 0 \leqslant t \leqslant T\}$ is a Brownian motion on $\ker X(x_0)$;*
(iii) *for each $t \geqslant 0$, $\tilde{/\!/}_t : C_{x_0}M \to O(m)$ is a measurable map into the orthogonal group of $\mathbf{R}^m$ with $\tilde{/\!/}_t(\sigma)[\ker X(x_0)] = \ker X(\sigma_t)$ for μ_{x_0} almost all $\sigma \in C_{x_0}M$.*

N.B. We will regularly consider random variables on $C_{x_0}M$, such as $\tilde{/\!/}_t$, to be random variables on $C_0\mathbf{R}^m$ by taking their composition with $\mathcal{I}$. For example the stochastic equation (3.5) above is to be interpreted that way. Moreover let $\mathcal{F}^{x_0}$ be the σ-algebra on $C_0\mathbf{R}^m$ generated by $\mathcal{I}$ with $\{\mathcal{F}_t^{x_0},\ 0 \leqslant t \leqslant T\}$ the filtration generated by $(x_s\colon\ 0 \leqslant s \leqslant T)$. Then we can, and often will, consider $\mathcal{F}^{x_0}$-measurable functions as functions, defined up to equivalence, on $C_{x_0}M$.

Let $\mathcal{F}^{\beta}$ be the σ-algebra generated by $\{\beta_t\colon\ 0 \leqslant t \leqslant T\}$, and $\mathcal{F}^{\tilde{B}}$ that generated by $\{\tilde{B}_t\colon\ 0 \leqslant t \leqslant T\}$. From Elworthy and Yor [38], Elworthy, LeJan and Li [30] we know that

(a) $\mathcal{F}^{\beta}$ and $\mathcal{F}^{\tilde{B}}$ are independent and
(b) $\mathcal{F}^{\tilde{B}} = \mathcal{F}^{x_0}$;
(c) Eq. (3.3) can be written as the Itô equation

$$Dv_t = \nabla_{v_t} X(\tilde{/\!/}_t\, d\beta_t) - \frac{1}{2}\operatorname{Ric}^{\#}(v_t)\, dt + X(x_t)\dot{h}_t\, dt \tag{3.6}$$

where $\operatorname{Ric}_x^{\#}\colon T_xM \to T_xM$ corresponds to the Ricci curvature by $\langle \operatorname{Ric}_x^{\#}(u^1), u^2\rangle = \operatorname{Ric}(u^1, u^2)$ for u^1, u^2 in T_xM.

We shall often write covariant derivatives such as $\nabla_v X$ as $\nabla X(v)$ so $\nabla X(v) \circ dB_t$ is just $\nabla_v X \circ dB_t$.

C. We first show that $\bigwedge^q T_\omega\mathcal{I}$ take values in the exterior product space $\bigwedge^q TC_{x_0}$ rather than just in $\bigwedge_\epsilon^q TC_{x_0}$. Recall that a continuous linear map of H to a separable Banach space E is *γ-radonifying* if it maps the canonical Gaussian cylinder set measure of H to a Borel measure on E. The 2-*summing norm*, $\pi_2(A)$, of an operator $A\colon E \to F$ is given by

$$\pi_2(A)^2 = \sup_{\{x_n\}\subset E} \frac{\sum \|Ax_n\|^2}{\sup_{\|u\|=1,\, u\in E^*} \sum (u(x_n))^2}$$

where $\{x_n\}$ is a finite subset of E. When E and F are Hilbert spaces A has finite two summing norm if and only if A is Hilbert–Schmidt. See for example Pietsch [66].

Lemma 3.2. *For almost all $\omega \in C_0\mathbf{R}^m$ the map*

$$T_\omega\mathcal{I}\colon H \to T_{\mathcal{I}(\omega)}C_{x_0}$$

is γ-radonifying. Its operator norm $\|T\mathcal{I}\|$ is in $L^p(C_0\mathbf{R}^m)$ for $1 \leqslant p < \infty$ as is the 2-summing norm of its adjoint.

Proof. Note that $\alpha\colon h \mapsto \int_0^{\cdot} (T\xi_s)^{-1} X(x_s)(\dot{h}_s)\, ds$ maps H to $L_0^{2,1}(T_{x_0}M)$ and is continuous linear; almost surely. The inclusion $i\colon L_0^{2,1}(T_{x_0}M) \to C_0T_{x_0}M$ is γ-radonifying. Write $T\mathcal{I} = T\xi_\cdot \circ i \circ \alpha$. Then the first result follows by composition properties of γ-radonifying maps and continuity of $T_{x_0}\xi_\cdot\colon C_0T_{x_0}M \to T_{x_\cdot(\omega)}C_{x_0}$. The pth power integrability of the operator norms come from the corresponding properties of $T\xi_t$ and $(T\xi_t)^{-1}$, e.g. see Kifer [49]. For the 2-summing norm apply Schwartz's duality theorem [68] to see that the adjoint of the γ-radonifying map i

is 2-summing with norm $\pi_2(i)$. Then use the composition properties of 2-summing operators to estimate the 2-summing norm

$$\pi_2\big((T\mathcal{I})^*\big) \leqslant \|\alpha^*\|\pi_2\big(i^*\big)\big\|(T\xi_t)^*\big\|, \quad \text{a.s.}$$

Then apply the integrability results again to see the norm is in L^p. $\square$

Theorem 3.3. *For almost all ω the map $\bigwedge^q T_\omega I$ can be considered as a continuous linear map from the Hilbert space completion of the qth exterior power of H to the projective exterior power of the tangent space $(T_{\mathcal{I}(\omega)}C_{x_0})$*

$$\bigwedge^q (T_\omega\mathcal{I}) : \bigwedge^q \big(L_0^{2,1}\mathbf{R}^m\big) \to \bigwedge^q (T_{\mathcal{I}(\omega)}C_{x_0}).$$

Moreover the operator norms lie in $L^p(C_0\mathbf{R}^m)$ for $1 \leqslant p < \infty$.

Proof. This follows from Lemma 3.2 and results of Carmona and Chevet [13] especially their Proposition 3.1 and Lemma 3.1 a version of which is stated below as Lemma 3.4. Although they only deal with tensor products of two maps the lemma shows that the result holds for general q by induction. $\square$

Denote by $E \otimes_\pi F$ the completion of the tensor product space of two Banach spaces E and F using projective tensor product norm, cf. notation (i) in Section 2A.

Lemma 3.4 *(Carmona and Chevet). Consider separable Hilbert spaces H and K and separable Banach spaces E and F. Let $T : H \to E$ be γ-radonifying and $S : K \to F$ bounded linear. Then $S \otimes T : H \otimes K \to E \otimes_\pi F$ is a bounded linear map into the projective tensor product. Moreover*

$$\|S \otimes T\|_{\mathbb{L}(H\otimes K; E\otimes_\pi F)} \leqslant \pi_2\big(T^*\big)\|S\|$$

where $\pi_2(T^*)$ denotes the 2-summing norm of the adjoint of T.

The conditional expectations of these operators can be defined as in Elworthy and Yor [38], Elworthy, LeJan and Li [30], to give bounded linear maps, defined almost surely,

$$\overline{\bigwedge^q (T\mathcal{I})}(\omega) : \bigwedge^q H \to \bigwedge^q (T_{\mathcal{I}(\omega)}C_{x_0}).$$

For example

$$\overline{\bigwedge^q (T\mathcal{I})}(\omega) := \mathbf{E}\big\{\bigwedge^q (T_\omega\mathcal{I})\big|\mathcal{F}^{x_0}\big\}(\omega)$$

is given by

$$\overline{\bigwedge^q (T\mathcal{I})}(\omega)(h)_t = \big(\bigwedge^q /\!\!/_t\big)\mathbf{E}\big\{\bigwedge^q \big(/\!\!/_t^{-1}\big)\bigwedge^q \big(T\mathcal{I}_t(h)\big)\big|\mathcal{F}^{x_0}\big\}(\omega).$$

For μ_{x_0} almost all $\sigma \in C_{x_0}M$ we have also

$$\big(\overline{\bigwedge^q (T\mathcal{I})}\big)_\sigma : \bigwedge^q H \to \bigwedge^q (T_\sigma C_{x_0})$$

given by

$$\left(\overline{\bigwedge^q(T\mathcal{I})}\right)_\sigma(h) := \mathbf{E}\left\{\bigwedge^q(T.\mathcal{I})(h)\,\middle|\,\mathcal{I}=\sigma\right\}.$$

Note the inequalities

$$\begin{aligned}\left\|\overline{\bigwedge^q(T\mathcal{I})}(\omega)(h)\right\| &\leqslant \mathbf{E}\left\{\left\|\bigwedge^q(T.\mathcal{I})(h)\right\|\,\middle|\,\mathcal{F}^{x_0}\right\}(\omega) \quad \text{a.s.}\\ &\leqslant \mathbf{E}\left\|\bigwedge^q(T_\omega\mathcal{I})\right\|\|h\| \quad \text{a.s.}\end{aligned}$$

which give L^p bounds for operator norms of $\overline{\bigwedge^q(T\mathcal{I})}$, $q=1,2,\ldots.$

D. *Definition of $\mathcal{H}^q_\sigma$, H-q-vector fields and H-q-forms.* We can now define $\mathcal{H}^q_\sigma$, for almost all $\sigma\in C_{x_0}M$, to be the image of $\overline{\bigwedge^q(T\mathcal{I})_\sigma}$ in $\bigwedge^q T_\sigma C_{x_0}$ together with the inner product induced by the linear bijection

$$\overline{\bigwedge^q T\mathcal{I}_\sigma}\Big|_{[\ker\overline{\bigwedge^q T\mathcal{I}_\sigma}]^\perp} : \left[\ker\overline{\bigwedge^q(T\mathcal{I})_\sigma}\right]^\perp \to \mathcal{H}^q_\sigma.$$

Thus the $\mathcal{H}^q_\sigma$ are Hilbert spaces with natural continuous linear inclusions ι_σ, say, into the $\bigwedge^q T_\sigma C_{x_0}$.

Denote by $\mathcal{H}^q=\bigcup_\sigma\mathcal{H}^q_\sigma$ the "vector bundle over $C_{x_0}M$" with fibres $\mathcal{H}^q_\sigma$, and $(\mathcal{H}^q)^*$ the corresponding dual "bundle." Set $\mathcal{H}=\mathcal{H}^1$. Since these are only almost sure defined it is not strictly speaking correct to consider them as bundles over $C_{x_0}M$ though some vector bundle structure is given to $\mathcal{H}$ in [35] see also Section 9 below. The space of L^2 sections of $\mathcal{H}^q$ and $\mathcal{H}^{q*}$ are denoted by $L^2\Gamma\mathcal{H}^q$ and $L^2\Gamma\mathcal{H}^{q*}$. Sections of $(\mathcal{H}^q)^*$ or of $(\mathcal{H}^q)$ will be called H-q-forms (or admissible q-forms), or H-q-vector fields, respectively. Note that any q-form on $C_{x_0}M$ restricts to give an H-q-form.

4. Characterization of $\mathcal{H}^1_\cdot$ and $\mathcal{H}^2_\cdot$

A. 'Damped parallel translations' $W^{(q)}_t$ will play an essential role. For a q-vector $v\in\bigwedge^q T_{x_0}M$, define $W^{(q)}_t(V)\in\bigwedge^q T_{x_t}M$ to be the random q-vector satisfying

$$\frac{D}{dt}W^{(q)}_t(V) = -\frac{1}{2}\mathcal{R}^q\,W^{(q)}_t(V),\quad 0\leqslant t\leqslant T, \tag{4.1}$$

where $\mathcal{R}^q\in\mathrm{Hom}(\bigwedge^q TM;\bigwedge^q TM)$ is the Weitzenbock curvature term defined by $\mathcal{R}=\Delta-\operatorname{trace}\nabla^2$, see e.g. Airault [3], Elworthy [26], Ikeda and Watanabe [46], Elworthy, LeJan and Li [30], Elworthy, Li and Rosenberg [36], Malliavin [61]. Here (4.1) is a covariant equation along the paths of our solution $\{x_t\colon 0\leqslant t\leqslant T\}$ to (3.2).

For $q=1$ write W_t for $W^{(1)}_t$. Then $W_t : T_{x_0}M\to T_{x_t}M$ is the Dohrn–Guerra translation given by

$$\frac{D}{dt}W_t(V) = -\frac{1}{2}\mathrm{Ric}^{\#}_{x_t}\big(W_t(V)\big),\quad 0\leqslant t\leqslant T.$$

Write

$$\frac{\mathbb{D}}{dt} = W_t \frac{D}{dt} W_t^{-1}$$

acting on suitably regular vector fields $\{v_t\colon 0 \leqslant t \leqslant T\}$ along the paths of $\{x_t\colon 0 \leqslant t \leqslant T\}$. Then

$$\frac{\mathbb{D}}{dt} = \frac{D}{dt} + \frac{1}{2}\mathrm{Ric}^{\#},$$

cf. Fang, formula (1.3), in Fang [41] and Norris [63].

Definition 4.1. For almost all paths ω, define the L^2 tangent space $L^2 T_{\sigma.} C_{x_0}$ to consist of those paths $u : [0, T] \to TM$ over σ with

$$/\!/_{\cdot}^{-1} u_{\cdot} \in L^2\big([0, T]; T_{x_0} M\big)$$

together with its natural Hilbert space structure.

It was shown in [28], see also [30,32] that

$$\overline{T\mathcal{I}}_t(h) = \mathcal{W}_t\big(X(x_{\cdot})\dot{h}_{\cdot}\big) \tag{4.2}$$

where

$$\mathcal{W} : L^2 T_{x.} C_{x_0} \to T_{x.} C_{x_0}$$

is defined by

$$\big(\mathcal{W}(u)\big)_t = W_t \int_0^t (W_r)^{-1}(u_r)\, dr. \tag{4.3}$$

Note that

$$\frac{\mathbb{D}}{dt}\big(\mathcal{W}(u)\big)_t = u_t, \quad u \in L^2 T_{x.} C_{x_0}. \tag{4.4}$$

Thus, as shown in [30,32],

$$\mathcal{H}_\sigma^1 = \big\{ v \in T_\sigma C_{x_0}\colon /\!/_{\cdot}^{-1} v_{\cdot} \in L_0^{2,1}(T_{x_0} M) \big\} \tag{4.5}$$

with inner product

$$\langle v^1, v^2 \rangle_{\mathcal{H}^1} = \int_0^T \left\langle \frac{\mathbb{D}}{ds} v_s^1, \frac{\mathbb{D}}{ds} v_s^2 \right\rangle ds \tag{4.6}$$

so that $\frac{\mathbb{D}}{d\cdot} : \mathcal{H}_\sigma^1 \to L^2 T_\sigma C_{x_0}$ is an isometric isomorphism with inverse $\mathcal{W}$ for almost all $\sigma \in C_{x_0} M$. Thus it agrees as a Hilbertable space with the usual Bismut tangent space, though the inner product is not the one originally used. Using the same notation, by Section 2D we note that a vector u of $\bigwedge^2 T_\sigma C_{x_0} M$ is in $\bigwedge^2 \mathcal{H}_\sigma$ if and only if there exists $\underline{k} \in \bigwedge^2 L^2 T_\sigma C_{x_0} M$ so that

$$u_{s,t} = \left(\bigwedge^2 \mathcal{W}\right)_{s,t}\underline{k},$$

or written in full,

$$u_{s,t} = \left(W_s \int_0^s (W_{r_1})^{-1}(-)\,dr_1 \otimes W_t \int_0^t (W_{r_2})^{-1}(-)\,dr_2\right)\underline{k}_{r_1,r_2}. \tag{4.7}$$

If so $\underline{k}_{s,t} = \frac{\mathbb{D}}{\partial s} \otimes \frac{\mathbb{D}}{\partial t} u$ or equally $\underline{k} = \bigwedge^2 \frac{\mathbb{D}}{d\cdot} u$.

B. More generally let $L^2(\bigwedge^q TM)_\sigma$ and $C_0(\bigwedge^q TM)_\sigma$ denote respectively the spaces of L^2 and continuous paths vanishing at 0, $u : [0, T] \to \bigwedge^q TM$ over σ. Define

$$\mathcal{W}^{(q)} : L^2\left(\bigwedge^q TM\right)_\sigma \to C_0\left(\bigwedge^q TM\right)_\sigma$$

by

$$\left(\mathcal{W}^{(q)}(V_\cdot)\right)_t = W_t^{(q)} \int_0^t \left(W_r^{(q)}\right)^{-1}(V_r)\,dr \tag{4.8}$$

$$= \int_0^t W^{(q)}{}_t^r(V_r)\,dr \tag{4.9}$$

where

$$W^{(q)}{}_t^s = W_t^{(q)}\left(W_s^{(q)}\right)^{-1}$$

is the solution to

$$\frac{D}{dt} W^{(q)}{}_t^s(V) = -\frac{1}{2}\mathcal{R}^q\left(W^{(q)}{}_t^s(V)\right), \quad s \leqslant t \in [0, T], \tag{4.10}$$

with $W^{(q)}{}_s^s = \mathrm{Id} : \bigwedge^q T_{\sigma_s} M \to \bigwedge^q T_{\sigma_s} M$. Write W_t^s for $W^{(1)}{}_t^s$ and observe that $\mathcal{W}^{(1)} = \mathcal{W}$. For simplicity we shall write $\mathcal{W}_t^{(q)}(V_\cdot)$ for $(\mathcal{W}^{(q)}(V_\cdot))_t$.

Set

$$\frac{\mathbb{D}^{(q)}}{dt} = \left(\frac{D}{dt}\right) + \frac{1}{2}\mathcal{R}^q, \tag{4.11}$$

acting on q-vectors on M along a sample path σ. Then as for $q = 1$, and for $W_t^{(q)}$ defined by (4.10):

$$\frac{\mathbb{D}^{(q)}}{dt} V_{t,\ldots,t} = W_t^{(q)} \frac{d}{dt}\left(W_t^{(q)}\right)^{-1} V_{t,\ldots,t}$$

and the inverse of $\frac{\mathbb{D}^{(q)}}{d\cdot}$ is

$$\left(\frac{\mathbb{D}^{(q)}}{d\cdot}\right)^{-1} = W_{\cdot}^{(q)} \int_0^{\cdot} W_r^{(q)}(\mathrm{ev}_r -)\,dr = \mathcal{W}^q$$

where ev_r, generically, denotes the evaluation operator at r. Furthermore let $\mathcal{R} : \bigwedge^2 TM \to \bigwedge^2 TM$ be the curvature operator. Then the second Weitzenbock curvature $\mathcal{R}^2$ is given by

$$\mathcal{R}^2 = d\bigwedge^2\left(\mathrm{Ric}^\#\right) - 2\mathcal{R}.$$

Here the operator $d\bigwedge^2(\mathrm{Ric}^\#)$, also $(d\bigwedge^2)(\frac{\mathbb{D}}{d\cdot})$ below, is defined using formula (2.3). Therefore using (2.2), for $V \in \bigwedge^2 T_\sigma C_{x_0} M$,

$$\frac{\mathbb{D}^{(2)}}{dt} V_{t,t} = \left(\left(\left(d\bigwedge^2\right)\left(\frac{\mathbb{D}}{d\cdot}\right)\right) V\right)_{t,t} - \mathcal{R}(V_{t,t}), \tag{4.12}$$

whenever all the terms involved make sense. In the above we have identified $\frac{D}{dt} V_{t,t}$ with $(d\bigwedge^2 \frac{D}{d\cdot})(V)_{t,t}$ where the first refers to covariant differentiation of the 2-vector field $\{V_{t,t}: 0 \leqslant t \leqslant T\}$ along σ obtained from the element V in $\bigwedge^2 T_\sigma C_{x_0}$.

C. In this section we shall discuss a system of equations related to the conditional expectation of the Itô map. First note that the curvature operator $\mathcal{R}$ on the manifold M induces a linear map Q_σ on $\bigwedge^2_\epsilon T_\sigma C_{x_0}$ given by

$$Q_\sigma(G)_{s,t} = \left(\mathbf{1} \otimes W_t^s\right) W_s^{(2)} \int_0^s \left(W_r^{(2)}\right)^{-1}\left(\mathcal{R}_{\sigma_r}(G_{r,r})\right) dr, \quad s \leqslant t. \tag{4.13}$$

Equivalently,

$$Q_\sigma(G)_{s,t} = (W_s \otimes W_t)\left(\bigwedge^2\left(W_{\cdot}^{-1}\right) W_{\cdot}^{(2)} \int_0^{\cdot} \left(W_r^{(2)}\right)^{-1}\left(\mathcal{R}_{\sigma_r}(G_{r,r})\right) dr\right)_{\min(s,t)}.$$

Clearly

$$\begin{cases} \left(\mathbf{1} \otimes \dfrac{\mathbb{D}}{dt}\right) Q(G)_{s,t} = 0, & s < t, \\ \dfrac{\mathbb{D}^{(2)}}{ds} Q(G)_{s,s} = \mathcal{R}(G_{s,s}). \end{cases} \tag{4.14}$$

The second equation is equivalent to $(d\bigwedge^2 \frac{\mathbb{D}}{ds})Q(G)_{s,s} = \mathcal{R}((I+Q)G)_{s,s}$. Define $j_G : [0,T] \to T_{x_0}M \otimes T_{x_0}M$ by

$$j_G(s) = \left(W_s^{-1} \otimes W_s^{-1}\right) W_s^{(2)} \int_0^s \left(W_r^{(2)}\right)^{-1} \left(\mathcal{R}_{\sigma_r}(G_{r,r})\right) dr. \tag{4.15}$$

Then j_G is C^1 and, writing $s \wedge t$ for $\min\{s,t\}$,

$$\left(W_s^{-1} \otimes W_t^{-1}\right) Q_\sigma(G)_{s,t} = j_G(s \wedge t). \tag{4.16}$$

If we set

$$D\left(\bigwedge^2 T_\sigma C_{x_0}\right) = \left\{ \begin{array}{l} u \in \bigwedge_\epsilon^2 T_\sigma C_{x_0} \text{ such that} \\ (1)\ \text{for each } 0 \leqslant s < T,\ t \mapsto (/\!/_s^{-1} \otimes /\!/_t^{-1}) u_{s,t} \text{ is} \\ \qquad \text{absolutely continuous on } (s,T]; \\ (2)\ r \mapsto \bigwedge^2 (/\!/_r^{-1}) u_{r,r} \text{ is absolutely continuous on } [0,T] \end{array} \right\}$$

then $Q(G)$ clearly lies in $D(\bigwedge^2 T_\sigma C_{x_0})$. There is another linear map $\mathbb{R}$ on $\bigwedge_\epsilon^2 T_\sigma C_{x_0}$ defined by

$$\mathbb{R}(Z)_{s,t} = (W_s \otimes W_t) \int_0^s \left(\bigwedge^2 W_r{}^{-1}\right)\left(\mathcal{R}_{\sigma_r}(Z_{r,r})\right) dr, \quad s \leqslant t, \tag{4.17}$$

which also sends $\bigwedge_\epsilon^2 T_\sigma C_{x_0} M$ to $D(\bigwedge^2 T_\sigma C_{x_0} M)$. Furthermore, from Eq. (4.12)

$$\begin{cases} \left(\mathbf{1} \otimes \dfrac{\mathbb{D}}{dt}\right) \mathbb{R}(Z)_{s,t} = 0, \quad s < t, \\ \dfrac{\mathbb{D}^{(2)}}{ds} \mathbb{R}(Z)_{s,s} = \mathcal{R}_{\sigma_s}\left(Z_{s,s} - \mathbb{R}(Z)_{s,s}\right). \end{cases} \tag{4.18}$$

In fact $\mathbf{1} + Q$ and $\mathbf{1} - \mathbb{R}$ are inverse of each other as described in the following lemma. It will be shown later, Section 9D, that $\mathbb{R}$ restricted to $\bigwedge^2 \mathcal{H}^1$ is the curvature operator of the damped Markovian connection on $\mathcal{H}^1$ which is induced by the map $\frac{\mathbb{D}}{d\cdot}$ from the pointwise connection on the L^2 tangent bundle $L^2 T C_{x_0}$.

Lemma 4.2. (i) *Given $G \in D(\bigwedge^2 T_\sigma C_{x_0})$, there is a unique solution $Z \in D(\bigwedge^2 T_\sigma C_{x_0})$ to the following equations*

$$\begin{cases} \left(\mathbf{1} \otimes \dfrac{\mathbb{D}}{dt}\right) Z_{s,t} = \left(\mathbf{1} \otimes \dfrac{\mathbb{D}}{dt}\right) G_{s,t}, \quad s < t, \\ \dfrac{\mathbb{D}^{(2)}}{ds} Z_{s,s} = \left(\left(\left(d\bigwedge^2\right)\left(\dfrac{\mathbb{D}}{d\cdot}\right)\right) G\right)_{s,s}, \\ Z_{0,0} = G_{0,0}. \end{cases} \tag{4.19}$$

The solution is

$$Z_{s,t} = G_{s,t} + Q_\sigma(G)_{s,t}.$$

Conversely for each $Z \in D(\bigwedge^2 T_\sigma C_{x_0})$ *the unique solution to* (4.19) *is given by*

$$G = Z - \mathbb{R}(Z). \tag{4.20}$$

(ii) *As operators on* $\bigwedge_\epsilon^2 T_\sigma C_{x_0} M$ *both* Q *and* $\mathbb{R}$ *are compact and* $\mathbf{1} + Q$ *and* $\mathbf{1} - \mathbb{R}$ *are mutual inverses. In particular for all* v *in* $\bigwedge_\epsilon^2 T_\sigma C_{x_0} M$,

$$Q(v) = \mathbb{R}\big(v + Q(v)\big),$$
$$Q(\mathbf{1} + Q)^{-1} v = \mathbb{R}(v).$$

(iii) *The following holds on* $D(\bigwedge^2 T_\sigma C_{x_0})$:

$$\big(\textstyle\bigwedge^2 W^{-1} Z\big)_{s,t} - \big(\textstyle\bigwedge^2 W^{-1} Z\big)_{s\wedge t} = \big(\textstyle\bigwedge^2 W^{-1} G\big)_{s,t} - \big(\textstyle\bigwedge^2 W^{-1} G\big)_{s\wedge t}, \tag{4.21}$$

which is equivalent to, for $r \leqslant s \leqslant t$,

$$Z_{r,t} - \big(\mathbf{1} \otimes W_t^s\big) Z_{r,s} = G_{r,t} - \big(\mathbf{1} \otimes W_t^s\big) Z_{r,s}.$$

Proof. Given $G \in D(\bigwedge^2 T_\sigma C_{x_0})$, $Z = (\mathbf{1} + Q_\sigma)(G)$ certainly solves (4.19). For uniqueness let Z be any solution in $D(\bigwedge^2 T_\sigma C_{x_0})$. Solve the first equation in (4.19) to get

$$Z_{s,t} = G_{s,t} + (W_s \otimes W_t)\big(\tilde{j}(s)\big), \quad s \leqslant t, \tag{4.22}$$

some $\tilde{j}(s) \in \bigwedge^2 T_{\sigma_0} M$. Then

$$Z_{s,s} = G_{s,s} + (W_s \otimes W_s)\big(\tilde{j}(s)\big). \tag{4.23}$$

In particular $(W_s \otimes W_s)(\tilde{j}(s))$ is absolutely continuous in s. Substitute the above equation (4.23) into (4.19) and use (4.12) to see

$$\frac{\mathbb{D}^{(2)}}{ds}(W_s \otimes W_s)\big(\tilde{j}(s)\big) = \mathcal{R}(G_{s,s}),$$

giving

$$(W_s \otimes W_s)\big(\tilde{j}(s)\big) = W_s^{(2)} \int_0^s \big(W_r^{(2)}\big)^{-1}\big(\mathcal{R}_{\sigma_r}(G_{r,r})\big)\,dr.$$

Thus $\tilde{j}(s) = j_G(s)$ and uniqueness holds by formula (4.16).

Similarly given $Z \in D(\bigwedge^2 T_\sigma C_{x_0})$, $(\mathbf{1} - \mathbb{R})(Z)$ is seen to satisfy (4.19) given $Z \in D(\bigwedge^2 T_\sigma C_{x_0})$.

Now using the isometry between $\bigotimes_\epsilon^2 C_0T_{x_0}M$ and $C_0([0,T]^2; \bigotimes^2 T_{x_0}M)$ and the Arzèla–Ascoli theorem applied to $(s,t)\mapsto j(s,t)$ for a bounded set of G, we see that $Q:\bigwedge_\epsilon^2 T_\sigma C_{x_0}M \to \bigwedge_\epsilon^2 T_\sigma C_{x_0}M$ is compact. Therefore $\mathbf{1}+Q$ has closed range. Since we have just seen that its range contains all Z in the dense subspace $D(\bigwedge^2 T_\sigma C_{x_0})$ it is surjective and so an isomorphism. By Eq. (4.20) its inverse is $\mathbf{1}-\mathbb{R}$ and so $\mathbb{R}$ is compact. The rest of parts (i) and (ii) follows directly.

Part (iii) follows from (4.22) and (4.23). □

See Section 6 below for a more detailed examination of $Q(V)$.

D. The following theorem gives alternative descriptions of the space $\mathcal{H}_\sigma^2$.

Theorem 4.3. *For any $h^1, h^2 \in L_0^{2,1}\mathbf{R}^m$, set $\underline{h} = h^1\wedge h^2$. Then*

$$\overline{\bigwedge^2 T\mathcal{I}}(\underline{h}) = (\mathbf{1}+Q)\bigwedge^2 \overline{T\mathcal{I}}(\underline{h}). \tag{4.24}$$

In particular the space $\mathcal{H}_\sigma^2 = \{\overline{\bigwedge^2 T\mathcal{I}_\sigma}(h),\ h\in\bigwedge^2 H\}$ can be characterised by any one of the following:

(i)
$$\mathcal{H}_\sigma^2 = \left\{ \begin{array}{l} u\in D(\bigwedge^2 T_\sigma C_{x_0}), \textit{such that there exists } G\in \mathcal{H}_\sigma^1\wedge\mathcal{H}_\sigma^1, \\ \quad \textit{with } ((\mathbf{1}\otimes \frac{\mathbb{D}}{d\cdot})u)_{s,t} = ((\mathbf{1}\otimes\frac{\mathbb{D}}{d\cdot})G)_{s,t},\ s<t,\ \textit{and} \\ \quad \frac{\mathbb{D}^{(2)}}{ds}u_{s,s} = (((d\bigwedge^2)\frac{\mathbb{D}}{d\cdot})G)_{s,s},\ 0\leqslant s\leqslant T \end{array} \right\}.$$

(ii)
$$\mathcal{H}_\sigma^2 = \left\{u\in \bigwedge\nolimits_\varepsilon^2 T_\sigma C_{x_0}, \textit{such that } u = v + Q_\sigma(v),\ \textit{some } v\in\mathcal{H}_\sigma^1\wedge\mathcal{H}_\sigma^1\right\},$$

and for $v_1, v_2 \in \bigwedge^2\mathcal{H}_\sigma^1$, by definition,

$$\left\langle v^1 + Q_\sigma\left(v^1\right), v^2 + Q_\sigma\left(v^2\right)\right\rangle_{\mathcal{H}_\sigma^2} = \left\langle v^1, v^2\right\rangle_{\bigwedge^2\mathcal{H}_\sigma^1}. \tag{4.25}$$

(iii) *$u\in\mathcal{H}^2$ if and only if $u-\mathbb{R}(u)\in\bigwedge^2\mathcal{H}^1$. If so*

$$\|u\|_{\mathcal{H}^2} = \left\|u-\mathbb{R}(u)\right\|_{\bigwedge^2\mathcal{H}^1}.$$

In particular $\mathcal{H}_\sigma^2$ depends on the Riemannian structure of M but not *the choice of stochastic differential equation* (3.2) *provided its LeJan–Watanabe connection in the sense of Elworthy, LeJan and Li* [29] *is the Levi-Civita connection.*

Proof. For $h^1\wedge h^2\in L_0^{2,1}(R^m)$, write $V^1\wedge V^2 = (\bigwedge^2 T\mathcal{I})(h^1\wedge h^2)$. Then applying Itô's formula in t for $0\leqslant s<t\leqslant T$ with D_t referring to covariant stochastic differentiation in t,

$$\begin{aligned}
&(\mathbf{1}\otimes D_t)\left(V^1\wedge V^2\right)_{s,t} \\
&\quad = \frac{1}{2}V_s^1\otimes\left(\nabla X\left(V_t^2\right)\circ dB_t + X(x_t)\left(\dot h_t^2\right)dt\right) \\
&\qquad - \frac{1}{2}V_s^2\otimes\left(\nabla X\left(V_t^1\right)\circ dB_t + X(x_t)\left(\dot h_t^1\right)dt\right) \\
&\quad = \frac{1}{2}V_s^1\otimes\left(\nabla X\left(V_t^2\right)/\!/_t\, d\beta_t - \frac{1}{2}\mathrm{Ric}^\#\left(V_t^2\right)dt + X(x_t)\left(\dot h_t^2\right)dt\right)
\end{aligned}$$

$$- \frac{1}{2} V_s^2 \otimes \left(\nabla X\big(V_t^1\big) /\!\!/_t \, d\beta_t - \frac{1}{2} \operatorname{Ric}^\#\big(V_t^1\big)\, dt + X(x_t)\big(\dot h_t^1\big)\, dt \right)$$

$$= \big(\mathbf{1} \otimes \nabla X(-) /\!\!/_t \, d\beta_t\big)\big(V^1 \wedge V^2\big)_{s,t} - \left(\mathbf{1} \otimes \frac{1}{2} \operatorname{Ric}^\#(-)\right)\big(V^1 \wedge V^2\big)_{s,t}\, dt$$

$$+ \frac{1}{2}\big(V_s^1 \otimes X(x_t)\big(\dot h_t^2\big) - V_s^2 \otimes X(x_t)\big(\dot h_t^1\big)\big)\, dt.$$

Write $\overline{V^1 \wedge V^2}$ for the conditional expectation of $V^1 \wedge V^2$ with respect to $\mathcal{F}^{x_0}$, and similarly let $\overline{V}^i$ stand for the conditional expectation of V^i with respect to $\mathcal{F}^{x_0}$. Then by (3.6), following Elworthy and Yor [38], and (4.4)

$$(\mathbf{1} \otimes D_t)\big(\overline{V^1 \wedge V^2}\big)_{s,t} = -\left(\mathbf{1} \otimes \frac{1}{2} \operatorname{Ric}^\#(-)\right)\big(\overline{V^1 \wedge V^2}\big)_{s,t}\, dt$$

$$+ \frac{1}{2}\big(\overline{V}_s^1 \otimes X\big(\dot h_t^2\big) - \overline{V}_s^2 \otimes X\big(\dot h_t^1\big)\big)\, dt.$$

This is equivalent to

$$\left(\mathbf{1} \otimes \frac{\mathbb{D}}{d\cdot}\right)\big(\overline{V^1 \wedge V^2}\big)_{s,t} = \frac{1}{2}\big(\overline{V}_s^1 \otimes X\big(\dot h_t^2\big) - \overline{V}_s^2 \otimes X\big(\dot h_t^1\big)\big)$$

$$= \left(\mathbf{1} \otimes \frac{\mathbb{D}}{d\cdot}\right)\big(\overline{V}^1 \wedge \overline{V}^2\big)_{s,t}.$$

On the other hand, Itô's formula, applied to the 2-vector field $\{V_t^1 \wedge V_t^2,\ 0 \leqslant t \leqslant T\}$ along σ in M, gives

$$D_t\big(V_t^1 \wedge V_t^2\big) = V_t^1 \wedge \big(\nabla X\big(V_t^2\big) \circ dB_t + X(x_t)\big(\dot h_t^2\big)\, dt\big)$$

$$+ \big(\nabla X\big(V_t^1\big) \circ dB_t + X(x_t)\big(\dot h_t^1\big)\, dt\big) \wedge V_t^2.$$

Change to Itô differentials and decompose the noise recalling that ∇X vanishes on $[\ker X]^\perp$:

$$D_t\big(V_t^1 \wedge V_t^2\big) = V_t^1 \wedge \left(\nabla X\big(V_t^2\big)(/\!\!/_t \, d\beta_t) - \frac{1}{2} \operatorname{Ric}^\#\big(V_t^2\big)\, dt + X(x_t)\big(\dot h_t^2\big)\, dt\right)$$

$$+ \left(\nabla X\big(V_t^1\big)(/\!\!/_t \, d\beta_t) - \frac{1}{2} \operatorname{Ric}^\#\big(V_t^1\big)\, dt + X(x_t)\big(\dot h_t^1\big)\, dt\right) \wedge V_t^2$$

$$+ \frac{1}{2} \sum_{i=1}^m \big(\nabla X^i \wedge \nabla X^i\big)\big(V_t^1 \wedge V_t^2\big)$$

$$= \Big(d{\textstyle\bigwedge}^2\big(\nabla X(-)(/\!\!/_t \, d\beta_t)\big)\Big)\big(V_t^1 \wedge V_t^2\big)$$

$$- \left(d{\textstyle\bigwedge}^2\left(\frac{1}{2} \operatorname{Ric}^\#(-)\right)\right)\big(V_t^1 \wedge V_t^2\big)\, dt + \sum_{i=1}^m \big(\nabla X^i \wedge \nabla X^i\big)\big(V_t^1 \wedge V_t^2\big)$$

$$+ \big(V_t^1 \wedge X(x_t)\big(\dot h_t^2\big) + X(x_t)\big(\dot h_t^1\big) \wedge V_t^2\big)\, dt.$$

But

$$-\left(d\bigwedge^2\left(\frac{1}{2}\operatorname{Ric}^{\#}(-)\right)\right)+\sum_{i=1}^{m}\nabla X^i\wedge\nabla X^i=-\frac{1}{2}\mathcal{R}^2, \tag{4.26}$$

as in Elworthy [27] for gradient systems, see also Elworthy, LeJan and Li [30] for the general situation. Again use the technique of Elworthy and Yor [38], taking conditional expectations to get

$$\frac{D}{dt}\overline{V_t^1\wedge V_t^2}=-\frac{1}{2}\mathcal{R}^2\big(\overline{V_t^1\wedge V_t^2}\big)+\overline{V}_t^1\wedge X(x_t)\big(\dot{h}_t^2\big)+X(x_t)\big(\dot{h}_t^1\big)\wedge\overline{V}_t^2.$$

Thus

$$\begin{aligned}\frac{\mathbb{D}^{(2)}}{dt}\overline{V_t^1\wedge V_t^2}&=\overline{V}_t^1\wedge X(x_t)\big(\dot{h}_t^2\big)+X(x_t)\big(\dot{h}_t^1\big)\wedge\overline{V}_t^2\\&=\left(d\bigwedge^2\left(\frac{\mathbb{D}}{dt}\right)\right)\big(\overline{V}^1\wedge\overline{V}^2\big)_{t,t}.\end{aligned}$$

We have shown given $u=\overline{\bigwedge^2 T\mathcal{I}}(h^1\wedge h^2)$, it is related to $\overline{T\mathcal{I}}(h^1)\wedge\overline{T\mathcal{I}}(h^2)$ by Eq. (4.19). Solve the equation to obtain

$$\overline{\bigwedge^2 T\mathcal{I}}(\underline{h})_{s,t}=\bigwedge^2\overline{T\mathcal{I}}(\underline{h})_{s,t}+\big(\mathbf{1}\otimes W_t^s\big)W_s^{(2)}\int_0^s\big(W_r^{(2)}\big)^{-1}\big(\mathcal{R}\big(\bigwedge^2\overline{T\mathcal{I}}(\underline{h})_{r,r}\big)\big)\,dr,$$

that is, the desired identity (4.24). On the other hand, given u satisfying (4.19) for $G=\bigwedge^2\overline{T\mathcal{I}}(\underline{h})$, $\underline{h}\in\bigwedge^2 L_0^{2,1}(\mathbf{R}^m)$, then $u=\overline{\bigwedge^2 T\mathcal{I}}(\underline{h})$ by uniqueness of the solution. This proves the first equivalence. The second equivalence follows from Lemma 4.2. Part (iii) follows straightforwardly from the previous lemma. □

5. *H*-one-forms: Exterior differentiation and Hodge decomposition

A. *Differentiation of functions.* For scalar analysis in our context and with this notation, we refer to [35] or for the basic facts to [30]. As emphasised in [35] it is necessary to fix an initial domain, $\operatorname{Dom}(d_{\mathcal{H}})\subset L^2(C_{x_0}M;\mathbf{R})$ for the H-derivative operator $d_{\mathcal{H}}$. We shall choose this to be a subspace which contains the smooth cylindrical functions and consists of BC^2 functions in the Fréchet sense, using the natural Finsler structure of $C_{x_0}M$, see [37]. For example the space of all smooth cylindrical functions. (We will require two derivatives in order to be able to prove that exact H-one-forms are closed.) It is standard, going back to Driver [20], that then $d_{\mathcal{H}}:\operatorname{Dom}(d_{\mathcal{H}})\subset L^2(C_{x_0}M)\to L^2\Gamma\mathcal{H}^*$ is closable. We will denote its closure by $\bar{d}^0$ to show it is acting on zero forms, or simply by $\bar{d}$, and let $\mathbb{D}^{2,1}$ be its domain with graph norm. There is the analogous result for functions with values in a separable Hilbert space G. In this case the domain will be written as $\mathbb{D}^{2,1}(G)$ or $\mathbb{D}^{2,1}(C_{x_0}M;G)$ and for almost all $\sigma\in C_{x_0}M$ the derivative $\bar{d}f_\sigma$ of f at the path σ will be in the space of Hilbert–Schmidt maps $\mathcal{L}_2(\mathcal{H}_\sigma;G)$. As usual for real-valued functions there is the corresponding gradient operator $\nabla:\mathbb{D}^{2,1}\to L^2\Gamma\mathcal{H}$. The negative of its adjoint we write as

$$\operatorname{div}: \operatorname{Dom}(\operatorname{div}) \subset L^2\Gamma\mathcal{H} \to L^2(C_{x_0}M; \mathbf{R}),$$

so if V is an H-vector field in Dom(div) and $f \in \mathbb{D}^{2,1}$ then

$$\int_{C_{x_0}M} \bar{d}f(V)\,d\mu = \int_{C_{x_0}M} \langle \nabla(f)(\sigma), V(\sigma)\rangle_{\mathcal{H}_\sigma}\,d\mu(\sigma)$$
$$= -\int_{C_{x_0}M} f(\sigma)\operatorname{div}(V)(\sigma)\,d\mu(\sigma). \tag{5.1}$$

This divergence operator is closed and the standard Riesz correspondence $\phi \mapsto \phi^\#$ with inverse $V \mapsto V^\#$ between H-one-forms and H-vector fields maps the domain of the adjoint d^* of $\bar{d}$ to that of the divergence with $d^*\phi = -\operatorname{div}(\phi^\#)$.

For $1 \leqslant p < \infty$ there are the spaces $\mathbb{D}^{p,1}$ defined in the same way as for $p = 2$ but using L^p norms. Spaces of "weakly differentiable" functions $\mathbb{W}^{p,1}(C_{x_0}M; G)$, $1 \leqslant p < \infty$, were also given in [35], loosely following [23]. Here we shall also denote those weak derivatives by $\bar{d}$. Whether $\mathbb{W}^{p,1} = \mathbb{D}^{p,1}$, as occurs on $C_0\mathbf{R}^m$, is an open question. We note the following from [35], cf. [32]. Parts (a) and (b) are essentially equivalent and (a) is a vital step in the proof of the closability of the exterior derivative used below.

Theorem 5.1.

(a) *The map $\overline{T\mathcal{I}(-)}_{\cdot}$ from $L^2(C_0\mathbf{R}^m; H)$ to vector fields on $C_{x_0}M$ given by*

$$\overline{T\mathcal{I}(V)}_\sigma = \mathbb{E}\{\omega \mapsto T\mathcal{I}_\omega(V(\omega)) \,|\, \mathcal{I}(\omega) = \sigma\} \tag{5.2}$$

gives a continuous linear map $\overline{T\mathcal{I}(-)}_{\cdot} : L^2(C_0\mathbf{R}^m; H) \to L^2\Gamma\mathcal{H}$.

(b) *The pull back operation $\phi \mapsto \mathcal{I}^*(\phi)$ defined from one-forms on $C_{x_0}M$ to H-one-forms on $C_0\mathbf{R}^m$ by $(\mathcal{I}^*\phi)_\omega = \phi_{\mathcal{I}(\omega)} \circ T_\omega\mathcal{I}$ extends to give a continuous linear map $\mathcal{I}^* : L^2\Gamma\mathcal{H}^* \to L^2(C_0\mathbf{R}^m; H^*)$.*

(c) *If $f \in \mathbb{D}^{p,1}(C_{x_0}M; G)$ then the composition $f \circ \mathcal{I}$ is in $\mathbb{D}^{p,1}(C_0\mathbf{R}^m; G)$ and then $\bar{d}(f \circ \mathcal{I}) = \mathcal{I}^*(\bar{d}f)$.*

(d) *A measurable function $f : C_{x_0}M \to G$ has $f \in \mathbb{W}^{p,1}(C_{x_0}M; G)$ iff the composition $f \circ \mathcal{I}$ is in $\mathbb{D}^{p,1}(C_0\mathbf{R}^m; G)$ and then the weak derivative $\bar{d}f$ satisfies $\bar{d}(f \circ \mathcal{I}) = \mathcal{I}^*(\bar{d}f)$.*

B. *Exterior differentiation of H-one-forms.* For any C^1 one-form ϕ on $C_{x_0}M$ there is the usual exterior derivative $d\phi$ given by formula (1.2). This can be restricted to give an H-2-form, $d^1_{\mathcal{H}}\phi$ say. Thus $d^1_{\mathcal{H}}\phi_\sigma$ is the composition of $d\phi_\sigma$ with the, continuous, inclusion of $\mathcal{H}^2_\sigma$ in $\bigwedge^2 T_\sigma C_{x_0}M$. As for functions we choose an initial domain $\operatorname{Dom}(d^1_{\mathcal{H}})$ to give an operator

$$d^1_{\mathcal{H}} : \operatorname{Dom}(d^1_{\mathcal{H}}) \subset L^2\Gamma(\mathcal{H}^1)^* \to L^2\Gamma(\mathcal{H}^2)^*.$$

The domain must consist of C^2 one-forms ϕ on $C_{x_0}M$ which satisfy

(i) as an H-one-form, $\phi \in L^\infty \Gamma \mathcal{H}^*$.
(ii) The exterior derivative $d\phi$ when restricted to $\mathcal{H}^2$ is essentially bounded, i.e. $d^1_{\mathcal{H}}\phi \in L^\infty \Gamma \mathcal{H}^{2*}$.
(iii) (Module structure) If $f \in \mathrm{Dom}(d_{\mathcal{H}})$ and $\phi \in \mathrm{Dom}(d^1_{\mathcal{H}})$ then $f\phi \in \mathrm{Dom}(d^1_{\mathcal{H}})$.
(iv) The domain of $d_{\mathcal{H}}$ is mapped into the domain of $d^1_{\mathcal{H}}$ by $d_{\mathcal{H}}$.

All these hold if we use smooth cylindrical functions and forms as initial domains, or C^2 functions and C^1 forms which are bounded together with their exterior derivative using the natural Finsler metric on $C_{x_0}M$. In fact it is shown in [35] that $\mathbb{D}^{2,1}$ is independent of the choice of $\mathrm{Dom}(d_{\mathcal{H}})$ under these restrictions, so we may as well assume that the latter is the space of smooth cylindrical functions.

Under these assumptions we have

Theorem 5.2. *(See [32].) The exterior derivative considered as an operator*

$$d^1_{\mathcal{H}} : \mathrm{Dom}\big(d^1_{\mathcal{H}}\big) \subset L^2\Gamma\big(\mathcal{H}^1\big)^* \to L^2\Gamma\big(\mathcal{H}^2\big)^*$$

is closable.

Since the proof was given in full in [32] and the analogous proof for two-forms is in Part II it will be omitted here. However we note that the main step is to obtain a simple integration by parts formula for elements of $\mathrm{Dom}(d^1_{\mathcal{H}})$ by considering their pull backs, and that of their exterior derivatives to Wiener space by the Itô map. The pull back operation commutes with exterior differentiation, and a simple integration by parts formula for Wiener space can be applied to give the standard closability argument when combined with part (a) of Theorem 5.1. The crucial point is that, for $h \in \bigwedge^2 H$,

$$\int\limits_{C_{x_0}M} d^1_{\mathcal{H}}\phi\big(\overline{\textstyle\bigwedge^2(T\mathcal{I})(h)}\big)\, d\mu_{x_0} = \int\limits_{C_0\mathbf{R}^m} d^1_{\mathcal{H}}\phi\big(\textstyle\bigwedge^2(T\mathcal{I})(h)\big)\, d\mathbb{P}.$$

Let $\bar{d}^1$ denote the closure of $d^1_{\mathcal{H}}$.

Theorem 5.3. *(See [32].) The derivative $\bar{d}^0 f$ of any function $f \in \mathbb{D}^{2,1}$ lies in the domain of $\bar{d}^1$ and*

$$\bar{d}^1\bar{d}^0 f = 0.$$

The derivation property $\bar{d}^1(f\phi) = f\bar{d}^1\phi + \bar{d}^0 f \wedge \phi$ is given meaning and proved in Theorem 7.1 below.

C. *The first L^2 de Rham cohomology group and a Hodge decomposition for H-one-forms.* From the results above we can define the first L^2-cohomology group of $C_{x_0}M$ to be the quotient of the kernel of $\bar{d}^1$ by the image of $\bar{d}^0$. An important result here is due to Fang:

Theorem 5.4. *(See Fang [39].) The image of $\bar{d}^0$ is a closed subspace of $L^2\Gamma\mathcal{H}^*$.*

It is almost a formality now to define the self-adjoint Hodge–Kodaira operator $\triangle$ or $\triangle^1$ by

$$\triangle^1 = \bar{d}^{1*}\bar{d}^1 + \bar{d}^0\bar{d}^{0*}$$

and to obtain the Hodge decomposition. For the details we refer to [32] or Part II.

Theorem 5.5. *(See [32].) There is the orthogonal decomposition*

$$L^2\Gamma\mathcal{H} = \mathrm{Image}\big(\bar{d}^0\big) + \overline{\mathrm{Image}}\big(\bar{d}^{1*}\big) + \ker\triangle^1$$

where $\overline{\mathrm{Image}(\bar{d}^{1*})}$ *denotes the closure of the image of the adjoint of* $\bar{d}^1$.

6. Tensor products as operators: Algebraic operations on *H*-one-forms

To show that the exterior product of H-one-forms can be defined as an H-two-form (by a pointwise construction) and to obtain a better understanding of the spaces $\mathcal{H}^2_\sigma$ we will give an interpretation of H-two-vectors in terms of linear maps from $\mathcal{H}^1_\sigma$ to itself. We will also give an example on flat linear Wiener space to show how a theory of tangent processes would lead to analogues of the elements in $\mathcal{H}^2_\sigma$.

A. First we establish our notation and review the well-known results identifying various completions of the algebraic tensor product $H \otimes_0 H$, with spaces of linear maps, and the dualities between the spaces. For example see Ruston [67], though our conventions are slightly different. Here H will be a separable real Hilbert space. Identify $H \otimes_0 H$ with finite rank operators on H by

$$H \otimes_0 H \to \mathcal{L}(H; H)$$

given by

$$(u \otimes v)(h) = \langle v, h\rangle u. \tag{6.1}$$

This extends to an identification of the projective tensor product (the "smallest") $H \otimes_\pi H$ with the space $\mathcal{L}_1(H; H)$ of trace class operators, of our usual $H \otimes H$ with the Hilbert–Schmidt operators $\mathcal{L}_2(H; H)$, and of the inductive, the 'largest reasonable,' completion $H \otimes_\varepsilon H$ with the space of compact operators $\mathcal{L}_c(H; H)$ in $\mathcal{L}(H; H)$:

$$\begin{tikzcd}
H \otimes_\pi H \arrow[r] \arrow[d, "\simeq"] & H \otimes H \arrow[r] \arrow[d, "\simeq"] & H \otimes_\varepsilon H \arrow[d, "\simeq"] & \\
\mathcal{L}_1(H; H) \arrow[r] & \mathcal{L}_2(H; H) \arrow[r] & \mathcal{L}_c(H; H) \arrow[r, hook] & \mathcal{L}(H; H).
\end{tikzcd}$$

The vertical arrows above are isometries, the inner product on $\mathcal{L}_2(H; H)$ being given by

$$\langle S, T\rangle_{\mathcal{L}_2} := \mathrm{trace}\, T^*S = \sum_{i=1}^{\infty} \langle Se_i, Te_i\rangle_H \tag{6.2}$$

for $\{e_i\}_{i=1}^{\infty}$ an orthonormal base of H. So $\operatorname{trace}(u \otimes v) = \langle u, v\rangle$ and

$$\|u \otimes v\|_{\mathcal{L}_2} = \|u\|\|v\| = \|u \otimes v\|_{H\otimes H}.$$

These conventions lead to the following isomorphism with the space of bounded bilinear maps

$$\mathcal{L}(H; H) \to \mathcal{L}(H, H; \mathbf{R}),$$
$$T \mapsto \tilde{T}$$

being given by

$$\tilde{T}(h_1, h_2) = \langle h_1, Th_2\rangle \tag{6.3}$$

with resulting isomorphism, as $\mathcal{L}(H, H; \mathbf{R}) \simeq (H \otimes_\pi H)^*$,

$$\mathcal{L}(H; H) \xrightarrow{D_1} \big(\mathcal{L}_1(H; H)\big)^*$$

expressed by

$$D_1(T)(S) = \operatorname{trace} T^* S. \tag{6.4}$$

This construction shows that D_1 restricts to an isomorphism

$$\mathcal{L}_{skew}(H; H) \xrightarrow{D_1} \big(\textstyle\bigwedge_\pi^2 H\big)^*$$

where $\mathcal{L}_{skew}(H; H)$ refers to the skew adjoint elements of $\mathcal{L}(H; H)$. We shall see later that our operator Q can be considered as a map from $\bigwedge^2 \mathcal{H}^1$ to $\mathcal{L}_{skew}(\mathcal{H}; \mathcal{H})$.

B. We will need the 'double duality' map $\breve{\theta} = D_1^* \circ i$ with i the canonical inclusion $\mathcal{L}_1(H; H) \to \mathcal{L}_1(H; H)^{**}$:

$$\mathcal{L}_1(H; H) \xrightarrow{\breve{\theta}} \mathcal{L}(H; H)^*,$$
$$\breve{\theta}(T)(S) := \operatorname{trace} S^* T,$$

$T \in \mathcal{L}_1(H; H)$, $S \in \mathcal{L}(H; H)$. Through the isomorphism $\mathcal{L}_1(H; H) \simeq H \otimes_\pi H$, it corresponds to the continuous bilinear map

$$\theta : H \times H \to \mathcal{L}(H; H)^*$$

given by

$$\theta\big(h^1, h^2\big) = \breve{\theta}\big(h^1 \otimes h^2\big)$$

so that

$$\theta\big(h^1, h^2\big)(S) = \big\langle h^1, Sh^2\big\rangle. \tag{6.5}$$

C. Let $H = L_0^{2,1}\mathbf{R}^m$. If V belongs to the inductive tensor product $H \otimes_\varepsilon H \hookrightarrow \bigotimes_\varepsilon^2 C_0\mathbf{R}^m$ we see, by taking V primitive, that the corresponding element S^V, say, in $\mathcal{L}(H; H)$ is given by

$$S^V(h)_s \equiv V(h)_s = \int_0^T \left(\frac{\partial}{\partial t} V_{s,t}\right)(\dot{h}_t)\, dt, \tag{6.6}$$

identifying $\frac{\partial}{\partial t} V_{s,t} \in \mathbf{R}^m \otimes \mathbf{R}^m$ with the corresponding element of $\mathcal{L}(\mathbf{R}^m; \mathbf{R}^m)$. For more general kernels $V \in \bigotimes_\varepsilon^2 C_0\mathbf{R}^m$ this can be used to define a linear operator S^V and we let $\mathcal{K}\mathbf{R}^m$ denote the set of such V for which $\frac{\partial}{\partial t} V_{s,t}$ exists for almost all t for each $s \in [0, T]$ and (6.6) determines an element S^V of $\mathcal{L}(H; H)$.

As our main example of an element of $\mathcal{K}\mathbf{R}^m$ let

$$j : [0, T] \to \mathbf{R}^m \otimes \mathbf{R}^m$$

be absolutely continuous with essentially bounded derivative and $j(0) = 0$. Set $V_{s,t} = j(s \wedge t)$. Then V belongs to $\mathcal{K}\mathbf{R}^m$

$$S^V(h)_s = \int_0^T \frac{\partial}{\partial t} j(s \wedge t)(\dot{h}_t)\, dt = \int_0^s j'(r)(\dot{h}_r)\, dr$$

and there is a conjugacy

$$\begin{array}{ccc} L^2([0,T];\mathbf{R}^m) & \xrightarrow{M^{j'}} & L^2([0,T];\mathbf{R}^m) \\ \uparrow \frac{d}{d\cdot} & & \uparrow \frac{d}{d\cdot} \\ L_0^{2,1}\mathbf{R}^m & \xrightarrow{S^V} & L_0^{2,1}\mathbf{R}^m \end{array}$$

to the multiplication (i.e. zero order) operator $M^{j'}$ given by

$$M^{j'}(f)(t) = j'(t) f(t)$$

for $j'(t)$ considered to be in $\mathcal{L}(\mathbf{R}^m; \mathbf{R}^m)$. In particular we see that in general such V do not correspond to compact operators, let alone to elements of $H \otimes H$. Also for $\theta : H \times H \to \mathcal{L}(H; H)^*$ defined in Section 6C we see from (6.5) that

$$\theta\big(h^1, h^2\big)\big(S^V\big) = \int_0^T \big\langle \dot{h}_s^1, j'(s)\big(\dot{h}_s^2\big)\big\rangle_{\mathbf{R}^m}\, ds. \tag{6.7}$$

Theorem 6.1. *For V in $\mathcal{H}_\sigma^1 \wedge \mathcal{H}_\sigma^1$ let $Q(V) \in \bigwedge_\epsilon^2 T_\sigma C_{x_0}$ be defined by* (4.13). *Then considered as a kernel it determines an element $S^{Q(V)}$ of $\mathcal{L}(\mathcal{H}_\sigma^1; \mathcal{H}_\sigma^1)$ which is conjugate to a multiplication operator M on $L^2 T_\sigma C_{x_0} M$:*

$$\begin{tikzcd} L^2T_\sigma C_{x_0}M \arrow[r, "M"] & L^2T_\sigma C_{x_0}M \\ \mathcal{H}^1_\sigma \arrow[u, "\frac{\mathbb{D}}{\partial t}"'] \arrow[r, "S^{Q(V)}"] & \mathcal{H}^1_\sigma . \arrow[u, "\frac{\mathbb{D}}{\partial t}"'] \end{tikzcd}$$

Here $M(u)_t = W_t j'_V(t)(W_t^{-1}u_t)$ *for* j_V *given by Eq.* (4.15) (*and so* j'_V *by* (6.9) *below*), $u \in L^2TC_{x_0}M$.

Proof. Set $\tilde{V}_{s,t} = (W_s^{-1} \otimes W_t^{-1})V_{s,t}$. Let $\tilde{Q} : \bigwedge^2 L_0^{2,1}T_{x_0}M \to \bigwedge^2 C_0T_{x_0}M$ be given by

$$\tilde{Q}(U)_{s,t} = \left(W_s^{-1} \otimes W_t^{-1}\right)Q\left(\bigwedge^2(U)\right)_{s,t}. \tag{6.8}$$

Then from Eq. (4.16)

$$\tilde{Q}(\tilde{V})_{s,t} = j_V(s \wedge t).$$

As earlier $S^{\tilde{Q}(\tilde{V})}$ is conjugate, by $\frac{d}{dt}$, to $M^{j'_V}$ acting on $L^2([0,T]; T_{x_0}M)$.

For $h \in \mathcal{H}^1_\sigma$ we have $S^{Q(V)}(h)_t = W_t(S^{\tilde{Q}(\tilde{V})}(W_\cdot^{-1}h_\cdot))_t$ so

$$\begin{aligned} \frac{\mathbb{D}}{dt}S^{Q(V)}(h)_t &= W_t\frac{d}{dt}\left(S^{\tilde{Q}(\tilde{V})}\left(W_\cdot^{-1}h_\cdot\right)\right) = W_t\left(M^{j'_V}\left(\frac{d}{dt}W_\cdot^{-1}h_\cdot\right)\right)_t \\ &= W_t\left(M^{j'_V}\left(W_\cdot^{-1}\frac{\mathbb{D}}{dt}h_\cdot\right)\right)_t = W_t j'_V(t)W_t^{-1}\frac{\mathbb{D}}{dt}h \end{aligned}$$

proving the conjugacy. □

Thus $Q(V)_\sigma$ corresponds to an element of $\mathcal{L}_{skew}(\mathcal{H}_\sigma; \mathcal{H}_\sigma)$, and so of $(\mathcal{H}_\sigma^* \otimes_\pi \mathcal{H}_\sigma^*)^*$, but is not compact and in particular does not belong to $\bigwedge^2 \mathcal{H}^1_\sigma$. This yields

Corollary 6.2. *There is a natural inclusion of* $\mathcal{H}^2_\sigma$ *in* $\mathcal{L}_{skew}(\mathcal{H}_\sigma; \mathcal{H}_\sigma)$ *given by* $V \mapsto S^V$.

Note that by the definition (4.15) and formula (4.12)

$$\begin{aligned} j'_V(t) &= \left(W_t^{-1} \otimes W_t^{-1}\right)\left(\frac{\mathbb{D}^{(2)}}{dt} + \mathcal{R}_{\sigma_t}\right)W_t^{(2)}\int_0^t \left(W_r^{(2)}\right)^{-1}\mathcal{R}_{\sigma_r}\left(\bigwedge^2(W_r)V_{r,r}\right)dr \\ &= \left(W_t^{-1} \otimes W_t^{-1}\right)\left(\mathcal{R}_{\sigma_t}\left(\bigwedge^2(W_t)V_{t,t}\right)\right) \\ &\quad + \left(W_t^{-1} \otimes W_t^{-1}\right)\left(\mathcal{R}_{\sigma_t}W_t^{(2)}\int_0^t \left(W_r^{(2)}\right)^{-1}\mathcal{R}_{\sigma_r}\left(\bigwedge^2(W_r)V_{r,r}\right)dr\right). \end{aligned} \tag{6.9}$$

Remark 6.3. The inclusion can also be seen geometrically from the fact that if $U \in \mathcal{H}^2_\sigma$ then $U - \mathbb{R}(U) \in \bigwedge^2 \mathcal{H}_\sigma \subset \mathcal{L}_{skew}(\mathcal{H}_\sigma; \mathcal{H}_\sigma)$ where $\mathbb{R}$ is the curvature operator of the damped Markovian connection which takes values in $\mathcal{L}_{skew}(\mathcal{H}_\sigma; \mathcal{H}_\sigma)$; see Section 9D below.

D. *Interior and exterior products.* For any separable Hilbert space H define the interior product by an element h of H by

$$\iota_h : H \otimes_0 H \to H, \quad h \in H,$$
$$\iota_h\big(h^1 \otimes h^2\big) := \big\langle h^1, h\big\rangle h^2 = S^*(h),$$

where $S \in \mathcal{L}(H; H)$ corresponds to $h^1 \otimes h^2$. Thus ι_h extends to a continuous linear map over all the completed tensor products we use and even can be defined consistently as

$$\iota_h : \mathcal{L}(H; H) \to H, \quad \text{by}$$
$$\iota_h(S) := S^*(h).$$

E. The first part of the following lemma is standard, but the conventions are important, see Appendix A.

Lemma 6.4. (i) *The maps $\iota_h : H \otimes H \to H$ and $h\otimes : H \to H \otimes H$ are mutually adjoint as are the maps $\iota_h : \bigwedge^2 H \to H$ and $h\wedge : H \to \bigwedge^2 H$.*

(ii) *The adjoint of $h\otimes : H \to H \otimes_\pi H$ is $\iota_h : \mathcal{L}(H; H) \to H$, identifying $(H \otimes_\pi H)^*$ with $\mathcal{L}(H; H)$ by D_1 as in* (6.3). *Similarly the adjoint of $h\wedge : H \to \bigwedge_\pi^2 H$ is the restriction of ι_h to the skew-symmetric elements $\mathcal{L}_{skew}(H; H)$ of $\mathcal{L}(H; H)$, using the restrictions of D_1* (*see Section* 6B *above*).

Proof of (ii). If $S \in \mathcal{L}(H; H)$ and $h_1 \in H$ then

$$\big\langle \iota_h(S), h_1\big\rangle = \big\langle S^*(h), h_1\big\rangle = \operatorname{trace}\big[S^* \circ (h \otimes h_1)\big]$$
$$= D_1(S)(h \otimes h_1) = D_1(S)(h \otimes \cdot)(h_1)$$

while if S is skew-symmetric

$$D_1(S)(h \otimes h_1) = \langle h, Sh_1\rangle = \frac{1}{2}\big\{\langle h, Sh_1\rangle - \langle Sh, h_1\rangle\big\} = D_1(S)(h \wedge h_1). \qquad \square$$

F. Now take $H = L_0^{2,1} T_{x_0} M$ and consider $\tilde{Q} : \bigwedge^2 H \to \bigwedge^2 C_0 T_{x_0} M$ given as in (6.8). The inclusion $H \hookrightarrow C_0 T_{x_0} M$ has an injective adjoint with dense range $(C_0 T_{x_0} M)^* \to H$. Let $\phi^\#$ denote the image of $\phi \in (C_0 T_{x_0} M)^*$ under this map. There is the interior product

$$\iota_\phi : \bigwedge^2 C_0 T_{x_0} M \to C_0 T_{x_0} M$$

given by

$$\iota_\phi\big(u^1 \wedge u^2\big) = \frac{1}{2}\big(\phi\big(u^1\big)u^2 - \phi\big(u^2\big)u^1\big).$$

Lemma 6.5. *For $\underline{h} \in \bigwedge^2 H$ consider $S^{\tilde{Q}_\sigma(\underline{h})} \in \mathcal{L}(H; H)$. Then for $\phi \in (C_0 T_{x_0} M)^*$ we have*

$$\iota_\phi\big(\tilde{Q}_\sigma(\underline{h})\big) = \iota_{\phi^\#} S^{\tilde{Q}_\sigma(\underline{h})} = -S^{\tilde{Q}_\sigma(\underline{h})}\big(\phi^\#\big).$$

Proof. Write ϕ in terms of a $T_{x_0}M$-valued countably additive measure, m^ϕ, of finite variation on $[0,T]$ so

$$\phi(w) = \int_0^T \langle w_s, dm^\phi(s)\rangle, \quad w \in C_0 T_{x_0} M.$$

Then, if $\underline{u} = u^1 \wedge u^2 \in \bigwedge^2 C_0 T_{x_0} M$ has $u_{s,t} = u_{t,s}$,

$$\iota_\phi(\underline{u})_t = \frac{1}{2}\int_0^T \langle u_s^1, dm^\phi(s)\rangle u_t^2 - \langle u_s^2, dm^\phi(s)\rangle u_t^1 = -\int_0^T \underline{u}_{t,s}\big(dm^\phi(s)\big)$$

treating $\underline{u}_{t,s} \in \bigwedge^2 T_{x_0}M$ as an element of $\mathcal{L}_{skew}(T_{x_0}M; T_{x_0}M)$. Thus

$$\big(\iota_\phi\big[\tilde{Q}_\sigma(\underline{h})\big]\big)_t = -\int_0^T j_{\underline{h}}(s\wedge t)\big(dm^\phi(s)\big) = -\int_0^t \left(\frac{d}{ds} j_{\underline{h}}(s)\right)\left(\int_s^T dm^\phi(r)\right) ds. \tag{6.10}$$

On the other hand, if $k \in H$,

$$\int_0^T \langle \dot{\phi}_s^\#, \dot{k}_s\rangle\, ds = \langle \phi^\#, k\rangle_H = \int_0^T \langle k_s, dm^\phi(s)\rangle\, ds = \int_0^T \left\langle \dot{k}_s, \int_s^T dm^\phi(r)\right\rangle ds.$$

Thus $\phi_t^\# = \int_0^t (\int_s^T dm^\phi(r))\, ds$ (a well-known result in Wiener space theory). This, using (6.10) and then Section 6C above, gives

$$\begin{aligned}\big(\iota_\phi\big[\tilde{Q}_\sigma(\underline{h})\big]\big)_t &= -\int_0^t \frac{d}{ds} j_{\underline{h}}(s)\,\big(\dot{\phi}_s^\#\big)\, ds = -S^{\tilde{Q}_\sigma(\underline{h})}\big(\phi^\#\big)\\ &= \iota_{\phi^\#} S^{\tilde{Q}_\sigma(\underline{h})}\end{aligned}$$

by definition (see Section 6E). □

Remark. The same calculation shows that the analogous result holds with general elements of $\mathcal{K}T_{x_0}M$, see Section 6C, replacing $\tilde{Q}_\sigma(h)$.

G. Set

$$\tilde{\mathcal{H}}_\sigma^2 = (1+\tilde{Q}_\sigma)\big[\textstyle\bigwedge^2 H\big] \subset \textstyle\bigwedge^2 C_0 T_{x_0} M.$$

From Section 6D above we can consider elements of $\tilde{\mathcal{H}}_\sigma^2$ as skew-symmetric bounded linear operators on H. This can be exploited to extend the definition of exterior products:

Lemma 6.6. *The mapping*

$$(C_0T_{x_0}M)^* \times (C_0T_{x_0}M)^* \to \big(\tilde{\mathcal{H}}^2_\sigma\big)^*$$

given by

$$\big(\phi^1, \phi^2\big) \to \phi^1 \wedge \phi^2\big|_{\tilde{\mathcal{H}}^2_\sigma}$$

extends to a continuous, antisymmetric, bilinear map

$$H \times H \xrightarrow{\triangle} \big(\tilde{\mathcal{H}}^2_\sigma\big)^*$$

inducing a bounded linear map $\tilde{\theta}_\sigma : \bigwedge^2_\pi H \to (\tilde{\mathcal{H}}^2_\sigma)^*$ *which agrees with the map* $\breve{\theta}$ *of* Section 6C:

$$\begin{tikzcd} & \mathcal{L}(H;H)^* \arrow[d] \\ \bigwedge^2_\pi H \arrow[ur, "\breve{\theta}"] \arrow[r, "\tilde{\theta}_\sigma"] & (\tilde{\mathcal{H}}^2_\sigma)^* \end{tikzcd}$$

using the inclusion of $\tilde{\mathcal{H}}^2_\sigma$ *into* $\mathcal{L}(H;H)$.

Proof. For $S \equiv S^{\tilde{Q}_\sigma(\underline{h})} \in \mathcal{L}_{skew}(H;H)$ corresponding to $\tilde{Q}_\sigma(\underline{h})$ as above, if $\phi^1, \phi^2 \in (C_0T_{x_0}M)^*$ then using Lemma 6.5,

$$\begin{aligned}\big(\phi^1 \wedge \phi^2\big)\big(\tilde{Q}_\sigma(h)\big) &= \phi^2\big(\iota_{\phi^1}\big(\tilde{Q}_\sigma(\underline{h})\big)\big) = -\phi^2\big(S\big({\phi^1}^\#\big)\big) \\ &= -\big\langle {\phi^2}^\#, S\big({\phi^1}^\#\big)\big\rangle_H. \end{aligned} \tag{6.11}$$

Also

$$\|S\|_{\mathcal{L}(H;H)} = \sup_{0\leqslant s\leqslant T} \big|\alpha_{\underline{h}}(s)\big| \leqslant \text{const} \cdot \sup_r |h_{rr}| \leqslant \text{const} \cdot \|\underline{h}\|_{\bigwedge^2 H} \tag{6.12}$$

for $\alpha_{\underline{h}}$ the multiplication operator corresponding to S as in Section 6C, i.e. $\alpha_{\underline{h}}(t) = \frac{d}{dt} j_{\underline{h}}(t)$ given by Eq. (6.9). Therefore

$$\big|\big\langle {\phi^2}^\#, S{\phi^1}^\#\big\rangle\big| \leqslant \text{const} \cdot \|\underline{h}\|_{\bigwedge^2 H} \cdot \big\|{\phi^2}^\#\big\|_H \cdot \big\|{\phi^1}^\#\big\|_H.$$

This shows we have $\tilde{\theta}_\sigma \in \mathcal{L}(\bigwedge^2_\pi H; (\tilde{\mathcal{H}}^2_\sigma)^*)$. This agrees with $\breve{\theta}$, as required, by equality (6.5). □

H. We now interpret these result in terms of $\mathcal{H}$-forms and $\mathcal{H}$ vectors on $C_{x_0}M$.

Theorem 6.7. (i) *For $v \in \mathcal{H}^1_\sigma$ there is an interior product* (*annihilation operator*)

$$\iota_v : \mathcal{H}^2_\sigma \to \mathcal{H}^1_\sigma$$

which is continuous linear, and agrees with the usual ι_ϕ for $\phi \in (T_\sigma C_{x_0} M)^*$ when $v = \phi^\#$. The map $(v, U) \mapsto \iota_v(U)$ is in $\mathcal{L}(\mathcal{H}^1_\sigma, \mathcal{H}^2_\sigma; \mathcal{H}^1_\sigma)$ and is bounded uniformly in σ.

(ii) *The map*

$$(T_\sigma C_{x_0} M)^* \times (T_\sigma C_{x_0} M)^* \to \big(\mathcal{H}^2_\sigma\big)^*,$$
$$\big(\phi^1, \phi^2\big) \mapsto \big(\phi^1 \wedge \phi^2\big)\big|_{\mathcal{H}^2_\sigma}$$

extends to give a continuous linear map

$$\lambda_\sigma : \big(\mathcal{H}^1_\sigma\big)^* \wedge_\pi \big(\mathcal{H}^1_\sigma\big)^* \to \big(\mathcal{H}^2_\sigma\big)^*$$

which is bounded uniformly in σ as an element of $\mathcal{L}((\mathcal{H}^1_\sigma)^* \wedge_\pi (\mathcal{H}^1_\sigma)^*; (\mathcal{H}^2_\sigma)^*)$.

(iii) *Moreover, if $v \in \mathcal{H}^1_\sigma$, $\ell \in (\mathcal{H}^1_\sigma)^*$ and $U \in \mathcal{H}^2_\sigma$,*

$$\lambda_\sigma\big(v^\# \wedge \ell\big)(U) = \ell(\iota_v U).$$

Proof. (i) The existence of ι_v and its properties come from Lemma 6.5 and the bounds on S noted in Eq. (6.12).

(ii) Lemma 6.6 provides the proof of (ii) with λ_σ being conjugate by $\bigwedge^2(W_\cdot)$ to the map $\tilde{\theta}_\sigma$ of Lemma 6.6. We see from there that $\tilde{\theta}_\sigma$ is bounded uniformly in σ if the inclusion $\mathcal{H}^2_\sigma \to \mathcal{L}(H; H)$ is. However this is essentially the map $\underline{h} \mapsto S^{\tilde{Q}_\sigma(\underline{h})}$ again.

For (iii) approximate $v^\#$ and ℓ by elements coming from $(T_\sigma C_{x_0})^*$. By Lemma 6.5, if $U = V + Q(V)$

$$\iota_v(U) = \iota_v(V) - S^{Q(V)}\big(v^\#\big)$$

so

$$\begin{aligned} \ell\big(\iota_v(U)\big) &= \ell\big(\iota_v(V)\big) - \big\langle \ell^\#, S^{Q(V)}\big(v^\#\big)\big\rangle_{\mathcal{H}^1_\sigma} \\ &= \big(v^\# \wedge \ell\big)(V) + \big(v^\# \wedge \ell\big)\big(Q_\sigma(V)\big), \quad \text{by (6.11).} \qquad \square \end{aligned}$$

We shall write $\lambda_\sigma(\phi \wedge \psi)$ as $\phi \wedge_\pi \psi$ when no confusion can arise.

Remark 6.8. The map λ_σ is independent of the choice of the Hilbert space inner product given to $\mathcal{H}^1_\sigma$, or $\mathcal{H}^2_\sigma$. Its adjoint gives a continuous map

$$\lambda^*_\sigma : \mathcal{H}^2_\sigma \to \big(\big(\mathcal{H}^1_\sigma\big)^* \wedge_\pi \big(\mathcal{H}^1_\sigma\big)^*\big)^*$$

of $\mathcal{H}^2_\sigma$ into the skew-symmetric bi-forms on $(\mathcal{H}^1_\sigma)^*$.

7. The derivation property for $\bar{d}^1$

A. We can now formulate and prove the derivation property of $\bar{d}^1$.

Theorem 7.1. *Suppose* $f : C_{x_0}M \to \mathbf{R}$ *is in* $\mathrm{Dom}(\bar{d}^0)$ *and* $\phi \in \mathrm{Dom}(\bar{d}^1) \cap L^\infty\Gamma(\mathcal{H}^1)^*$ *with* $\bar{d}^1\phi \in L^\infty\Gamma(\mathcal{H}^2)^*$. *Then* $f\phi \in \mathrm{Dom}(\bar{d}^1)$ *and*

$$\bar{d}^1(f\phi) = \bar{d}^0 f \wedge_\pi \phi + f\big(\bar{d}^1\phi\big)$$

where $\wedge_\pi$ *is defined above by Theorem* 6.7.

Proof. Let $\{\phi_j\}_{j=1}^\infty$ be a sequence in $\mathrm{Dom}(d^1_{\mathcal{H}})$ with $\phi_j \to \phi$ in $L^2\Gamma(\mathcal{H}^1)^*$ and $d^1\phi_j \to \bar{d}^1\phi$ in $L^2\Gamma(\mathcal{H}^2)^*$. Assume first that $f \in \mathrm{Dom}(d_{\mathcal{H}})$. Then $f\phi_j \to f\phi$ in $L^2\Gamma(\mathcal{H}^1)^*$ by the module structure of $\mathrm{Dom}(d^1_{\mathcal{H}})$, and by standard calculus

$$d(f\phi_j) = df \wedge \phi_j + f\, d\phi_j,$$

therefore

$$d(f\phi_j)|_{\mathcal{H}^2_\sigma} = \lambda_\sigma(df|_{\mathcal{H}^1_\sigma} \wedge \phi_j|_{\mathcal{H}^1_\sigma}) + f(d\phi_j)|_{\mathcal{H}^2_\sigma}$$

in the notation of Theorem 6.7. By the uniform bound on λ_σ from that theorem, and taking a subsequence if necessary to assume $\phi_j|_{\mathcal{H}^1_\sigma} \to \phi|_{\mathcal{H}^1_\sigma}$ for almost all σ, we see

$$\lambda_\sigma(df|_{\mathcal{H}^1_\sigma} \wedge \phi_j|_{\mathcal{H}^1_\sigma}) \to \lambda_\sigma\big(\bar{d}^0 f_\sigma \wedge \phi_\sigma\big)$$

almost surely and so in L^2 by the dominated convergence theorem. Since $f(d\phi_j) \to f\bar{d}^1\phi$ and $f\phi_j \in \mathrm{Dom}(d^1_{\mathcal{H}})$ the result follows for $f \in \mathrm{Dom}(d_{\mathcal{H}})$.

For general $f \in \mathrm{Dom}(\bar{d}^0)$ take $\{f_j\}_{j=1}^\infty$ in $\mathrm{Dom}(d_{\mathcal{H}})$ with $f_j \to f$ in L^2 and $\bar{d}f_j \to \bar{d}f$ in $L^2\Gamma(\mathcal{H}^1)^*$. From above we know that $f_j\phi \in \mathrm{Dom}(\bar{d}^1)$ with

$$\bar{d}^1(f_j\phi) = \bar{d}f_j \wedge_\pi \phi - f_j\big(\bar{d}^1\phi\big), \quad j = 1 \text{ to } \infty.$$

Now ϕ and $\bar{d}^1\phi$ are bounded so as before we see $\bar{d}f_j \wedge_\pi \phi \to df \wedge_\pi \phi$ and $f_j\bar{d}^1\phi \to f\bar{d}^1\phi$, both in $L^1\Gamma(\mathcal{H}^2)^*$, completing the proof. □

8. Infinitesimal rotations as divergences

We will say that a p-vector field V on $C_{x_0}M$ (or similarly on $C_0(\mathbf{R}^m)$), *has a divergence* if there exists $\mathrm{div}\, V \in L^1\Gamma\bigwedge^{p-1} TC_{x_0}M$ such that for all smooth, bounded, cylindrical $(p-1)$-forms ϕ we have

$$\int_{C_{x_0}M} d\phi(V)\, d\mu_{x_0} = -\int_{C_{x_0}M} \phi(\mathrm{div}\, V)\, d\mu_{x_0}. \tag{8.1}$$

For $p=1$ from Driver [20] we know that not only do sufficiently regular elements of $L^2\Gamma\mathcal{H}^1$ have divergences but so do the *infinitesimal rotations* $R^\alpha \in L^2\Gamma\bigwedge^2 TC_{x_0}M$ given by

$$R^\alpha_t = /\!/_t \int_0^t /\!/_s^{-1}\alpha_s\,dx_s \tag{8.2}$$

where $\alpha_s : C_{x_0}M \to \mathcal{L}_{skew}(T_{x_s}M; T_{x_s}M)$, $0 \leqslant s \leqslant T$, is in L^2 and progressively measurable. Indeed

$$\operatorname{div} R^\alpha_\cdot = 0.$$

For more examples of one-vector fields with divergences see Bell [9], Cruzeiro and Malliavin [19], Fang [40], and Hu, Üstünel and Zakai [45] and for p-vector fields see [33]. As in finite dimensions if a p-vector field V has a divergence $\operatorname{div} V$, when $p>1$, then $\operatorname{div} V$ has a vanishing divergence. In view of the looseness of the definition and the homotopical triviality of $C_{x_0}M$ we would expect that a field with a divergence which is zero would necessarily be a divergence, and we will give some evidence for this which also sheds light on the structure of our modified de Rham complex.

First we observe that the exterior product of suitably regular H-vector fields in Dom(div) has a divergence. For this let $V^1, V^2 \in L^2\Gamma\mathcal{H}^1$. Then we have an L^2 section $V^1\wedge V^2$ of $\mathcal{H}^1\wedge\mathcal{H}^1$. If ϕ is a smooth (bounded) cylindrical 1-form, then as discussed in Appendix B,

$$2\,d\phi\big(V^1\wedge V^2\big) = \iota_{V^1}d\iota_{V^2}(\phi) - \iota_{V^2}d\iota_{V^1}(\phi) - 2\phi\big(\big[V^1,V^2\big]\big)$$

provided V^1, V^2 are sufficiently regular. Give such a regularity

$$\begin{aligned}
&2\int_{C_{x_0}M} d\phi\big(V^1\wedge V^2\big)\,d\mu_{x_0}\\
&\quad= \int_{C_{x_0}M} \iota_{V^1}(\phi)\operatorname{div} V^2\,d\mu_{x_0} - \int_{C_{x_0}M} \iota_{V^2}(\phi)\operatorname{div} V^1\,d\mu_{x_0} - \int_{C_{x_0}M} \phi\big(\big[V^1,V^2\big]\big)\,d\mu_{x_0}.
\end{aligned}$$

Thus $V^1\wedge V^2$ has a divergence with

$$2\operatorname{div}\big(V^1\wedge V^2\big) = -\big(\operatorname{div} V^2\big)V^1 + \big(\operatorname{div} V^1\big)V^2 + \big[V^1,V^2\big]. \tag{8.3}$$

The first two terms are sections of $\mathcal{H}^1$ but as is well known, Cruzeiro and Malliavin [18], Driver [21], the bracket involves a stochastic integral of the form I for

$$I_t = /\!/_t \int_0^t /\!/_s^{-1}\mathcal{R}\big(V^1_s\wedge V^2_s\big)\,dx_s, \tag{8.4}$$

i.e. an infinitesimal rotation. The above applies in particular to $V^i = \overline{T\mathcal{I}}(h^i)$ for $h^i \in W^{2,1}(C_{x_0}M; H)$, $i=1,2$.

Also if $\underline{h}: C_{x_0}M \to \bigwedge^2 H$ is in $W^{2,1}$, the 2-vector field $\overline{\bigwedge^2 T\mathcal{I}(\underline{h})}$ has a divergence with $\operatorname{div}\overline{\bigwedge^2 T\mathcal{I}(\underline{h})} = \overline{T\mathcal{I}(\operatorname{div}(\underline{h}\circ\mathcal{I}))}$. Indeed for ϕ a smooth cylindrical one-form

$$\int_{C_{x_0}M} d\phi\big(\overline{\textstyle\bigwedge^2 T\mathcal{I}(\underline{h})}\big)\,d\mu_{x_0} = \int_{C_0\mathbf{R}^m} \mathcal{I}^*(d\phi)(\underline{h}\circ\mathcal{I})\,dP$$

$$= \int_{C_0\mathbf{R}^m} d\big(\mathcal{I}^*\phi\big)(\underline{h}\circ\mathcal{I})\,dP = -\int_{C_0\mathbf{R}^m} \mathcal{I}^*\phi(\operatorname{div}\underline{h}\circ\mathcal{I})\,dP$$

$$= -\int_{C_0\mathbf{R}^m} \phi\big(\overline{T\mathcal{I}\big(\operatorname{div}(\underline{h}\circ\mathcal{I})\big)}\big)\,dP.$$

Here we use the fact that since $\underline{h} \in W^{2,1}$, we have $\underline{h}\circ\mathcal{I} \in \mathbb{D}^{2,1} \subset \operatorname{Dom}(\operatorname{div})$. Consequently,

$$\operatorname{div}\big(\overline{\textstyle\bigwedge^2 T\mathcal{I}(\underline{h})}\big) = \overline{T\mathcal{I}(\operatorname{div}\underline{h})}. \tag{8.5}$$

(For another version of this result see Section 8E.) On the other hand,

$$\overline{\textstyle\bigwedge^2 T\mathcal{I}(\underline{h})} = \textstyle\bigwedge^2 \overline{T\mathcal{I}}(\underline{h}) + Q\big(\bigwedge^2 \overline{T\mathcal{I}}(\underline{h})\big).$$

Thus:

Proposition 8.1. *For $\underline{h} = h^1 \wedge h^2$ with $h^i \in W^{2,1}(C_{x_0}M; H)$, $i = 1, 2$, the two-vector field $Q(\bigwedge^2 \overline{T\mathcal{I}}(\underline{h}))$ has a divergence with*

$$\operatorname{div} Q\big(\textstyle\bigwedge^2 \overline{T\mathcal{I}}(\underline{h})\big) = \overline{T\mathcal{I}(\operatorname{div}\underline{h})} - \operatorname{div}\big(\textstyle\bigwedge^2 \overline{T\mathcal{I}}(\underline{h})\big).$$

Since $\overline{T\mathcal{I}}(\operatorname{div}\underline{h}) \in \Gamma\mathcal{H}^1$ we see that $\operatorname{div} Q(\bigwedge^2 \overline{T\mathcal{I}}(\underline{h}))$ must cancel out the infinitesimal rotation term I in $\operatorname{div}(\bigwedge^2 \overline{T\mathcal{I}}(\underline{h}))$. A geometrical interpretation of this is given below, see Section 9. The following result concerning the flat Wiener space case shows how this can happen. It should be considered together with formula (4.16) for Q and the discussion in Section 6C.

Proposition 8.2. *Every two-vector field $V : C_0(\mathbf{R}^m) \to \bigwedge^2 C_0(\mathbf{R}^m)$ given by $V_{s,t} = j(s\wedge t)$ for $j(t) = \int_0^t \alpha_r\,dr$, where $\alpha_\cdot : [0,T]\times C_0(\mathbf{R}^m) \to \mathcal{L}_{skew}(\mathbf{R}^m;\mathbf{R}^m)$ is progressively measurable with $\int_{C_0(\mathbf{R}^m)}\int_0^T |\alpha_s|^2\,ds\,d\mu_{x_0} < \infty$, has a divergence. It is given by*

$$\operatorname{div} V = \int_0^{\cdot} \alpha_s\,dB_s,$$

i.e. $\operatorname{div} V = R^\alpha$.

Proof. Let $f : C_0(\mathbf{R}^m) \to \mathbf{R}$ be bounded and C^∞ and let $\ell \in C_0(\mathbf{R}^m)^*$. Define the 1-form ϕ on $C_0(\mathbf{R}^m)$ by

$$\phi_\omega(v) = f(\omega)\ell(v).$$

Bounded cylindrical 1-forms can be written as sums of such forms. Then $d\phi = df \wedge \ell$.

Let k be the image of ℓ under the inclusions $C_0(\mathbf{R}^m)^* \to L_0^{2,1}(\mathbf{R}^m)$ adjoint to the inclusion of $L_0^{2,1}$ in C_0.

From (6.7) above we see

$$d\phi(V) = \int_0^T \big\langle \overbrace{(\nabla_H f)}^{\cdot}{}_s, \alpha_s \dot{k}_s \big\rangle_{\mathbf{R}^m} \, ds = df\left(\int_0^{\cdot} \alpha_s \dot{k}_s \, ds \right).$$

Thus

$$\begin{aligned}
\int_{C_0\mathbf{R}^m} d\phi(V)\, dP &= \int_{C_0\mathbf{R}^m} f(\omega) \int_0^T \langle \alpha_s \dot{k}_s, dB_s \rangle_{\mathbf{R}^m} \, dP(\omega) \\
&= -\int_{C_0\mathbf{R}^m} f(\omega) \int_0^T \langle \dot{k}_s, \alpha_s \, dB_s \rangle_{\mathbf{R}^m} \, dP(\omega) \\
&= -\int_{C_0\mathbf{R}^m} f(\omega) \ell\left(\int_0^{\cdot} \alpha_s \, dB_s \right) dP(\omega)
\end{aligned}$$

as required. (The last equality being obvious in the (most relevant) case when $\ell(v) = \lambda(v_{t_0})$ some $\lambda \in (\mathbf{R}^m)^*$, some $0 \leqslant t_0 \leqslant T$, in which case $\dot{k}_s = \chi_{[0,t_0]}(s)\lambda$.) $\square$

9. Differential geometry of the space $\mathcal{H}^2$ of two-vectors

In this section we will give a bundle structure to the Bismut tangent bundle $\mathcal{H}$ and interpret the quantities Q and $\mathbb{R}$ which define $\mathcal{H}^2$ in terms of a natural connection on $\mathcal{H}$.

A. *The L^2 tangent bundle and its frame bundle.* Let $\pi : OM \to M$ be the orthonormal frame bundle of M. Our Banach manifold $C_{x_0}M$ has natural structural group $C_{id}([0,T]; O(n))$ with frame bundle identified with the space of paths $C_{\pi^{-1}(x_0)}([0,T]; OM)$ in the frame bundle OM of M, starting at any frame over x_0. Let

$$\tilde{\pi} : C_{\pi^{-1}(x_0)}(OM) \to C_{x_0}M$$

be the projection. Note that $C_{id}(O(n))$ has an orthogonal representation on $L^2([0,T]; \mathbf{R}^n)$, acting pointwise

$$C_{id}\big(O(n)\big) \xrightarrow{\rho} O\big(L^2\big([0,T];\mathbf{R}^n\big)\big),$$
$$\rho(\alpha)(f)(t) = \alpha(t)\big(f(t)\big).$$

For α, β in $C_{id}(O(n))$,

$$\begin{aligned}
\big\|\rho(\alpha)-\rho(\beta)\big\|_{\mathbb{L}(L^2([0,T];\mathbf{R}^n);L^2([0,T];\mathbf{R}^n))} &= \sup_{\|f\|_{L_2}\leqslant 1}\sqrt{\int_0^T \big|\alpha(s)f(s)-\beta(s)f(s)\big|^2\,ds} \\
&\leqslant \sup_{\|f\|_{L_2}\leqslant 1}\sqrt{\int_0^T \big|f(s)\big|^2 \sup_{0\leqslant s\leqslant T}\big|\alpha(s)-\beta(s)\big|^2\,ds} \\
&\leqslant \sup_{0\leqslant s\leqslant T}\big|\alpha(s)-\beta(s)\big| = d(\alpha,\beta).
\end{aligned}$$

Thus ρ is continuous into the uniform topology and we see it is even C^∞ with derivative map $T_\alpha\rho$ at α:

$$T_\alpha\rho : T_\alpha C_{id}O(n) \to TO\big(L^2\big([0,T];\mathbf{R}^n\big)\big) \subset \mathcal{L}\big(L^2\big([0,T];\mathbf{R}^n\big);L^2\big([0,T];\mathbf{R}^n\big)\big)$$

given by $T_\alpha\rho(V)(f)(t) = V(t)f(t)$.

From this we see that the L^2 tangent bundle $L^2TC_{x_0}M$ has the structure of a C^∞ bundle associated to $C_{\pi^{-1}(x_0)}(OM)$, whose elements u act as frames on it by

$$u : L^2\big([0,T];\mathbf{R}^n\big) \to L^2T_\sigma C_{x_0}M, \quad \sigma = \tilde{\pi}u,$$
$$u(f)_t = u_t\big(f(t)\big).$$

This construction determines $L^2TC_{x_0}M$ as a C^∞ bundle over $C_{x_0}M$. It tells us what its smooth sections (in the Fréchet sense) are. (For example see Remark 9.1 below.)

B. *The pointwise connection.* Let $\tilde{\nabla}$ denote the *pointwise connection* on $C_{x_0}M$, as described in greater generality by Eliasson [25]. It is defined on the bundle $L^2TC_{x_0}M \to C_{x_0}M$ by

$$(\tilde{\nabla}_V U)_t = \frac{D}{ds}U\big(\exp_{\sigma_\cdot}(sV_\cdot)\big)_t\Big|_{s=0} \tag{9.1}$$

where $\frac{D}{ds}$ and exp come from the Levi-Civita connection on TM. Thus

$$\begin{aligned}
(\tilde{\nabla}_V U)_t &= X(\sigma_t)\frac{d}{ds}\Big(Y\big(\exp_{\sigma_t}(sV_t)\big)U\big(\exp_{\sigma_\cdot}(sV_\cdot)\big)_t\Big)\Big|_{s=0} \\
&= X(\sigma_t)d\big[\tilde{Y}(\cdot)U(\cdot)\big](V)_t,
\end{aligned}$$

where the L^2-valued one-form $\tilde{Y} : L^2TC_{x_0}M \to L^2([0,T];\mathbf{R}^m)$ is the lift of Y, i.e.

$$\tilde{Y}_\sigma(V)(t) = Y_{\sigma(t)}\big(V(t)\big).$$

This says that the pointwise connection is the LW connection in the sense of [30], for the lift $\tilde{X}$ of X to $C_{x_0}M$.

This connection is torsion free and is metric for the L^2 metric.

Remark 9.1. The pointwise derivative $\tilde{\nabla}Y : TC_{x_0}M \times L^2TC_{x_0}M \to L^2([0,T];\mathbf{R}^m)$ is C^∞.

To see this let Υ be a locally defined C^∞ frame field for $L^2TC_{x_0}M$ giving a local trivialisation over an open subset U of $C_{x_0}M$

$$\Upsilon : U \times L^2\big([0,T];\mathbf{R}^m\big) \to L^2TC_{x_0}M.$$

Then

$$\big[\tilde{Y}_\sigma \Upsilon(\sigma)(f)\big]_t = Y_{\sigma_t}\big(\Upsilon(\sigma)_t f(t)\big).$$

Its derivative is

$$(\nabla_{v_t}\tilde{Y})\Upsilon(\sigma)_t f(t) + Y_{\sigma_t}\big(\tilde{\nabla}_v \Upsilon\big(f(t)\big)\big).$$

C. *The bundle structure of $\mathcal{H}$ and its damped Markovian connection.* Let $C^0_{x_0}M$ be a set of paths of full measure along each element of which the Levi-Civita parallel translation, $/\!\!/$, is defined and satisfies its basic composition properties. Then $\mathcal{H}_\sigma$ is defined for each $\sigma \in C^0_{x_0}M$ by formula (4.5) with an isometry $\mathcal{W}_\sigma : L^2T_\sigma C_{x_0}M \to \mathcal{H}_\sigma$, with inverse $\frac{\mathbb{D}}{d\cdot}$. Thus we get an induced smooth vector bundle structure on $\mathcal{H}^1$, over $C^0_{x_0}M$ by

$$\frac{\mathbb{D}}{ds} : \mathcal{H}^1 \leftrightarrows L^2TC_{x_0}M.$$

We can use this isomorphism to pull back the pointwise connection to get a metric connection $\mathbb{V}$ on $\mathcal{H}^1$. This is the damped Markovian connection defined in a different way by Cruzeiro and Fang in [15,16], Cruzeiro, Fang and Malliavin [17]. The basis for a covariant Sobolev calculus using it is given in [35]. In particular we have a closed covariant derivative operator $\mathbb{V}$ with domain, denoted by $\mathbb{D}^{2,1}\mathcal{H}^1$, in the space of L^2 sections of $\mathcal{H}^1$ mapping to the L^2 sections of $\mathcal{L}_2(\mathcal{H}^1;\mathcal{H}^1)$. In general we shall not distinguish between $C^0_{x_0}M$ and $C_{x_0}M$.

Since the inverse map to $\frac{\mathbb{D}}{d\cdot}$ is $\mathcal{W}$ it follows from Eq. (4.2) that this connection is the LW connection associated to $\overline{T\mathcal{I}}$ in the sense of [30]. With this in mind define

$$\begin{aligned} \mathbb{X} &: C_{x_0}M \times H \to \mathcal{H}^1, \\ \mathbb{X}(\sigma)(h) &= \overline{T\mathcal{I}}(h). \end{aligned} \tag{9.2}$$

As noted in [35] the adjoint of $\mathbb{X}$ is the H-valued H-one-form $\mathbb{Y}$ given by

$$\mathbb{Y}_\sigma(V) = \int_0^\cdot Y^*_{\sigma(r)} \frac{\mathbb{D}}{dr} V_r \, dr.$$

This is also a right inverse to $\mathbb{X}$. Suppose that u^1 and u^2 are in $\mathbb{D}^{2,1}\mathcal{H}$. For $j=1,2$, set $h^j(\sigma) = \mathbb{Y}_\sigma(u^j(\sigma))$. Then, by [35], $h^j \in \mathbb{D}^{2,1}(C_{x_0}M; H)$ and

$$\begin{aligned}\boldsymbol{\nabla}_{u^1(\sigma)}u^2 &= \mathbb{X}(\sigma)\,\bar{d}\big[\mathbb{Y}(u^2)\big]\big(u^1(\sigma)\big)\\ &= \mathbb{X}(\sigma)\,\bar{d}h^2\big(\overline{T\mathcal{I}}\big(h^1(\sigma)\big)\big) = \mathbb{X}(\sigma)\big(\overline{\bar{d}\big(h^2\circ\mathcal{I}\big)}_\sigma\big(h^1(\sigma)\big)\big).\end{aligned} \tag{9.3}$$

We saw in Proposition 8.1 that for certain v^1 and v^2 the two-vector field $Q(v^1\wedge v^2)$ has a divergence. After the following lemma we can identify that divergence:

Lemma 9.2. *Suppose $h: C_0\mathbf{R}^m \to H$ is adapted. Then*

$$\overline{T\mathcal{I}(h)} = \overline{T\mathcal{I}}(\bar{h}).$$

Proof. Set $v_t = T\mathcal{I}_t(h)$. Then, since h is adapted we have as for Eq. (3.6)

$$Dv_t = \nabla_{v_t}X(\tilde{/\!/}_t\,d\beta_t) - \frac{1}{2}\operatorname{Ric}^{\#}(v_t)\,dt + X(x_t)\dot{h}_t\,dt.$$

Now take conditional expectations as usual to get the result. $\square$

Theorem 9.3. *For any $\mathcal{F}^{x_0}_\star$ adapted vector fields $u^i \in L^p\Gamma\mathcal{H}^1$, $i=1,2$, some $p>2$,*

$$\operatorname{div} Q\big(u^1\wedge u^2\big) = \frac{1}{2}\mathbb{T}\big(u^1,u^2\big), \tag{9.4}$$

where $\mathbb{T}$ is the torsion of the damped Markovian connection $\boldsymbol{\nabla}$.

Proof. As above set $h^j = \mathbb{Y}(u^j)$, $j=1,2$. Define the adapted H-vector fields $\tilde{h}^j$, $j=1,2$, on $C_0\mathbf{R}^m$ by $\tilde{h}^j = h^j\circ\mathcal{I}$. First assume that each u^j, and so h^j and $\tilde{h}^j$, belong to $\mathbb{D}^{p,1}$.

By the integration by parts formulae, as for the proof of (8.5) for two-vector fields in Section 8, and using the fact that $\dot{\tilde{h}}^j(\omega)_s \perp \ker X(x_s(\omega))$ a.s.,

$$\begin{aligned}\operatorname{div}\big(u^j\big)\circ\mathcal{I} &= \mathbf{E}\big\{\operatorname{div}(\tilde{h}^j)\big|\mathcal{F}^{x_0}\big\} = -\mathbf{E}\left\{\int_0^T\langle\dot{h}_s, dB_s\rangle\Big|\mathcal{F}_{x_0}\right\}\\ &= -\int_0^T\big\langle\dot{\tilde{h}}^j_s, X(x_s)\,dB_s\big\rangle = \operatorname{div}\big(\tilde{h}^j\big).\end{aligned}$$

In particular $\operatorname{div}(\tilde{h}^j)$ is $\mathcal{F}^{x_0}$-measurable. Consequently, from Proposition 8.1 and formula (8.3),

$$\begin{aligned}2\operatorname{div} Q\big(u^1\wedge u^2\big) &= 2\overline{T\mathcal{I}\big(\operatorname{div}\big(\tilde{h}^1\wedge\tilde{h}^2\big)\big)} - 2\operatorname{div}\big(u^1\wedge u^2\big)\\ &= \overline{T\mathcal{I}\big(-\tilde{h}^1\operatorname{div}\big(\tilde{h}^2\big) + \tilde{h}^2\operatorname{div}\big(\tilde{h}^1\big) + \big[\tilde{h}^1,\tilde{h}^2\big]\big)}\\ &\quad - \big(\operatorname{div} u^1\big)u^2 + u^1\operatorname{div}\big(u^2\big) - \big[u^1,u^2\big]\\ &= \overline{T\mathcal{I}\big(\big[\tilde{h}^1,\tilde{h}^2\big]\big)} - \big[u^1,u^2\big].\end{aligned}$$

Also from (9.3)

$$\begin{aligned}
\left[u^1, u^2\right](\sigma) &= \mathbb{X}(\sigma)\left(\overline{(\bar{d}\tilde{h}^2)}_\sigma\left(h^1(\sigma)\right) - \overline{(\bar{d}\tilde{h}^1)}_\sigma\left(h^2(\sigma)\right)\right) - \mathbb{T}\left(u^1, u^2\right)(\sigma) \\
&= \mathbb{X}(\sigma)\left(\overline{\left[\tilde{h}^1, \tilde{h}^2\right]}\right)_\sigma - \mathbb{T}\left(u^1, u^2\right)(\sigma) \\
&= \overline{T\mathcal{I}}_\sigma\left(\overline{\left[\tilde{h}^1, \tilde{h}^2\right]}_\sigma\right) - \mathbb{T}\left(u^1, u^2\right)(\sigma)
\end{aligned}$$

giving

$$2\operatorname{div} Q\left(u^1 \wedge u^2\right)(\sigma) = \overline{T\mathcal{I}\left(\left[\tilde{h}^1, \tilde{h}^2\right]\right)}_\sigma - \overline{T\mathcal{I}}_\sigma\left(\overline{\left[\tilde{h}^1, \tilde{h}^2\right]}_\sigma\right) + \mathbb{T}\left(u^1, u^2\right)(\sigma).$$

For adapted vector fields the first two terms cancel by the previous lemma, so we have (9.4) for adapted $\mathbb{D}^{p,1}$ vector fields.

If u^1, u^2 are adapted but not in $\mathbb{D}^{p,1}$ we can choose, cf. Lemma 9.4, sequences of adapted processes $\{u_n^j\}_{n=1}^\infty$, $j = 1, 2$, in $\mathbb{D}^{p,1}\mathcal{H}$, converging to u^1, u^2 in L^p. Then as $n \to \infty$,

$$\mathbb{T}\left(u_n^1, u_n^2\right) \to \mathbb{T}\left(u^1, u^2\right)$$

in $L^1 T C_{x_0} M$, by the formula

$$\mathbb{T}\left(V^1, V^2\right) = \tilde{X}\left((\nabla_{V^2}\tilde{Y})V^1 - (\nabla_{V^1}\tilde{Y})V^2\right)$$

given in Appendix B. On the other hand, for any C^∞ cylindrical 1-form ϕ,

$$\int \phi\left(\mathbb{T}\left(u_n^1, u_n^2\right)\right) = -2\int d\phi\left(Q\left(u_n^1 \wedge u_n^2\right)\right) \to -2\int d\phi\left(Q\left(u^1 \wedge u^2\right)\right).$$

Thus for all adapted L^p vector fields u^i, we have

$$\operatorname{div} Q\left(u^1 \wedge u^2\right) = \frac{1}{2}\mathbb{T}\left(u^1, u^2\right). \qquad \square$$

Lemma 9.4. *If u is an $\mathcal{F}_\star^{x_0}$-adapted H-vector field in $L^p \Gamma \mathcal{H}^1$ for some $p > 1$, there is a sequence $u_n \in \mathbb{D}^{p,1}\mathcal{H}^1$ of $\mathcal{F}_\star^{x_0}$ adapted H-vector fields such that u_n converges to u in L^p.*

Proof. Set $\tilde{h} = \mathbb{Y}(\frac{d}{d\cdot}u) \circ \mathcal{I} \in L^p(C_0\mathbf{R}^m; L^2([0,T]; \mathbf{R}^m))$. As finite chaos expansions are dense in L^p, let $\{\tilde{h}_n\}$ be a sequence of functions with finite chaos expansion converging to $\tilde{h}$ in $L^p(C_0\mathbf{R}^m; L^2([0,T]; \mathbf{R}^m))$. Define $v_n : C_{x_0} \to L^2([0,T]; \mathbf{R}^m)$ by

$$(v_n \circ \mathcal{I})_t = \mathbf{E}\left\{\tilde{h}_n \middle| \mathcal{F}_t^{x_0}\right\}.$$

Then v_n belongs to $\mathbb{D}^{p,1}$, see [35]. Set $u_n = \mathbb{X}(\int_0^\cdot (v_n)_s\, ds)$ then u_n converges in L^p to u. $\square$

Remark 9.5.

(1) It is noted in Cruzeiro and Fang [16] that the divergence of $\mathbb{T}(v^1, v^2)$ vanishes for a class of adapted H-vector fields v^1 and v^2.

(2) The conclusion of the theorem does not hold for general smooth nonadapted vector fields. In fact for a smooth, cylindrical $f : C_{x_0}M \to \mathbf{R}$ we have $\mathbb{T}(f\bar{v}^1, \bar{v}^2) = f\mathbb{T}(\bar{v}^1, \bar{v}^2)$. But

$$\operatorname{div} Q\big(\big(f\bar{v}^1\big) \wedge \bar{v}^2\big) = \operatorname{div}\big(f\,Q\big(\bar{v}^1 \wedge \bar{v}^2\big)\big) = f\ \operatorname{div}\big(Q\big(\bar{v}^1 \wedge \bar{v}^2\big)\big) + \iota_{\nabla f} Q\big(\bar{v}^1 \wedge \bar{v}^2\big).$$

Though we state the following for Brownian motion measures and the damped Markovian connections note that it applies in considerable generality, for example for any metric connection on a finite dimensional Riemannian manifold with smooth measure. In it we consider the closed covariant derivative operator

$$\mathbb{V} : \mathbb{D}^{2,1} \subset L^2\Gamma\mathcal{H}^1 \to L^2\Gamma\mathcal{L}_2\big(\mathcal{H}^1; \mathcal{H}^1\big)$$

with L^2-adjoint $\mathbb{V}^* : L^2\Gamma\mathcal{L}(\mathcal{H}^1; \mathcal{H}^1) \to L^2\Gamma\mathcal{H}^1$.

Proposition 9.6. *Let $U, V \in L^\infty\Gamma\mathcal{H}^1$. Suppose $U \in \mathbb{D}^{2,1}$ and $V \in \operatorname{Dom}(\operatorname{div})$. Then $\sigma \mapsto U(\sigma) \otimes V(\sigma)$ as an element, $U \otimes V$, of $L^2\Gamma(\mathcal{H}^1 \otimes \mathcal{H}^1)$ is in $\operatorname{Dom}(\mathbb{V}^*)$ and*

$$\mathbb{V}^*(U \otimes V)(\sigma) = -(\operatorname{div} V)(\sigma)U(\sigma) - \mathbb{V}_{V(\sigma)}U. \tag{9.5}$$

In particular this holds if U and V are both essentially bounded and in $\mathbb{D}^{2,1}$ in which case:

$$\mathbb{V}^*(U \wedge V) = \operatorname{div}(U \wedge V) + \frac{1}{2}\mathbb{T}(U, V). \tag{9.6}$$

Proof. Let $Z \in \mathbb{D}^{2,1}\mathcal{H}^1$. By (6.1) and (6.2),

$$\begin{aligned}
&\int_{C_{x_0}M} \big\langle (\mathbb{V}Z)_\sigma, U \otimes V(\sigma)\big\rangle_{\mathcal{H}^1_\sigma \otimes \mathcal{H}^1_\sigma}\, d\mu_{x_0}(\sigma)\\
&\quad= \int_{C_{x_0}M} \big\langle (\mathbb{V}Z)_\sigma, U \otimes V(\sigma)\big\rangle_{\mathcal{L}_2(\mathcal{H}^1;\mathcal{H}^1)}\, d\mu_{x_0}(\sigma)\\
&\quad= \int_{C_{x_0}M} \sum_{i=1}^\infty \big\langle (\mathbb{V}_{e_i}Z)_\sigma, U(\sigma)\big\langle V(\sigma), e_i\big\rangle\big\rangle_{\mathcal{H}^1_\sigma}\, d\mu_{x_0}(\sigma)\\
&\quad= \int_{C_{x_0}M} \big\langle \mathbb{V}_{V(\sigma)}Z, U(\sigma)\big\rangle_{\mathcal{H}^1_\sigma}\, d\mu_{x_0}(\sigma)\\
&\quad= \int_{C_{x_0}M} d\langle Z, U\rangle_{\mathcal{H}^1}\big(V(\sigma)\big)\, d\mu_{x_0}(\sigma) - \int_{C_{x_0}M} \langle Z, \mathbb{V}_{V(\sigma)}U\rangle_{\mathcal{H}^1_\sigma}\, d\mu_{x_0}(\sigma),
\end{aligned}$$

since $\mathbb{V}$ is a metric connection. This proves the first part.

For the second part first note from [35] that H-vector fields which are in $\mathbb{D}^{2,1}$ are in Dom(div). Then plug $U \wedge V = \frac{1}{2}\{U \otimes V - V \otimes U\}$ into Eq. (9.5) and use formula (8.3) to see

$$\mathbb{V}^*(U \wedge V) = \frac{1}{2}\{-(\operatorname{div} V)U + (\operatorname{div} U)V - \mathbb{V}_V U + \mathbb{V}_U V\}$$
$$= \operatorname{div}(U \wedge V) + \frac{1}{2}\mathbb{T}(U, V). \qquad \square$$

By formula (9.4) this immediately gives

Corollary 9.7. *For U, V as in Proposition* 9.6

$$\mathbb{V}^*(U \wedge V) = \operatorname{div}(I + Q)(U \wedge V) \tag{9.7}$$

provided U, V are non-anticipating. In particular for h^1, h^2 in $L_0^{2,1}(\mathbf{R}^m)$ non-random

$$\operatorname{div}\big(\overline{\textstyle\bigwedge^2 T\mathcal{I}}\big(h^1 \wedge h^2\big)\big) = \nabla^*\big(\overline{T\mathcal{I}}\big(h^1\big) \wedge \overline{T\mathcal{I}}\big(h^2\big)\big). \tag{9.8}$$

Note that for Z as above, if $f : C_{x_0}M \to \mathbf{R}$ is smooth and cylindrical then

$$\int_{C_{x_0}M} \langle \mathbb{V}Z, fU \wedge V\rangle_{\mathcal{H}^1 \otimes \mathcal{H}^1}\, d\mu_{x_0}$$
$$= \int_{C_{x_0}M} \big\langle \mathbb{V}(fZ) - Z \otimes \nabla f, U \wedge V\big\rangle_{\mathcal{H}^1 \otimes \mathcal{H}^1}\, d\mu_{x_0}$$
$$= \int_{C_{x_0}M} \left\{\big\langle Z, f\,\mathbb{V}^*(U \wedge V)\big\rangle - \frac{1}{2}\langle Z, U\rangle\, df(V) + \frac{1}{2}\langle Z, V\rangle\, df(U)\right\} d\mu_{x_0}.$$

So

$$\mathbb{V}^*[fU \wedge V] = f\,\mathbb{V}^*(U \wedge V) - \frac{1}{2}\{U\, df(V) - V\, df(U)\}$$
$$= f\,\mathbb{V}^*(U \wedge V) + \iota_{\nabla f}(U \wedge V)$$

whereas

$$\operatorname{div}(I + Q)(fU \wedge V) = f \operatorname{div}(I + Q)(U \wedge V) + \iota_{\nabla f}(U \wedge V) + \iota_{df}\, Q(U \wedge V).$$

Thus the formula is not true, if ‘non-anticipating’ is dropped.

D. *The curvature operator.* The curvature operator $\mathbb{R}$ of the damped Markovian connection $\mathbb{V}$ on $\Gamma\mathcal{H}^1$ is conjugate to the curvature operator

$$\tilde{\mathcal{R}} : \textstyle\bigwedge^2 TC_{x_0}M \to \mathcal{L}_{skew}\big(L^2TC_{x_0}M; L^2TC_{x_0}M\big)$$

of the pointwise connection on the L^2 tangent bundle via the map $\frac{\mathbb{D}}{dt}$. In fact

$$\mathbb{R} : \textstyle\bigwedge^2 T_\sigma C_{x_0}M \to \mathcal{L}_{skew}\big(\mathcal{H}^1_\sigma; \mathcal{H}^1_\sigma\big)$$

is given by

$$\big(\mathbb{R}(U)h\big)_t = \mathcal{W}_t\left(\tilde{\mathcal{R}}_\sigma\big(U(\sigma)\big)\left(\frac{\mathbb{D}}{d\cdot}h.\right)\right),$$

that is

$$\big(\mathbb{R}(U)h\big)_t = W_t \int_0^t W_s^{-1}\mathcal{R}_{\sigma_s}(U_{s,s})\left(\frac{\mathbb{D}}{d\cdot}h_s\right) ds. \tag{9.9}$$

We shall show that this agrees with the definition given in Eq. (4.17).

Our convention that $(a \otimes b)(u) = \langle b, u\rangle a$ makes clear the correspondence between the curvature operator $\mathcal{R}$ of M considered as a map $\mathcal{R}: \bigwedge^2 TM \to \mathcal{L}(TM; TM)$ and it considered as a map $\mathcal{R}: \bigwedge^2 TM \to \bigwedge^2 TM$. Note also that for a linear map T

$$\big[(T \otimes \mathbf{1})(a \otimes b)\big](u) = T\big((a \otimes b)(u)\big).$$

Then

$$\begin{aligned}
\mathbb{R}(U)(h)_t &= W_t \int_0^t W_r^{-1}\mathcal{R}(U_{rr})\left(\frac{\mathbb{D}}{dr}h_r\right) dr = W_t \int_0^t \big[\big(W_r^{-1} \otimes \mathbf{1}\big)\mathcal{R}(U_{rr})\big]\left(\frac{\mathbb{D}}{dr}h_r\right) dr \\
&= \int_0^t \big[\big(W_t(W_r)^{-1} \otimes \mathbf{1}\big)\mathcal{R}(U_{rr})\big]\left(\frac{\mathbb{D}}{dr}h_r\right) dr \\
&= \int_0^T \chi_{[0,t)}(r)(W_t \otimes W_r)\textstyle\bigwedge^2\big(W_r^{-1}\big)\mathcal{R}(U_{rr})\left(\frac{\mathbb{D}}{dr}h_r\right) dr.
\end{aligned}$$

Proposition 9.8. *As a linear map from $\bigwedge^2 T_\sigma C_{x_0}M$ to $\bigwedge^2 TC_{x_0}M$, the curvature operator of the damped Markovian connection on $\mathcal{H}^1$ is given by*

$$\mathbb{R}(U)_{s,t} = (W_s \otimes W_t) \int_0^t \textstyle\bigwedge^2(W_r)^{-1}\mathcal{R}(U_{rr})\, dr, \quad t < s. \tag{9.10}$$

Proof. Since $\mathbb{R}(U)$ is regular, its integral representation is

$$\mathbb{R}(U)(h.)_t = \int_0^T \left(\mathbf{1} \otimes \frac{\mathbb{D}}{dr}\right)\mathbb{R}(U)_{t,r}\left(\frac{\mathbb{D}h_r}{dr}\right) dr.$$

Compare this with the integral representation above the proposition to see the result. □

E. *The domain of* $\bar{d}^{1*}$. An important result for functions on $C_0\mathbf{R}^m$ was that the domain of the divergence acting on H-vector fields contains $\mathbb{D}^{2,1}(C_0\mathbf{R}^m; H)$, in particular H-vector fields which are in $\mathbb{D}^{2,1}$ are Skorohod integrable [51]. For $C_{x_0}M$ the analogous result was proved in [35] using the damped Markovian connection. We have not yet given a "bundle structure" or connection to $\mathcal{H}^2$ or its dual, but $\bigwedge^2 L^2TC_{x_0}M$ is a smooth bundle and inherits a connection from the pointwise connection. This will be the LW connection for $\bigwedge^2 \tilde{X}$. As discussed, in general, in [35] a section Z of $\bigwedge^2 L^2TC_{x_0}M$ is in $\mathbb{D}^{2,1}\bigwedge^2 L^2TC_{x_0}M$ if $\bigwedge^2 \tilde{Y}(Z)$ is in $\mathbb{D}^{2,1}(C_{x_0}M; \bigwedge^2 L^2([0,T]; \mathbf{R}^m))$. Where defined, the map

$$(\mathbf{1}+Q)\bigwedge^2 \mathcal{W}: \bigwedge^2 L^2TC_{x_0}M \to \mathcal{H}^2$$

is an isometry and it would be natural to use this to give a connection on $\mathcal{H}^2$. In this sense the following shows that the results mentioned above extend to our H-two-forms (or equivalently for the divergence operator on H-two-vectors). It is stated in terms of the weak Sobolev class $W^{2,1}$ for, possibly, greater generality.

Theorem 9.9. 1. *Let* $\phi \in L^2\Gamma\mathcal{H}^2$. *If*

$$\phi \circ (\mathbf{1}+Q) \circ \bigwedge^2 \mathcal{W} \in W^{2,1}\Gamma \bigwedge^2 \big(L^2TC_{x_0}M\big)^*$$

then $\phi \in \mathrm{Dom}(\bar{d}^{1*})$.

2. *More generally* $\phi \in \mathrm{Dom}(\bar{d}^{1*})$ *if the conditional expectation of its pull back by the Itô map*

$$\mathbf{E}\big\{\mathcal{I}^*(\phi)\big|\mathcal{F}^{x_0}\big\}: C_0\mathbf{R}^m \to \big(\bigwedge^2 H\big)^*$$

is in the domain of $\bar{d}^{1*}$ *on* $C_0\mathbf{R}^m$. *If so, for almost all* $\sigma \in C_{x_0}M$ *the* H*-vector field* $\mathrm{div}\,\phi^{\#}$ *is given by*

$$\mathrm{div}\big(\phi^{\#}\big) = \overline{T\mathcal{I}\big(\mathrm{div}\big(\mathbf{E}\big\{\mathcal{I}^*(\phi)\big|\mathcal{F}^{x_0}\big\}\big)^{\#}\big)}.$$

Proof. Set

$$g(\sigma) = \phi \circ (\mathbf{1}+Q) \circ \bigwedge^2 \mathcal{W} \circ \bigwedge^2 \tilde{X}(\sigma) \circ \bigwedge^2 \left(\frac{d}{d\cdot}\right)$$

for $\sigma \in C_{x_0}M$. Then our first condition implies that $g \in W^{2,1}(C_{x_0}M; \bigwedge^2 H)$. Note that $g = \phi \circ (\mathbf{1}+Q) \circ \bigwedge^2 \mathcal{W} \circ \bigwedge^2 \mathbb{X}$ and so

$$g \circ \mathcal{I} = \mathbf{E}\big\{\mathcal{I}^*(\phi)\big|\mathcal{F}^{x_0}\big\}.$$

By [35] $g \circ \mathcal{I} \in \mathbb{D}^{2,1}$ on $C_0\mathbf{R}^m$. By [69] this implies that as an H-two-form $g \circ \mathcal{I}$ is in the domain of d^{1*}. Now let $\psi \in \mathrm{Dom}(d^1_{\mathcal{H}})$, cylindrical one-form on $C_{x_0}M$. Then we have

$$\int_{C_{x_0}M} \langle d^1_{\mathcal{H}}\psi, \phi\rangle_{\mathcal{H}^{2*}} = \int_{C_{x_0}} \big\langle d^1_{\mathcal{H}}\psi\big(\overline{\bigwedge^2 T\mathcal{I}}(-)\big), \phi\big(\overline{\bigwedge^2 T\mathcal{I}}(-)\big)\big\rangle_{(\bigwedge^2 H)^*}$$

$$
\begin{aligned}
&= \int\limits_{C_0\mathbf{R}^m} \bigl\langle d^1_{\mathcal{H}}\psi \textstyle\bigwedge^2 T\mathcal{I}(-), \mathbf{E}\bigl\{\mathcal{I}^*(\phi)\big|\mathcal{F}^{x_0}\bigr\}\bigr\rangle_{\bigwedge^2 H^*} \\
&= \int\limits_{C_0\mathbf{R}^m} \bigl\langle \mathcal{I}^*\bigl(d^1_{\mathcal{H}}\psi\bigr), \mathbf{E}\bigl\{\mathcal{I}^*(\phi)\big|\mathcal{F}^{x_0}\bigr\}\bigr\rangle_{\bigwedge^2 H^*} \\
&= \int\limits_{C_0\mathbf{R}^m} \bigl\langle \bar{d}^1\mathcal{I}^*(\psi), \mathbf{E}\bigl\{\mathcal{I}^*(\phi)\big|\mathcal{F}^{x_0}\bigr\}\bigr\rangle_{\bigwedge^2 H^*} \\
&= \int\limits_{C_0\mathbf{R}^m} \bigl\langle I^*(\psi), \bigl(\bar{d}^1\bigr)^*\mathbf{E}\bigl\{\mathcal{I}^*(\phi)\big|\mathcal{F}^{x_0}\bigr\}\bigr\rangle_{\bigwedge^2 H^*}.
\end{aligned}
$$

From this the results follow. □

Corollary 9.10. *Every C^1 cylindrical 2-form on $C_{x_0}M$ is in the domain of $\bar{d}^{1*}$.*

Proof. Let $M^{(k)} = M \times M \times \cdots \times M$ be the Cartesian product of k copies of M and for $0 \leqslant t_1 \leqslant \cdots \leqslant t_k \leqslant T$ define $\rho_{\underline{t}}: C_{x_0}M \to M^{(k)}$ by $\rho_{\underline{t}}(\sigma) = (\sigma(t_1), \ldots, \sigma(t_k))$. Suppose $\phi = \rho_{\underline{t}}^*(\theta)$ for θ a C^1 two-form on $M^{(k)}$. Then

$$
\begin{aligned}
\mathbf{E}\bigl\{\mathcal{I}^*(\phi)\big|\mathcal{F}^{x_0}\bigr\} &= \phi_{\mathcal{I}(\cdot)} \circ (\mathbf{1} + \mathcal{Q}_{\mathcal{I}(\cdot)}) \circ \textstyle\bigwedge^2 \mathbb{X}\bigl(\mathcal{I}(\cdot)\bigr) \\
&= \theta \circ \textstyle\bigwedge^2 X^{(k)}\bigl(\mathcal{I}(\rho_{\underline{t}(\cdot)})\bigr) \circ \textstyle\bigwedge^2 Y^{(k)} \textstyle\bigwedge^2 T\rho_{\underline{t}} \circ (\mathbf{1} + \mathcal{Q}_{\mathcal{I}(\cdot)}) \circ \textstyle\bigwedge^2 \mathbb{X}\bigl(\mathcal{I}(\cdot)\bigr)
\end{aligned}
$$

where $X^{(k)}(z_1, \ldots, z_k) = \bigoplus_{j=1}^k X(z_j) : \bigoplus^k \mathbf{R}^m \to T_z M^{(k)}$ for $z = (z_1, \ldots, z_k) \in M^{(k)}$. Now, from the differentiability of θ and X it is clear that $\theta \circ \bigwedge^2 X^{(k)}(\mathcal{I}(\rho_{\underline{t}(\cdot)}))$ is in $\mathbb{D}^{p,1}$ for all $1 \leqslant p < \infty$, while it follows from standard approximation arguments that so is $\bigwedge^2 Y^{(k)} \bigwedge^2 T\rho_{\underline{t}} \circ (\mathbf{1} + \mathcal{Q}_{\mathcal{I}(\cdot)}) \circ \bigwedge^2 \mathbb{X}(\mathcal{I}(\cdot))$, for example as in [2]. Thus we can apply the theorem as required. □

Appendix A. Conventions

In the past we have used different conventions on the exterior product of a differential form, inner product of two antisymmetric tensor vectors, and the interior product of a vector with another. Here we were driven by the insistence that exterior product spaces are subspaces of the corresponding tensor products. To make these differences more transparent and easier for the reader to compare to their own conventions, we list in this section the conventions we use. It is only necessary to state them for uncompleted tensor products.

A. Let E, F be a real linear spaces. Any multilinear $\psi : E \times E \times \cdots \times E \to F$ determines a linear map $\tilde{\psi} : E \otimes_0 E \otimes_0 \cdots \otimes_0 E \to F$ with

$$\tilde{\psi}(u_1 \otimes \cdots \otimes u_q) = \psi(u_1, \ldots, u_q).$$

Denote by $\bigwedge_0^q E$ the subspace of anti-symmetric tensors and use the convention

$$u_1 \wedge \cdots \wedge u_q = \frac{1}{q!} \sum_\pi (-1)^\pi u_{\pi(1)} \otimes \cdots \otimes u_{\pi(q)} \tag{A.1}$$

where the summation is over all permutations π of $\{1,2,\dots,q\}$ and $(-1)^\pi$ is the sign of the permutation. This convention ensures that if ψ is anti-symmetric then

$$\tilde{\psi}(u_1 \wedge \cdots \wedge u_q) = \psi(u_1, \dots, u_q).$$

An inner product $\langle -,- \rangle$ on E determines one on the tensor products

$$\langle u_1 \otimes \cdots \otimes u_q, v_1 \otimes \cdots \otimes v_q \rangle = \prod_{i=1}^{q} \langle u_i, v_i \rangle, \tag{A.2}$$

any $u_i, v_i \in E$. In turn this determines one on the exterior powers by restriction, giving

$$\langle u_1 \wedge \cdots \wedge u_q, v_1 \wedge \cdots \wedge v_q \rangle = \frac{1}{q!} \det \begin{pmatrix} \langle u_1, v_1 \rangle & \langle u_1, v_2 \rangle & \dots & \langle u_1, v_q \rangle \\ \dots & \dots & \dots & \dots \\ \langle u_q, v_1 \rangle & \langle u_q, v_2 \rangle & \dots & \langle u_q, v_q \rangle \end{pmatrix}. \tag{A.3}$$

Now suppose there is a pairing $\langle\langle -,- \rangle\rangle : E' \times E \to \mathbf{R}$ between E and a linear space E'. We are thinking of the cases $E = E'$ with inner product pairing or E' being the dual space of E with respect to some topology, with $\langle\langle l, e \rangle\rangle = l(e)$. Then if $l \in E'$, there is the standard interior product, or annihilation operator ι_l,

$$\iota_l(u_1 \otimes \cdots \otimes u_q) = \langle\langle l, u_1 \rangle\rangle (u_2 \otimes \cdots \otimes u_q). \tag{A.4}$$

This gives

$$\iota_l(u_1 \wedge \cdots \wedge u_q) = \frac{1}{q} \sum_{j=1}^{q} (-1)^{j+1} \langle\langle l, u_j \rangle\rangle u_1 \wedge \cdots \wedge \hat{u}_j \wedge \cdots \wedge u_q \tag{A.5}$$

where $\hat{u}$ means the omission of the vector u. Note that:

(i) If $E = E'$ with inner product pairing then for each $v \in E$ the operator $\iota_v : \bigwedge_0^q E \to \bigwedge_0^{q-1} E$ is adjoint to the map determined by $u_1 \wedge \cdots \wedge u_{q-1} \to v \wedge u_1 \wedge \cdots \wedge u_{q-1}$.

(ii) The interior product is now not a skew-derivation, cf. [50, p. 65]. For example if $q = 2$ we have

$$\iota_l(u_1 \wedge u_2) = \frac{1}{2} \big\{ \langle\langle l, u_1 \rangle\rangle u_2 - \langle\langle l, u_2 \rangle\rangle u_1 \big\}.$$

Keeping the duality between the interior product and the "creation operator" $v \wedge -$, for ψ as above and $v \in E$ define:

$$\iota_v \psi : \mathsf{X}^{(q-1)} E \to \mathbf{R}$$

by

$$\iota_v(\psi)(u_1, \dots, u_{q-1}) = \psi(v, u_1, \dots, u_{q-1}),$$

so that if ψ is skew-symmetric we have

$$\iota_v(\psi)(u_1 \wedge \cdots \wedge u_{q-1}) = \psi(v \wedge u_1 \wedge \cdots \wedge u_{q-1}).$$

If ϕ_1 and ϕ_2 are in a dual space to E then $\phi_1 \wedge \phi_2$ is defined on $\bigwedge_0^2 E$ by

$$\phi_1 \wedge \phi_2(u_1 \wedge u_2) = \frac{1}{2}\big[\phi_1(u_1)\phi_2(u_2) - \phi_2(u_1)\phi_1(u_2)\big].$$

This is in agreement with $\iota_v(\phi_1 \wedge \phi_2) := \frac{1}{2}(\phi_1(v)\phi_2 - \phi_2(v)\phi_1)$.

B. More generally if $S : E_1 \to E_2$ and $T : F_1 \to F_2$ are two linear maps of Banach spaces, there is the induced linear map

$$S \otimes T : E_1 \otimes_0 F_1 \to E_2 \otimes_0 F_2.$$

If $E_1 = F_1$ and $E_2 = F_2$ set $S \wedge T = \frac{1}{2}(S \otimes T + T \otimes S)$ so $S \otimes S$ agrees with $S \wedge S$ as a linear operator on $\bigwedge^2 E_1$. This reduces to the previous definitions when $E_2 = F_2 = \mathbf{R}$ after identifying $\mathbf{R} \otimes \mathbf{R}$ with $\mathbf{R}$.

C. Consider now the tangent bundle TM of a smooth manifold M. The exterior differentiation $d : \bigwedge^q TM \to \bigwedge^{q+1} TM$ is defined by

$$\begin{aligned} &d\phi\big(V^1 \wedge \cdots \wedge V^{q+1}\big) \\ &\quad = \frac{1}{(q+1)} \sum_{i=1}^{q+1} (-1)^{i+1} L_{V^i}\big[\phi\big(V^1 \wedge \cdots \wedge \widehat{V^i} \wedge \cdots \wedge V^{q+1}\big)\big] \\ &\qquad + \frac{1}{(q+1)} \sum_{1 \leqslant i < j \leqslant q+1} (-1)^{i+j} \phi\big(\big[V^i, V^j\big] \wedge V^1 \wedge \cdots \wedge \widehat{V^i} \wedge \cdots \wedge \widehat{V^j} \wedge \cdots \wedge V^{q+1}\big) \end{aligned} \tag{A.6}$$

where L_{V^i} denotes Lie differentiation in the direction of v^i. This differs from the convention used in our previous research paper, e.g. [30,31,33] where we did not add any constants before d and d^*. This lead to a change in the divergence of q-vector fields by a factor of q

$$\mathrm{div}_{old}(V) = q\, \mathrm{div}_{new}(V). \tag{A.7}$$

By our conventions if f is a function on M,

$$\langle df \wedge \phi, \psi\rangle = \langle \phi, \iota_{df}\psi\rangle, \tag{A.8}$$

$$d(f\phi) = df \wedge \phi + f\, d\phi, \tag{A.9}$$

$$\mathrm{div}(fV) = f(\mathrm{div}\, V) + \iota_V(df). \tag{A.10}$$

Appendix B. Brackets of vector fields, torsion, and $d\phi(v^1 \wedge v^2)$

Lie brackets of H-vector fields have been discussed in many places, e.g. [19,21,55], for completeness, and definitiveness, we give a definition and some properties here. The torsion of the damped Markovian connection is also described, for explicit formulae see [16]. We refer to [35] for the Sobolev calculus of sections of $\mathcal{H}$, related bundles, and smooth bundles such as $L^2TC_{x_0}M$. The latter will always be taken here with its pointwise connection.

Proposition B.1. *The inclusion map of $\mathcal{H}$ into $L^2TC_{x_0}M$ is in $\mathbb{D}^{p,1}$ for $1 \leqslant p < \infty$ as a section of $\mathcal{L}_2(\mathcal{H}; L^2TC_{x_0}M)$ and any H-vector field V in $D^{p,1}\mathcal{H}$, or $\mathbb{W}^{p,1}\mathcal{H}$, is a $\mathbb{D}^{p,1}$, or $\mathbb{W}^{p,1}$, section of $L^2TC_{x_0}M$. Moreover for such V the pointwise (weak) covariant derivative $\tilde{\nabla}_- V$ is an L^p section of $\mathcal{L}(\mathcal{H}; TC_{x_0}M)$.*

Proof. For the first assertion it suffices to show that the map

$$\Theta : C_{x_0}M \to \mathcal{L}_2\big(H; L^2\big([0,T]; \mathbf{R}^m\big)\big)$$

given by

$$\Theta(\sigma)(h) = \tilde{Y}_\sigma \overline{T\mathcal{I}}_\sigma(h)$$

is in $\mathbb{D}^{p,1}$. However $\Theta(\sigma)(h)_t = Y_{\sigma(t)} W_t \int_0^t W_s^{-1} X(\sigma(s))(\dot{h}_s)\,ds$ and so the result holds from standard arguments, as in [2]. For the claim about sections we can apply the corresponding arguments to $\sigma \mapsto \Theta(\sigma)(U(\sigma))$ for $U \in \mathbb{D}^{p,1}(C_{x_0}M; H)$, or in $\mathbb{W}^{p,1}(C_{x_0}M; H)$; in the latter case it is only necessary to consider the composition with $\mathcal{I}$, see Theorem 5.1. In particular the final assertion comes from standard results giving the continuity in t of the derivative of $(\Theta \circ \mathcal{I})(U \circ \mathcal{I})_t : C_0\mathbf{R}^m \to \mathbf{R}^m$, e.g. as [64, p. 106]. Alternatively the derivative can be calculated explicitly as in [2]. $\square$

Definition B.2. If V^1 and V^2 are in $\mathbb{W}^{p,1}\mathcal{H}$ define their Lie bracket by

$$\big[V^1, V^2\big] = \tilde{\nabla}_{V^1} V^2 - \tilde{\nabla}_{V^2} V^1,$$

where $\tilde{\nabla}$ is the pointwise connection defined by formula (9.1).

By Proposition B.1, $[V^1, V^2]$ is then a measurable vector field, i.e. section of $TC_{x_0}M$. Since the pointwise connection restricts to a torsion free connection on $TC_{x_0}M$ this definition agrees with the usual one. Moreover if $f : C_{x_0}M \to \mathbf{R}$ is smooth and cylindrical we have

$$\bar{d}\big(\bar{d}f\big(V^2\big)\big)V^1 = \tilde{\nabla}_{V^1}(\bar{d}f)V^2 + \bar{d}f\big(\tilde{\nabla}_{V^1} V^2\big)$$

so that

$$\bar{d}\big(\bar{d}f\big(V^2\big)\big)V^1 - \bar{d}\big(\bar{d}f\big(V^1\big)\big)V^2 = \bar{d}f\big(\big[V^1, V^2\big]\big)$$

as usual. The torsion $\mathbb{T}(V^1, V^2)$ is defined as a measurable vector field by

$$\mathbb{T}\big(V^1, V^2\big) = \boldsymbol{\nabla}_{V^1} V^2 - \boldsymbol{\nabla}_{V^2} V^1 - \big[V^1, V^2\big].$$

To see the torsion as an "H-tensor field" use the LW characterisation of the pointwise connection to observe first that

$$\mathbb{T}(V^1, V^2) = \mathbb{V}_{V^1} V^2 - \mathbb{V}_{V^2} V^1 - \tilde{\nabla}_{V^1} V^2 + \tilde{\nabla}_{V^2} V^1$$
$$= \mathbb{V}_{V^1} V^2 - \tilde{X}\bar{d}(\tilde{Y} V^2) V^1 - \mathbb{V}_{V^2} V^1 + \tilde{X}\bar{d}(\tilde{Y} V^1) V^2.$$

Now consider the restriction of $\tilde{Y}$ to $\mathcal{H}$ as a section of $\mathcal{L}_2(\mathcal{H}; L^2([0,T]; \mathbf{R}^m))$. As above it lies in $\mathbb{D}^{p,1}$, $1 \leqslant p < \infty$, with $\mathbb{V}\tilde{Y}$ a section of $\mathcal{L}_2(\mathcal{H}; \mathcal{L}_2(\mathcal{H}; L^2([0,T]; \mathbf{R}^m)))$. Then

$$\mathbb{V}_{V^1} V^2 - \tilde{X}\bar{d}(\tilde{Y} V^2) V^1 = -\tilde{X}(\mathbb{V}_{V^1}\tilde{Y}) V^2$$

and so

$$\mathbb{T}(V^1, V^2) = \tilde{X}\big((\mathbb{V}_{V^2}\tilde{Y}) V^1 - (\mathbb{V}_{V^1}\tilde{Y}) V^2\big).$$

From this we see we can consider the torsion as a section of $\mathcal{L}_2(\bigwedge^2 \mathcal{H}; TC_{x_0}M)$. Alternatively noting that $\tilde{Y}$ maps $\mathcal{H}$ into $C_0([0,T]; \mathbf{R}^m)$ and arguing as before we see that it gives a section of $\mathcal{L}_{skew}(\mathcal{H}, \mathcal{H}; TC_{x_0}M)$. In both cases the sections are in L^p for all $1 \leqslant p < \infty$.

Finally we give the result used in Section 8.

Proposition B.3. *If ϕ is a smooth cylindrical 1-form and V^1, V^2 are in $\mathbb{W}^{p,1}\mathcal{H}$ then, almost surely,*

$$2\, d\phi(V^1 \wedge V^2) = \iota_{V^1}\bar{d}\iota_{V^2}\phi - \iota_{V^2}\bar{d}\iota_{V^1}\phi - \phi([V^1, V^2]).$$

Proof. Using the pointwise connection on the sections of $T^*C_{x_0}M$:

$$\iota_{V^1}\bar{d}\iota_{V^2}\phi - \iota_{V^2}\bar{d}\iota_{V^1}\phi - \phi([V^1, V^2])$$
$$= (\tilde{\nabla}_{V^1}\phi)(V^2) + \phi(\tilde{\nabla}_{V^1})V^2 - (\tilde{\nabla}_{V^2}\phi)(V^1) - \phi(\tilde{\nabla}_{V^2} V^1) - \phi([V^1, V^2])$$
$$= \tilde{\nabla}_{V^1}\phi(V^2) - \tilde{\nabla}_{V^2}\phi(V^1)$$
$$= \frac{1}{2} d\phi(V^1, V^2)$$

by the standard formula, as the pointwise connection $\tilde{\nabla}$ has no torsion. □

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A Kadison transitivity theorem for C^*-semigroups

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Abstract

We prove a semigroup analogue of the Kadison Transitivity Theorem for C^*-algebras. Specifically, we show that a closed, homogeneous, self-adjoint, topologically transitive, semigroup of operators acting on a separable Hilbert space is (strictly) transitive if the semigroup contains a non-zero compact operator. Additional structural information about such semigroups is obtained, and examples are provided to demonstrate that the theorem is the best possible in the semigroup case.

Keywords: Kadison Transitivity Theorem; Self-adjoint semigroups of operators; Transitive semigroups; Compact operators

In 1957, Kadison [2] showed that a closed, self-adjoint algebra of operators $\mathcal{S}$ which acts topologically transitively on a Hilbert space is actually transitive. We consider the analogous problem with most of the linear structure of $\mathcal{S}$ removed: *if a multiplicative semigroup $\mathcal{S}$ of bounded operators acting on a Hilbert space is closed, homogeneous, self-adjoint and topologically transitive, is $\mathcal{S}$ transitive*?

We shall adopt standard notation and let H denote a complex, separable, infinite-dimensional Hilbert space. Let $B(H)$ denote the C^*-algebra of all bounded operators on H, and let $K(H)$ denote the ideal of all compact operators in $B(H)$.

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As mentioned above, the objects we are considering are like C^*-algebras, but no closure property for addition is assumed, and we shall assume only that $\mathcal{S}$ is closed under multiplication by non-negative real scalars. We shall call such objects C^*-semigroups, and the precise definition follows.

Definition 1. A C^*-semigroup is a subset $\mathcal{S}$ of $B(H)$ which has the following properties:

(1) $\mathcal{S}$ is closed under multiplication: that is, if S and T are in $\mathcal{S}$, then ST is in $\mathcal{S}$;
(2) $\mathcal{S}$ is closed in the norm topology of $B(H)$;
(3) $\mathcal{S}$ is homogeneous: that is, if S is in $\mathcal{S}$ and $r \in \mathbb{R}^+ = [0, \infty)$ is a non-negative real number, then rS is in $\mathcal{S}$;
(4) $\mathcal{S}$ is self-adjoint: that is, if S is in $\mathcal{S}$, then S^* is in $\mathcal{S}$.

We wish to investigate the question of whether topologically transitive C^*-semigroups are transitive. A subset $\mathcal{T}$ of $B(H)$ is *transitive* if for any two vectors $\mathbf{x}$ and $\mathbf{y}$ in H with $\mathbf{x} \neq \mathbf{0}$, there exists T in $\mathcal{T}$ such that $T\mathbf{x} = \mathbf{y}$ (or equivalently, for each non-zero $\mathbf{x}$ in H, $\mathcal{T}\mathbf{x} = H$). A subset $\mathcal{T}$ of $B(H)$ is *topologically transitive* if for any two vectors $\mathbf{x}$ and $\mathbf{y}$ in H with $\mathbf{x} \neq \mathbf{0}$, there exists a sequence T_n in $\mathcal{T}$ such that $T_n\mathbf{x}$ converges to $\mathbf{y}$ (or equivalently, $\mathcal{T}\mathbf{x}$ is dense in H).

In the case where a C^*-semigroup acts on a finite-dimensional space, topological transitivity does imply transitivity. This is shown in [3] and as with algebras, the hypothesis of self-adjointness of the semigroup is not required in finite dimensions.

We consider the case where our C^*-semigroup contains a non-zero compact operator, and establish a Kadison Transitivity Theorem in this setting. Next, using this transitivity theorem, we are able to extract additional structural information about C^*-semigroups containing non-trivial compact operators. For example, we show that the minimal non-zero rank in a given topologically transitive C^*-semigroup is either one or two. Using this, we show that every transitive C^*-semigroup either contains all rank one operators (if the minimal non-zero rank is one), or—up to unitary equivalence—contains a certain minimal transitive C^*-semigroup $\mathcal{S}_{\mathbb{H}} = \{(\alpha_i \beta_j U_i V_j)\}$ where α_i, β_j are appropriate non-negative scalars and U_i, V_j lie in the group $\mathcal{U}_{\mathbb{H}}$ of unitary quaternions (see Example 7, Proposition 13 and Theorem 16). $\mathcal{U}_{\mathbb{H}}$ is exactly the group $SU_2(\mathbb{C})$ of all unitary 2×2 matrices with determinant 1.

Finally, we look at the limits of our transitivity theorem and construct a number of examples with the goal of showing that the theorem is best possible. These examples will show that if our topologically transitive semigroup lacks any of the properties mentioned in the hypotheses (being norm-closed, being homogeneous, being self-adjoint, or having non-zero intersection with the set of compact operators) then the semigroup need not be transitive. We close with an open question regarding the non-homogeneous case.

1. A Kadison Transitivity Theorem for C^*-semigroups

In this section we shall assume that our C^*-semigroup $\mathcal{S}$ contains non-zero compact operators.

Suppose $K \neq 0$ is a compact operator in a C^*-semigroup $\mathcal{S}$. While we may not have access to all of the spectral projections of K^*K, as we would in a C^*-algebra, we do have access to the

spectral projection corresponding to all of the eigenvalues of maximum modulus, since

$$\left(\frac{1}{\|K^*K\|}K^*K\right)^n$$

converges in norm, as $n \to \infty$, to this non-zero projection P in $\mathcal{S}$. By the compactness of K, P must be of finite rank. Let

$$\rho = \min\{rank(P)\colon 0 \neq P^2 = P^* = P \in \mathcal{S}\}.$$

The same argument (and lower semicontinuity of rank) shows that if a C^*-semigroup $\mathcal{S}$ contains a non-zero operator of finite rank r then it contains a non-zero projection of rank at most r.

We summarize this as our first lemma.

Lemma 1. *If a C^*-semigroup $\mathcal{S}$ in $B(H)$ contains a non-zero compact operator then it contains (some) non-zero operators of finite rank including a projection of the smallest non-zero rank present.*

Lemma 2. *If $\mathcal{S}$ is a topologically transitive C^*-semigroup in $B(H)$ and P is a projection of finite rank in $\mathcal{S}$, then the semigroup $P\mathcal{S}P$ is transitive on PH.*

Proof. The semigroup is closed, homogeneous and topologically transitive on a finite-dimensional space, so the lemma follows from Theorem 6 of [3]. $\square$

Let P_ρ be a projection of minimal rank ρ in $\mathcal{S}$ and consider the semigroup ideal $\mathcal{S}P_\rho\mathcal{S}$. This semigroup is also topologically transitive. (Lemma 4 of [3] shows that non-trivial ideals of topologically transitive semigroups are topologically transitive, and although the proof given there is for semigroups acting on finite-dimensional spaces, the same proof is valid in infinite-dimensions.) If we can show this semigroup ideal is transitive we will have our analogue of the Kadison Transitivity Theorem. We begin with the following structure theorem.

Theorem 3. *Let P_ρ be a projection of minimal rank ρ in a topologically transitive C^*-semigroup $\mathcal{S}$. Consider the decomposition $H = P_\rho H \oplus (I - P_\rho)H$.*

(1) *There is a closed group $\mathcal{U}_\rho$ of unitaries in $\mathbb{M}_\rho(\mathbb{C})$ such that*

$$P_\rho\mathcal{S}P_\rho|_{P_\rho H} = \mathbb{R}^+\mathcal{U}_\rho.$$

(2) *If $\left[\begin{smallmatrix} tU & 0 \\ T & 0\end{smallmatrix}\right] \in \mathcal{S}$ for some $t \geqslant 0$, then $T^*T \in \mathbb{R}^+ I_\rho$ (in other words, T is a multiple of an isometry).*

(3) *Every $Z \in \mathcal{S}P_\rho$ can be factored within $\mathcal{S}P_\rho$ as*

$$Z = \begin{bmatrix} zI_\rho & 0 \\ Z_3 & 0\end{bmatrix}\begin{bmatrix} U & 0 \\ 0 & 0\end{bmatrix},$$

where $U \in \mathcal{U}_\rho$ and $z \geqslant 0$.

(4) *If $\left[\begin{smallmatrix} rI_\rho & 0 \\ R & 0\end{smallmatrix}\right], \left[\begin{smallmatrix} tI_\rho & 0 \\ T & 0\end{smallmatrix}\right] \in \mathcal{S}$ for some $r, t \geqslant 0$, then $(rt)I_\rho + T^*R \in \mathbb{R}^+\mathcal{U}_\rho$.*

Proof. Elements of $\mathcal{S}$ can be represented as 2×2 operator matrices with respect to the decomposition $H = P_\rho H \oplus (I - P_\rho)H$. Since $P_\rho \mathcal{S} P_\rho$ is a C^*-subsemigroup of $\mathcal{S}$ and all of its non-zero elements have minimal rank, it follows that $P_\rho \mathcal{S} P_\rho|_{P_\rho H}$ is a C^*-semigroup of multiples of unitaries in $\mathbb{M}_\rho(\mathbb{C})$ (otherwise we could do as we did with K above and contradict the minimality of rank). $P_\rho \mathcal{S} P_\rho|_{P_\rho H}$ is topologically transitive because $\mathcal{S} P_\rho H$ is dense in H. By Lemma 2, $P_\rho \mathcal{S} P_\rho|_{P_\rho H}$ is transitive.

Let $\mathcal{U}_\rho$ be the set of unitaries in $P_\rho \mathcal{S} P_\rho|_{P_\rho H}$. Clearly $\mathcal{U}_\rho$ is a closed self-adjoint semigroup of unitaries, Hence it is a closed group, and

$$P_\rho \mathcal{S} P_\rho|_{P_\rho H} = \mathbb{R}^+ \mathcal{U}_\rho$$

follows from the definition.

Every element Z of SP_ρ satisfies the equation $Z = ZP_\rho$, and therefore has the form

$$\begin{bmatrix} zU & 0 \\ T & 0 \end{bmatrix}$$

with $z \in \mathbb{R}^+$ and $U \in \mathcal{U}_\rho$. In particular

$$\begin{bmatrix} U & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{S}$$

and

$$Z = \begin{bmatrix} zU & 0 \\ T & 0 \end{bmatrix} \begin{bmatrix} U^* & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} zI_\rho & 0 \\ TU^* & 0 \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & 0 \end{bmatrix},$$

which gives a required factorization.

On the other hand $z^2 I_\rho + T^*T$ is the north-west block of Z^*Z and consequently must be a non-negative multiple of a unitary, which in view of the fact that $z^2 I_\rho + T^*T$ is a non-negative operator on $\mathbb{C}^\rho$ implies that $z^2 I_\rho + T^*T$ is a scalar multiple of the identity. Thus $T^*T \in \mathbb{R}^+ I_\rho$.

If

$$\begin{bmatrix} rI & 0 \\ R & 0 \end{bmatrix}, \begin{bmatrix} tI & 0 \\ T & 0 \end{bmatrix} \in \mathcal{S}$$

for some $r, t \geqslant 0$, then

$$\begin{bmatrix} (rt)I_\rho + R^*T & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} rI & 0 \\ R & 0 \end{bmatrix}^* \begin{bmatrix} tI & 0 \\ T & 0 \end{bmatrix} \in \mathcal{S}$$

so that $(rt)I_\rho + R^*T \in \mathbb{R}^+ \mathcal{U}_\rho$. $\square$

We need the following lemma, which is a generalization of problem 20 in [4].

We refer the reader to pages 11 to 21 of [1] for the basic facts about the trace, trace-class and Hilbert–Schmidt operators which will be used below.

Let $\operatorname{tr}(A)$ denote the trace of a trace-class operator A in $B(H)$ and let $\Re(z)$ denote the real part of a complex number z.

Lemma 4. *Let $r \in \mathbb{N}$ and suppose that $\{Q_n\}_{n=1}^{\infty}$ is a sequence of finite-rank projections in $B(H)$ which satisfy*:

(1) *rank*$(Q_n) = r$ *for* $n = 1, 2, \dots,$
(2) *for all* $\varepsilon > 0$, *there exists* $N \in \mathbb{N}$ *such that* $\operatorname{tr}(Q_m Q_n) > r - \varepsilon$ *when* $n, m \geqslant N$.

Then Q_n converges to a projection Q (of rank r) in Hilbert–Schmidt norm (and hence in operator norm).

Proof. Consider the Hilbert–Schmidt norm of $Q_n - Q_m$:

$$\begin{aligned}\|Q_n - Q_m\|_{HS}^2 &= \operatorname{tr}\big((Q_n - Q_m)^*(Q_n - Q_m)\big)\\ &= \operatorname{tr}\big(Q_n^* Q_n\big) - 2\Re\big(\operatorname{tr}\big(Q_m^* Q_n\big)\big) + \operatorname{tr}\big(Q_m^* Q_m\big)\\ &= \operatorname{tr}(Q_n) - 2\operatorname{tr}(Q_m Q_n) + \operatorname{tr}(Q_m)\\ &= 2\big(r - \operatorname{tr}(Q_m Q_n)\big).\end{aligned}$$

It is clear that our hypotheses imply that $\{Q_n\}_{n=1}^{\infty}$ is a Cauchy sequence in Hilbert–Schmidt norm, which dominates the operator norm. Hence $\{Q_n\}_{n=1}^{\infty}$ converges in norm, and the norm limit of a convergent sequence of projections of rank r is itself a rank r projection. □

Theorem 5. *If $\mathcal{S}$ is a topologically transitive C^*-semigroup in $B(H)$ with minimal (non-zero) rank $\rho \in \mathbb{N}$, then there exist (orthogonal) projections $\{P_i\}_{i=1}^{\infty}$ in $\mathcal{S}$ such that*

(1) *the rank of each P_i is ρ,*
(2) *if $i \neq j$ then $P_i P_j = 0$, and*
(3) *$\{P_i H\colon i = 1, 2, \dots\}$ spans H.*

Proof. Let P_1 be a projection of minimal rank in $\mathcal{S}$.

Consider two unit vectors $\mathbf{x}$ and $\mathbf{y}$ with $\mathbf{x}$ in the range of P_1 and $\mathbf{y}$ in the kernel of P_1. By topological transitivity there exists a sequence of operators $\{S_n\}_{n=1}^{\infty}$ in $\mathcal{S}$ such that $S_n\mathbf{x}$ converges to $\mathbf{y}$, so that $S_n P_1 \mathbf{x}$ also converges to $\mathbf{y}$. As a block matrix with respect to the decomposition $H = P_1 H \oplus (I - P_1)H$, we may write

$$S_n P_1 = \begin{bmatrix} r_n U_n & 0 \\ Y_n & 0 \end{bmatrix},$$

where (by Theorem 3) $\{r_n\}$ is a sequence in $\mathbb{R}^+$ converging to 0 and $\{U_n\}$ is a sequence of unitaries in the closed group $\mathcal{U}_\rho$ associated with P_1.

Consideration of $(S_n P_1)^*(S_n P_1)$ and Theorem 3 shows that we must have $Y_n^* Y_n = \beta_n I_\rho$ where β_n is a sequence of real scalars converging to 1. Hence $\|S_n P_1\|$ converges to 1 and by replacing S_n by $\frac{1}{\|S_n P_1\|} S_n$, we still have that $\frac{1}{\|S_n P_1\|} S_n P_1 \mathbf{x}$ converges to $\mathbf{y}$. As such, with no loss of generality, we may assume that each $S_n P_1$ is a partial isometry.

Now for n and m in $\mathbb{N}$, consider

$$X_{m,n} = (S_m P_1)^* S_n P_1 = \begin{bmatrix} r_m r_n U_m^* U_n + Y_m^* Y_n & 0 \\ 0 & 0 \end{bmatrix}.$$

Since $X_{m,n}$ lives in $P_1\mathcal{S}P_1$, its compression to P_1H must be a positive multiple of a unitary. Clearly this multiple is less than or equal to 1.

On the other hand, letting $\mathbb{N}^2$ to be directed by the relation

$$(m,n) \leqslant (k,l) \quad \Longleftrightarrow \quad m \leqslant k,\ n \leqslant l,$$

we have that

$$\langle X_{m,n}\mathbf{x}, \mathbf{x}\rangle = \langle S_n P_1\mathbf{x}, S_m P_1\mathbf{x}\rangle$$

is a net approaching $\langle \mathbf{y}, \mathbf{y}\rangle = 1$. Thus, when n and m are large, $X_{m,n}$ is almost unitary (again, when restricted to P_1H). Next, consider

$$X_{m,n}^* X_{m,n} = \begin{bmatrix} Y_n^* Y_m Y_m^* Y_n + E_{m,n} & 0 \\ 0 & 0 \end{bmatrix},$$

where $E_{m,n}$ is a net of operators which converges to 0. It follows that $Y_n^* Y_m Y_m^* Y_n$ approaches I_ρ. Thus, given $\varepsilon > 0$, there exists N such that when $n, m \geqslant N$, $\operatorname{tr}(Y_n^* Y_m Y_m^* Y_n) > \rho - \epsilon$.

Now consider

$$Q_n = (S_n P_1)(S_n P_1)^* = \begin{bmatrix} r_n^2 I_\rho & r_n U_n Y_n^* \\ r_n Y_n U_n^* & Y_n Y_n^* \end{bmatrix}.$$

For each $n \in \mathbb{N}$, $S_n P_1$ is a partial isometry. Thus Q_n is a projection (of minimal rank), and

$$Q_n = \begin{bmatrix} 0 & 0 \\ 0 & Y_n Y_n^* \end{bmatrix} + \Delta_n$$

where each Δ_n is an operator of rank at most 2ρ such that $\|\Delta_n\| \to 0$ as n approaches infinity.

Choose N_1 in $\mathbb{N}$ such that $\|\Delta_n\| < 1$ when $n \geqslant N_1$. Then

$$Q_m Q_n = \begin{bmatrix} 0 & 0 \\ 0 & Y_m Y_m^* Y_n Y_n^* \end{bmatrix} + \Delta_{m,n}$$

where $\Delta_{m,n}$ is an operator of rank at most 2ρ such that $\|\Delta_{m,n}\|$ approaches 0. Thus, given $\varepsilon > 0$, there exists an N_2 for which $n, m \geqslant N_2$ implies that

$$\operatorname{tr}(Q_m Q_n) \geqslant \operatorname{tr}\bigl(Y_m Y_m^* Y_n Y_n^*\bigr) - \frac{\varepsilon}{2}.$$

But $\operatorname{tr}(Y_m Y_m^* Y_n Y_n^*) = \operatorname{tr}(Y_n^* Y_m Y_m^* Y_n)$ and, by the above computation with $X_{n,m}$, there exists $N_3 \in \mathbb{N}$ such that

$$\operatorname{tr}\bigl(Y_n^* Y_m Y_m^* Y_n\bigr) > \rho - \frac{\varepsilon}{2}.$$

Set $N = \max\{N_1, N_2, N_3\}$. Then when $n, m \geqslant N$,

$$\operatorname{tr}(Q_m Q_n) > \rho - \varepsilon.$$

By Lemma 4, Q_n converges in Hilbert–Schmidt norm, and hence in operator norm, to a projection P_2 in $\mathcal{S}$, and clearly $P_1 P_2 = 0$.

We can inductively construct $P_k, P_{k+1}, \dots$ by repeating this argument, but at each stage choosing $\mathbf{y}$ in the kernel of $P_1, P_2, \dots, P_k$ to obtain the desired sequence. $\square$

We have seen that we can decompose $H = \bigoplus_{i=1}^{\infty} H_i$ where each H_i has dimension ρ, and the orthogonal projection onto each H_i is in $\mathcal{S}$. Let us refer to such a decomposition as a *minimal block decomposition* of H relative to $\mathcal{S}$. An operator S in $\mathcal{S}$ then admits a block matrix decomposition $[S_{i,j}]_{i,j=1}^{\infty}$ where each $S_{i,j}$ is a multiple of an $\rho \times \rho$ unitary matrix. This follows easily from Lemma 1 via the fact that $P_i S P_j \in \mathcal{S}$ and has rank at most ρ.

We are now ready to prove one of our main results, a Kadison Transitivity Theorem for self-adjoint, homogeneous, closed semigroups.

Theorem 6 *(A Kadison Transitivity Theorem for C^*-semigroups). If $\mathcal{S}$ in $B(H)$ is a topologically transitive C^*-semigroup which contains a non-zero compact operator, then $\mathcal{S}$ is transitive.*

Proof. Let $\mathbf{x}$ and $\mathbf{y}$ be vectors in H with $\|\mathbf{x}\| = 1$. To prove the theorem, it suffices to construct S in $\mathcal{S}$ such that $S\mathbf{x} = \mathbf{y}$. If $\mathbf{y} = 0$, then $0 \in \mathcal{S}$ will accomplish this. Therefore we restrict our attention to the case where $\mathbf{y} \neq 0$.

By Theorem 5, $\{P_i H\colon i = 1, 2, \dots\}$ spans H, and therefore there exists P_i such that $P_i \mathbf{x} \neq 0$. With no loss of generality we may assume that $P_1 \mathbf{x} \neq 0$. Then by topological transitivity, there exist $\{S_k\}_{k=1}^{\infty}$ in $\mathcal{S}$ such that $(S_k P_1)\mathbf{x} \to \mathbf{y}$ as $k \to \infty$.

With respect to the block decomposition mentioned above,

$$S_k P_1 = \begin{bmatrix} r_1^{(k)} U_1^{(k)} & 0 & \cdots & 0 \\ r_2^{(k)} U_2^{(k)} & 0 & \cdots & 0 \\ r_3^{(k)} U_3^{(k)} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \end{bmatrix}$$

where for each $i = 1, 2, \dots$, $\{r_i^{(k)}\}_{k=1}^{\infty}$ is a sequence of non-negative real numbers and $\{U_i^{(k)}\}_{k=1}^{\infty}$ is a sequence of $\rho \times \rho$ unitary matrices.

Note that

$$\|S_k P_1 \mathbf{x}\|^2 = \sum_{i=1}^{\infty} \bigl(r_i^{(k)}\bigr)^2 \bigl\|U_i^{(k)} \mathbf{x}\bigr\|^2 = \sum_{i=1}^{\infty} \bigl(r_i^{(k)}\bigr)^2$$

and

$$\|S_k P_1\|_{\mathrm{HS}}^2 = \sum_{i=1}^{\infty} \bigl(r_i^{(k)}\bigr)^2 \bigl\|U_i^{(k)}\bigr\|_{\mathrm{HS}}^2 = \sum_{i=1}^{\infty} \bigl(r_i^{(k)}\bigr)^2 r_{\mathcal{S}}$$

so that $S_k P_1$ is a Hilbert–Schmidt operator for each $k \geqslant 1$.

It follows that $\lim_{k\to\infty} r_i^{(k)} = \|P_i\mathbf{y}\|$. Using the compactness of the unitary group in $M_\rho(\mathbb{C})$, for each $i \geqslant 1$, we can find a convergent subsequence of $\{U_i^{(k)}\}_{k=1}^{\infty}$. By passing to subsequences, we may assume without loss of generality that for each $i = 1, 2, \ldots,$

$$\lim_{k\to\infty} r_i^{(k)} = r_i = \|P_i\mathbf{y}\| \quad \text{and} \quad \lim_{k\to\infty} U_i^{(k)} = U_i.$$

We have that $\sum_{i=1}^{\infty} r_i^2 = \|\mathbf{y}\|^2$. Thus, if we define the operator S as the block matrix with $r_i U_i$ in the ith entry of the first column and zeroes elsewhere, we have that $S\mathbf{x} = \mathbf{y}$ and $\|S\|_{\mathrm{HS}}^2 = \sum_{i=1}^{\infty} r_i^2 r_{\mathcal{S}} = \|\mathbf{y}\|^2 r_{\mathcal{S}}$. Thus S is also a Hilbert–Schmidt operator.

The theorem will be proven if we can show that S is in $\mathcal{S}$. We shall do this by demonstrating that $S_k P_1$ converges to S in the Hilbert–Schmidt norm.

We know that $S_k P_1\mathbf{x}$ converges to $\mathbf{y}$ in norm, and since $\|S_k P_1\|_{\mathrm{HS}} = \sqrt{r_{\mathcal{S}}}\|S_k P_1\mathbf{x}\|$, it follows that $\|S_k P_1\|_{\mathrm{HS}}$ converges to $\sqrt{r_{\mathcal{S}}}\|y\|$, which equals $\|S\|_{\mathrm{HS}}$.

Recalling that $r_i^{(k)} U_i^{(k)}$ converges to $r_i U_i$ in norm for every i, is easy to see that $\langle S_k P_1, F\rangle_{\mathrm{HS}} \to \langle S, F\rangle_{\mathrm{HS}}$ for any operator F having only finitely many non-zero block-entries with respect to our decomposition. Since $S_k P_1$ is a bounded sequence it follows that $S_k P_1$ converges to S weakly in the Hilbert space of Hilbert–Schmidt operators on H.

In Hilbert spaces $\|v_n\| \to \|v\|$ coupled with weak convergence of v_n to v is equivalent to norm convergence. Hence $S_k P_1$ converges to S with respect to the Hilbert–Schmidt norm.

Thus S is a limit in the Hilbert–Schmidt norm of elements of $\mathcal{S}$ and *a fortiori* it is the operator norm limit of those same elements. Since $\mathcal{S}$ is closed, it follows that $S \in \mathcal{S}$ and therefore that $\mathcal{S}$ is transitive. $\square$

2. Structure of topologically transitive C^*-semigroups

We can obtain more structural information about topologically transitive C^*-semigroups, but first let us look at a few examples.

Of course, any topologically transitive C^*-algebra is such an example, but if we remove the additive structure there are still examples. The simplest of these is the semigroup of rank one operators on a Hilbert space H:

$$\mathcal{R}_1 = \bigl\{S \in B(H)\colon\ rank(S) \leqslant 1\bigr\}.$$

Another example with similar structure is the following.

Example 7. Let

$$\mathbb{H} = \left\{ \begin{bmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{bmatrix} \colon\ \alpha, \beta \in \mathbb{C} \right\}$$

denote the set of quaternions as a subset of $M_2(\mathbb{C})$. Then $\mathbb{H}$ is a real algebra. Let $\mathcal{U}_{\mathbb{H}}$ denote the unitary matrices in $\mathbb{H}$. These are just the matrices of the above type in $\mathbb{H}$ for which $|\alpha|^2 + |\beta|^2 = 1$. It is easy to check that $\mathcal{U}_{\mathbb{H}} = SU_2(\mathbb{C})$. Consider the Hilbert space $H = \bigoplus_{i=1}^{\infty} \mathbb{C}^2$

and the following semigroup acting on H:

$$\mathcal{S}_{\mathbb{H}} = \left\{ \left[u_i v_j U_i V_j^* \right]_{i,j=1}^{\infty} \colon u_i, v_j \in [0, \infty),\ U_i, V_j \in \mathcal{U}_{\mathbb{H}},\ \sum_{i=1}^{\infty} |u_i|^2 < \infty,\ \sum_{j=1}^{\infty} |v_j|^2 < \infty \right\}.$$

It is relatively straightforward to verify that $\mathcal{S}_{\mathbb{H}}$ is a transitive C^*-semigroup, and that all non-zero elements of $\mathcal{S}_{\mathbb{H}}$ have rank two.

What minimal non-zero ranks are possible in a topologically transitive C^*-semigroup $\mathcal{S}$? What does the ideal of elements of $\mathcal{S}$ of minimal non-zero rank (together with 0) look like? Is it possible that the two examples above, where the minimal ranks are one and two, are the only possibilities? In what follows, we consider this question.

Lemma 8. *Let $\mathcal{S}$ be a (topologically) transitive C^*-semigroup in $B(H)$, and let $[r_i]_{i=1}^{\infty}$ be a square-summable sequence of positive numbers. Then there exists a unitary operator $X \in B(H)$ and an operator $T \in X^{-1}\mathcal{S}X$ such that $T_{i,1} = r_i I_\rho$ for all i with respect to our block matrix decomposition.*

Proof. Choose $\mathbf{y} \in H$ with $\|P_i \mathbf{y}\| = r_i$ for all i. Theorem 6 implies that $\mathcal{S}$ is transitive, and so there exists $S \in \mathcal{S}$ such that $S\mathbf{e}_1 = \mathbf{y}$. In particular, $S_{i,1}$ is r_i times a unitary. Post-multiplying by the element of $\mathcal{S}$ that has the inverse of $S_{1,1}$ in the $(1,1)$ position and zeros in the other entries, we may, with no loss of generality, assume that $S_{1,1} = I$.

Let X be the block-diagonal unitary matrix $[X_{i,j}]$ defined by

$$X_{i,i} = \begin{cases} I & \text{if } i = 1, \\ \frac{1}{\|S_{i,1}\|} S_{i,1} & \text{otherwise} \end{cases}$$

then $T = X^{-1}SX$ has the required form. $\square$

Let $\mathcal{S}_{i,j} = \{S_{i,j} \colon S = [S_{i,j}] \in \mathcal{S}\}$.

For a $\rho \times \rho$ matrix A, let $\mathcal{Z}_{i,j}(A)$ be the operator in $B(H)$ with A in the (i,j)th entry and all other entries 0. By Lemma 8, $\mathcal{Z}_{i,j}(I_\rho)$ is in $\mathcal{S}$ if either $i = 1$ or $j = 1$.

Theorem 9. *If $\mathcal{S}$ is a (topologically) transitive C^*-semigroup which contains a non-zero compact operator, then there exists a unitary group $\mathcal{U}_\rho$ in $M_\rho(\mathbb{C})$ such that $\mathbb{R}^+\mathcal{U}_\rho$ acts transitively on $\mathbb{C}^\rho$ and $\mathcal{S}_{i,j} = \mathbb{R}^+\mathcal{U}_\rho$ for $i, j = 1, 2, \ldots$.*

Proof. By Theorem 3, there exists a unitary group $\mathcal{U}_\rho$ in $M_\rho(\mathbb{C})$ such that $\mathbb{R}^+\mathcal{U}_\rho$ acts transitively on $\mathbb{C}^\rho$ and $\mathbb{R}^+\mathcal{U}_\rho = \mathcal{S}_{1,1}$. By consideration of certain products of elements $\mathcal{S}$ we show that all other block entries $\mathcal{S}_{i,j}$ also equal $\mathbb{R}^+\mathcal{U}_\rho$.

Note that for $S \in \mathcal{S}$,

$$\mathcal{Z}_{1,j}(I_\rho)\mathcal{Z}_{j,j}(S_{j,j}) = \mathcal{Z}_{1,j}(S_{j,j})$$

so $\mathcal{S}_{j,j} \subseteq \mathcal{S}_{1,j}$. Also,

$$\mathcal{Z}_{j,1}(I_\rho)\mathcal{Z}_{1,j}(S_{1,j}) = \mathcal{Z}_{j,j}(S_{1,j})$$

so $\mathcal{S}_{1,j} \subseteq \mathcal{S}_{j,j}$. Therefore $\mathcal{S}_{1,j} = \mathcal{S}_{j,j}$.

Similarly,

$$\mathcal{Z}_{j,1}(I_\rho)\mathcal{Z}_{1,1}(S_{1,1}) = \mathcal{Z}_{j,1}(S_{1,1})$$

so $\mathcal{S}_{1,1} \subseteq \mathcal{S}_{j,1}$. Also,

$$\mathcal{Z}_{1,j}(I_\rho)\mathcal{Z}_{j,1}(S_{j,1}) = \mathcal{Z}_{1,1}(S_{j,1})$$

so $\mathcal{S}_{j,1} \subseteq \mathcal{S}_{1,1}$. Therefore $\mathcal{S}_{j,1} = \mathcal{S}_{1,1}$.

Now $\mathcal{S}_{1,1}$ and $\mathcal{S}_{j,j}$ are self-adjoint and hence, so are $\mathcal{S}_{j,1}$ and $\mathcal{S}_{1,j}$. Self-adjointness of $\mathcal{S}$ gives us that $\mathcal{S}_{j,1}^* = \mathcal{S}_{1,j}$. Putting it all together, we get that

$$\mathcal{S}_{1,1} = \mathcal{S}_{j,j} = \mathcal{S}_{1,j} = \mathcal{S}_{j,1}$$

for all $j = 1, 2, \ldots$.

To handle the remaining $\mathcal{S}_{i,j}$, consider

$$\mathcal{Z}_{i,j}(S_{i,j})\mathcal{Z}_{j,1}(I_\rho) = \mathcal{Z}_{i,1}(S_{i,j})$$

so $\mathcal{S}_{i,j} \subseteq \mathcal{S}_{i,1}$ and

$$\mathcal{Z}_{i,1}(S_{i,1})\mathcal{Z}_{1,j}(I_\rho) = \mathcal{Z}_{i,j}(S_{i,1})$$

so $\mathcal{S}_{i,1} \subseteq \mathcal{S}_{i,j}$. Thus $\mathcal{S}_{1,1} = \mathcal{S}_{i,1} = \mathcal{S}_{i,j}$ for all $i, j = 1, 2, \ldots$. $\square$

What properties must this unitary group $\mathcal{U}_\rho$ in $M_\rho(\mathbb{C})$ have? By Lemma 8, after applying unitary similarity the operator

$$J = \begin{bmatrix} I & 0 & \ldots \\ r_2 I & 0 & \ldots \\ r_3 I & 0 & \ldots \\ \vdots & \vdots & \vdots \end{bmatrix}$$

can be assumed to be in $\mathcal{S}$, where $r_n = 1/(n-1)$ if $n \geqslant 2$. *For the remainder of the section we shall assume that the similarity has been applied and that $J \in \mathcal{S}$.* If

$$S = \begin{bmatrix} U & 0 & \ldots \\ V & 0 & \ldots \\ 0 & 0 & \ldots \\ \vdots & \vdots & \vdots \end{bmatrix}$$

is also in $\mathcal{S}$, then $J^* S$ is in $\mathcal{S}$ and so $U + V$ is in $\mathbb{R}^+\mathcal{U}_\rho$. Thus we see that a form of weak linearity is forced upon $\mathcal{U}_\rho$. While weak, this linearity severely restricts the possibilities for $\mathcal{U}_\rho$.

Some notation: given $\mathbf{x}_i$ in $\mathbf{C}^\rho$ with $\sum \|\mathbf{x}_i\|^2 < \infty$, let $\mathbf{x} = [\mathbf{x}_1, \mathbf{x}_2, \ldots]^T$ denote the vector in H such that, with respect to our decomposition $H = \bigoplus_{i=1}^n H_i$, $P_i\mathbf{x} = \mathbf{x}_i$.

The weak linearity will be most usefully expressed in the following form.

Lemma 10. *Let $\mathcal{S}$ be a (topologically) transitive C^*-semigroup which contains a non-zero compact operator. Then, with respect to the above decomposition, for each unit vector $\mathbf{y}$ in $\mathbb{C}^\rho$ there exists $R_\mathbf{y}$ in $\mathcal{U}_\rho$ with the following properties*:

(1) *the first column of $R_\mathbf{y}$ is $\mathbf{y}$*;

(2)
$$\begin{bmatrix} I & 0 & \dots \\ R_\mathbf{y} & 0 & \dots \\ 0 & 0 & \dots \\ \vdots & \vdots & \vdots \end{bmatrix} \in \mathcal{S};$$

(3) *$I + R_\mathbf{y}$ is in $\mathbb{R}^+\mathcal{U}_\rho$*;

(4) *$R_\mathbf{y}$ satisfies a quadratic equation of the form $R_\mathbf{y}^2 - \lambda R_\mathbf{y} + I = 0$ for some real scalar λ.*

Proof. By Theorem 6 our semigroup $\mathcal{S}$ is transitive, and so for any unit vector $\mathbf{y}$ in $\mathbb{C}^\rho$ there exists S in $\mathcal{S}$ such that $S[\mathbf{e}_1, \mathbf{0}, \mathbf{0}, \mathbf{0}, \dots]^T = [\mathbf{e}_1, \mathbf{y}, \mathbf{0}, \mathbf{0}, \dots]^T$. So

$$SP_1 = \begin{bmatrix} U & 0 & \dots \\ V & 0 & \dots \\ 0 & 0 & \dots \\ \vdots & \vdots & \vdots \end{bmatrix}$$

where U and V are unitaries in $\mathcal{U}_\rho$ with the first column of U being $\mathbf{e}_1$ and the first column of V being $\mathbf{y}$. Since $\mathbf{e}_1$ is an eigenvector of U with corresponding eigenvalue 1, and hence also an eigenvector of U^* with respect to the same eigenvalue, $VU^*\mathbf{e}_1 = \mathbf{y}$. By postmultiplying by $\mathcal{Z}_{1,1}(U^*) \in \mathcal{S}$, we obtain that for any unit vector $\mathbf{y}$, there exists a unitary $R_\mathbf{y} = VU^*$ in $\mathcal{U}_\rho$ whose first column is $\mathbf{y}$ and such that

$$W = \begin{bmatrix} I & 0 & \dots \\ R_\mathbf{y} & 0 & \dots \\ 0 & 0 & \dots \\ \vdots & \vdots & \vdots \end{bmatrix}$$

is in $\mathcal{S}$.

For such a $R_\mathbf{y}$, $J^*W \in P_1\mathcal{S}P_1$ and therefore $I + R_\mathbf{y}$ must also be in $\mathbb{R}^+\mathcal{U}_\rho$, and so the spectrum of $R_\mathbf{y}$ must lie both on a circle of radius one, centered at zero, and on some circle (possibly a degenerate circle of radius 0, i.e. a point) centered at -1. This implies that $R_\mathbf{y}$ is either $\pm I$, or the spectrum of $R_\mathbf{y}$ consists of just two conjugate eigenvalues. In either case $R_\mathbf{y}$ satisfies a quadratic equation of the form $R_\mathbf{y}^2 - \lambda R_\mathbf{y} + I = 0$ for some real scalar λ. □

Observation 11. *If a unitary U satisfies an equation of the form $U^2 - \lambda U + I = 0$, and $\langle Ux, x\rangle = 0$ for some non-zero x, then $\lambda = 0$ and consequently*

$$U = -U^*.$$

Theorem 12. *If $\mathcal{S}$ is a (topologically) transitive C^*-semigroup which contains a non-zero compact operator, then the minimal rank of non-zero elements in $\mathcal{S}$ is either one or two.*

Proof. Suppose that the minimal rank ρ is greater than 2.

Consider the possible entries of $R_{\mathbf{e_2}}$ from Lemma 10. The first column of $R_{\mathbf{e_2}}$ is (by definition) $\mathbf{e_2}$, and by Observation 11 the second column of $R_{\mathbf{e_2}}$ must be $-\mathbf{e_1}$. Making a change of orthonormal basis to replace $\mathbf{e}_3$ with an eigenvector of $R_{\mathbf{e_2}}$ (which we shall henceforth refer to as $\mathbf{e}_3$), we have that $R_{\mathbf{e_2}}$ has the following form:

$$R_{\mathbf{e_2}} = \left[\begin{array}{ccc|ccc} 0 & -1 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & \alpha & 0 & 0 & \dots \\ \hline 0 & 0 & 0 & * & * & \dots \\ 0 & 0 & 0 & * & * & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right]$$

where $\alpha \in \{-i, i\}$, and $*$ denotes an entry whose value is unknown.

Next, consider the possible entries of $R_{\mathbf{e_3}}$. The first column of $R_{\mathbf{e_3}}$ is (by definition) $\mathbf{e_3}$, and by Observation 11 the third column of $R_{\mathbf{e_2}}$ must be $-\mathbf{e_1}$. It follows that $R_{\mathbf{e_3}}$ has the following form:

$$R_{\mathbf{e_3}} = \left[\begin{array}{ccc|ccc} 0 & 0 & -1 & 0 & 0 & \dots \\ 0 & \beta & 0 & * & * & \dots \\ 1 & 0 & 0 & 0 & 0 & \dots \\ \hline 0 & * & 0 & * & * & \dots \\ 0 & * & 0 & * & * & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right]$$

where β is a complex scalar of modulus at most 1, and again $*$ denotes an entry whose value is unknown.

We now arrive at our contradiction by considering

$$\begin{bmatrix} I & 0 & \dots \\ R_{\mathbf{e_2}} & 0 & \dots \\ 0 & 0 & \dots \\ \vdots & \vdots & \vdots \end{bmatrix}^* \begin{bmatrix} I & 0 & \dots \\ R_{\mathbf{e_3}} & 0 & \dots \\ 0 & 0 & \dots \\ \vdots & \vdots & \vdots \end{bmatrix}.$$

The $(1, 1)$ block matrix entry of this product is $I + R_{\mathbf{e_2}}^* R_{\mathbf{e_3}}$. Both $R_{\mathbf{e_2}}$ and $R_{\mathbf{e_3}}$ are unitaries in $\mathcal{U}_\rho$ and hence so is $R_{\mathbf{e_2}}^* R_{\mathbf{e_3}}$. By spectral considerations as above we are led to the conclusion that $R_{\mathbf{e_2}}^* R_{\mathbf{e_3}}$ must satisfy a quadratic of the form presented above, but doing the matrix multiplication we obtain that

$$R_{\mathbf{e_2}}^* R_{\mathbf{e_3}} = \left[\begin{array}{ccc|ccc} 0 & \beta & 0 & * & * & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ \bar{\alpha} & 0 & 0 & 0 & 0 & \dots \\ \hline 0 & * & 0 & * & * & \dots \\ 0 & * & 0 & * & * & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right].$$

Since $R_{\mathbf{e}_2}^* R_{\mathbf{e}_3}\mathbf{e}_1 = \bar{\alpha}\mathbf{e}_3$, by Observation 11 $R_{\mathbf{e}_2}^* R_{\mathbf{e}_3}\mathbf{e}_3$ (which equals $\mathbf{e}_2$) must be equal to $-\frac{1}{\bar{\alpha}}\mathbf{e}_1$. This is a contradiction. □

Proposition 13. *Let $\mathcal{S}$ be a (topologically) transitive C^*-semigroup, and suppose that $\mathcal{S}$ contains an operator of rank one. Then $\mathcal{S}$ contains the set $\mathcal{R}_1$ of all operators of rank zero or one in $B(H)$. Furthermore, $\mathcal{R}_1$ is itself a transitive C^*-semigroup.*

Proof. By the Transitivity Theorem (Theorem 6), $\mathcal{S}$ is transitive. By Lemma 1, there exists a projection P_1 of rank one in $\mathcal{S}$. Choose $\mathbf{e}_1 \in P_1 H$. Let $\mathbf{x}, \mathbf{y} \neq \mathbf{0}$ in H. Since $\mathcal{S}$ is transitive, we can find $S \in \mathcal{S}$ such that $S\mathbf{x} = \mathbf{e}_1$. Next, find $T \in \mathcal{S}$ so that $T\mathbf{e}_1 = \mathbf{y}$. Then $S_1 = P_1 S$, $T_1 = TP_1 \in \mathcal{S}$ are rank one operators and $T_1 S_1 \mathbf{x} = \mathbf{y}$. From this it easily follows that $\mathcal{S}$ contains all rank one operators. Since $\mathcal{S}$ is a C^*-semigroup, it also contains the zero operator.

That $\mathcal{R}_1$ is a transitive C^*-semigroup is a routine calculation. □

Thus when the minimal rank of a topologically transitive C^*-semigroup is one, it is trivial that the group of unitaries $\mathcal{U}_\rho$ consists of all unitaries (which in this case are just the scalars of modulus one). When the minimal rank is two, the situation is just slightly more complicated.

As we saw in Example 7, the C^*-semigroup

$$\mathcal{S}_{\mathbb{H}} = \left\{ \left[u_i v_j U_i V_j^*\right]_{i,j=1}^{\infty}\colon\ u_i, v_j \in [0,\infty),\ U_i, V_j \in \mathcal{U}_{\mathbb{H}},\ \sum_{i=1}^{\infty} |u_i|^2 < \infty,\ \sum_{j=1}^{\infty} |v_j|^2 < \infty \right\}$$

is a transitive C^*-semigroup of constant rank 2 (except for the zero element).

Theorem 14. *If $\mathcal{S}$ is a (topologically) transitive C^*-semigroup which contains a non-zero compact operator, and if the minimal rank of non-zero elements in $\mathcal{S}$ is two, then $\mathcal{U}_{\mathbb{H}} \subseteq \mathcal{U}_\rho$. Thus $\mathbb{T}\mathcal{U}_\rho$ consists of the entire unitary group $U_2(\mathbb{C})$ in $M_2(\mathbb{C})$, where $\mathbb{T} = \{z \in \mathbb{C}\colon\ |z| = 1\}$.*

Proof. By Lemma 10, given a unit vector $\mathbf{x} = \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ in $\mathbb{C}^2$, there exists $R_{\mathbf{x}}$ in $\mathcal{U}_\rho$ such that

$$\begin{bmatrix} I & 0 & \dots \\ R_{\mathbf{x}} & 0 & \dots \\ 0 & 0 & \dots \\ \vdots & \vdots & \vdots \end{bmatrix}$$

is in $\mathcal{S}$. Obviously $R_{\mathbf{x}}$ must have the form

$$R_{\mathbf{x}} = \begin{bmatrix} x & \gamma\bar{y} \\ y & -\gamma\bar{x} \end{bmatrix}$$

for some γ of modulus one. From Lemma 10, $I + R_{\mathbf{x}}$ is a multiple of a unitary and hence has orthogonal columns

$$I + R_{\mathbf{x}} = \begin{bmatrix} 1+x & \gamma\bar{y} \\ y & 1-\gamma\bar{x} \end{bmatrix}$$

so $(1+x)\bar{\gamma}y + y(1-\bar{\gamma}x) = 0$ which implies that $(\bar{\gamma}+1)y = 0$. So whenever $y \neq 0$, $\gamma = -1$ and so

$$V_{\mathbf{x}} = \begin{bmatrix} x & -\bar{y} \\ y & \bar{x} \end{bmatrix}$$

is in $\mathcal{U}_\rho$ whenever $y \neq 0$. However, $\mathcal{U}_\rho$ is closed so this holds even if $y = 0$. Thus $\mathcal{U}_\rho$ contains $\mathcal{U}_{\mathbb{H}}$.

Since the quaternion group $\mathcal{U}_{\mathbb{H}}$ coincides with the unitary group $SU_2(\mathbb{C})$ of 2×2 complex matrices with determinant equal to 1, it easily follows that $U_2(\mathbb{C}) \subseteq \mathbb{T}\mathcal{U}_\rho$. □

Since $\mathcal{U}_{\mathbb{H}} = SU_2(\mathbb{C})$, every unitary matrix W in $M_2(\mathbb{C})$ can be written as γA, with $A \in \mathcal{U}_{\mathbb{H}}$ and $\gamma \in \mathbb{T}$.

Given a (topologically) transitive C^*-semigroup $\mathcal{S}$ such that the minimal rank of non-zero elements in $\mathcal{S}$ is two, Theorem 14 dictates that $\mathcal{U}_\rho \supseteq \mathcal{U}_{\mathbb{H}}$, and therefore $\gamma I \in \mathcal{U}_\rho$ whenever $\gamma A \in \mathcal{U}_\rho$ for some $A \in \mathcal{U}_{\mathbb{H}}$.

Let $\mathcal{G} = \{\gamma\colon\ \gamma I \in \mathcal{U}_\rho\}$. Then $\mathcal{G}$ is a closed subgroup of $\mathbb{T}$ and so it must either be all of $\mathbb{T}$ or the finite group of the mth roots of unity for some m. Clearly $\mathcal{U}_\rho = \mathcal{G}\mathcal{U}_{\mathbb{H}}$.

Furthermore, given any closed subgroup $\mathcal{K}$ of $\mathbb{T}$, consider the semigroup $\mathcal{S}_{\mathcal{K}}$ which is constructed just as $\mathcal{S}_{\mathbb{H}}$ (of Example 7) was, except that $\mathcal{U}_{\mathbb{H}}$ is replaced by $\mathcal{K}\mathcal{U}_{\mathbb{H}}$ in the construction. Then $\mathcal{S}_{\mathcal{K}}$ is a transitive C^*-semigroup where every non-zero element has rank two, and

$$(\mathcal{S}_{\mathcal{K}})_{i,j} = \mathbb{R}^+\mathcal{K}\mathcal{U}_{\mathbb{H}}$$

for all i, j.

In summary, we have demonstrated the following:

Theorem 15. *A group $\mathcal{U}$ of unitaries in $M_2(\mathbb{C})$ appears as $\mathcal{U}_\rho$ for some* (*topologically*) *transitive C^*-semigroup $\mathcal{S}$ with the minimal rank of non-zero elements equal to two if and only if either $\mathcal{U} = U_2(\mathbb{C})$ or $\mathcal{U} = \mathcal{G}\mathcal{U}_{\mathbb{H}}$ where $\mathcal{G}$ is the group of mth roots of unity for some $m \in \mathbb{N}$.*

The next result shows that the semigroup $\mathcal{S}_{\mathbb{H}}$ of Example 7 is, up to unitary equivalence, the unique minimal topologically transitive C^*-semigroup for which the minimal rank of non-zero members is two.

Theorem 16. *The semigroup $\mathcal{S}_{\mathbb{H}}$ is, up to unitary equivalence, contained in any* (*topologically*) *transitive C^*-semigroup for which the minimal rank of non-zero members is two.*

Proof. We need to show that if $\mathcal{S}$ is any (topologically) transitive C^*-semigroup for which $\min\{\operatorname{rank} S\colon\ 0 \neq S \in \mathcal{S}\} = 2$, then there exists a unitary operator $U \in B(H)$ so that $\mathcal{S}_{\mathbb{H}} \subseteq U^*\mathcal{S}U$.

The unitary operator in question is simply the operator required to ensure that $J \in \mathcal{S}$, where J is the operator mentioned in the comments following Theorem 9. Let us therefore show that if $\mathcal{S}$ is a topologically transitive C^*-semigroup that contains J, then $\mathcal{S}$ contains $\mathcal{S}_{\mathbb{H}}$.

By Theorem 6, $\mathcal{S}$ is in fact transitive. By Theorems 14 and 9, $\mathcal{S}$ contains $P_i\mathcal{S}_{\mathbb{H}}P_j$ for all $i, j \geqslant 1$. Let $\{\mathbf{e}_{2k-1}, \mathbf{e}_{2k}\}$ be an orthonormal basis for P_kH, so that $\{\mathbf{e}_k\}_{k=1}^{\infty}$ is an orthonormal basis for H.

Our objective is to prove that $\mathcal{S}$ contains every operator K of the form

$$\begin{bmatrix} K_1 & 0 & 0 & \dots \\ K_2 & 0 & 0 & \dots \\ K_3 & 0 & 0 & \dots \\ \vdots & 0 & 0 & \dots \end{bmatrix},$$

where

(i) $K_n = \left(\begin{smallmatrix} \alpha_n & -\overline{\beta_n} \\ \beta_n & \overline{\alpha_n} \end{smallmatrix}\right) \in \mathbb{H}$;
(ii) $\{\alpha_1\overline{\alpha_n}, \beta_1\overline{\beta_n}\}$ is linearly independent over $\mathbb{R}$.
(iii) $\sum_{n=1}^{\infty}(|\alpha_n|^2 + |\beta_n|^2) < \infty$.

Since $\mathcal{S}$ is self-adjoint and closed, it follows from this that $\mathcal{S}_{\mathbb{H}} \subseteq \mathcal{S}$.

Step One. For arbitrary choices of $x_n > 0$ with $\sum_{n=1}^{\infty} x_n^2 < \infty$,

$$X = \begin{bmatrix} x_1 I_2 & 0 & \dots \\ x_2 I_2 & 0 & \dots \\ x_3 I_2 & 0 & \dots \\ \vdots & \vdots & \ddots \end{bmatrix} \in \mathcal{S}.$$

Indeed, let $(x_n)_{n=1}^{\infty}$ be chosen as above, and let X denote the corresponding operator. Set $\mathbf{x} = \sum_{n=1}^{\infty} x_n \mathbf{e}_{2n-1} \in H$. Since $\mathcal{S}$ is transitive, there exists $M = MP_1 \in \mathcal{S}$ so that $M\mathbf{e}_1 = \mathbf{x}$. Let $L_n = P_n M P_1$. Since each L_n is a multiple of a unitary matrix, by considering the first column of M, we see that $L_n = \left[\begin{smallmatrix} x_n & 0 \\ 0 & x_n\lambda_n \end{smallmatrix}\right]$ for some $\lambda_n \in \mathbb{T}$, $n \geqslant 1$. Now $J \in \mathcal{S}$ (see the discussion after Theorem 9) and so $J^*M = P_1(J^*M)P_1 \in \mathcal{S}$. Furthermore, the compression of J^*M to $P_1 H$ looks like

$$\begin{bmatrix} \sum_{n=1}^{\infty}(r_n x_n) & 0 \\ 0 & \sum_{n=1}^{\infty}(r_n x_n \lambda_n) \end{bmatrix}.$$

Since $r_n, x_n > 0$ for all $n \geqslant 1$, and since the compression of J^*M too must be a multiple of a unitary, we must have

$$\sum_{n=1}^{\infty}(r_n x_n) = \left|\sum_{n=1}^{\infty}(r_n x_n \lambda_n)\right|.$$

By the equality case of the triangle inequality this is impossible unless $\lambda_i = \lambda_j$ for all $i, j \geqslant 1$. Now $\left[\begin{smallmatrix} x_1 & 0 \\ 0 & x_1\lambda_1 \end{smallmatrix}\right] \in P_1\mathcal{S}P_1|_{P_1 H}$, $\left[\begin{smallmatrix} x_1 & 0 \\ 0 & x_1 \end{smallmatrix}\right] \in \mathbb{H} \subseteq P_1\mathcal{S}P_1|_{P_1 H}$, so $\left[\begin{smallmatrix} 1 & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 0 \\ 0 & \bar{\lambda} \end{smallmatrix}\right] \subseteq P_1\mathcal{S}P_1|_{P_1 H}$. Choose $Z = P_1 Z P_1 \in \mathcal{S}$ so that $Z|_{P_1 H} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right]$. Then $MZ = X \in \mathcal{S}$.

Step Two. Consider the set $\mathcal{D}$ of vectors of the form $\mathbf{z} = \sum_{n=1}^{\infty}(\alpha_n \mathbf{e}_{2n-1} + \beta_n \mathbf{e}_{2n})$, where

(i) $\{\alpha_1\overline{\alpha_n}, \beta_1\overline{\beta_n}\}$ is linearly independent over $\mathbb{R}$,
(ii) $\sum_{n=1}^{\infty}(|\alpha_n|^2 + |\beta_n|^2) < \infty$.

It is easy to verify that $\mathcal{D}$ is dense in H. Fix $\mathbf{z}$ as above.

By transitivity of $\mathcal{S}$, we can find an operator $B = BP_1 \in \mathcal{S}$ so that $B\mathbf{e}_1 = \mathbf{z}$. Letting $B_n = P_n B P_1|_{P_1 H}$, we may write $B_n = \left[\begin{smallmatrix} \alpha_n & -\overline{\beta_n}\lambda_n \\ \beta_n & \overline{\alpha_n}\lambda_n \end{smallmatrix}\right]$, where $|\lambda_n| = 1$ for all $n \geqslant 1$.

We claim that $\lambda_n = \lambda_1$ for all $n \geqslant 1$.

Indeed, by Step One, for each $n \geqslant 2$,

$$Q_n = \begin{array}{c} \begin{bmatrix} I_2 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ I_2 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \end{array} \begin{array}{c} P_1 H \\ P_2 H \\ \vdots \\ P_{n-1} H \\ P_n H \\ P_{n+1} H \\ \vdots \end{array} \in \mathcal{S}P_1,$$

and so $Q_n^* B = B_1 + B_n = \left[\begin{smallmatrix} \alpha_1+\alpha_n & -\overline{\beta_1}\lambda_1-\overline{\beta_n}\lambda_n \\ \beta_1+\beta_n & \overline{\alpha_1}\lambda_1+\overline{\alpha_n}\lambda_n \end{smallmatrix}\right]$ is a multiple of a unitary. This implies that for each $n \geqslant 2$,

$$\begin{aligned} |\alpha_1 + \alpha_n| &= |\overline{\alpha_1}\lambda_1 + \overline{\alpha_n}\lambda_n| \quad \text{and} \\ |\beta_1 + \beta_n| &= |\overline{\beta_1}\lambda_1 + \overline{\beta_n}\lambda_n|. \end{aligned}$$

Set $\gamma_n = \overline{\lambda_1}\lambda_n$, $n \geqslant 2$. Squaring these two equations and cancelling common terms yields

$$\begin{aligned} \alpha_1\overline{\alpha_n}(1-\gamma_n) &= -\overline{\alpha_1\overline{\alpha_n}(1-\gamma_n)} \quad \text{and} \\ \beta_n\overline{\beta_n}(1-\gamma_n) &= -\overline{\beta_n\overline{\beta_n}(1-\gamma_n)}. \end{aligned}$$

Equivalently

$$\begin{cases} \alpha_1\overline{\alpha_n}(1-\gamma_n) \in i\mathbb{R}, \\ \beta_1\overline{\beta_n}(1-\gamma_n) \in i\mathbb{R}. \end{cases}$$

If $\gamma_n \neq 1$, then $\{\alpha_1\overline{\alpha_n},\ \beta_1\overline{\beta_n}\}$ is linearly dependent over $\mathbb{R}$, a contradiction.

Hence $\lambda_n = \lambda_1$, $n \geqslant 1$. Now $B_1 = \left[\begin{smallmatrix} \alpha_1 & -\overline{\beta_1} \\ \beta_1 & \overline{\alpha_1} \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & 0 \\ 0 & \lambda_1 \end{smallmatrix}\right]$. Since the first term is invertible, $W = P_1 W P_1 \in \mathcal{S}$, where $W|_{P_1 H} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & \overline{\lambda_1} \end{smallmatrix}\right]$. Finally, with $K_n = B_n\left[\begin{smallmatrix} 1 & 0 \\ 0 & \overline{\lambda_1} \end{smallmatrix}\right]$, we see that $K := BW \in \mathcal{S}P_1 \subseteq \mathcal{S}$ is precisely of the form announced at the beginning of the proof. $\square$

Let $\mathcal{A}_{\mathbb{H}}$ be the closed, *real* algebra generated by $\mathcal{S}_{\mathbb{H}}$. Since $\mathbb{H}$ is a closed real algebra, the block-entries (with respect to our usual decomposition) of every element of $\mathcal{A}_{\mathbb{H}}$ lie in $\mathbb{H}$.

Since $\mathcal{A}_{\mathbb{H}}$ contains every operator $[T_{i,j}]$ with finitely many non-zero $T_{i,j}$ each of which is an element of $\mathbb{H}$, it follows that $\mathcal{A}_{\mathbb{H}}$ also contains every compact operator $[T_{i,j}]$ with $T_{i,j} \in \mathbb{H}$.

Let $\mathbb{C}\mathcal{A}_{\mathbb{H}}$ be the *semigroup* of all complex multiples of the elements of $\mathcal{A}_{\mathbb{H}}$ (of course $\mathbb{C}\mathcal{A}_{\mathbb{H}}$ need not be an algebra over $\mathbb{C}$).

Theorem 17. *Up to unitary equivalence $\mathbb{C}\mathcal{A}_{\mathbb{H}}$ is the unique maximal transitive C^*-semigroup of compact operators for which the minimal rank of non-zero members is two.*

Proof. That $\mathbb{C}\mathcal{A}_{\mathbb{H}}$ is a C^*-semigroup with minimal non-zero rank 2 is clear.

Suppose that $\mathcal{S}$ is a maximal C^*-semigroup of compacts in which the minimal non-zero rank is 2. By Theorem 16 we may assume that $\mathcal{S}_{\mathbb{H}} \subseteq \mathcal{S}$.

Let $T = [T_{i,j}]$ be an element of $\mathcal{S}$ which is not in $\mathbb{C}\mathcal{A}_{\mathbb{H}}$. By Theorem 9 each $T_{i,j}$ is a multiple of a unitary in $M_2(\mathbb{C})$. As we have mentioned previously $\mathcal{U}_{\mathbb{H}} = SU_2(\mathbb{C})$, and consequently $\mathbb{T}\mathbb{H} = \mathbb{R}\mathcal{U}_{\mathbb{H}} = \mathbb{R}U_2(\mathbb{C})$. In particular $T_{i,j} \in \mathbb{T}\mathbb{H}$ for all i, j. If there is an $\alpha \in \mathbb{T}$ such that $T_{i,j} \in \alpha\mathbb{H}$ for all i, j, then $T \in \mathbb{T}\mathcal{A}_{\mathbb{H}}$ (by the observations immediately preceding this theorem) contrary to our assumption. Hence we must assume that no such α exists. Passing to T^* if necessary we may therefore assume that there is some p such that $T_{p,q} = \lambda_1 H_1$ and $T_{p,q+k} = \lambda_2 H_2$ for some $q, k > 0$, $H_1, H_2 \in \mathbb{H}\backslash\{0\}$, and some $\lambda_1, \lambda_2 \in \mathbb{T}$ which are linearly independent over $\mathbb{R}$.

Now, $\mathcal{Z}_{1,p}(I)$ lies in $\mathcal{S}$ (see notation of Theorem 9). Let $R = \mathcal{Z}_{1,p}(I)T$. Then $R = P_1 R = [R_{i,j}] \in \mathcal{S}$ and $R_{1,q} = \lambda_1 H_1$ and $R_{1,q+k} = \lambda_2 H_2$.

Since $\mathcal{S}_{\mathbb{H}} \subset \mathcal{S}$ we can choose $M = MP_1 = [M_{i,j}]$ in $\mathcal{S}$ with $M_{q,1} = \frac{1}{\|H_1\|^2} H_1^*$ and $M_{q+k,1} = I$.

Then $(RM)_{1,1} = \lambda_1 I_2 + \lambda_2 H_2$ must be a multiple of a unitary matrix. Therefore $\overline{\lambda_1} RM = I_1 + \overline{\lambda_1}\lambda_2 H_2$ is also a multiple of a unitary. Yet

$$I_1 + \overline{\lambda_1}\lambda_2 H_2 = \begin{bmatrix} 1 + \mu x & -\mu\bar{y} \\ \mu y & 1 + \mu\bar{x} \end{bmatrix},$$

where $\mu = \overline{\lambda_1}\lambda_2$. Since the columns of a unitary are orthogonal and $\mu \notin \mathbb{R}$, an easy calculation shows that $y = 0$.

Similarly, we can show that the (2, 1)-entry of H_1 must be zero, so that H_1 and H_2 are diagonal (non-zero) quaternions $\left[\begin{smallmatrix} x_1 & 0 \\ 0 & \overline{x_1} \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} x_2 & 0 \\ 0 & \overline{x_2} \end{smallmatrix}\right]$, respectively.

For any pair K_1, K_2 of elements of $\mathbb{H}$ there exists $N = [N_{i,j}] \in \mathcal{S}_{\mathbb{H}}$ such that $N_{q,1} = K_1$ and $N_{q+k,1} = K_2$. Then $(RN)_{1,1} = \lambda_1 H_1 K_1 + \lambda_2 H_2 K_2$ must be a multiple of a unitary, and so the same is true for $H_1 K_1 + \mu H_2 K_2$. Hence $H_1 K_1 + \mu H_2 K_2$ is a multiple of a unitary for every $K_1, K_2 \in \mathbb{H}$.

Consider $K_1 = I$ and $K_2 = \left[\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right]$. Then

$$H_1 K_1 + \mu H_2 K_2 = \begin{bmatrix} x_1 + \mu x_2 & -\mu x_2 \\ \mu\overline{x_2} & \overline{x_1} + \mu\overline{x_2} \end{bmatrix}.$$

The columns of this matrix must be orthogonal, and that (via an easy calculation) leads to $\mu \in \mathbb{R}$, which is a contradiction.

Hence $\mathcal{S} \subseteq \mathbb{C}\mathcal{A}_{\mathbb{H}}$, and by maximality, the two are equal. $\square$

As we have mentioned already, the proof of Lemma 4 of [3] shows that a non-trivial semigroup ideal is (topologically) transitive exactly when the semigroup itself is (topologically) transitive. When a C^*-semigroup $\mathcal{S}$ contains a non-zero compact operator (and therefore some operators of rank one or two by Theorem 12), the set of compact operators $\mathcal{K}_{\mathcal{S}}$ in $\mathcal{S}$ is a C^*-semigroup that is a semigroup ideal in $\mathcal{S}$. Therefore $\mathcal{S}$ is transitive if and only if $\mathcal{K}_{\mathcal{S}}$ is transitive, so that it is rewarding to study transitive C^*-semigroups consisting entirely of compact operators.

In such a case, Theorems 12, 13, 16 and 17 (including its proof) can be combined to yield the following result.

Theorem 18. *If $\mathcal{S}$ is a (topologically) transitive C^*-semigroup in $B(H)$ consisting entirely of compact operators, then either $\mathcal{S}$ contains the set $\mathcal{R}_1$ of all operators of rank zero or one, or the minimal rank of non-zero members of $\mathcal{S}$ is two and up to simultaneous unitary equivalence*

$$\mathcal{S}_{\mathbb{H}} \subseteq \mathcal{S} \subseteq \mathbb{C}\mathcal{A}_{\mathbb{H}}.$$

3. Counterexamples in transitive semigroups

In this section we consider the conditions on our semigroup in Theorem 6 and construct examples of semigroups which satisfy all but one of the conditions and are topologically transitive but not transitive. In this way we show that the hypotheses of the theorem cannot be weakened if we wish to obtain transitivity.

The conditions on a semigroup $\mathcal{S}$ in Theorem 6 are:

- $\mathcal{S}$ is closed,
- $\mathcal{S}$ is self-adjoint,
- $\mathcal{S}$ contains a non-zero compact operator,
- $\mathcal{S}$ is homogeneous.

The example in the case where $\mathcal{S}$ is not closed is easily constructed.

In the following example $\mathbb{Q}[i]$ denotes all complex numbers whose real and imaginary parts are rational numbers.

Example 19 *($\mathcal{S}$ not closed).* The semigroup

$$\mathcal{S} = \left\{ rA \in K\left(\ell^2(\mathbb{N})\right):\ A = [a_{ij}]_{i,j=1}^{\infty},\ a_{i,j} \in \mathbb{Q}[i],\ r \in \mathbb{R}^+ \right\}$$

is clearly homogeneous, self-adjoint and contains non-zero compact operators but is not closed. Also, it is easy to see that $\mathcal{S}$ is topologically transitive but not transitive, since the entries of elements of $\mathcal{S}$ are linearly dependent over $\mathbb{Q}$.

The first difficult case to consider is where our semigroup does not contain any compacts. In the case of C^*-algebras, the Kadison Transitivity Theorem shows that topological transitivity still implies transitivity. This is not true for C^*-semigroups, as we shall now see.

Notation. Given an $k \times k$ matrix A, $A^{(\infty)}$ shall denote the *inflation* of A. This shall be an operator acting on $\ell^2(\mathbb{N})$, defined as follows. Let $\{\delta_i\}_{i=1}^{\infty}$ denote the standard basis for $\ell^2(\mathbb{N})$, and for $n = 0, 1, 2, \ldots,$ let H_n be the n-dimensional subspace of $\ell^2(\mathbb{N})$ spanned by $\{\delta_{nk+1}, \delta_{nk+2}, \ldots \delta_{(n+1)k}\}$. Then $\ell^2(\mathbb{N}) = \bigoplus_{n=0}^{\infty} H_n$ and each H_n can be identified with $\mathbb{C}^k$ in the obvious way. With respect to this decomposition, $A^{(\infty)} = \bigoplus_{n=0}^{\infty} A$.

Example 20 *($\mathcal{S}$ does not contain non-zero compact operators).* For each $m = 1, 2, \ldots,$ let

$$\mathcal{S}_m = \left\{ \begin{bmatrix} A & 0 \\ 0 & nA \end{bmatrix}^{(\infty)} :\ A \in M_{2^m}(\mathbb{C}),\ n \in \mathbb{N},\ n \geqslant 2^m \right\}.$$

It is clear that each $\mathcal{S}_m$ is a closed, self-adjoint, homogeneous, semigroup. Also, if $m_1 \leqslant m_2$ then $\mathcal{S}_{m_1}\mathcal{S}_{m_2} \subseteq \mathcal{S}_{m_2}$ and $\mathcal{S}_{m_2}\mathcal{S}_{m_1} \subseteq \mathcal{S}_{m_2}$ so $\mathcal{S} = \bigcup_{m=1}^{\infty} \mathcal{S}_m$ is also a self-adjoint, homogeneous semigroup. This semigroup is closed since if $\{S_j\}_{j=1}^{\infty}$ is a convergent sequence of operators in $\mathcal{S}$, then we must have that either $\{S_j\}_{j=1}^{\infty}$ converges to 0, which is in $\mathcal{S}$, or there exists M such that $S_j \in \mathcal{S}_M$ for all $j = 1, 2, \ldots$. Since $\mathcal{S}_M$ is closed, the limit of the sequence $\{S_j\}_{j=1}^{\infty}$ is in $\mathcal{S}$ and hence $\mathcal{S}$ is closed.

$\mathcal{S}$ is topologically transitive since it is weakly dense in $B(\ell^2(\mathbb{N}))$, but it is not transitive since $\mathcal{S}\mathbf{e}_1$ consists of all vectors in $\ell^2(\mathbb{N})$ which have finitely many nonzero entries.

This does not contradict Theorem 6 since $\mathcal{S}$ contains no compact operators.

What about the self-adjointness condition? Can it be dropped and the results of Theorem 6 maintained? Clearly the condition required can be relaxed to requiring only that $\mathcal{S}$ be similar to a self-adjoint set, as this is equivalent to a renorming of the Hilbert space. If the self-adjointness condition is relaxed much beyond this, transitivity can be lost, as the following example shows.

Example 21 *($\mathcal{S}$ not self-adjoint).* Let $\mathcal{S}_{\mathbb{H}}$ be as in Example 7, which acts on $\ell^2(\mathbb{N})$ and let

$$D = \bigoplus_{n=1}^{\infty} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{n} \end{bmatrix}$$

act on the same space. Consider

$$\mathcal{S} = \bigl\{X \in B\bigl(\ell^2(\mathbb{N})\bigr):\ XD = DS \text{ for some } S \in \mathcal{S}_{\mathbb{H}}\bigr\}.$$

It is immediate that $\mathcal{S}$ is homogeneous and consists of compact operators (in fact all non-zero operators in $\mathcal{S}$ have rank 2). It is also not hard to show directly that $\mathcal{S}$ is not self-adjoint, but this will follow from Theorem 6 and what follows. We claim that $\mathcal{S}$ is closed and topologically transitive, but not transitive.

$\mathcal{S}$ is closed. Suppose $\{X_n\}_{n=1}^{\infty}$ is a sequence of operators in $\mathcal{S}$ which converges to an operator X. Then $\{X_nD\}_{n=1}^{\infty}$ converges to XD, and there exist $\{S_n\}_{n=1}^{\infty}$ a sequence of operators in $\mathcal{S}_{\mathbb{H}}$ such that $X_nD = DS_n$ for all $n \geqslant 1$. Thus $\{DS_n\}_{n=1}^{\infty}$ converges to XD. However, elements of $\mathcal{S}_{\mathbb{H}}$ have a very special structure: the $(2n-1)$th row determines the $(2n)$th row, and the correspondence is given by an isometric function, namely

$$f(x_1, x_2, x_3, x_4, \ldots) = (-\overline{x_2}, \overline{x_1}, -\overline{x_4}, \overline{x_3}, \ldots).$$

If we let P_{odd} denote the projection in $B(\ell^2(\mathbb{C}))$ onto the span of $\{\mathbf{e}_{2n+1}\}_{n=0}^{\infty}$, then $P_{\mathrm{odd}}DS_n = P_{\mathrm{odd}}S_n$ and so $\{P_{\mathrm{odd}}S_n\}_{n=1}^{\infty}$ converges. But the row correspondence mentioned above implies that $\{S_n\}_{n=1}^{\infty}$ converges to an operator S in $\mathcal{S}_{\mathbb{H}}$. Hence $\{DS_n\}_{n=1}^{\infty}$ converges to DS. From above it also converges to XD. Hence $XD = DS$, so that $X \in \mathcal{S}$ and $\mathcal{S}$ is closed.

$\mathcal{S}$ is not transitive. It is easy to see that $\mathcal{S}$ is not transitive, as the intertwining equation which defines $\mathcal{S}$ implies that the range of D is invariant for $\mathcal{S}$. Thus, no vector $\mathbf{x}$ in $Ran(D)$ could be mapped to a vector $\mathbf{y}$ not in $Ran(D)$ by an operator in $\mathcal{S}$.

$\mathcal{S}$ is topologically transitive. For each $n = 1, 2, \dots,$ let P_{2n} be the projection onto the span of $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{2n}\}$. The structure of $\mathcal{S}_{\mathbb{H}}$ is such that if S is in $\mathcal{S}_{\mathbb{H}}$, so is its "upper-left corners" $P_{2n} S P_{2n}$. Since each P_{2n} commutes with D, the intertwining equation that defines $\mathcal{S}$ gives that if X is in $\mathcal{S}$, then so is its "upper-left corners" $P_{2n} X P_{2n}$. Considering the subsemigroup of $\mathcal{S}$ consisting of these "upper-left corners," it is straightforward to show that any non-zero vector $\mathbf{x}$ can be mapped to any vector $\mathbf{y}$ with finitely many non-zero entries by some such "upper-left corner" in $\mathcal{S}$, so $\mathcal{S}$ is topologically transitive.

It is interesting to note that while $\mathcal{S}$ is not transitive, $\mathcal{S}^*$ is transitive.

What about the homogeneity condition?

Example 22 *($\mathcal{S}$ not homogeneous, finite dimensions).* For $n \in \mathbb{N}$, let

$$\mathcal{Q} = \left\{ Q \in M_n\left(\mathbb{Q}[i]\right):\ \deg(Q) \leqslant \left|\det(Q)\right| \right\}$$

where

$$\deg(Q) = \min\left\{ r \in \mathbb{N}:\ rQ \in M_n\left(\mathbb{Z}[i]\right) \right\}.$$

This is a generalization of an example from [3], where the above example is constructed in the two-dimensional real case.

Following the analysis in [3], it can be shown that $\mathcal{Q}$ is a discrete (and therefore closed), self-adjoint semigroup in $M_n(\mathbb{C})$ which is topologically transitive but not transitive.

From this we see that the homogeneity condition is required when our Hilbert space is finite-dimensional. It is not clear whether it is required when the Hilbert space is infinite-dimensional. As shown in [3], the full force of homogeneity is not required in finite-dimensions to force transitivity. Merely the existence of a single contraction in a closed topologically transitive semigroup can sometimes be enough. In infinite dimensions, any compact operator acts like a contraction on "most" of its domain, and so it seems a very difficult problem to show that the homogeneity condition is required in this case.

We give one alternate example in the non-compact case in infinite dimensions.

Example 23 *($\mathcal{S}$ not homogenous, $\mathcal{S}$ contains no nonzero compacts).* For $n = 1, 2, \dots,$ define the following on $\ell^2(\mathbb{N})$:

$$\mathcal{S}_n = \left\{ \bigoplus_{j=0}^{\infty} X_j:\ X_j \in M_{2^n}(\mathbb{C}) \right\}$$

where X_0 is an arbitrary invertible matrix in $M_{2^n}(\mathbb{C})$, and each X_j is chosen to be either X_0 or $(X_0^*)^{-1}$ according to the following rule: after the first 2^k X_j are chosen, the next 2^k are chosen as the adjoint of the inverse of the first 2^k. (This should look familiar to those acquainted with Thue sequences.)

Then let $\mathcal{S}$ denote the norm closure of $\bigcup_{n=1}^{\infty} \mathcal{S}_n$.

The construction gives that $\mathcal{S}_n \subset \mathcal{S}_m$ when $n < m$, and each $\mathcal{S}_n$ is a semigroup, so $\mathcal{S}$ is a semigroup. It is clear that $\mathcal{S}$ is closed, topologically transitive and self-adjoint.

It is also straightforward that if $S \in \bigcup_{n=1}^{\infty} \mathcal{S}_n$ is such that $\|S\mathbf{e}_1\| < \epsilon$, then $\|S\| > \frac{1}{\epsilon}$. Thus $\mathcal{S}$ cannot be transitive since if there exists $S \in \mathcal{S}$ which mapped $\mathbf{e}_1$ to $\mathbf{0}$, S would be the limit of S_n in $\bigcup_{n=1}^{\infty} \mathcal{S}_n$ with $\|S\mathbf{e}_1\|$ approaching 0. From above we get that this implies that $\|S_n\|$ approaches ∞, a contradiction.

Unfortunately this example contains no compacts except for the zero operator.

So we are still left with the following:

Open Question 24. If $\mathcal{S}$ is a closed, self-adjoint semigroup of compact operators which is topologically transitive on a separable, *infinite-dimensional* Hilbert space H, is $\mathcal{S}$ transitive?

Remark 1. While the results in this paper are stated for a C^*-semigroup acting on a separable Hilbert space, analogous results are true on non-separable spaces. In particular, Theorems 5 and 6 hold, with the only modification being that the maximal family of orthogonal projections in $\mathcal{S}$ is no longer countable, but indexed by some set of higher cardinality.

In closing, it should be noted that the Kadison Transitivity Theorem for C^*-algebras actually gives n-transitivity for all $n \in \mathbb{N}$. In its strongest form, it states that if $\mathcal{A}$ is a topologically transitive C^*-algebra acting on a Hilbert space H, then given $\epsilon > 0$, any finite-dimensional subspace M of H and an operator F acting on M, there exists $A \in \mathcal{A}$ such that $A|_M = F$ and $\|A\| \leqslant \|F\| + \epsilon$. There is no possibility of extending the conclusion of Kadison Transitivity Theorem to C^*-semigroups to give n-transitivity, as is demonstrated by the C^*-semigroup $\mathcal{R}_1 = \{S \in B(H)\colon\ rank(S) \leqslant 1\}$. However, as easily seen from the proof of Theorem 6, the conclusion could be strengthened to give an operator $S \in \mathcal{S}$ such that $S\mathbf{x} = \mathbf{y}$ and $\|S\| \leqslant \frac{\|\mathbf{y}\|}{\|\mathbf{x}\|}$.

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Reinforcement of an inequality due to Brascamp and Lieb

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Abstract

In this paper, we obtain a reinforcement of an inequality due to Brascamp and Lieb and a reinforcement of Poincaré's inequality for general logarithmical concave measures on $\mathbb{R}^d$. The formula used in the proof is related to theorems concerning the integration of log-concave functions (such as results of Prékopa and of Ball, Barthe and Naor). We also obtain a lower bound for the variance of the same family of measures.

Keywords: Logarithmically concave measure; Poincaré inequality; Operators; Semigroups

Contents

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1. Introduction

In a paper devoted to the Li–Yau's inequality, Bakry and Ledoux [2] noticed that it was possible to improve Poincaré's inequality for the Gaussian measure. If we denote by μ the standard Gaussian measure on $\mathbb{R}^d$, they obtained, for all regular function f:

$$Var_\mu(f) \leqslant \int \|\nabla f\|^2 \, d\mu - \frac{1}{2d}\left(\int \Delta f \, d\mu\right)^2,$$

where $Var_\mu(f) = \int (f - \int f \, d\mu)^2 \, d\mu$.

On the other hand, it is possible to prove:

$$\left\| \int \nabla f \, d\mu \right\|^2 + \frac{1}{2d}\left(\int \Delta f \, d\mu\right)^2 \leqslant Var_\mu(f).$$

Those inequalities are easy to prove with the use of Hermite's polynomials (see the end of this paper).

The aim of this paper is to generalize those inequalities to any measure with a regular log-concave density with respect to the Lebesgue measure on $\mathbb{R}^d$. The upper bound obtained here strengthens another inequality which is due to Brascamp and Lieb [4].

More precisely, we consider a C^2 function $\varphi : \mathbb{R}^d \to \mathbb{R}$ such that:

$$\forall x \in \mathbb{R}^d, \quad \operatorname{Hess} \varphi(x) > 0 \quad \text{and} \quad \int e^{-\varphi} \, dx < +\infty.$$

When the opposite will be not specified, functions φ of this paper will satisfy those hypothesis.

We denote by $\langle x, y \rangle$ the usual scalar product on $\mathbb{R}^d$ and by $\|x\|$ the norm associated to this scalar product. We define:

$$d\mu_\varphi(x) = \frac{e^{-\varphi(x)} \, dx}{\int e^{-\varphi(z)} \, dz}$$

(so $d\mu = d\mu_{\|x\|^2/2}$) and for $f \in L^2(\mu_\varphi)$:

$$Var_{\mu_\varphi}(f) = \int \left(f - \int f \, d\mu_\varphi\right)^2 d\mu_\varphi.$$

The inequality of Brascamp and Lieb is the following:

$$\forall f \in C^1\big(\mathbb{R}^d\big) \cap L^2(\mu_\varphi), \quad Var_{\mu_\varphi}(f) \leqslant \int \big\langle (\operatorname{Hess} \varphi)^{-1} \nabla f, \nabla f \big\rangle \, d\mu_\varphi.$$

Furthermore, we define:

$$a(\varphi) = \inf_{x \in \mathbb{R}^d} \min\big\{\lambda \text{ eigenvalue of } \operatorname{Hess} \varphi(x)\big\},$$

$$b(\varphi) = \sup_{x \in \mathbb{R}^d} \max\big\{\lambda \text{ eigenvalue of } \operatorname{Hess} \varphi(x)\big\}$$

(we have $0 \leqslant a(\varphi) \leqslant b(\varphi) \leqslant +\infty$ and $b(\varphi) > 0$). When the context will be clear, we will write those quantities a and b.

Let $H_1(\mu_\varphi)$ be the set of functions f in $L^2(\mu_\varphi)$ such that ∇f (in the distribution sense) is in $L^2(\mu_\varphi)$. We endow $H_1(\mu_\varphi)$ with the norm:

$$\|f\|_{H_1} = \left[\int \left(f^2 + \|\nabla f\|^2\right) d\mu_\varphi\right]^{1/2}.$$

Finally, we define, for f in $L^2(\mu_\varphi)$ and if φ is in $L^2(\mu_\varphi)$:

$$c_\varphi(f) = \int \varphi f \, d\mu_\varphi - \int \varphi \, d\mu_\varphi \int f \, d\mu_\varphi.$$

Theorem 1. *We assume that φ is in $L^2(\mu_\varphi)$.*
Then, for all f in $H_1(\mu_\varphi)$:

$$Var_{\mu_\varphi}(f) \leqslant \int \left\langle (\operatorname{Hess}\varphi)^{-1} \nabla f, \nabla f \right\rangle d\mu_\varphi - \frac{(1 + a/b)}{d} c_\varphi(f)^2.$$

In the case $\varphi(x) = \frac{\|x\|^2}{2}$, we rediscover the inequality of Bakry and Ledoux. Actually, with the Ornstein–Uhlenbeck operator $L_0 = \Delta - \langle x, \nabla \rangle$, we write:

$$\begin{aligned} \int \Delta f \, d\mu &= \int \langle x, \nabla f \rangle \, d\mu = -\int L_0\left(\frac{\|x\|^2}{2}\right) f \, d\mu \\ &= \int \left(\|x\|^2 - d\right) f \, d\mu = 2\left(\int \varphi f \, d\mu - \int \varphi \, d\mu \int f \, d\mu\right). \end{aligned}$$

One of the consequences of the Brascamp and Lieb inequality is the Poincaré inequality for measures μ_φ if $a(\varphi) > 0$ (it is not the only method, see for example [6]). It is obvious with the following inequality:

$$\int \left\langle (\operatorname{Hess}\varphi)^{-1} \nabla f, \nabla f \right\rangle d\mu_\varphi \leqslant \frac{1}{a(\varphi)} \int \|\nabla f\|^2 \, d\mu_\varphi.$$

Consequently, Theorem 1 gives a reinforcement of the Poincaré inequality under the hypothesis $a(\varphi) > 0$.

On the other hand, the inequality of Brascamp and Lieb allows to recover the following result due to Prékopa [8] (see [4]). If we consider the function Φ defined by

$$e^{-\Phi(y)} = \int e^{-\varphi(x,y)} \, dx,$$

then Prékopa proved that Φ is convex if φ is convex. Also, Theorem 1 has a link with this result. More precisely, the proof of Theorem 1 is based on a new expression for $Var_{\mu_\varphi}(f)$ (formula (4)) when f belongs to a dense subset of $H_1(\mu_\varphi)$ (see later). This expression allows us to rediscover, without improvement, results of Ball, Barthe and Naor [3] and Cordero-Erausquin, Fradelizi and Maurey [5], which are related to Prékopa's theorem. See Section 3 for a more complete explanation.

Furthermore, in this paper, we obtain another formula for $Var_{\mu_\varphi}(f)$ (formula (3)) which allows us to give a lower bound for the variance.

In the next theorem, we use the following notation. If A is a symmetric and positive $d \times d$ matrix, we define, for x in $\mathbb{R}^d$:

$$|||x|||_A^2 = \langle Ax, x\rangle.$$

Theorem 2. *We assume $\varphi \in L^2(\mu_\varphi)$ and $\operatorname{Hess}\varphi \in L^1(\mu_\varphi)$. Then, for all f in $L^2(\mu_\varphi)$:*

$$Var_{\mu_\varphi}(f) \geqslant \left|\left|\left| \int f \nabla\varphi \, d\mu_\varphi \right|\right|\right|^2_{(\int \operatorname{Hess}\varphi \, d\mu_\varphi)^{-1}} + \frac{1}{d} c_\varphi(f)^2.$$

We immediately see that this theorem does not allow to reobtain the inequality for the Gaussian measure. We will show that is possible to recover the right inequality under the hypothesis $a(\varphi) > 0$.

Remark 1. Under the hypothesis of the theorem, $\int f \nabla\varphi \, d\mu_\varphi$ is well defined because we will prove later: $\operatorname{Hess}\varphi \in L^1(\mu_\varphi) \Rightarrow \nabla\varphi \in L^2(\mu_\varphi)$ (see Lemma 8).

Remark 2. If we assume f is in $H_1(\mu_\varphi)$, then we have: $\int f \nabla\varphi \, d\mu_\varphi = \int \nabla f \, d\mu_\varphi$. This equality is obvious for f in C_c^∞ (the set of C^∞ functions with compact support) and it is known that C_c^∞ is a dense subset of $(H_1(\mu_\varphi), \| \, \|_{H_1})$ [9, Lemma 3.1.12].

2. Upper and lower bounds for the variance

2.1. Preliminary results

In this section, we give some lemmas concerning heat kernels in $\mathbb{R}^d$.

We consider a C^2 function $\varphi : \mathbb{R}^d \to \mathbb{R}$ such that:

$$\forall x \in \mathbb{R}^d, \quad \operatorname{Hess}\varphi(x) > 0 \quad \text{and} \quad \int e^{-\varphi} \, dx < +\infty.$$

Define $L = \Delta - \langle \nabla\varphi, \nabla\rangle$ and $D(L)$ his domain in $L^2(\mu_\varphi)$. The properties of L we will use could be find in Bakry [1] and Royer [9]. We recall that $D(L) \subset H_1(\mu_\varphi)$ [9, Theorem 3.1.13] and if f and g belong to $D(L)$, then:

$$\int f Lg \, d\mu_\varphi = -\int \langle \nabla f, \nabla g\rangle \, d\mu_\varphi.$$

We denote P_t the semigroup associated to L. $P_t f(x) = E(f(X_t^x))$ with:

$$X_t^x = x + \sqrt{2} B_t - \int_o^t \nabla\varphi\big(X_s^x\big) \, ds,$$

and B_t a d dimensional Brownian motion issued from 0.

The hypothesis $a(\varphi) \geqslant 0$ allows to prove the existence of $E(f(X_t^x))$ for all t and all x [9, Theorem 2.2.19]. We recall that:

$$D(L) = \left\{ f \in L^2(\mu_\varphi),\ \frac{P_t f - f}{t} \text{ has a limit in } L^2(\mu_\varphi) \text{ when } t \text{ goes to } 0 \right\},$$

$$Lf = \lim_{t \to 0} \frac{P_t f - f}{t}.$$

Furthermore [9, Theorem 2.2.27]:

$$D(L) = \left\{ f \in L^2(\mu_\varphi) \cap W^{2,2}_{\mathrm{loc}}(\mathbb{R}^d),\ \text{in distribution sense } Lf \in L^2(\mu_\varphi) \right\}.$$

We need two results of density.

The following lemma appears in the proof given by Maurey of the result of Ball, Barthe et Naor (proof of Proposition 4.1 in [7]) and it will be useful to prove Theorem 2.

Lemma 3. *Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a C^2 function, not necessarily convex, such that $\int e^{-\phi}\,dx < +\infty$. Define $L = \Delta - \langle \nabla\phi, \nabla \rangle$, then*:

$$\overline{L(C_c^\infty)} = \left\{ f \in L^2(\mu_\phi),\ \int f\,d\mu_\phi = 0 \right\} \quad \textit{for the norm } \| \ \|_{L^2(\mu_\phi)}.$$

On the other hand, for the proof of Theorem 1, we need a result of density in $H_1(\mu_\varphi)$. The first step is the following lemma.

We denote by C_b^k the set of C^k bounded functions as well as all of their derivatives.

Lemma 4. *For f in $L^2(\mu_\varphi)$, we define*:

$$g_t(x) = -\int_0^t P_s f(x)\,ds.$$

Then $g_t \in D(L)$ and:

$$Lg_t = f - P_t f.$$

Furthermore, if we assume $a(\varphi) > 0$ and if $f \in C_b^1$, then $Lg_t \in H_1(\mu_\varphi)$ and Lg_t goes to $f - \int f\,d\mu_\varphi$ in $H_1(\mu_\varphi)$ when t goes to $+\infty$; moreover, $g_t \in C_b^1$.

Proof. The first claim of the lemma is well known (the proof uses some of the arguments used in Lemma 9, see farther).

If we assume $a = a(\varphi) > 0$, we know that $f - P_t f$ goes to $f - \int f\,d\mu_\varphi$ in $L^2(\mu_\varphi)$ when t goes to $+\infty$ (see for example Theorem 3.2.1 of [9]). Moreover, for f in C_b^1, $P_t f \in C_b^1$ (it is easy to prove, for example by studying X_t^x as a function of x), $\nabla Lg_t = \nabla f - \nabla P_t f$ and we have the following inequality [6, Lemma 1.2]:

$$\|\nabla P_t f\|^2 \leqslant e^{-2at} P_t\left(\|\nabla f\|^2\right). \tag{1}$$

Consequently:

$$\int \|\nabla P_t f\|^2\,d\mu_\varphi \leqslant e^{-2at} \int P_t\left(\|\nabla f\|^2\right) d\mu_\varphi \leqslant e^{-2at} \int \|\nabla f\|^2\,d\mu_\varphi,$$

which shows that $\nabla L g_t$ goes to ∇f in $L^2(\mu_\varphi)$ when t goes to $+\infty$. Moreover, $g_t \in C_b^1$ because

$$\|g_t\|_\infty \leqslant \int_0^t \|f\|_\infty \, ds < +\infty$$

and

$$\|\nabla g_t\| \leqslant \int_0^t e^{-as} \sqrt{P_s\big(\|\nabla f\|^2\big)} \, ds,$$

so $\|\nabla g_t\|_\infty < +\infty$. □

Lemma 5. *Let φ be a C^2 function such that $a(\varphi) > 0$ and $\operatorname{Hess}\varphi \in L^2(\mu_\varphi)$. Then, for all f in C_b^1, there exits $(g_n)_{n\in\mathbb{N}}$ in $(C_c^\infty)^{\mathbb{N}}$ such that*:

$$\lim_{n\to+\infty} L g_n = f - \int f \, d\mu_\varphi \quad \textit{for the norm } \| \ \|_{H_1}.$$

Proof. We denote $a = a(\varphi)$.

We use the previous lemma. We define $g_n = -\int_0^n P_s f \, ds$; $g_n \in D(L) \cap C_b^1$ and $L g_n$ goes to $f - \int f \, d\mu_\varphi$ in $H_1(\mu_\varphi)$.

We will prove that $\operatorname{Hess} g_n$ belongs to $L^2(\mu_\varphi)$. For a moment, we forget the subscript n.

For h in C_c^∞, we have (Lemma 7, see farther):

$$\begin{aligned}\int (Lh)^2 \, d\mu_\varphi = Var_{\mu_\varphi}(Lh) &= \int \operatorname{Tr}\big[(\operatorname{Hess} h)^2\big] \, d\mu_\varphi + \int \langle \operatorname{Hess}\varphi \nabla h, \nabla h\rangle \, d\mu_\varphi \\ &\geqslant \int \operatorname{Tr}\big[(\operatorname{Hess} h)^2\big] \, d\mu_\varphi. \end{aligned} \tag{2}$$

C_c^∞ is a dense subset of $D(L)$ for the norm $(\|g\|^2_{L^2(\mu_\varphi)} + \|Lg\|^2_{L^2(\mu_\varphi)})^{1/2}$, consequently, it is possible to find a sequence $h_n \in C_c^\infty$ such that h_n goes to g for this norm. Moreover, $D(L) \subset W^{2,2}_{\mathrm{loc}}(\mathbb{R}^d)$; we deduce from the closed graph theorem, that h_n goes to g in $W^{2,2}_{\mathrm{loc}}(\mathbb{R}^d)$. Then, for all bounded and open subset G of $\mathbb{R}^d$, it is possible to find a subsequence h_{n_k} of h_n such that $\frac{\partial^2 h_{n_k}}{\partial x_i \partial x_j}$ goes to $\frac{\partial^2 g}{\partial x_i \partial x_j}$ almost everywhere on G (when k goes to infinity). Using Fatou's lemma, we obtain:

$$\int (Lg)^2 \, d\mu_\varphi \geqslant \int \operatorname{Tr}\big[(\operatorname{Hess} g)^2\big] \, d\mu_\varphi.$$

We deduce $\operatorname{Hess} g$ belongs to $L^2(\mu_\varphi)$.

So, we have approached in $H_1(\mu_\varphi)$, $f - \int f \, d\mu_\varphi$ with a sequence $L g_n$, where $g_n \in D(L) \cap C_b^1$, $L g_n \in C_b^1$, $\operatorname{Hess} g_n \in L^2(\mu_\varphi)$. To conclude, it is sufficient to approach in $H_1(\mu_\varphi)$, all function Lg, where $g \in D(L) \cap C_b^1$, $Lg \in C_b^1$, $\operatorname{Hess} g \in L^2(\mu_\varphi)$ with a member of $L(C_c^\infty)$.

We fix $\zeta \in C_c^\infty$ such that $0 \leqslant \zeta \leqslant 1$, $\zeta = 1$ on $B(0,1)$, $\zeta = 0$ on $B(0,2)^c$. Let $\zeta_n(x) = \zeta(\frac{x}{n})$. We choose $\rho \in C_c^\infty$, $\rho \geqslant 0$, such that $\int \rho(x)\,dx = 1$ and we define $\rho_q(x) = q^d \rho(qx)$. Define:

$$u_n = \zeta_n g \quad \text{and} \quad u_{n,q} = \rho_q * u_n.$$

We will show $\lim_{n\to+\infty} Lu_n = Lg$ and $\lim_{q\to+\infty} Lu_{n,q} = Lu_n$ in $H_1(\mu_\varphi)$.

- $Lu_n = \zeta_n Lg + gL\zeta_n + 2\langle \nabla g, \nabla \zeta_n\rangle$.

We have, with formula (2):

$$\int (L\zeta_n)^2\,d\mu_\varphi = \int \operatorname{Tr}\big[(\operatorname{Hess}\zeta_n)^2\big]\,d\mu_\varphi + \int \langle \operatorname{Hess}\varphi \nabla\zeta_n, \nabla\zeta_n\rangle\,d\mu_\varphi.$$

Consequently, the property $\operatorname{Hess}\varphi \in L^1(\mu_\varphi)$ allows us to prove:

$$\lim_{n\to+\infty} \int (L\zeta_n)^2\,d\mu_\varphi = 0.$$

So, it is easy to see that Lu_n goes to Lg in $L^2(\mu_\varphi)$. Moreover:

$$\nabla Lu_n = Lg\nabla\zeta_n + \zeta_n\nabla Lg + L\zeta_n\nabla g + g\nabla L\zeta_n + 2\operatorname{Hess} g\nabla\zeta_n + 2\operatorname{Hess}\zeta_n\nabla g,$$

and

$$\nabla L\zeta_n = \nabla\Delta\zeta_n - \operatorname{Hess}\varphi\nabla\zeta_n - \operatorname{Hess}\zeta_n\nabla\varphi.$$

Consequently, we obtain, because $\operatorname{Hess}\varphi \in L^2(\mu_\varphi)$ (and $\nabla\varphi \in L^2(\mu_\varphi)$, see farther Lemma 8):

$$\lim_{n\to+\infty} \int (\nabla L\zeta_n)^2\,d\mu_\varphi = 0.$$

The property $\operatorname{Hess} g \in L^2(\mu_\varphi)$ allows us to prove:

$$\lim_{n\to+\infty} \int \|\operatorname{Hess} g\nabla\zeta_n\|^2\,d\mu_\varphi = 0.$$

Then, we obtain ∇Lu_n goes ∇Lg in $L^2(\mu_\varphi)$, which gives $\lim_{n\to+\infty} Lu_n = Lg$ in $H_1(\mu_\varphi)$.

- $Lu_{n,q} = \rho_q * \Delta u_n - \langle \nabla\varphi, \rho_q * \nabla u_n\rangle$. Using the property that supports of $Lu_{n,q}$ are included in a compact set independent of q, we prove that $Lu_{n,q}$ goes to Lu_n in $L^2(\mu_\varphi)$. In the same way, we obtain that $\nabla Lu_{n,q}$ goes to ∇Lu_n in $L^2(\mu_\varphi)$. □

Corollary 6. *Let φ be a C^2 function such that $a(\varphi) > 0$ and $\operatorname{Hess}\varphi \in L^2(\mu_\varphi)$. Then $L(C_c^\infty)$ is a dense subset of $\{f \in H_1(\mu_\varphi), \int f\,d\mu_\varphi = 0\}$ for the norm $\|\ \|_{H_1}$.*

Proof. It is sufficient to use the density of C_c^∞ in $H_1(\mu_\varphi)$ and to apply the previous lemma. □

2.2. Proofs of Theorems 1 and 2

We begin with the following lemma which contains the main information. In particular, it is easy to see that formula (5) allows us to give a new proof of the inequality of Brascamp and Lieb.

Lemma 7. *Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be a C^2 function, not necessarily convex, such that $\int e^{-\varphi}\,dx < +\infty$. Let $f \in L^2(\mu_\varphi)$ such that there exists $g \in C_c^\infty$ verifying*

$$Lg = f - \int f\,d\mu_\varphi$$

then:

$$Var_{\mu_\varphi}(f) = \int \mathrm{Tr}\big[(\mathrm{Hess}\, g)^2\big] d\mu_\varphi + \int \langle \mathrm{Hess}\,\varphi \nabla g, \nabla g\rangle\, d\mu_\varphi, \tag{3}$$

and

$$Var_{\mu_\varphi}(f) = -\int \big\{2\langle \nabla f, \nabla g\rangle + \mathrm{Tr}\big[(\mathrm{Hess}\, g)^2\big] + \langle \mathrm{Hess}\,\varphi \nabla g, \nabla g\rangle\big\}\, d\mu_\varphi. \tag{4}$$

Furthermore, if we assume $\mathrm{Hess}\,\varphi(x) > 0$ *for all x, then*:

$$\begin{aligned} Var_{\mu_\varphi}(f) &= \int \big\langle (\mathrm{Hess}\,\varphi)^{-1}\nabla f, \nabla f\big\rangle\, d\mu_\varphi \\ &\quad - \int \big\{\mathrm{Tr}\big[(\mathrm{Hess}\, g)^2\big] + \big\|\!\big|\nabla f + \mathrm{Hess}\,\varphi \nabla g\big|\!\big\|^2_{(\mathrm{Hess}\,\varphi)^{-1}}\big\}\, d\mu_\varphi. \end{aligned} \tag{5}$$

Proof.

$$Var_{\mu_\varphi}(f) = \int \left(f - \int f\,d\mu_\varphi\right) Lg\, d\mu_\varphi = -\int \langle \nabla f, \nabla g\rangle\, d\mu_\varphi. \tag{6}$$

With

$$\nabla f = \nabla Lg = L\nabla g - \mathrm{Hess}\,\varphi \nabla g,$$

we obtain:

$$\begin{aligned} & Var_{\mu_\varphi}(f) = -\int \langle \nabla g, L\nabla g\rangle\, d\mu_\varphi + \int \langle \mathrm{Hess}\,\varphi \nabla g, \nabla g\rangle\, d\mu_\varphi \\ \Rightarrow\quad & Var_{\mu_\varphi}(f) = \int \mathrm{Tr}\big[(\mathrm{Hess}\, g)^2\big]\, d\mu_\varphi + \int \langle \mathrm{Hess}\,\varphi \nabla g, \nabla g\rangle\, d\mu_\varphi. \end{aligned} \tag{7}$$

Then we write $Var_{\mu_\varphi}(f) = 2 * (6)–(7)$, consequently:

$$Var_{\mu_\varphi}(f) = -\int \big\{2\langle \nabla f, \nabla g\rangle + \mathrm{Tr}\big[(\mathrm{Hess}\, g)^2\big] + \langle \mathrm{Hess}\,\varphi \nabla g, \nabla g\rangle\big\}\, d\mu_\varphi.$$

If we assume $\operatorname{Hess}\varphi(x) > 0$ for all x, we deduce:

$$
\begin{aligned}
Var_{\mu_\varphi}(f) &= \int \big\langle (\operatorname{Hess}\varphi)^{-1}\nabla f, \nabla f\big\rangle \, d\mu_\varphi \\
&\quad - \int \big\{ \operatorname{Tr}\big[(\operatorname{Hess} g)^2\big] + |||\nabla f + \operatorname{Hess}\varphi \nabla g|||^2_{(\operatorname{Hess}\varphi)^{-1}} \big\} \, d\mu_\varphi. \qquad \square
\end{aligned}
$$

Remark 3. Although formula (5) is the one which will allow us to prove Theorem 1, perhaps it is not the most important formula in the lemma. It seems that the most interesting formula is (4). This formula is valid even if φ is not convex and we will use it to recover results of [3,5] (see Section 3).

Proof of Theorem 1.

First step. We assume that φ is a C^2 function such that $\int e^{-\varphi}\,dx < +\infty$, $\varphi \in L^2(\mu_\varphi)$ and $\operatorname{Hess}\varphi(x) > 0$ for all x. Let f be a function in $L^2(\mu_\varphi)$ such that there exists g in C_c^∞ verifying:

$$
f - \int f \, d\mu_\varphi = Lg
$$

(we will present a regularization procedure for general f at the end of the proof).

The equality (5) gives:

$$
\begin{aligned}
Var_{\mu_\varphi}(f) &= \int \big\langle (\operatorname{Hess}\varphi)^{-1}\nabla f, \nabla f\big\rangle \, d\mu_\varphi \\
&\quad - \int \big\{ \operatorname{Tr}\big[(\operatorname{Hess} g)^2\big] + |||\nabla f + \operatorname{Hess}\varphi \nabla g|||^2_{(\operatorname{Hess}\varphi)^{-1}} \big\} \, d\mu_\varphi.
\end{aligned}
$$

We write:

$$
\int \operatorname{Tr}\big[(\operatorname{Hess} g)^2\big] \, d\mu_\varphi \geqslant \frac{1}{d} \int \big[\operatorname{Tr}(\operatorname{Hess} g)\big]^2 \, d\mu_\varphi \geqslant \frac{1}{d} \left(\int \Delta g \, d\mu_\varphi \right)^2.
$$

On the other hand, $\nabla f + \operatorname{Hess}\varphi \nabla g = L\nabla g$, consequently:

$$
|||\nabla f + \operatorname{Hess}\varphi \nabla g|||^2_{(\operatorname{Hess}\varphi)^{-1}} = |||L\nabla g|||^2_{(\operatorname{Hess}\varphi)^{-1}} \geqslant \frac{1}{b} \|L\nabla g\|^2.
$$

Furthermore:

$$
\begin{aligned}
\int \|L\nabla g\|^2 \, d\mu_\varphi &= \sum_i \int L\left(\frac{\partial g}{\partial x_i}\right) L\left(\frac{\partial g}{\partial x_i}\right) d\mu_\varphi \\
&= -\sum_{i,j} \int \frac{\partial^2 g}{\partial x_i \partial x_j} \frac{\partial}{\partial x_j} L\left(\frac{\partial g}{\partial x_i}\right) d\mu_\varphi \\
&= -\sum_{i,j} \int \frac{\partial^2 g}{\partial x_i \partial x_j} \left[L\left(\frac{\partial^2 g}{\partial x_i \partial x_j}\right) - \sum_k \frac{\partial^2 \varphi}{\partial x_k \partial x_j} \frac{\partial^2 g}{\partial x_k \partial x_i} \right] d\mu_\varphi \\
&= \sum_{i,j,k} \int \left(\frac{\partial^3 g}{\partial x_i \partial x_j \partial x_k} \right)^2 d\mu_\varphi + \int \operatorname{Tr}\big[\operatorname{Hess}\varphi (\operatorname{Hess} g)^2\big] \, d\mu_\varphi
\end{aligned}
$$

$$\Rightarrow \quad \int \|L\nabla g\|^2 \, d\mu_\varphi \geqslant a \int \mathrm{Tr}\big[(\mathrm{Hess}\, g)^2\big] \, d\mu_\varphi \geqslant \frac{a}{d}\bigg(\int \Delta g \, d\mu_\varphi\bigg)^2.$$

Finally:

$$Var_{\mu_\varphi}(f) \leqslant \int \big\langle (\mathrm{Hess}\,\varphi)^{-1}\nabla f, \nabla f\big\rangle \, d\mu_\varphi - \frac{(1+a/b)}{d}\bigg(\int \Delta g \, d\mu_\varphi\bigg)^2.$$

The last term is:

$$\int \Delta g \, d\mu_\varphi = \int \langle \nabla g, \nabla\varphi\rangle \, d\mu_\varphi = -\int \varphi L g \, d\mu_\varphi = -\int \varphi\bigg(f - \int f \, d\mu_\varphi\bigg) d\mu_\varphi.$$

So, Theorem 1 is proved for f in $L^2(\mu_\varphi)$ such that there exists g in C_c^∞ verifying:

$$f - \int f \, d\mu_\varphi = Lg.$$

Second step. Now, we use Corollary 6 to prove the result for all f in $H_1(\mu_\varphi)$. With this aim in view, we need that φ satisfies hypothesis $a(\varphi) > 0$ and $\mathrm{Hess}\,\varphi \in L^2(\mu_\varphi)$. Consequently, we have to give regularization procedures for f and φ.

For the moment, φ only is a C^2 function such that $\int e^{-\varphi} \, dx < +\infty$, $\varphi \in L^2(\mu_\varphi)$ and $\mathrm{Hess}\,\varphi(x) > 0$ for all x; $a = a(\varphi)$ may be equal to 0. Define $\varphi_\varepsilon(x) = \varphi(x) + \varepsilon\frac{\|x\|^2}{2}$ with $\varepsilon > 0$. It is easy to see that $H_1(\mu_\varphi) \subset H_1(\mu_{\varphi_\varepsilon})$. Furthermore, the hypothesis $\varphi \in L^2(\mu_\varphi)$ implies $\varphi \in L^2(\mu_{\varphi_\varepsilon})$ and so $\varphi_\varepsilon \in L^2(\mu_{\varphi_\varepsilon})$ ($\|x\|^2 \in L^2(\mu_{\varphi_\varepsilon})$ because $\mathrm{Hess}\,\varphi_\varepsilon \geqslant \varepsilon I$).

If the result is proved for φ_ε and for all f in $H_1(\mu_{\varphi_\varepsilon})$, we have, for all f in $H_1(\mu_\varphi)$:

$$\begin{aligned} Var_{\mu_{\varphi_\varepsilon}}(f) \leqslant{} & \int \big\langle (\mathrm{Hess}\,\varphi_\varepsilon)^{-1}\nabla f, \nabla f\big\rangle \, d\mu_{\varphi_\varepsilon} \\ & - \frac{(1 + a(\varphi_\varepsilon)/b(\varphi_\varepsilon))}{d}\bigg(\int \varphi_\varepsilon f \, d\mu_{\varphi_\varepsilon} - \int \varphi_\varepsilon \, d\mu_{\varphi_\varepsilon} \int f \, d\mu_{\varphi_\varepsilon}\bigg)^2. \end{aligned}$$

$\mathrm{Hess}\,\varphi_\varepsilon \geqslant \mathrm{Hess}\,\varphi > 0$ so $\langle (\mathrm{Hess}\,\varphi_\varepsilon)^{-1}\nabla f, \nabla f\rangle \leqslant \langle (\mathrm{Hess}\,\varphi)^{-1}\nabla f, \nabla f\rangle$. Moreover:

$$\int \big\langle (\mathrm{Hess}\,\varphi)^{-1}\nabla f, \nabla f\big\rangle \, d\mu_{\varphi_\varepsilon} \leqslant \frac{\int \langle (\mathrm{Hess}\,\varphi)^{-1}\nabla f, \nabla f\rangle \, d\mu_\varphi}{\int \exp(-\varepsilon\frac{\|x\|^2}{2}) \, d\mu_\varphi},$$

consequently

$$\limsup_{\varepsilon\to 0^+} \int \big\langle (\mathrm{Hess}\,\varphi_\varepsilon)^{-1}\nabla f, \nabla f\big\rangle \, d\mu_{\varphi_\varepsilon} \leqslant \int \big\langle (\mathrm{Hess}\,\varphi)^{-1}\nabla f, \nabla f\big\rangle \, d\mu_\varphi.$$

It is easy to prove:

$$\lim_{\varepsilon\to 0^+} Var_{\mu_{\varphi_\varepsilon}}(f) = Var_{\mu_\varphi}(f).$$

Furthermore, $a(\varphi_\varepsilon) \geqslant a(\varphi) + \varepsilon$ and $b(\varphi_\varepsilon) \leqslant b(\varphi) + \varepsilon$. So, the result for φ_ε implies the result for φ.

We work now with a C^2 function φ such that $a > 0$. We know that C_c^∞ is a dense subset of $H_1(\mu_\varphi)$. Let f belongs to $H_1(\mu_\varphi)$ and let f_n belongs to $H_1(\mu_\varphi)$ such that f_n goes to f in $H_1(\mu_\varphi)$. If we assume that the result is proved for f_n, we have:

$$\int \big\langle (\operatorname{Hess}\varphi)^{-1}\nabla(f-f_n), \nabla(f-f_n)\big\rangle \, d\mu_\varphi \leqslant \frac{1}{a}\int \big\|\nabla(f-f_n)\big\|^2 \, d\mu_\varphi. \tag{8}$$

We deduce:

$$\lim_{n\to+\infty}\int \big\langle (\operatorname{Hess}\varphi)^{-1}\nabla f_n, \nabla f_n\big\rangle \, d\mu_\varphi = \int \big\langle (\operatorname{Hess}\varphi)^{-1}\nabla f, \nabla f\big\rangle \, d\mu_\varphi,$$

which enables us to prove the result for f.

So, it is sufficient to assume $f \in C_c^\infty$. To apply Corollary 6, we need that φ verifies $\operatorname{Hess}\varphi \in L^2(\mu_\varphi)$. There are two possibilities. If $b(\varphi) < +\infty$ then this hypothesis is satisfied because:

$$\operatorname{Hess}\varphi(x) \leqslant bI,$$
$$\text{and} \quad e^{-\varphi(x)} \leqslant e^{-\varphi(0)-\langle\nabla\varphi(0),x\rangle - a\|x\|^2/2}.$$

In the case $b(\varphi) = +\infty$, we approach φ by a function which verifies hypothesis of Corollary 6.

Because φ is a C^2 function such that $a > 0$, we see that φ goes to $+\infty$ when x goes to infinity. Consequently, φ has a minimum at a point x_0. We define a new function ψ with $\psi(x) = \varphi(x) - a\|x - x_0\|^2/2$. ψ is a C^2 convex function and has a minimum at x_0. Denote $c = \psi(x_0)(= \varphi(x_0))$. Define:

$$\psi_n(x) = \sup_{\|y\|\leqslant n}\big\{\big\langle x-y, \nabla\psi(y)\big\rangle + \psi(y)\big\}.$$

ψ_n is convex and verifies:

- if $n \geqslant \|x_0\|$, then, for all x: $\psi_n(x) \geqslant c$,
- if $\|x\| \leqslant n$ then $\psi_n(x) = \psi(x)$,
- $\exists(\alpha_n, \beta_n) \in (\mathbb{R}_+)^2$, $\forall x \in \mathbb{R}^d$, $\psi_n(x) \leqslant \alpha_n\|x\| + \beta_n$.

We choose $\rho \in C_c^\infty$, $\rho \geqslant 0$, $\operatorname{Supp}\rho \subset B(0,1)$ such that $\int \rho(x)\,dx = 1$ and we define $\rho_n(x) = n^d\rho(nx)$. Let $\Phi_n = \rho_n * \psi_n + a\|x - x_0\|^2/2$.

$\rho_n * \psi_n$ is a convex and C^∞ function. Furthermore, when $n \geqslant \|x_0\|$, we can find positive real numbers α_n', β_n' such that:

$$\begin{aligned}\big\|\operatorname{Hess}(\rho_n * \psi_n)(x)\big\| = \big\|\operatorname{Hess}\big(\rho_n * (\psi_n - c)\big)(x)\big\| &\leqslant \int \big\|\operatorname{Hess}\rho_n(y)\big\|\big(\psi_n(x-y) - c\big)\,dy \\ &\leqslant \alpha_n'\|x\| + \beta_n'.\end{aligned}$$

Because $\rho_n * \psi_n$ is bounded below by c if $n \geqslant \|x_0\|$, we see, for $n \geqslant \|x_0\|$ and for all $p \geqslant 1$:

$$\int \|\operatorname{Hess}\Phi_n\|^p e^{-\Phi_n}\,dx < +\infty.$$

In particular, $\operatorname{Hess}\Phi_n$ belongs to $L^2(\mu_{\Phi_n})$. With the same argument, it is easy to see that Φ_n belongs to $L^2(\mu_{\Phi_n})$.

Now, we study the convergence of Φ_n. Let K be a compact subset of $\mathbb{R}^d$ and let $N \in \mathbb{N}$ such that $K \subset B(0, N-1)$. For all x in K and for $n \geqslant N$, we have:

$$\rho_n * (\psi_n - \psi)(x) = \int_{\|y\| \leqslant n} \rho_n(x-y)(\psi_n - \psi)(y)\,dy = 0.$$

We deduce that Φ_n and $\operatorname{Hess}\Phi_n$ converge uniformly on every compact set to φ and $\operatorname{Hess}\varphi$ respectively.

Now, assume that we have proved, for all f in C_c^∞:

$$\begin{aligned} Var_{\mu_{\Phi_n}}(f) \leqslant{} & \int \big\langle (\operatorname{Hess}\Phi_n)^{-1}\nabla f, \nabla f\big\rangle\,d\mu_{\Phi_n} \\ & - \frac{(1 + a(\Phi_n)/b(\Phi_n))}{d}\left(\int \Phi_n f\,d\mu_{\Phi_n} - \int \Phi_n\,d\mu_{\Phi_n}\int f\,d\mu_{\Phi_n}\right)^2. \end{aligned}$$

Firstly, we remark $a(\Phi_n)/b(\Phi_n) \geqslant 0 = a(\varphi)/b(\varphi)$.

Define $d\upsilon = d\mu_{a\|x-x_0\|^2/2}$.

We obtain with the dominated convergence theorem (and because $\exp(-\rho_n * \psi_n(x)) \leqslant \exp(-c)$ for $n \geqslant \|x_0\|$):

$$\lim_{n\to+\infty}\int \exp\big(-\rho_n * \psi_n(x)\big)\,d\upsilon(x) = \int \exp\big(-\psi(x)\big)\,d\upsilon(x).$$

Then, we have, because $(\operatorname{Hess}\Phi_n)^{-1} \leqslant \frac{1}{a}I$ and because f has compact support:

$$\lim_{n\to+\infty} Var_{\mu_{\Phi_n}}(f) = Var_{\mu_\varphi}(f),$$

$$\lim_{n\to+\infty}\int f\,d\mu_{\Phi_n} = \int f\,d\mu_\varphi, \qquad \lim_{n\to+\infty}\int f\Phi_n\,d\mu_{\Phi_n} = \int f\varphi\,d\mu_\varphi,$$

$$\lim_{n\to+\infty}\int \big\langle (\operatorname{Hess}\Phi_n)^{-1}\nabla f, \nabla f\big\rangle\,d\mu_{\Phi_n} = \int \big\langle (\operatorname{Hess}\varphi)^{-1}\nabla f, \nabla f\big\rangle\,d\mu_\varphi.$$

Now, we have to study $\int \Phi_n\,d\mu_{\Phi_n}$. We write:

$$\int \Phi_n\,d\mu_{\Phi_n} = \frac{\int \Phi_n \exp(-\rho_n * \psi_n)\,d\upsilon}{\int \exp(-\rho_n * \psi_n)\,d\upsilon}.$$

We have to show that: $\lim_{n\to+\infty}\int \Phi_n \exp(-\rho_n * \psi_n)\,d\upsilon = \int \varphi\exp(-\psi)\,d\upsilon$. Again, we prove it with the dominated convergence theorem and with the following inequality (for $n \geqslant \|x_0\|$):

$$0 \leqslant \big(\rho_n * \psi_n(x) - c + 1\big)\exp\big(-\rho_n * \psi_n(x) + c - 1\big) \leqslant e^{-1}.$$

So, the result for Φ_n and for all f in C_c^∞ implies the result for φ.

Consequently, it is sufficient to prove the result for a C^∞ function φ, verifying $a > 0$, $\operatorname{Hess}\varphi \in L^2(\mu_\varphi)$, $\varphi \in L^2(\mu_\varphi)$ and for $f \in C_c^\infty$.

With the help of an argument already used (inequality (8)) and Corollary 6, we see that it is possible to assume there exists $g \in C_c^\infty$ such that $Lg = f - \int f\, d\mu_\varphi$.

The proof of Theorem 1 is achieved. $\square$

Now, we give the proof of Theorem 2. We begin with a lemma.

Lemma 8. *Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be a C^2 function such that $\int e^{-\varphi}\, dx < +\infty$ and* $\operatorname{Hess}\varphi \in L^1(\mu_\varphi)$. *Then $\nabla\varphi$ belongs to $L^2(\mu_\varphi)$.*

Proof. Again, we choose $\zeta \in C_c^\infty$ such that $0 \leqslant \zeta \leqslant 1$, $\zeta = 1$ on $B(0,1)$, $\zeta = 0$ on $B(0,2)^c$. We define for $r \geqslant 1$: $\zeta_r(x) = \zeta(\frac{x}{r})$. We have ($c$ is a constant independent of r):

$$\begin{aligned}
\int \zeta_r \|\nabla\varphi\|^2\, d\mu_\varphi &= -\int \zeta_r \bigl\langle \nabla\varphi, \nabla\bigl(e^{-\varphi}\bigr)\bigr\rangle\, dx \frac{1}{\int e^{-\varphi}\, dx} \\
&= \int \bigl[\langle \nabla\zeta_r, \nabla\varphi\rangle + \zeta_r \Delta\varphi\bigr]\, d\mu_\varphi \\
&= \int [-L\zeta_r + \Delta\zeta_r + \zeta_r \Delta\varphi]\, d\mu_\varphi \\
&= \int [\Delta\zeta_r + \zeta_r \Delta\varphi]\, d\mu_\varphi,
\end{aligned}$$

consequently:

$$\int \zeta_r \|\nabla\varphi\|^2\, d\mu_\varphi \leqslant c + \int \|\operatorname{Hess}\varphi\|\, d\mu_\varphi.$$

We deduce that $\nabla\varphi$ belongs to $L^2(\mu_\varphi)$. $\square$

Proof of Theorem 2. We consider here a C^2 function φ such that $\varphi \in L^2(\mu_\varphi)$ and $\operatorname{Hess}\varphi \in L^1(\mu_\varphi)$. We see with Lemma 3, that it is possible to assume there exists $g \in C_c^\infty$ such that $Lg = f - \int f\, d\mu_\varphi$. Equality (3) gives:

$$Var_{\mu_\varphi}(f) = \int \operatorname{Tr}\bigl[(\operatorname{Hess} g)^2\bigr]\, d\mu_\varphi + \int |||\nabla g|||^2_{\operatorname{Hess}\varphi}\, d\mu_\varphi.$$

As in the proof of Theorem 1, we obtain:

$$\int \operatorname{Tr}\bigl[(\operatorname{Hess} g)^2\bigr]\, d\mu_\varphi \geqslant \frac{1}{d} c_\varphi(f)^2.$$

Define $U = \int \nabla f\, d\mu_\varphi$ and $\theta(x) = g(x) + \langle (\int \operatorname{Hess}\varphi\, d\mu_\varphi)^{-1} U, x\rangle$. We have $\nabla\theta = \nabla g + (\int \operatorname{Hess}\varphi\, d\mu_\varphi)^{-1} U$, consequently:

$$\|\nabla g\|_{\operatorname{Hess}\varphi}^2 = \|\nabla\theta\|_{\operatorname{Hess}\varphi}^2 + \left\|\left(\int \operatorname{Hess}\varphi\, d\mu_\varphi\right)^{-1} U\right\|_{\operatorname{Hess}\varphi}^2 - 2\left\langle \operatorname{Hess}\varphi\nabla\theta, \left(\int \operatorname{Hess}\varphi\, d\mu_\varphi\right)^{-1} U\right\rangle.$$

Moreover:

$$\int \operatorname{Hess}\varphi\nabla\theta\, d\mu_\varphi = \int \operatorname{Hess}\varphi\nabla g\, d\mu_\varphi + U = \int \operatorname{Hess}\varphi\nabla g + \nabla f\, d\mu_\varphi = \int L\nabla g\, d\mu_\varphi = 0.$$

So:

$$\begin{aligned} Var_{\mu_\varphi}(f) &\geqslant \frac{1}{d}c_\varphi(f)^2 + \int \left\|\left(\int \operatorname{Hess}\varphi\, d\mu_\varphi\right)^{-1} U\right\|_{\operatorname{Hess}\varphi}^2 d\mu_\varphi \\ &\geqslant \frac{1}{d}c_\varphi(f)^2 + \left\|\int \nabla f\, d\mu_\varphi\right\|_{(\int \operatorname{Hess}\varphi\, d\mu_\varphi)^{-1}}^2 \\ &\geqslant \frac{1}{d}c_\varphi(f)^2 + \left\|\int f\nabla\varphi\, d\mu_\varphi\right\|_{(\int \operatorname{Hess}\varphi\, d\mu_\varphi)^{-1}}^2. \qquad \square \end{aligned}$$

Example 1. We fix h in $\mathbb{R}^d$, we assume x belongs to $L^2(\mu_\varphi)$ and we apply Theorems 1 and 2 to $f(x) = \langle h, x\rangle$. We use the same notation for $y \in \mathbb{R}^d$ and for the column

$$\begin{pmatrix} y_1 \\ \vdots \\ y_d \end{pmatrix}$$

of his components in the canonical basis of $\mathbb{R}^d$. We find:

$$Var_{\mu_\varphi}(f) = \langle Mh, h\rangle - \langle Y, h\rangle^2 \quad \text{where } M = \int xx^*\, d\mu_\varphi \text{ and } Y = \int x\, d\mu_\varphi,$$

$$c_\varphi(f) = \langle Z, h\rangle \quad \text{where } Z = \int x\varphi\, d\mu_\varphi - \int \varphi\, d\mu_\varphi \int x\, d\mu_\varphi.$$

We obtain:

$$\left(\int \operatorname{Hess}\varphi\, d\mu_\varphi\right)^{-1} + \frac{1}{d}ZZ^* \leqslant M - YY^* \leqslant \int (\operatorname{Hess}\varphi)^{-1}\, d\mu_\varphi - \frac{1+a/b}{d}ZZ^*,$$

which is a generalization of Example 1, p. 379 of [4].

If $\varphi(x) = \|x - x_0\|_A^2/2$, then the two inequalities are equalities and $M - YY^* = A^{-1}$.

2.3. Improvement of the lower bound of the variance

The purpose of this section is to obtain a lower bound for the variance which recovers the case of the Gaussian measure. We first prove that L is a surjective mapping on $\{f \in L^2(\mu_\varphi), \int f\,d\mu_\varphi = 0\}$ if $a(\varphi) > 0$.

Lemma 9. *We assume $a(\varphi) > 0$. For f in $L^2(\mu_\varphi)$, we define*:

$$g(x) = -\int_0^{+\infty} \left(P_t f(x) - \int f\,d\mu_\varphi\right) dt.$$

Then, g exits almost surely, $g \in D(L)$ and we have:

$$Lg = f - \int f\,d\mu_\varphi.$$

Proof. We denote $a = a(\varphi)$.

The Poincaré inequality implies (see for example [9, théorème 3.2.1]):

$$\left\| P_t f - \int f\,d\mu_\varphi \right\|_{L^2(\mu_\varphi)} \leqslant e^{-at} \left\| f - \int f\,d\mu_\varphi \right\|_{L^2(\mu_\varphi)}. \tag{9}$$

Define $d\sigma(s) = a\exp(-as)\,ds$.

We have:

$$\begin{aligned}
\int \left[\int_0^\infty \left| P_s f(x) - \int f\,d\mu_\varphi \right| ds\right]^2 d\mu_\varphi &= \frac{1}{a^2}\int \left[\int_0^\infty e^{as}\left| P_s f(x) - \int f\,d\mu_\varphi \right| d\sigma(s)\right]^2 d\mu_\varphi \\
&\leqslant \frac{1}{a^2}\int \left[\int_0^\infty e^{2as}\left| P_s f(x) - \int f\,d\mu_\varphi \right|^2 d\sigma(s)\right] d\mu_\varphi \\
&\leqslant \frac{1}{a^2}\left\| f - \int f\,d\mu_\varphi \right\|^2_{L^2(\mu_\varphi)}.
\end{aligned}$$

Consequently, g is well defined and belongs to $L^2(\mu_\varphi)$. Moreover:

$$\mathcal{F} : L^2(\mu_\varphi) \to L^2(\mu_\varphi),$$
$$f \mapsto g$$

is a linear and continuous map on $L^2(\mu_\varphi)$.

$P_s g(x) = E[-\int_0^{+\infty}(P_t f(X_s^x) - \int f\,d\mu_\varphi)\,dt] = -\int_0^{+\infty}(P_{t+s} f(x) - \int f\,d\mu_\varphi)\,dt$ (we can commute the integrals thanks to (9)), so:

$$P_s g = \mathcal{F}(P_s f).$$

Furthermore, using the spectral decomposition of L, we see that $P_s g$ and $P_s f$ belongs to $D(L)$ for $s > 0$; consequently, because $\mathcal{F}$ is a continuous map:

$$L P_s g = \mathcal{F}(L P_s f) = -\int_0^{+\infty} L P_{t+s} f\,dt = P_s f - \int f\,d\mu_\varphi \quad \text{for } s > 0.$$

We obtain:

$$\lim_{s\to 0} P_s g = g \quad \bigl(\text{in } L^2(\mu_\varphi)\bigr),$$
$$\lim_{s\to 0} L P_s g = f - \int f\,d\mu_\varphi \quad \bigl(\text{in } L^2(\mu_\varphi)\bigr).$$

We deduce, because L is a closed operator: $g \in D(L)$ and $Lg = f - \int f\,d\mu$. $\square$

We assume $a(\varphi) > 0$. We define $\varphi_1 = \varphi$. Using the previous lemma, we see that, for all $p \geqslant 1$, there exists φ_{p+1} in $D(L)$ such that:

$$L\varphi_{p+1} = \varphi_p - \int \varphi_p\,d\mu_\varphi$$

(φ_{p+1} is defined up to a constant).

For $f \in L^2(\mu_\varphi)$, we define:

$$c_p(f) = \int \varphi_p f\,d\mu_\varphi - \int \varphi_p\,d\mu_\varphi \int f\,d\mu_\varphi \quad \bigl(c_1(f) = c_\varphi(f)\bigr).$$

And if $\operatorname{Hess}\varphi \in L^1(\mu_\varphi)$, we define:

$$V(f) = \left(\int \operatorname{Hess}\varphi\,d\mu_\varphi\right)^{-1} \int f\nabla\varphi\,d\mu_\varphi.$$

Theorem 10. *If $a = a(\varphi) > 0$, $\varphi \in L^2(\mu_\varphi)$ and if $\operatorname{Hess}\varphi \in L^1(\mu_\varphi)$, then, for all f in $L^2(\mu_\varphi)$:*

$$Var_{\mu_\varphi}(f) \geqslant \frac{1}{d}\left(c_1(f)^2 + a^2\left(2 + \frac{a}{b}\right)\sum_{k=0}^{+\infty} a^{2k} c_{k+2}\bigl[f - \bigl\langle V(f), \nabla\varphi\bigr\rangle\bigr]^2\right) + \left\|\int f\nabla\varphi\,d\mu_\varphi\right\|^2_{(\int \operatorname{Hess}\varphi\,d\mu_\varphi)^{-1}}.$$

Remark 4. If φ is an even function, then for all f in $L^2(\mu_\varphi)$, $c_{k+2}[f - \langle V(f), \nabla\varphi\rangle] = c_{k+2}[f]$. Actually, because φ is even, we have:

$$X_t^x = x + \sqrt{2}B_t - \int_0^t \nabla\varphi\big(X_s^x\big)\,ds$$

$$\Rightarrow \quad -X_t^x = -x - \sqrt{2}B_t - \int_0^t \nabla\varphi\big(-X_s^x\big)\,ds.$$

So: $-X_t^x \overset{\text{law}}{\sim} X_t^{-x}$. We deduce, for all even function h:

$$P_t h(-x) = E\big(h\big(X_t^{-x}\big)\big) = E\big(h\big(-X_t^x\big)\big) = P_t h(x).$$

Using the formula of Lemma 9, we find, for all $p \geqslant 1$, that φ_p is an even function. Consequently, $c_{k+2}[\frac{\partial\varphi}{\partial x_i}] = 0$ and $c_{k+2}[f - \langle V(f), \nabla\varphi\rangle] = c_{k+2}[f]$.

Example 2. If we apply Theorem 10 to $\varphi(x) = \|x\|^2/2$, we recover the lower bound given in the introduction for the Gaussian measure. Actually, we have:

$$\varphi_p = \left(-\frac{1}{2}\right)^{p-1}\varphi,$$

so $c_p[f - \langle V(f), \nabla\varphi\rangle] = c_p[f] = (-\frac{1}{2})^{p-1}c_\varphi(f)$. We deduce:

$$\begin{aligned} c_1(f)^2 + a^2\left(2 + \frac{a}{b}\right)\sum_{k=0}^{+\infty} a^{2k} c_{k+2}\big[f - \big\langle V(f), \nabla\varphi\big\rangle\big]^2 &= c_\varphi(f)^2 + 3\sum_{k=0}^{+\infty}\left(-\frac{1}{2}\right)^{2k+2} c_\varphi(f)^2 \\ &= 2c_\varphi(f)^2 \\ &= \frac{1}{2}\left(\int \Delta f\,d\mu\right)^2. \end{aligned}$$

Lemma 11. *If $a > 0$ and $\varphi \in L^2(\mu_\varphi)$, then, for all $p \geqslant 1$ and for all f in $L^2(\mu_\varphi)$,*

$$Var_{\mu_\varphi}(f) \geqslant \frac{1}{d}\left(c_1(f)^2 + \left(2 + \frac{a}{b}\right)\sum_{k=2}^{p} a^{2(k-1)} c_k(f)^2\right).$$

Remark 5. It is possible to compare result of Lemma 11 and result of Theorem 10 in the following way. In the case φ is an even function, we obtain, using Remark 4 and Theorem 10:

$$Var_{\mu_\varphi}(f) \geqslant \frac{1}{d}\left(c_1(f)^2 + a^2\left(2 + \frac{a}{b}\right)\sum_{k=0}^{+\infty} a^{2k} c_{k+2}(f)^2\right) + \left\|\int f\nabla\varphi\,d\mu_\varphi\right\|^2_{(\int \mathrm{Hess}\,\varphi\,d\mu_\varphi)^{-1}},$$

whereas Lemma 11 only gives:

$$Var_{\mu_\varphi}(f) \geqslant \frac{1}{d}\left(c_1(f)^2 + \left(2+\frac{a}{b}\right)\sum_{k=2}^{+\infty} a^{2(k-1)} c_k(f)^2\right)$$
$$\geqslant \frac{1}{d}\left(c_1(f)^2 + a^2\left(2+\frac{a}{b}\right)\sum_{k=0}^{+\infty} a^{2k} c_{k+2}(f)^2\right).$$

Proof of Lemma 11. We proceed by induction on p. The case $p=1$ follows from Theorem 2. Assume that the lemma is proved for $p \geqslant 1$. Using Lemma 3, we can assume there exists $g \in C_c^\infty$ such that $Lg = f - \int f\, d\mu_\varphi$. As in the proof of Theorem 2, we write:

$$Var_{\mu_\varphi}(f) \geqslant \frac{1}{d} c_1(f)^2 + \int |||\nabla g|||^2_{\mathrm{Hess}\varphi}\, d\mu_\varphi.$$

Theorem 1 gives:

$$\int |||\nabla g|||^2_{\mathrm{Hess}\,\varphi}\, d\mu_\varphi \geqslant a^2 \int |||\nabla g|||^2_{(\mathrm{Hess}\,\varphi)^{-1}}\, d\mu_\varphi \geqslant a^2\left(Var_{\mu_\varphi}(g) + \frac{1}{d}\left(1+\frac{a}{b}\right) c_1(g)^2\right).$$

By induction:

$$Var_{\mu_\varphi}(g) \geqslant \frac{1}{d}\left(c_1(g)^2 + \left(2+\frac{a}{b}\right)\sum_{k=2}^{p} a^{2(k-1)} c_k(g)^2\right).$$

Then:

$$Var_{\mu_\varphi}(f) \geqslant \frac{1}{d}\left(c_1(f)^2 + a^2 c_1(g)^2 + a^2\left(2+\frac{a}{b}\right)\sum_{k=2}^{p} a^{2(k-1)} c_k(g)^2 + a^2\left(1+\frac{a}{b}\right)c_1(g)^2\right)$$
$$\geqslant \frac{1}{d}\left(c_1(f)^2 + \left(2+\frac{a}{b}\right)\sum_{k=2}^{p+1} a^{2(k-1)} c_{k-1}(g)^2\right).$$

Moreover:

$$c_{k-1}(g) = \int \left(\varphi_{k-1} - \int \varphi_{k-1}\, d\mu_\varphi\right) g\, d\mu_\varphi = \int L(\varphi_k) g\, d\mu_\varphi = \int \varphi_k L g\, d\mu_\varphi = c_k(f), \tag{10}$$

which allows us to conclude. □

Proof of Theorem 10. Applying Lemma 11, we obtain, for all f in $L^2(\mu_\varphi)$:

$$Var_{\mu_\varphi}(f) \geqslant \frac{1}{d}\left(c_1(f)^2 + \left(2+\frac{a}{b}\right)\sum_{k=2}^{\infty} a^{2(k-1)} c_k(f)^2\right). \tag{11}$$

In particular, this series converges and it also converges for $f - \langle V(f), \nabla\varphi\rangle$ instead of f because $f - \langle V(f), \nabla\varphi\rangle \in L^2(\mu_\varphi)$.

Again with Lemma 3, it is possible to assume there exists $g \in C_c^\infty$ such that $Lg = f - \int f\,d\mu_\varphi$. As in the proof of Theorem 2, we obtain:

$$Var_{\mu_\varphi}(f) \geqslant \frac{1}{d}c_\varphi(f)^2 + \int |||\nabla g|||^2_{\operatorname{Hess}\varphi}\,d\mu_\varphi.$$

Again, we define $U = \int \nabla f\,d\mu_\varphi$ and $\theta(x) = g(x) + \langle (\int \operatorname{Hess}\varphi\,d\mu_\varphi)^{-1}U, x\rangle$. Then:

$$\begin{aligned} Var_{\mu_\varphi}(f) &\geqslant \frac{1}{d}c_\varphi(f)^2 + \int |||\nabla\theta|||^2_{\operatorname{Hess}\varphi}\,d\mu_\varphi + \int \left|\left|\left|\left(\int \operatorname{Hess}\varphi\,d\mu_\varphi\right)^{-1}U\right|\right|\right|^2_{\operatorname{Hess}\varphi}d\mu_\varphi \\ &\geqslant \frac{1}{d}c_\varphi(f)^2 + \int |||\nabla\theta|||^2_{\operatorname{Hess}\varphi}\,d\mu_\varphi + \left|\left|\left|\int \nabla f\,d\mu_\varphi\right|\right|\right|^2_{(\int \operatorname{Hess}\varphi\,d\mu_\varphi)^{-1}}. \end{aligned}$$

Theorem 1 applied to θ gives:

$$\int |||\nabla\theta|||^2_{\operatorname{Hess}\varphi}d\mu_\varphi \geqslant a^2\int |||\nabla\theta|||^2_{(\operatorname{Hess}\varphi)^{-1}}\,d\mu_\varphi \geqslant a^2\left(Var_{\mu_\varphi}(\theta) + \frac{1}{d}\left(1+\frac{a}{b}\right)c_1(\theta)^2\right),$$

and using inequality (11):

$$\int |||\nabla\theta|||^2_{\operatorname{Hess}\varphi}\,d\mu_\varphi \geqslant \frac{a^2}{d}\left(2+\frac{a}{b}\right)\left(c_1(\theta)^2 + a^2\sum_{k=0}^{\infty}a^{2k}c_{k+2}(\theta)^2\right).$$

Moreover, with $\theta = g + \langle V(f), x\rangle$ and with (10), we obtain:

$$\begin{aligned} c_{k+2}(\theta) &= c_{k+2}(g) + c_{k+2}\big(\langle V(f), x\rangle\big) \\ &= c_{k+3}(f) + c_{k+2}\big(\langle V(f), x\rangle\big). \end{aligned}$$

We have $Lx_i = -\frac{\partial\varphi}{\partial x_i}$ ($x_i \in D(L)$ because $\nabla\varphi \in L^2(\mu_\varphi)$), consequently, with (10), $c_{k+2}(x_i) = -c_{k+3}(\frac{\partial\varphi}{\partial x_i})$. We deduce:

$$c_{k+2}(\theta) = c_{k+3}(f) - c_{k+3}\big(\langle V(f), \nabla\varphi\rangle\big) = c_{k+3}\big[f - \langle V(f), \nabla\varphi\rangle\big],$$

which enables us to conclude. □

We finish this paragraph by given another formulation for Theorem 10 under an additional hypothesis. In the case φ belongs to $L^2(\mu_\varphi)$ and $a = a(\varphi) > 0$, we denote $\mathcal{H}$ the following hypothesis:

$$\exists\beta \in L^1\big(\mathbb{R}_+, \mathbb{R}_+^*\big),\ \forall t \geqslant 0, \qquad \left\|P_t\varphi - \int \varphi\,d\mu_\varphi\right\|_{L^2(\mu_\varphi)} \leqslant e^{-at}\beta(at). \tag{$\mathcal{H}$}$$

Let us remark that inequality (9) applied to φ only gives:

$$\left\|P_t\varphi - \int \varphi\,d\mu_\varphi\right\|_{L^2(\mu_\varphi)} \leqslant e^{-at}\left\|\varphi - \int \varphi\,d\mu_\varphi\right\|_{L^2(\mu_\varphi)}.$$

Remark 6. This hypothesis is satisfied by the Gaussian measure. For $\varphi(x) = \|x\|^2/2$, we have*:*

$$\begin{aligned}
P_t\varphi(x) &= \int \varphi\big(e^{-t}x + \sqrt{1-e^{-2t}}\,y\big)\,d\mu(y)\\
&= e^{-2t}\left(\frac{\|x\|^2}{2} - \frac{d}{2}\right) + \frac{d}{2}\\
&= e^{-2t}\left(\varphi(x) - \int \varphi\,d\mu\right) + \int \varphi\,d\mu\\
\Rightarrow \quad & \left\|P_t\varphi - \int \varphi\,d\mu_\varphi\right\|_{L^2(\mu)} = e^{-2t}\left\|\varphi - \int \varphi\,d\mu_\varphi\right\|_{L^2(\mu)}.
\end{aligned}$$

Remark 7. We will see later that any even function φ (such that $a(\varphi) > 0$ and $\varphi \in L^2(\mu_\varphi)$) satisfies $\mathcal{H}$ with $\beta(t) = \exp(-t)$ (see Corollary 17).

Define $\alpha(t) = \sum_{k=0}^{\infty} \frac{t^k}{(k!)^2}$ and:

$$K_\varphi(x,y) = \iint_{(\mathbb{R}_+)^2} \alpha\big(a^2uv\big)\left(P_u\varphi(x) - \int \varphi\,d\mu_\varphi\right)\left(P_v\varphi(y) - \int \varphi\,d\mu_\varphi\right)du\,dv.$$

Hypothesis $\mathcal{H}$ implies $K_\varphi \in L^2(\mu_\varphi \otimes \mu_\varphi)$. Actually, define:

$$d\eta(u,v) = \alpha\big(a^2uv\big)e^{-au}\beta(au)e^{-av}\beta(av)\,du\,dv.$$

We easily see with $\alpha(uv) \leqslant e^u e^v$ (for $u \geqslant 0$ and $v \geqslant 0$) that $\int_{(\mathbb{R}_+)^2} d\eta(u,v) < +\infty$ (because $a > 0$). So:

$$\begin{aligned}
&\iint K_\varphi(x,y)^2\,d\mu_\varphi(x)\,d\mu_\varphi(y)\\
&\quad= \iint \Bigg(\iint_{(\mathbb{R}_+)^2} e^{au}\beta(au)^{-1}\left(P_u\varphi(x) - \int \varphi\,d\mu_\varphi\right)\\
&\qquad\times e^{av}\beta(av)^{-1}\left(P_v\varphi(y) - \int \varphi\,d\mu_\varphi\right)d\eta(u,v)\Bigg)^2\,d\mu_\varphi(x)\,d\mu_\varphi(y)\\
&\quad\leqslant \iint_{(\mathbb{R}_+)^2} e^{2au}\beta(au)^{-2}\left\|P_u\varphi - \int \varphi\,d\mu_\varphi\right\|^2_{L^2(\mu_\varphi)} e^{2av}\beta(av)^{-2}\\
&\qquad\times \left\|P_v\varphi - \int \varphi\,d\mu_\varphi\right\|^2_{L^2(\mu_\varphi)} d\eta(u,v) \iint_{(\mathbb{R}_+)^2} d\eta(u,v)\\
&\quad< +\infty.
\end{aligned}$$

We define for f and g in $L^2(\mu_\varphi)$:

$$\langle f, g\rangle_{K_\varphi} = \iint K_\varphi(x,y) f(x) g(y)\, d\mu_\varphi(x)\, d\mu_\varphi(y),$$
$$\|f\|_{K_\varphi} = \sqrt{\langle f, f\rangle_{K_\varphi}},$$

(in the proof of the following theorem, we will see that $\langle f, f\rangle_{K_\varphi} \geqslant 0$ for all f). We obtain:

Theorem 12. *If $a > 0$, $\varphi \in L^2(\mu_\varphi)$, $\operatorname{Hess}\varphi \in L^1(\mu_\varphi)$ and if φ satisfies $\mathcal{H}$ then, for all f in $L^2(\mu_\varphi)$:*

$$Var_{\mu_\varphi}(f) \geqslant \frac{1}{d}\left(c_\varphi(f)^2 + a^2\left(2+\frac{a}{b}\right)\left\|f - \langle V(f), \nabla\varphi\rangle\right\|^2_{K_\varphi}\right) + \left\|\left|\int f \nabla\varphi\, d\mu_\varphi\right\|\right|^2_{(\int \operatorname{Hess}\varphi\, d\mu_\varphi)^{-1}}.$$

By Theorem 10, it is sufficient to prove:

Lemma 13.

$$\forall f \in L^2(\mu_\varphi), \quad \sum_{k=0}^{+\infty} a^{2k} c_{k+2}(f)^2 = \|f\|^2_{K_\varphi}.$$

Proof. We rewrite coefficients $c_{k+2}(f)$ with the semigroup P_t. We easily see that, for all f in $L^2(\mu_\varphi)$ and for $p \geqslant 2$:

$$\frac{dc_p(P_t f)}{dt} = c_{p-1}(P_t f) \quad \text{and} \quad \lim_{t\to+\infty} c_p(P_t f) = 0.$$

Inequality (9) gives:

$$\left|c_1(P_s f)\right| \leqslant e^{-as}\left\|f - \int f\, d\mu_\varphi\right\|_{L^2(\mu_\varphi)} \|\varphi\|_{L^2(\mu_\varphi)}.$$

We deduce:

$$c_p(P_t f) = (-1)^{p-1}\int_t^{+\infty} \frac{(s-t)^{p-2}}{(p-2)!} c_1(P_s f)\, ds.$$

Consequently:

$$\begin{aligned}\sum_{k=0}^{p} a^{2k} c_{k+2}(f)^2 &= \sum_{k=0}^{p} a^{2k}\left(\int_0^{+\infty} \frac{s^k}{k!} c_1(P_s f)\, ds\right)^2 \\ &= \sum_{k=0}^{p} a^{2k}\left(\int_0^{+\infty} \frac{u^k}{k!} c_1(P_u f)\, du\right)\left(\int_0^{+\infty} \frac{v^k}{k!} c_1(P_v f)\, dv\right)\end{aligned}$$

$$= \iint_{(\mathbb{R}_+)^2} \sum_{k=0}^{p} \frac{(a^2uv)^k}{(k!)^2} c_\varphi(P_u f) c_\varphi(P_v f)\, du\, dv.$$

Then, with hypothesis $\mathcal{H}$:

$$\begin{aligned}\left|c_\varphi(P_u f)\right| &= \left| \int \left(P_u\varphi - \int \varphi\, d\mu_\varphi\right) f\, d\mu_\varphi \right| \\ &\leqslant e^{-au}\beta(au)\|f\|_{L^2(\mu_\varphi)}.\end{aligned}$$

So:

$$\left| \sum_{k=0}^{p} \frac{(a^2uv)^k}{(k!)^2} c_\varphi(P_u f) c_\varphi(P_v f) \right| \leqslant \alpha(a^2uv) e^{-au}\beta(au) e^{-av}\beta(av)\|f\|^2_{L^2(\mu_\varphi)}.$$

We deduce, with the dominated convergence theorem:

$$\begin{aligned}\sum_{k=0}^{+\infty} a^{2k} c_{k+2}(f)^2 &= \iint_{(\mathbb{R}_+)^2} \alpha\big(a^2uv\big) c_\varphi(P_u f) c_\varphi(P_v f)\, du\, dv \\ &= \iint \iint_{(\mathbb{R}_+)^2} \alpha\big(a^2uv\big)\left(P_u\varphi(x) - \int \varphi\, d\mu_\varphi\right) \\ &\quad \times \left(P_u\varphi(y) - \int \varphi\, d\mu_\varphi\right) f(x) f(y)\, du\, dv\, d\mu_\varphi(x)\, d\mu_\varphi(y) \\ &= \|f\|^2_{K_\varphi}. \qquad \square\end{aligned}$$

Remark 8. It is an open question to give an interpretation of operator K_φ. Nevertheless, it is easy to see that α is a solution of the following equation:

$$t\alpha'' + \alpha' = \alpha$$

($J(t) = \alpha(-\frac{t^2}{4})$ is a Bessel function).

Example 3. For $\varphi(x) = \|x\|^2/2$:

$$\begin{aligned}K_\varphi(x, y) &= \frac{1}{3}\left(\varphi(x) - \int \varphi\, d\mu_\varphi\right)\left(\varphi(y) - \int \varphi\, d\mu_\varphi\right), \\ \|f\|^2_{K_\varphi} &= \frac{1}{3} c_\varphi(f)^2.\end{aligned}$$

3. Links with known results

3.1. Formula of Ball, Barthe and Naor

The purpose of this section is to establish a link between Lemma 7 (equality (4)) and a formula of Ball, Barthe and Naor [3] concerning the second derivative of Φ defined by:

$$e^{-\Phi(y)} = \int e^{-\varphi(x,y)}\,dx$$

where φ is a (not necessarily convex) function defined on $\mathbb{R}^d \times \mathbb{R}^p$.

We denote

$$d\upsilon_y(x) = \frac{e^{-\varphi(x,y)}\,dx}{\int e^{-\varphi(z,y)}\,dz}.$$

We fix $h \in \mathbb{R}^p$. With appropriate hypothesis on φ, we obtain:

$$\begin{aligned}
&\nabla\Phi(y) = \int \nabla_y\varphi\,d\upsilon_y(x),\\
&\operatorname{Hess}\Phi(y) = \nabla\Phi(y)\big(\nabla\Phi(y)\big)^* + \int\big[\operatorname{Hess}_y\varphi - \nabla_y\varphi(\nabla_y\varphi)^*\big]\,d\upsilon_y(x)\\
&\quad\Rightarrow\quad \big\langle\operatorname{Hess}\Phi(y)h, h\big\rangle = \int\langle\operatorname{Hess}_y\varphi h, h\rangle\,d\upsilon_y(x) - Var_{\upsilon_y}\big(\langle\nabla_y\varphi, h\rangle\big).
\end{aligned}\tag{12}$$

Denote:

$$\operatorname{Hess}\varphi = \begin{pmatrix} H_1 & H_3 \\ H_3^* & H_2 \end{pmatrix} \quad \text{where } H_1 = \operatorname{Hess}_x\varphi \text{ and } H_2 = \operatorname{Hess}_y\varphi.$$

The result of Ball, Barthe and Naor is the following (approximately as it is given in [7]).

Theorem 14. *We fix y and h in $\mathbb{R}^p$. We assume that* $\operatorname{Hess}\varphi$ *is defined on an open set like $\mathbb{R}^d \times B(y,r)$, that $\langle\operatorname{Hess}\Phi(y)h, h\rangle$ is given by formula* (12) *and that this formula is well defined* (*that is $\int e^{-\varphi(x,y)}\,dx < +\infty$ and $\operatorname{Hess}_y\varphi \in L^1(\upsilon_y)$*)*, and that $\varphi(.,y)$ is a C^2 function on $\mathbb{R}^d$. Then we have, without the hypothesis of convexity*:

$$\big\langle\operatorname{Hess}\Phi(y)h, h\big\rangle = \inf_{g\in C_c^\infty(\mathbb{R}^d)} \int \left\{\operatorname{Tr}\big[(\operatorname{Hess} g)^2\big] + \left\langle\operatorname{Hess}\varphi\begin{pmatrix}\nabla g\\ h\end{pmatrix}, \begin{pmatrix}\nabla g\\ h\end{pmatrix}\right\rangle\right\} d\upsilon_y(x).$$

Remark 9. By Lemma 8, we have: $\operatorname{Hess}_y\varphi \in L^1(\upsilon_y) \Rightarrow \nabla_y\varphi \in L^2(\upsilon_y)$.

Remark 10. The formula of Ball, Barthe and Naor gives an information on the second derivative of Φ and in particular, enables to prove Prékopa's theorem (by Theorem 14, if φ is a convex function on $\mathbb{R}^d \times \mathbb{R}^p$, it is easy to see that Φ is convex). This formula is important because it gives a way to prove Φ is convex even if φ is not. For example, using ideas of [3], Cordero-Erausquin,

Fradelizi and Maurey [5] proved the convexity of Φ for a very particular and nonconvex function φ. Then, they used this result to prove a conjecture called the (B) conjecture.

The function Φ studied in [5] is defined by:

$$e^{-\Phi(t)} = \int e^{-\frac{1}{2}\|e^t x\|^2} 1_C(x)\, dx,$$

where C is a symmetric convex set of $\mathbb{R}^d$ and $t \in \mathbb{R}$ (1_C is a log-concave function because C is convex). It is not possible to apply Prékopa's theorem to this function Φ because the function: $(t,x) \mapsto e^{-\frac{1}{2}\|e^t x\|^2} 1_C(x)$ is not a log-concave function. Nevertheless, Cordero-Erausquin, Fradelizi and Maurey proved that Φ is a convex function.

In their paper, instead of using directly the formula given in [3], Cordero-Erausquin, Fradelizi and Maurey chose to prove an inequality which is an improvement of Poincaré's inequality and which is related to the convexity of Φ. We will give a proof of this inequality in the next section.

Proof. Proof of Theorem 14 We fix y and h. We denote by L the operator: $L(F)(x) = \Delta F(x) - \langle \nabla_x \varphi(x,y), \nabla F(x)\rangle$. In the following, x is the variable of integration and of derivation (if it is not specified). We denote $f = \langle \nabla_y \varphi, h\rangle$.

For g in $C_c^\infty(\mathbb{R}^d)$, we obtain with equality (4):

$$Var_{\upsilon_y}(Lg) = -\int \bigl\{2\langle \nabla Lg, \nabla g\rangle + \mathrm{Tr}\bigl[(\mathrm{Hess}\, g)^2\bigr] + \langle \mathrm{Hess}_x\, \varphi \nabla g, \nabla g\rangle\bigr\}\, d\upsilon_y.$$

Moreover $H_3 h = \nabla_x f$, consequently:

$$\begin{aligned}
&\int \langle \mathrm{Hess}_y\, \varphi h, h\rangle\, d\upsilon_y(x) - Var_{\upsilon_y}(Lg)\\
&\quad = \int \bigl\{\langle H_2 h, h\rangle + 2\langle \nabla Lg, \nabla g\rangle + \mathrm{Tr}\bigl[(\mathrm{Hess}\, g)^2\bigr] + \langle H_1 \nabla g, \nabla g\rangle\bigr\}\, d\upsilon_y(x)\\
&\quad = \int \left\{\mathrm{Tr}\bigl[(\mathrm{Hess} g)^2\bigr] + \left\langle \mathrm{Hess}\, \varphi \begin{pmatrix} \nabla g \\ h\end{pmatrix}, \begin{pmatrix} \nabla g \\ h\end{pmatrix}\right\rangle\right\} d\upsilon_y(x)\\
&\qquad + 2\int \langle \nabla Lg - H_3 h, \nabla g\rangle\, d\upsilon_y(x)\\
&\quad = \int \left\{\mathrm{Tr}\bigl[(\mathrm{Hess} g)^2\bigr] + \left\langle \mathrm{Hess}\, \varphi \begin{pmatrix} \nabla g \\ h\end{pmatrix}, \begin{pmatrix} \nabla g \\ h\end{pmatrix}\right\rangle\right\} d\upsilon_y(x)\\
&\qquad - 2\int (Lg - f) Lg\, d\upsilon_y(x).
\end{aligned}$$

We deduce:

$$\begin{aligned}
\bigl\langle \mathrm{Hess}\, \Phi(y) h, h\bigr\rangle = {}& \int \left\{\mathrm{Tr}\bigl[(\mathrm{Hess}\, g)^2\bigr] + \left\langle \mathrm{Hess}\, \varphi \begin{pmatrix} \nabla g \\ h\end{pmatrix}, \begin{pmatrix} \nabla g \\ h\end{pmatrix}\right\rangle\right\} d\upsilon_y(x)\\
& - 2\int (Lg - f) Lg\, d\upsilon_y(x) + Var_{\upsilon_y}(Lg) - Var_{\upsilon_y}(f)
\end{aligned}$$

$$= \int \left\{ \mathrm{Tr}\big[(\mathrm{Hess}\, g)^2\big] + \left\langle \mathrm{Hess}\,\varphi \begin{pmatrix} \nabla g \\ h \end{pmatrix}, \begin{pmatrix} \nabla g \\ h \end{pmatrix} \right\rangle \right\} d\upsilon_y(x)$$
$$- Var_{\upsilon_y}(f - Lg).$$

Furthermore, with Lemma 3, we can find g_n in $C_c^\infty(\mathbb{R}^d)$ such that Lg_n goes to $\langle \nabla_y \varphi, h\rangle - \int \langle \nabla_y \varphi, h\rangle\, d\upsilon_y$ in $L^2(\upsilon_y)$, which leads to the conclusion. □

Now, wondering if Theorem 14 combined with equality (12) could give a new expression for the variance would lead to a positive answer. If we apply Theorem 14 to $\varphi(x, y) = yf(x) + \phi(x)$ (where $y \in \mathbb{R}$) and if we take $y = 0$, we obtain the following result. Nevertheless, we will give a direct proof thanks to equality (4).

Theorem 15. *We consider $\phi : \mathbb{R}^d \to \mathbb{R}$ a C^2 function, not necessarily convex, such that $\int e^{-\phi}\, dx < +\infty$. Then we have, for all f in $L^2(\mu_\phi)$:*

$$Var_{\mu_\phi}(f) = \sup_{g \in C_c^\infty(\mathbb{R}^d)} - \int \left\{ -2fLg + \mathrm{Tr}\big[(\mathrm{Hess}\, g)^2\big] + \langle \mathrm{Hess}\,\phi \nabla g, \nabla g\rangle \right\} d\mu_\phi,$$

where $L = \Delta - \langle \nabla\phi, \nabla\rangle$.

Proof. For g in $C_c^\infty(\mathbb{R}^d)$, we obtain with equality (4) applied to Lg:

$$\begin{aligned} Var_{\mu_\phi}(f) &= Var_{\mu_\phi}(Lg) + 2\int \left(f - \int f\, d\mu_\phi - Lg \right) Lg\, d\mu_\phi + Var_{\mu_\phi}(f - Lg) \\ &= -\int \left\{ -2(Lg)^2 + \mathrm{Tr}\big[(\mathrm{Hess}\, g)^2\big] + \langle \mathrm{Hess}\,\phi \nabla g, \nabla g\rangle \right\} d\mu_\phi \\ &\quad + 2\int (f - Lg) Lg\, d\mu_\phi + Var_{\mu_\phi}(f - Lg) \\ &= -\int \left\{ -2fLg + \mathrm{Tr}\big[(\mathrm{Hess}\, g)^2\big] + \langle \mathrm{Hess}\,\phi \nabla g, \nabla g\rangle \right\} d\mu_\phi + Var_{\mu_\phi}(f - Lg). \end{aligned}$$

We conclude with Lemma 3. □

Moreover, if we apply Theorem 15 to $\phi(x) = \varphi(x, y)$ and $f(x) = \langle \nabla_y \varphi, h\rangle$ (where y and h belong to $\mathbb{R}^p$), we recover Theorem 14. Consequently, using remark before Theorem 15, we see that Theorems 14 and 15 are equivalent.

3.2. *Inequality of Cordero-Erausquin, Fradelizi and Maurey*

We recover with Lemma 7 (and more precisely with formula (4)) the following result which is due to Cordero-Erausquin, Fradelizi and Maurey [5]. We consider again a C^2 function $\varphi : \mathbb{R}^d \to \mathbb{R}$ such that:

$$\forall x \in \mathbb{R}^d, \quad \mathrm{Hess}\,\varphi(x) > 0 \quad \text{and} \quad \int e^{-\varphi}\, dx < +\infty.$$

Theorem 16. *If $a = a(\varphi) > 0$, then, for all f in $H_1(\mu_\varphi)$ such that $\int \nabla f \, d\mu_\varphi = 0$:*

$$Var_{\mu_\varphi}(f) \leqslant \frac{1}{2a} \int \|\nabla f\|^2 \, d\mu_\varphi - a \left\| \int xf \, d\mu_\varphi - \int x \, d\mu_\varphi \int f \, d\mu_\varphi \right\|^2.$$

Remark 11. The last term in the inequality does not appear in [5] but it is possible to find it in the proof given in [5].

Proof. We will prove the following inequality which is equivalent to the one of the theorem.

With the same hypothesis and for all f in $H_1(\mu_\varphi)$:

$$\begin{aligned} Var_{\mu_\varphi}\left(f - \left\langle \int \nabla f \, d\mu_\varphi, x \right\rangle\right) &\leqslant \frac{1}{2a} \int \left\| \nabla f - \int \nabla f \, d\mu_\varphi \right\|^2 d\mu_\varphi \\ &- a \left\| \int x \left(f - \left\langle \int \nabla f \, d\mu_\varphi, x \right\rangle\right) d\mu_\varphi - \int x \, d\mu_\varphi \int \left(f - \left\langle \int \nabla f \, d\mu_\varphi, x \right\rangle\right) d\mu_\varphi \right\|^2. \end{aligned}$$

This formulation allows us to justify easily convergences we will use. We follow the proof of Theorem 1 (second step). We see it is sufficient to prove the result for f in C_c^∞ and for a C^∞ function φ such that $\operatorname{Hess} \varphi \in L^2(\mu_\varphi)$. Then, we use Corollary 6 and we see there exists $g_n \in C_c^\infty$ such that

$$\lim_{n \to +\infty} Lg_n = \left(f - \left\langle \int \nabla f \, d\mu_\varphi, x \right\rangle\right) - \int \left(f - \left\langle \int \nabla f \, d\mu_\varphi, x \right\rangle\right) d\mu_\varphi \quad \text{for the norm } \| \ \|_{H_1}.$$

Denote $f_n = Lg_n$. With equality (4), we obtain:

$$\begin{aligned} Var_{\mu_\varphi}(f_n) &= \frac{1}{2a} \int \|\nabla f_n\|^2 \, d\mu_\varphi \\ &- \int \left\{ \frac{1}{2a} \|\nabla f_n\|^2 + 2\langle \nabla f_n, \nabla g_n \rangle + \operatorname{Tr}\left[(\operatorname{Hess} g_n)^2\right] + \langle \operatorname{Hess} \varphi \nabla g_n, \nabla g_n \rangle \right\} d\mu_\varphi. \end{aligned}$$

We use the same kind of arguments as in [5]. We define: $\widetilde{g_n} = g_n - \langle c_n, x \rangle$ where $c_n = \int \nabla g_n \, d\mu_\varphi$ and we write:

$$\begin{aligned} &\int \left\{ \frac{1}{2a} \|\nabla f_n\|^2 + 2\langle \nabla f_n, \nabla g_n \rangle + \operatorname{Tr}\left[(\operatorname{Hess} g_n)^2\right] + \langle \operatorname{Hess} \varphi \nabla g_n, \nabla g_n \rangle \right\} d\mu_\varphi \\ &= \int \left\{ \frac{1}{2a} \|\nabla f_n + 2a \nabla \widetilde{g_n}\|^2 + \operatorname{Tr}\left[(\operatorname{Hess} \widetilde{g_n})^2\right] - a \|\nabla \widetilde{g_n}\|^2 + \|\nabla \widetilde{g_n} + c_n\|_{\operatorname{Hess} \varphi - aI}^2 \right\} d\mu_\varphi \\ &\quad + a\|c_n\|^2 + 2\left\langle \int \nabla f_n \, d\mu_\varphi, c_n \right\rangle. \end{aligned}$$

With the Brascamp and Lieb inequality applied to $\frac{\partial \widetilde{g_n}}{\partial x_i}$ and thanks to $(\operatorname{Hess} \varphi)^{-1} \leqslant \frac{1}{a} I$, we obtain:

$$\int \|\nabla \widetilde{g_n}\|^2 \, d\mu_\varphi \leqslant \frac{1}{a} \int \operatorname{Tr}\left[(\operatorname{Hess} \widetilde{g_n})^2\right] d\mu_\varphi.$$

Consequently:

$$Var_{\mu_\varphi}(f_n) \leqslant \frac{1}{2a}\int \|\nabla f_n\|^2\,d\mu_\varphi - a\|c_n\|^2 - 2\left\langle \int \nabla f_n\,d\mu_\varphi, c_n\right\rangle.$$

Furthermore:

$$c_n = \int \nabla g_n\,d\mu_\varphi = \int g_n \nabla\varphi\,d\mu_\varphi = -\int g_n L(x)\,d\mu_\varphi = -\int L(g_n)x\,d\mu_\varphi = -\int f_n x\,d\mu_\varphi.$$

We obtain the desired result when n goes to infinity. □

Theorem 16 allows us to obtain:

Corollary 17. *If $a = a(\varphi) > 0$, if f belongs to $L^2(\mu_\varphi)$ and if f and φ are even functions then*:

$$\forall t \geqslant 0,\quad \left\| P_t f - \int f\,d\mu_\varphi \right\|_{L^2(\mu_\varphi)} \leqslant e^{-2at} \left\| f - \int f\,d\mu_\varphi \right\|_{L^2(\mu_\varphi)}.$$

In particular, if φ is even and belongs to $L^2(\mu_\varphi)$ then φ satisfies hypothesis $\mathcal{H}$.

Proof. We consider $\xi(t) = e^{4at}\|P_t f - \int f\,d\mu_\varphi\|^2_{L^2(\mu_\varphi)}$.

$$\begin{aligned}
\xi'(t) &= 2e^{4at}\left(2a\left\|P_t f - \int f\,d\mu_\varphi\right\|^2_{L^2(\mu_\varphi)} + \int\left(P_t f - \int f\,d\mu_\varphi\right) L P_t f\,d\mu_\varphi\right)\\
&= 2e^{4at}\left(2a\left\|P_t f - \int f\,d\mu_\varphi\right\|^2_{L^2(\mu_\varphi)} - \int \|\nabla P_t f\|^2\,d\mu_\varphi\right).
\end{aligned}$$

With the proof of Remark 4, we see $P_t f$ is an even function, so $\int \nabla P_t f\,d\mu_\varphi = 0$. Theorem 16 gives $\xi'(t) \leqslant 0$, which allows us to conclude. □

4. Some possible extensions

In this paragraph, we introduce a reasoning process which allows to obtain best upper and lower bounds for the variance. We begin with the following lemma where we try to make the most of information contained in (5) and (3).

We always consider a C^2 function $\varphi : \mathbb{R}^d \to \mathbb{R}$ such that:

$$\forall x \in \mathbb{R}^d,\quad \operatorname{Hess}\varphi(x) > 0 \quad\text{and}\quad \int e^{-\varphi}\,dx < +\infty.$$

Lemma 18. *We assume φ belongs to $L^2(\mu_\varphi)$ and let f be a function in $L^2(\mu_\varphi)$ such that there exists g in C_c^∞ verifying*:

$$f - \int f\,d\mu_\varphi = Lg.$$

Then we have:

$$Var_{\mu_\varphi}(f) \leqslant \int \big\langle (\operatorname{Hess}\varphi)^{-1}\nabla f, \nabla f\big\rangle \, d\mu_\varphi - \frac{(1+a/b)}{d}\big(Var_{\mu_\varphi}(\Delta g) + c_\varphi(f)^2\big). \tag{13}$$

Moreover, if $\operatorname{Hess}\varphi \in L^1(\mu_\varphi)$, *then, with* $\theta = g + \langle V(f), x\rangle$, *we have*:

$$Var_{\mu_\varphi}(f) \geqslant \left|\!\left|\!\left| \int f \nabla\varphi \, d\mu_\varphi \right|\!\right|\!\right|^2_{(\int \operatorname{Hess}\varphi \, d\mu_\varphi)^{-1}} + \frac{1}{d} c_\varphi(f)^2 + a^2 \int |\!|\!|\nabla\theta|\!|\!|^2_{(\operatorname{Hess}\varphi)^{-1}} \, d\mu_\varphi + \frac{1}{d} Var_{\mu_\varphi}(\Delta g) \tag{14}$$

(*if* $a = 0$, *we agree that* $\int |\!|\!|\nabla\theta|\!|\!|^2_{(\operatorname{Hess}\varphi)^{-1}} \, d\mu_\varphi = 0$).

Proof. Equality (5) gives (as in the proof of Theorem 1):

$$\begin{aligned} Var_{\mu_\varphi}(f) &\leqslant \int \big\langle (\operatorname{Hess}\varphi)^{-1}\nabla f, \nabla f\big\rangle \, d\mu_\varphi - \left(1 + \frac{a}{b}\right) \int \operatorname{Tr}\big[(\operatorname{Hess} g)^2\big] \, d\mu_\varphi \\ &\leqslant \int \big\langle (\operatorname{Hess}\varphi)^{-1}\nabla f, \nabla f\big\rangle \, d\mu_\varphi - \frac{(1+a/b)}{d} \int (\Delta g)^2 \, d\mu_\varphi \\ &\leqslant \int \big\langle (\operatorname{Hess}\varphi)^{-1}\nabla f, \nabla f\big\rangle \, d\mu_\varphi - \frac{(1+a/b)}{d}\left(Var_{\mu_\varphi}(\Delta g) + \left(\int \Delta g \, d\mu_\varphi\right)^2\right). \end{aligned}$$

We conclude with the equality:

$$\int \Delta g \, d\mu_\varphi = -c_\varphi(f).$$

Equality (3) gives:

$$Var_{\mu_\varphi}(f) = \int \operatorname{Tr}\big[(\operatorname{Hess} g)^2\big] \, d\mu_\varphi + \int |\!|\!|\nabla g|\!|\!|^2_{\operatorname{Hess}\varphi} \, d\mu_\varphi.$$

Consequently:

$$Var_{\mu_\varphi}(f) \geqslant \frac{1}{d} Var_{\mu_\varphi}(\Delta g) + \frac{1}{d} c_\varphi(f)^2 + \int |\!|\!|\nabla g|\!|\!|^2_{\operatorname{Hess}\varphi} \, d\mu_\varphi. \tag{15}$$

Then, we follow the proof of Theorem 10 and we obtain (14). □

Previous lemma shows further terms in comparison with Theorems 1 and 2, they are:

$$\frac{1}{d} Var_{\mu_\varphi}(\Delta g) \quad \text{and} \quad a^2 \int |\!|\!|\nabla\theta|\!|\!|^2_{(\operatorname{Hess}\varphi)^{-1}} \, d\mu_\varphi.$$

To improve lower and upper bounds for $Var_{\mu_\varphi}(f)$, we have to bound below those two terms. For the first term, we use inequality (14). For the second term, we use (13) (with $f = \theta$) then we

work on $Var_{\mu_\varphi}(\theta)$ with (14). The difficulty is to rewrite new terms only with f. We have already used this method to prove Theorem 10 with the following inequality, which is a consequence of (13):

$$\int \|\nabla\theta\|^2_{(\mathrm{Hess}\,\varphi)^{-1}}\,d\mu_\varphi \geqslant Var_{\mu_\varphi}(\theta) + \frac{1}{d}\left(1+\frac{a}{b}\right)c_\varphi(\theta)^2. \tag{16}$$

In this section, we will show how to use $Var_{\mu_\varphi}(\Delta g)$ in (13) and (14) to improve Theorems 1, 2 and 10. The result we obtain here is certainly not the best one because it is obvious that inequality (13) and (14) could be combined indefinitely. How to use those two inequalities in an optimum way is an open question.

We define:

$$Cov_{\mu_\varphi}(h_1,h_2) = \int h_1h_2\,d\mu_\varphi - \int h_1\,d\mu_\varphi \int h_2\,d\mu_\varphi.$$

Now, for all $p \geqslant 1$, we assume that φ^p belongs to $L^2(\mu_\varphi)$. We define coefficients $\gamma_{k,p}$ (with $1 \leqslant k \leqslant p$) by:

$$\begin{cases} \gamma_{1,1} = 1, \\ \gamma_{1,p+1} = \gamma_{1,p} + \sum_{k=1}^{p} \gamma_{k,p} \int \frac{\varphi^k}{k!}\,d\mu_\varphi, \\ \gamma_{k,p+1} = \gamma_{k,p} - \gamma_{k-1,p} \quad \text{for } 2 \leqslant k \leqslant p, \\ \gamma_{p+1,p+1} = -\gamma_{p,p}. \end{cases}$$

For example, we obtain:

$$\gamma_{1,2} = 1 + \int \varphi\,d\mu_\varphi, z \qquad \gamma_{2,2} = -1,$$

$$\gamma_{1,3} = \left(1+\int \varphi\,d\mu_\varphi\right)^2 - \int \frac{\varphi^2}{2!}\,d\mu_\varphi, \qquad \gamma_{2,3} = -\left(2+\int\varphi\,d\mu_\varphi\right), \qquad \gamma_{3,3} = 1,$$

$$\gamma_{p,p} = (-1)^{p+1}.$$

Define:

$$u_p(\varphi) = \sum_{k=1}^{p} \gamma_{k,p}\frac{\varphi^k}{k!},$$

and, for all f in $L^2(\mu_\varphi)$:

$$\overline{c_p}(f) = Cov_{\mu_\varphi}\big(f, u_p(\varphi)\big) \quad \big(\overline{c_1}(f) = c_\varphi(f)\big).$$

Remark 12. It is possible to define $u_p(\varphi)$ as following. Define, for t in $\mathbb{R}$ and $p \geqslant 1$:

$$R_1(t) = t, \qquad R_{p+1}(t) = R_p(t) - \int_0^t R_p(s)\,ds + t\int R_p(\varphi)\,d\mu_\varphi.$$

Then we have: $u_p(\varphi) = R_p(\varphi)$.

As for K_φ, it is an open question to give an interpretation of $u_p(\varphi)$. If we calculate $u_p(\varphi)$ for $\varphi(x) = \|x\|^2/2$, we do not obtain a remarkable result.

Lemma 19. *We assume that φ^p belongs to $L^2(\mu_\varphi)$ for all p $(1 \leqslant p < +\infty)$. Then, for all f in $L^2(\mu_\varphi)$ and if $1 \leqslant p < +\infty$:*

$$Var_{\mu_\varphi}(f) \geqslant \sum_{k=1}^{p} \frac{1}{d^k} \overline{c_k}(f)^2.$$

Proof. The case $p = 1$ follows from Theorem 2. We assume the lemma proved for $p \geqslant 1$. Using Lemma 3, we see it is possible to assume there exists g in C_c^∞ such that $f - \int f \, d\mu_\varphi = Lg$. With (15), we obtain:

$$\begin{aligned} Var_{\mu_\varphi}(f) &\geqslant \frac{1}{d} Var_{\mu_\varphi}(\Delta g) + \frac{1}{d} c_\varphi(f)^2 \\ &\geqslant \sum_{k=1}^{p} \frac{1}{d^{k+1}} \overline{c_k}(\Delta g)^2 + \frac{1}{d} \overline{c_1}(f)^2 \end{aligned}$$

(in that particular case, it is more appropriate to use (15) instead of (14) because, for the first inequality, we do not nead to assume $\operatorname{Hess} \varphi \in L^1(\mu_\varphi)$).

Moreover:

$$\begin{aligned} \overline{c_k}(\Delta g) &= \overline{c_k}\big(\Delta g - \langle \nabla\varphi, \nabla g\rangle\big) + \overline{c_k}\big(\langle \nabla\varphi, \nabla g\rangle\big) \\ &= \overline{c_k}(f) + \sum_{i=1}^{k} \gamma_{i,k} Cov_{\mu_\varphi}\left(\langle \nabla\varphi, \nabla g\rangle, \frac{\varphi^i}{i!}\right), \end{aligned}$$

$$\begin{aligned} Cov_{\mu_\varphi}\left(\langle \nabla\varphi, \nabla g\rangle, \frac{\varphi^i}{i!}\right) &= \int \left\langle \nabla g, \nabla\left(\frac{\varphi^{i+1}}{(i+1)!}\right)\right\rangle d\mu_\varphi - \int \frac{\varphi^i}{i!} \, d\mu_\varphi \int \langle \nabla\varphi, \nabla g\rangle \, d\mu_\varphi \\ &= -\int \frac{\varphi^{i+1}}{(i+1)!} Lg \, d\mu_\varphi + \int \frac{\varphi^i}{i!} \, d\mu_\varphi \int \varphi Lg \, d\mu_\varphi \\ &= -Cov_{\mu_\varphi}\left(f, \frac{\varphi^{i+1}}{(i+1)!}\right) + \int \frac{\varphi^i}{i!} \, d\mu_\varphi c_\varphi(f). \end{aligned}$$

Consequently:

$$\begin{aligned} \overline{c_k}(\Delta g) &= Cov_{\mu_\varphi}\left(f, u_k(\varphi) - \sum_{i=1}^{k} \gamma_{i,k} \frac{\varphi^{i+1}}{(i+1)!} + \sum_{i=1}^{k} \gamma_{i,k} \int \frac{\varphi^i}{i!} \, d\mu_\varphi \varphi\right) \\ &= Cov_{\mu_\varphi}\big(f, u_{k+1}(\varphi)\big) \\ &= \overline{c_{k+1}}(f). \end{aligned}$$

We deduce:

$$Var_{\mu_\varphi}(f) \geqslant \sum_{k=1}^{p+1} \frac{1}{d^k} \overline{c_k}(f)^2. \qquad \square$$

Theorem 20. *We assume $\varphi^p \in L^2(\mu_\varphi)$ for all p $(1 \leqslant p < +\infty)$.*

(1) *If* $\operatorname{Hess} \varphi \in L^1(\mu_\varphi)$, *then, for all f in $L^2(\mu_\varphi)$*:

$$Var_{\mu_\varphi}(f) \geqslant \left\| \int f \nabla\varphi \, d\mu_\varphi \right\|^2_{(\int \operatorname{Hess}\varphi \, d\mu_\varphi)^{-1}} + \sum_{k=1}^{+\infty} \frac{1}{d^k} \overline{c_k}(f)^2.$$

Furthermore, if $a = a(\varphi) > 0$, then:

$$\begin{aligned} Var_{\mu_\varphi}(f) \geqslant{} & \left\| \int f \nabla\varphi \, d\mu_\varphi \right\|^2_{(\int \operatorname{Hess}\varphi \, d\mu_\varphi)^{-1}} + \sum_{k=1}^{+\infty} \frac{1}{d^k} \overline{c_k}(f)^2 \\ & + \frac{a^2}{d}\left(2 + \frac{a}{b}\right) \sum_{k=0}^{+\infty} a^{2k} c_{k+2}\big[f - \langle V(f), \nabla\varphi \rangle\big]^2. \end{aligned}$$

(2) *For all f in $H_1(\mu_\varphi)$, we have*:

$$Var_{\mu_\varphi}(f) \leqslant \int \big\langle (\operatorname{Hess}\varphi)^{-1} \nabla f, \nabla f \big\rangle \, d\mu_\varphi - \left(1 + \frac{a}{b}\right) \sum_{k=1}^{+\infty} \frac{1}{d^k} \overline{c_k}(f)^2.$$

Proof. With the previous lemma, we obtain, for all f in $L^2(\mu_\varphi)$:

$$Var_{\mu_\varphi}(f) \geqslant \sum_{k=1}^{+\infty} \frac{1}{d^k} \overline{c_k}(f)^2.$$

Fistly, we prove (20). It is sufficient to prove the result for functions f in $L^2(\mu_\varphi)$ such that there exists $g \in C_c^\infty$ verifying $f - \int f \, d\mu_\varphi = Lg$. Inequality (14) gives:

$$Var_{\mu_\varphi}(f) \geqslant \left\| \int f \nabla\varphi \, d\mu_\varphi \right\|^2_{(\int \operatorname{Hess}\varphi \, d\mu_\varphi)^{-1}} + \frac{1}{d} c_\varphi(f)^2 + \frac{1}{d} Var_{\mu_\varphi}(\Delta g).$$

Moreover:

$$\begin{aligned} Var_{\mu_\varphi}(\Delta g) &\geqslant \sum_{k=1}^{+\infty} \frac{1}{d^k} \overline{c_k}(\Delta g)^2 \\ &\geqslant \sum_{k=1}^{+\infty} \frac{1}{d^k} \overline{c_{k+1}}(f)^2 \quad \text{(see previous proof)}, \end{aligned}$$

which leads to the desired result. Now, if we assume $a = a(\varphi) > 0$, then inequalities (14) and (16) give:

$$Var_{\mu_\varphi}(f) \geqslant \left\| \int f \nabla\varphi \, d\mu_\varphi \right\|^2_{(\int \mathrm{Hess}\,\varphi \, d\mu_\varphi)^{-1}} + \frac{1}{d} c_\varphi(f)^2 + a^2 \left(Var_{\mu_\varphi}(\theta) + \frac{1}{d}\left(1 + \frac{a}{b}\right) c_\varphi(\theta)^2 \right) + \frac{1}{d} Var_{\mu_\varphi}(\Delta g),$$

and as in the proof of Theorem 10 (using Lemma 11), we obtain:

$$Var_{\mu_\varphi}(\theta) \geqslant \frac{1}{d}\left(2 + \frac{a}{b}\right) \sum_{k=0}^{+\infty} a^{2k} c_{k+2}\big[f - \langle V(f), \nabla\varphi \rangle\big]^2.$$

Now, we prove (2). We follow the proof of Theorem 1. We assume without difficulty $a > 0$ and f belongs to C_c^∞. As in the proof of this theorem, we use the sequence Φ_n. With this aim in view, we need to show, for all $k \geqslant 1$:

$$\lim_{n \to +\infty} \int \Phi_n^k \, d\mu_{\Phi_n} = \int \varphi^k \, d\mu_\varphi.$$

We prove this equality as we have done it in the proof of Theorem 1 for the case $k = 1$ and we use the fact that: $t \mapsto t^k \exp(-t)$ is a decreasing function for $t \geqslant k$. We deduce:

$$\lim_{n \to +\infty} Cov_{\mu_{\Phi_n}}\big(f, u_k(\Phi_n)\big) = Cov_{\mu_\varphi}\big(f, u_k(\varphi)\big).$$

So, we assume that φ verifies hypothesis of Corollary 6. Consequently, it is sufficient to prove (2) for f in $L^2(\mu_\varphi)$ such that $f - \int f \, d\mu_\varphi = Lg$ with $g \in C_c^\infty$. Now, we use inequality (13) to obtain:

$$Var_{\mu_\varphi}(f) \leqslant \int \big\langle (\mathrm{Hess}\,\varphi)^{-1} \nabla f, \nabla f \big\rangle \, d\mu_\varphi - \frac{(1 + a/b)}{d} \big(Var_{\mu_\varphi}(\Delta g) + c_\varphi(f)^2 \big).$$

As we shown before:

$$Var_{\mu_\varphi}(\Delta g) \geqslant \sum_{k=1}^{+\infty} \frac{1}{d^k} \overline{c_{k+1}}(f)^2,$$

which allows us to conclude. $\square$

Case (1) of this theorem could be modified to obtain a similar result as the one of Theorem 12 if we assume φ satisfies hypothesis $\mathcal{H}$.

We see that there exists a great number of possibilities to improve lower and upper bounds of the variance because we can use last theorem or Lemmas 11 and 19 to work on $Var_{\mu_\varphi}(\Delta g)$ and $\int \|\nabla\theta\|^2_{(\mathrm{Hess}\,\varphi)^{-1}} \, d\mu_\varphi$. Moreover, in equalities (13) and (14), it is possible to write $\int \mathrm{Tr}[(\mathrm{Hess}\, g)^2] \, d\mu_\varphi$ instead of $\frac{1}{d}(Var_{\mu_\varphi}(\Delta g) + c_\varphi(f)^2)$. So, we can use ideas developed in [5] in the proof of Theorem 16 (in particular a use of inequality of Brascamp and Lieb applied to $\frac{\partial g}{\partial x_i}$, which is a consequence of inequality (13)), which gives again new possibilities.

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Appendix A. Upper and lower bounds for the variance for the Gaussian measure

As indicated in [2], we use Hermite's polynomials H_n.
For $x \in \mathbb{R}$:

$$H_0(x) = 1, \qquad H_1(x) = x, \qquad H_2(x) = \frac{x^2 - 1}{\sqrt{2}}.$$

In $\mathbb{R}^d$, the pth chaos is generated by:

$$\mathcal{H}_\beta(x) = \prod_{i=1}^{d} H_{\beta_i}(x_i) \quad \text{where } x = (x_1, \dots, x_d),\ \beta = (\beta_1, \dots, \beta_d) \text{ and } \sum \beta_i = p.$$

So, for $f \in L^2(\mu)$ (where μ is the canonical Gaussian measure on $\mathbb{R}^d$):

$$f = \int f\, d\mu + \sum_i c_i x_i + \sum_i a_i H_2(x_i) + \sum_{i<j} b_{i,j} H_1(x_i) H_1(x_j) + G,$$

where G is orthogonal to chaos of order smaller than 2, and:

$$c_i = \int x_i f\, d\mu, \qquad a_i = \int \frac{x_i^2 - 1}{\sqrt{2}} f\, d\mu, \qquad b_{i,j} = \int x_i x_j f\, d\mu.$$

Denote $C = (c_1, \dots, c_d)$, $A_{i,i} = \sqrt{2} a_i$, $A_{i,j} = A_{j,i} = b_{i,j}$ if $i \neq j$. We have:

$$f = \int f\, d\mu + \langle C, x \rangle + \frac{1}{2} \langle Ax, x \rangle - \frac{1}{\sqrt{2}} \operatorname{Tr} A + G.$$

Consequently:

$$Var_\mu(f) = \|C\|^2 + \frac{1}{2} \operatorname{Tr}\left(A^2\right) + \int G^2\, d\mu,$$
$$\nabla f = C + Ax + \nabla G.$$

To obtain the upper bound for $Var_\mu(f)$, we write:

$$\int \|Ax\|^2\, d\mu = \operatorname{Tr}\left(A^2\right) \quad \text{and} \quad \int G^2\, d\mu \leqslant \int \|\nabla G\|^2\, d\mu \quad \text{because } \int G\, d\mu = 0,$$
$$\int \|\nabla f\|^2\, d\mu = \|C\|^2 + \int \|Ax\|^2\, d\mu + \int \|\nabla G\|^2\, d\mu.$$

We deduce:

$$Var_\mu(f) \leqslant \int \|\nabla f\|^2 \, d\mu - \frac{1}{2} \operatorname{Tr}(A^2) \leqslant \int \|\nabla f\|^2 \, d\mu - \frac{1}{2d} \big(\operatorname{Tr}(A)\big)^2,$$

and $\operatorname{Tr} A = \int (\|x\|^2 - d) f \, d\mu = \int \Delta f \, d\mu$.

To obtain the lower bound for $Var_\mu(f)$, we write:

$$\int \nabla f \, d\mu = C + \int \nabla G \, d\mu = C \quad \Rightarrow \quad \left\| \int \nabla f \, d\mu \right\|^2 = \|C\|^2.$$

So:

$$Var_\mu(f) \geqslant \left\| \int \nabla f \, d\mu \right\|^2 + \frac{1}{2} \operatorname{Tr}(A^2) \geqslant \left\| \int \nabla f \, d\mu \right\|^2 + \frac{1}{2d} \left(\int \Delta f \, d\mu \right)^2. \qquad \square$$

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Brownian Chen series and Atiyah–Singer theorem

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Communicated by C. Villani

This work is dedicated to my wife Alice

Abstract

The purpose of this work is to give a new and short proof of the Atiyah–Singer local index theorem for the Dirac operator on the spin bundle. This proof is obtained by using heat semigroups approximations based on the truncation of Brownian Chen series.

Keywords: Brownian motion; Chen series; Local index theorem

Contents

1. Introduction

The goal of this paper is to give a new and short proof of the local Atiyah–Singer index theorem by using approximations of heat semigroups. The heat equation approach to index theorems

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is not new: it was suggested by Atiyah, Bott [1] and McKean, Singer [21], and first carried out by Patodi [22] and Gilkey [14]. (See also Getzler [13].) Bismut in [9] introduces stochastic methods based on Feynman–Kac formula. For probabilistic approaches that are mainly based on Bismut's ideas, we also refer to [16] and [15, Chapter 7]. For a complete survey on (non-probabilistic) heat equation methods for index theorems, we refer to the book [8].

However, in our approach we will see that the A-genus appears in a natural way, from purely local computations on approximations of heat semigroups. Our method relies on explicit approximations of the holonomy over the heat equation on vector bundles and unlike the other probabilistic approaches does not involve the Feynman–Kac formula nor the techniques of stochastic differential geometry.

The idea is the following. Let $\mathbf{P}_t$ denote a heat semigroup. In recent works, see Baudoin [4] and Lyons, Victoir [18], by using Brownian Chen series, it has been pointed out that $\mathbf{P}_t$ admits a formal representation as the expectation of the exponential of a random Lie series. The truncation of this Lie series leads to explicit approximations of $\mathbf{P}_t$. More precisely, one gets a family of operators $\mathbf{P}_t^N$, $N \geqslant 1$, such that (in the sup norm)

$$\mathbf{P}_t = \mathbf{P}_t^N + O\left(t^{\frac{N+1}{2}}\right), \quad t \to 0. \tag{1.1}$$

This point of view has been used in Lyons, Victoir [18] to provide cubature formulae on Wiener space that give efficient numerical approximations of solutions of heat equations.

Assume now that $\mathbf{P}_t$ is the heat semigroup associated with the Dirac operator on the Clifford module over a compact d-dimensional spin manifold, d even. From (1.1), we will classically deduce

$$\mathbf{Str}\,\mathbf{P}_t = \mathbf{Str}\,\mathbf{P}_t^d + O\left(t^{\frac{1}{2}}\right), \quad t \to 0,$$

where $\mathbf{Str}$ denotes the supertrace. The Lie structure that explicitly appears in $\mathbf{P}_t^d$ now leads to algebraic cancellations that imply

$$\mathbf{Str}\,\mathbf{P}_t^d = \mathbf{Str}\,\mathbf{P}_t^2 + O\left(t^{\frac{1}{2}}\right), \quad t \to 0.$$

Since, from McKean–Singer theorem, the supertrace of $\mathbf{P}_t$ has to be constant and equal to the index of the Dirac operator, the local index theorem follows from the easy computation of $\mathbf{Str}\,\mathbf{P}_t^2$.

The paper is organized as follows. In Section 2, we survey results on random Chen series that are needed for the construction of approximations of heat semigroups. In Section 3, we use these series to construct explicit approximations of the holonomy above general heat equations on vector bundles. Finally, in Section 4, we develop the idea hinted above to provide a new short proof of the Atiyah–Singer local index theorem for the Dirac operator on the spin bundle.

2. Chen series

We introduce here notations that will be used throughout the paper and survey some results on Chen series that will be later needed. Basic background on Chen series with respect to regular paths can be found in [11] and background on Chen series with respect to Brownian paths can be found in [4, Chapter 1] (see also [12,18]). Let us note that the Chen series with respect to Brownian paths is also called, in the rough paths theory of Lyons, the signature of the Brownian motion.

Let $\mathbb{R}[\![X_0,\dots,X_d]\!]$ be the noncommutative algebra over $\mathbb{R}$ of the formal series with $d+1$ indeterminates, that is the set of series

$$Y=\sum_{k\geqslant 0}\sum_{i_1,\dots,i_k} a_{i_1,\dots,i_k}X_{i_1}\dots X_{i_k}.$$

The exponential of $Y\in\mathbb{R}[\![X_0,\dots,X_d]\!]$ is defined by

$$\exp(Y)=\sum_{k=0}^{+\infty}\frac{Y^k}{k!}.$$

We define the bracket between two elements U and V of $\mathbb{R}[\![X_0,\dots,X_d]\!]$ by

$$[U,V]=UV-VU.$$

If $I=(i_1,\dots,i_k)\in\{0,\dots,d\}^k$ is a word, we denote by X_I the commutator defined by

$$X_I=\big[X_{i_1},\big[X_{i_2},\dots,[X_{i_{k-1}},X_{i_k}]\dots\big].$$

We will denote by $\mathcal{S}_k$ the set of the permutations of $\{0,\dots,k\}$. If $\sigma\in\mathcal{S}_k$, we denote $e(\sigma)$ the cardinality of the set

$$\big\{j\in\{0,\dots,k-1\},\sigma(j)>\sigma(j+1)\big\},$$

and $\sigma(I)$ the word $(i_{\sigma(1)},\dots,i_{\sigma(k)})$.

Let us now consider a d-dimensional standard Brownian motion $(B_t)_{t\geqslant 0}=(B_t^1,\dots,B^d)_{t\geqslant 0}$. We use the convention that $B_t^0=t$. If $I=(i_1,\dots,i_k)\in\{0,\dots,d\}^k$ is a word with length k, the iterated Stratonovich integral

$$\int_{\Delta^k[0,t]}\circ dB^I=\int_{0\leqslant t_1\leqslant\dots\leqslant t_k\leqslant t}\circ dB_{t_1}^{i_1}\circ\dots\circ dB_{t_k}^{i_k},$$

can be defined as the limit in p-variation, $p>2$,

$$\lim_{n\to+\infty}\int_{\Delta^k[0,t]}\big(dB^n\big)^I,$$

where B^n denotes the piecewise linear interpolation of the paths of $(B_u)_{0\leqslant u\leqslant t}$ along the dyadic subdivision of $[0,t]$.

With the notation,

$$\Lambda_I(B)_t=\sum_{\sigma\in\mathcal{S}_k}\frac{(-1)^{e(\sigma)}}{k^2\binom{k-1}{e(\sigma)}}\int_{\Delta^k[0,t]}\circ dB^{\sigma^{-1}(I)},$$

we have the following theorem.

Theorem 2.1.

$$1+\sum_{k=1}^{+\infty}\sum_{I\in\{0,1,\dots,d\}^k}\left(\int_{\Delta^k[0,t]}\circ dB^I\right)X_{i_1}\dots X_{i_k}=\exp\left(\sum_{k\geqslant 1}\sum_{I\in\{0,1,\dots,d\}^k}\Lambda_I(B)_t\,X_I\right),\quad t\geqslant 0.$$

This theorem is due to Chen [11] and Strichartz [23] that prove that the above result holds for absolutely continuous paths. The result for Brownian paths is pointed out in Fliess [12]. And finally, Lyons [17], with rough paths theory, shows that it actually can be extended to very general paths.

If

$$Y=\sum_{k\geqslant 0}\sum_{I=(i_1,\dots,i_k)}a_{i_1,\dots,i_k}X_{i_1}\dots X_{i_k}$$

is a random series, that is if the coefficients are real random variables defined on a probability space, we will denote

$$\mathbb{E}(Y)=\sum_{k\geqslant 0}\sum_{I=(i_1,\dots,i_k)}\mathbb{E}(a_{i_1,\dots,i_k})X_{i_1}\dots X_{i_k}$$

as soon as this expression makes sense, that is as soon as for every $I=(i_1,\dots,i_k)$,

$$\mathbb{E}\left(|a_{i_1,\dots,i_k}|\right)<+\infty,$$

where $\mathbb{E}$ stands for the expectation.

Theorem 2.2. *(See [4,18].) We have*

$$\exp\left(t\left(X_0+\frac{1}{2}\sum_{i=1}^{d}X_i^2\right)\right)=\mathbb{E}\left(\exp\left(\sum_{k\geqslant 1}\sum_{I\in\{0,1,\dots,d\}^k}\Lambda_I(B)_tX_I\right)\right),\quad t\geqslant 0.$$

3. Approximation of elliptic heat kernels on vector bundles

In the spirit of Azencott [3], Ben Arous [7], Castell [10] and Lyons, Victoir [18], we use in this section Brownian Chen series in order to provide efficient approximations of heat semigroups and corresponding heat kernels. The idea is to truncate the Lie series that appear in the formal representation of the heat semigroup given by Theorem 2.2. As we shall see, this truncation that has already found applications for cubature formulae [18], is also particularly efficient to approximate the holonomy over the heat equation in a vector bundle.

Let $\mathbb{M}$ be a d-dimensional compact smooth Riemannian manifold and let $\mathcal{E}$ be a finite-dimensional vector bundle over $\mathbb{M}$. We denote by $\Gamma(\mathbb{M},\mathcal{E})$ the space of sections. Let now ∇ denote a connection on $\mathcal{E}$.

We consider the following linear partial differential equation:

$$\frac{\partial\Phi}{\partial t}=\mathcal{L}\Phi,\qquad \Phi(0,x)=f(x),\tag{3.2}$$

where $\mathcal{L}$ is an operator on $\mathcal{E}$ that can be written

$$\mathcal{L} = \nabla_0 + \frac{1}{2}\sum_{i=1}^{d} \nabla_i^2,$$

with

$$\nabla_i = \mathcal{F}_i + \nabla_{V_i}, \quad 0 \leqslant i \leqslant d,$$

the V_i's being smooth vector fields on $\mathbb{M}$ and the $\mathcal{F}_i$'s being smooth potentials (that is smooth sections of the bundle $\mathbf{End}(\mathcal{E})$). It is known that the solution of (3.2) can be written

$$\Phi(t,x) = \big(e^{t\mathcal{L}} f\big)(x) = \mathbf{P}_t f(x).$$

If $I \in \{0,1,\dots,d\}^k$ is a word, we denote

$$\nabla_I = \big[\nabla_{i_1}, \big[\nabla_{i_2}, \dots, [\nabla_{i_{k-1}}, \nabla_{i_k}]\dots\big]$$

and

$$d(I) = k + n(I),$$

where $n(I)$ is the number of 0 in the word I.

For $N \geqslant 1$, let us consider

$$\mathbf{P}_t^N = \mathbb{E}\left(\exp\left(\sum_{I, d(I)\leqslant N} \Lambda_I(B)_t \nabla_I\right)\right).$$

For instance

$$\mathbf{P}_t^1 = \mathbb{E}\left(\exp\left(\sum_{i=1}^{d} B_t^i \nabla_i\right)\right),$$

and

$$\mathbf{P}_t^2 = \mathbb{E}\left(\exp\left(\sum_{i=0}^{d} B_t^i \nabla_i + \frac{1}{2}\sum_{1\leqslant i<j\leqslant d} \int_0^t B_s^i\, dB_s^j - B_s^j\, dB_s^i [\nabla_i, \nabla_j]\right)\right).$$

The meaning of this last notation is the following. If $f \in \Gamma(\mathbb{M},\mathcal{E})$, then $(\mathbf{P}_t^N f)(x) = \mathbb{E}(\Psi(1,x))$, where $\Psi(\tau,x)$ is the solution of the first order partial differential equation with random coefficients:

$$\frac{\partial \Psi}{\partial \tau}(\tau,x) = \sum_{I, d(I)\leqslant N} \Lambda_I(B)_t (\nabla_I \Psi)(\tau,x), \qquad \Psi(0,x) = f(x).$$

Let us consider the following family of norms: if $f \in \Gamma(\mathbb{M}, \mathcal{E})$, for $k \geqslant 0$,

$$\|f\|_k = \sup_{0 \leqslant l \leqslant k} \sup_{0 \leqslant i_1, \dots, i_l \leqslant d} \sup_{x \in \mathbb{M}} \left\| \nabla_{i_1} \dots \nabla_{i_l} f(x) \right\|.$$

Theorem 3.1. *Let $N \geqslant 1$ and $k \geqslant 0$. For $f \in \Gamma(\mathbb{M}, \mathcal{E})$,*

$$\left\| \mathbf{P}_t f - \mathbf{P}_t^N f \right\|_k = O\left(t^{\frac{N+1}{2}}\right), \quad t \to 0.$$

Proof. The proof relies on a stochastic Taylor expansion and is close to Ben Arous [7], Castell [10] and Lyons, Victoir [18].

First, by using the scaling property of Brownian motion and expanding out the exponential with Taylor formula we obtain

$$\exp\left(\sum_{I, d(I) \leqslant N} \Lambda_I(B)_t \nabla_I \right) f = \left(\sum_{k=0}^{N} \frac{1}{k!} \left(\sum_{I, d(I) \leqslant N} \Lambda_I(B)_t \nabla_I \right)^k \right) f + t^{\frac{N+1}{2}} \mathbf{R}_N^1(t),$$

where the remainder term $\mathbf{R}_N^1(t)$ is such that $\mathbb{E}(\|\mathbf{R}_N^1(t)\|_k)$ is bounded when $t \to 0$. We now observe that, due to Theorem 2.1, the rearrangement of terms in the previous formula gives

$$\left(\sum_{k=0}^{N} \frac{1}{k!} \left(\sum_{I, d(I) \leqslant N} \Lambda_I(B)_t \nabla_I \right)^k \right) f = f + \sum_{I, d(I) \leqslant N} \int_{\Delta^{|I|}[0,t]} \circ dB^I \nabla_{i_1} \dots \nabla_{i_{|I|}} f + t^{\frac{N+1}{2}} \mathbf{R}_N^2(t),$$

where $\mathbb{E}(\|\mathbf{R}_N^2(t)\|_k)$ is bounded when $t \to 0$. Therefore

$$\exp\left(\sum_{I, d(I) \leqslant N} \Lambda_I(B)_t \nabla_I \right) f = f + \sum_{I, d(I) \leqslant N} \int_{\Delta^{|I|}[0,t]} \circ dB^I \nabla_{i_1} \dots \nabla_{i_{|I|}} f + t^{\frac{N+1}{2}} \mathbf{R}_N^3(t),$$

and

$$\mathbf{P}_t^N f = f + \sum_{I, d(I) \leqslant N} \mathbb{E}\left(\int_{\Delta^{|I|}[0,t]} \circ dB^I \right) \nabla_{i_1} \dots \nabla_{i_{|I|}} f + t^{\frac{N+1}{2}} \mathbb{E}\left(\mathbf{R}_N^3(t) \right),$$

where $\mathbb{E}(\|\mathbf{R}_N^3(t)\|_k)$ is bounded when $t \to 0$. We now have to compute the expectation of iterated Stratonovitch integrals. An easy computation (see [4, Chapter 1]) shows that if $\mathcal{I}_n$ is the set of words with length n obtained by all the possible concatenations of the words

$$\{0\}, \quad \{(i,i)\}, \quad i \in \{1, \dots, d\}.$$

1. If $I \notin \mathcal{I}_n$ then

$$\mathbb{E}\left(\int_{\Delta^n[0,t]} \circ dB^I \right) = 0;$$

2. If $I \in \mathcal{I}_n$ then

$$\mathbb{E}\left(\int_{\Delta^n[0,t]} \circ\, dB^I \right) = \frac{t^{\frac{n+n(I)}{2}}}{2^{\frac{n-n(I)}{2}} (\frac{n+n(I)}{2})!},$$

where $n(I)$ is the number of 0 in I (observe that since $I \in \mathcal{I}_n$, n and $n(I)$ necessarily have the same parity).

We conclude therefore

$$\left\| \mathbf{P}_t^N f - \sum_{k \leqslant \frac{N+1}{2}} \frac{t^k}{k!} \mathcal{L}^k f \right\|_k = O\big(t^{\frac{N+1}{2}}\big).$$

Since it is known that

$$\left\| \mathbf{P}_t f - \sum_{k \leqslant \frac{N+1}{2}} \frac{t^k}{k!} \mathcal{L}^k f \right\|_k = O\big(t^{\frac{N+1}{2}}\big),$$

the theorem is proved. □

Let us now assume that the operator $\mathcal{L}$ is elliptic at $x_0 \in \mathbb{M}$ in the sense that $(V_1(x_0), \ldots, V_d(x_0))$ is an orthonormal basis of the tangent space at x_0. In that case, $\mathbf{P}_t$ is known to admit a smooth Schwartz kernel at x_0. That is, there exists a smooth map

$$p(x_0, \cdot) : \mathbb{R}_{>0} \to \Gamma\big(\mathbb{M}, \mathbf{Hom}(\mathcal{E})\big)$$

such that for $f \in \Gamma(\mathbb{M}, \mathcal{E})$,

$$(\mathbf{P}_t f)(x_0) = \int_{\mathbb{M}} p_t(x_0, y) f(y)\, dy.$$

Theorem 3.2. *Let $N \geqslant 1$. There exists a map*

$$p^N(x_0, \cdot) : \mathbb{R}_{>0} \to \Gamma\big(\mathbb{M}, \mathbf{Hom}(\mathcal{E})\big)$$

such that for $f \in \Gamma(\mathbb{M}, \mathcal{E})$,

$$\big(\mathbf{P}_t^N f\big)(x_0) = \int_{\mathbb{M}} p_t^N(x_0, y) f(y)\, dy.$$

Moreover,

$$p_t(x_0, x_0) = p_t^N(x_0, x_0) + 0\big(t^{\frac{N+1-d}{2}}\big).$$

Proof. The proof is not simple. We shall proceed in several steps. In a first step, we shall show the existence of a kernel at x_0 for $\mathbf{P}_t^N$ acting on functions. In a second step we shall deduce by parallel transport, the existence of $p^N(x_0,\cdot)$. And finally, we shall prove the required estimate.

First step. Let us define,

$$\mathbf{Q}_t^N = \mathbb{E}\left(\exp\left(\sum_{I,d(I)\leqslant N} \Lambda_I(B)_t V_I\right)\right).$$

In order to show that $\mathbf{Q}_t^N$ admits a kernel at x_0, we show that for $t>0$, the stochastic process

$$Z_t^N = \exp\left(\sum_{I,d(I)\leqslant N} \Lambda_I(B)_t V_I\right)(x_0)$$

has a density with respect to the Riemannian measure of $\mathbb{M}$. To this end, from the well-known criterion of Malliavin (see [19,20]), we show that the Malliavin matrix of Z_t^N is invertible with probability one. A sufficient condition for that, is

$$\mathbb{D}_0^i Z_t^N, \quad i=1,\dots,d,$$

forms a basis of the tangent space at x_0 where $\mathbb{D}_0^i$ denotes the ith partial Malliavin's derivative taken at time 0. An easy computation shows that

$$\mathbb{D}_0^i Z_t^N = V_i(x_0), \quad t>0.$$

Our ellipticity assumption gives therefore the existence of $q^N(x_0,\cdot):\mathbb{R}_{>0}\times\mathbb{M}\to\mathbb{R}_{\geqslant 0}$, such that for every smooth $f:\mathbb{M}\to\mathbb{R}$,

$$\left(\mathbf{Q}_t^N f\right)(x_0) = \int_{\mathbb{M}} q_t^N(x_0,y) f(y)\,dy.$$

Second step. For $t>0$, let us consider the operator $\Theta_t^N(x_0)$ defined on $\Gamma(\mathbb{M},\mathcal{E})$ by the property that for $\eta\in\Gamma(\mathbb{M},\mathcal{E})$ and $y\in\mathcal{O}_{x_0}$,

$$\left(\Theta_t^N(x_0)\eta\right)(y) = \mathbb{E}\left(\left[\exp\left(\sum_{I,d(I)\leqslant N} \Lambda_I(B)_t \nabla_I\right)\eta\right](x_0) \,\Big|\, \exp\left(\sum_{I,d(I)\leqslant N} \Lambda_I(B)_t V_I\right)(x_0) = y\right).$$

We claim that $\Theta_t^N(x_0)$ is actually a potential, that is a section of the bundle $\mathbf{End}(\mathcal{E})$. For that, we have to show that for every smooth $f:\mathbb{M}\to\mathbb{R}$ and every $\eta\in\Gamma(\mathbb{M},\mathcal{E})$, $y\in\mathcal{O}_{x_0}$,

$$\left(\Theta_t^N(x_0) f\eta\right)(y) = f(y)\left(\Theta_t^N(x_0)\eta\right)(y).$$

If f is a smooth function on $\mathbb{M}$, we denote by $\mathcal{M}_f$ the operator on $\Gamma(\mathbb{M},\mathcal{E})$ that acts by multiplication by f. Due to the Leibniz rule for connections, we have for any word I:

$$[\nabla_I, \mathcal{M}_f] = \mathcal{M}_{V_I f}.$$

Consequently,

$$\left[\sum_{I,d(I)\leqslant N}\Lambda_I(B)_t\nabla_I,\mathcal{M}_f\right]=\mathcal{M}_{\sum_{I,d(I)\leqslant N}\Lambda_I(B)_tV_If}.$$

The above commutation property implies the following one:

$$\exp\left(\sum_{I,d(I)\leqslant N}\Lambda_I(B)_t\nabla_I\right)\mathcal{M}_f=\mathcal{M}_{\exp(\sum_{I,d(I)\leqslant N}\Lambda_I(B)_tV_I)f}\exp\left(\sum_{I,d(I)\leqslant N}\Lambda_I(B)_t\nabla_I\right).$$

Therefore,

$$\left[\Theta_t^N(x_0),\mathcal{M}_f\right]=0,$$

so that $\Theta_t^N(x_0)$ is a section of the bundle $\mathbf{End}(\mathcal{E})$. We can now conclude with the disintegration formula that for every $\eta\in\Gamma(\mathbb{M},\mathcal{E})$,

$$\left(\mathbf{P}_t^N\eta\right)(x_0)=\int_{\mathbb{M}}p_t^N(x_0,y)\eta(y)\,dy,$$

with

$$p_t^N(x_0,\cdot)=q_t^N(x_0,\cdot)\Theta_t^N(x_0).$$

Final step. Let us now turn to the proof of the pointwise estimate

$$p_t(x_0,x_0)=p_t^N(x_0,x_0)+O\left(t^{\frac{N+1-d}{2}}\right),\quad t\to 0.$$

Let $y\in\mathbb{M}$ be sufficiently close to x_0. Since $\mathcal{L}$ is elliptic at x_0, it is known (see for instance [8, Chapter 2]) that $p_t(x_0,y)$ admits a development

$$p_t(x_0,y)=\frac{e^{-\frac{d^2(x_0,y)}{2t}}}{(2\pi t)^{d/2}}\left(\sum_{k=0}^N\Psi_k(x_0,y)t^{\frac{k}{2}}+t^{\frac{N+1}{2}}\mathbf{R}_N(t,x_0,y)\right),\tag{3.3}$$

where the remainder term $\mathbf{R}_N(t,x_0,y)$ is bounded when $t\to 0$, $\Psi_k(x_0,\cdot)$ is a section of $\mathbf{End}(\mathcal{E})$ defined around x_0 and $d(\cdot,\cdot)$ is the distance defined around x_0 by the vector fields $V_1,\dots,V_d$. By using the fact that for every smooth $f:\mathbb{M}\to\mathbb{R}$,

$$\left(\mathbf{Q}_t^Nf\right)(x_0)=\mathbb{E}\left(f\left(\exp\left(\sum_{I,d(I)\leqslant N}\Lambda_I(B)_tV_I\right)(x_0)\right)\right),\quad t\geqslant 0,$$

and classical results for asymptotic development in small times of subelliptic heat kernels (see for instance [6] and [4, Chapter 3]), we get for $q_t^N(x_0,y)$ a development that is similar to (3.3). For $\Theta_t^N(x_0)$, the scaling property of Brownian motion implies that we have a short-time asymptotics

in powers $t^{k/2}$, $k \in \mathbb{N}$. Since,

$$p_t^N(x_0,\cdot) = q_t^N(x_0,\cdot)\Theta_t^N(x_0),$$

we deduce that

$$p_t^N(x_0,y) = \frac{e^{-\frac{d^2(x_0,y)}{2t}}}{(2\pi t)^{d/2}}\left(\sum_{k=0}^{N}\tilde{\Psi}_k(x_0,y)t^{\frac{k}{2}} + t^{\frac{N+1}{2}}\tilde{\mathbf{R}}_N(t,x_0,y)\right),$$

where the remainder term $\tilde{\mathbf{R}}_N(t,x_0,y)$ is bounded when $t \to 0$. With Theorem 3.1, we obtain that $\Psi_k = \tilde{\Psi}_k$, $k = 0,\dots,N$, and the required estimate easily follows. □

Remark 3.3. The question of the smoothness of p_t^N is not addressed here. It would require bounds on the inverse of the Malliavin matrix of Z_t^N.

Remark 3.4. Theorem 3.2 has been extended in a recent work (see [5]) to the case where a local Hörmander's condition is satisfied by the vector fields V_i.

From the previous theorem, we deduce an explicit asymptotic expansion of $p_t(x_0,x_0)$. If $I \in \{0,1,\dots,d\}^k$ is a word, we denote

$$\mathcal{F}_I = \nabla_I - \nabla_{V_I} \in \Gamma\big(\mathbb{M}, \mathbf{End}(\mathcal{E})\big).$$

Corollary 3.5. *For $N \geqslant 1$, when $t \to 0$,*

$$p_t(x_0,x_0) = d_t^N(x_0)\mathbb{E}\left(\exp\left(\sum_{I,d(I)\leqslant N}\Lambda_I(B)_t\mathcal{F}_I(x_0)\right)\,\Big|\,\sum_{I,d(I)\leqslant N}\Lambda_I(B)_tV_I(x_0)=0\right) + O\big(t^{\frac{N+1-d}{2}}\big),$$

where $d_t^N(x_0)$ is the density at 0 of the random variable $\sum_{I,d(I)\leqslant N}\Lambda_I(B)_tV_I(x_0)$.

Proof. Let us first observe that for the same reason than in the proof of Step 1 of the above theorem, the random process

$$\sum_{I,d(I)\leqslant N}\Lambda_I(B)_tV_I(x_0)$$

has a density $d_t^N(x_0,\cdot)$. Therefore, due to the disintegration formula, for every smooth $\eta \in \Gamma(\mathbb{M},\mathcal{E})$,

$$\begin{aligned}&\big(\mathbf{P}_t^N\eta\big)(x_0)\\ &\quad= \int_{\mathbf{T}_{x_0}\mathbb{M}}\mathbb{E}\left(\exp\left(\sum_{I,d(I)\leqslant N}\Lambda_I(B)_t\nabla_I\right)\eta(x_0)\,\Big|\,\sum_{I,d(I)\leqslant N}\Lambda_I(B)_tV_I(x_0)=0\right)d_t^N(x_0,y)\,dy,\end{aligned}$$

and the proof follows by letting η converge to Dirac distribution at x_0. □

4. The local index theorem for the Dirac operator on the spin bundle

The Atiyah–Singer index theorem for the Dirac operator on the spin bundle as proved in [2], is the following.

Theorem 4.1. *Let $\mathbb{M}$ be a compact, d-dimensional spin manifold, with d even. Let $\mathbf{D}$ be the Dirac operator on the spin bundle of $\mathbb{M}$. Then*

$$\mathbf{ind}(\mathbf{D}) = \left(\frac{1}{2i\pi}\right)^{d/2} \int_{\mathbb{M}} \left[A(\mathbb{M})\right]_d,$$

where $[A(\mathbb{M})]_d$ is the volume form on $\mathbb{M}$ obtained by taking the d-form piece of the A-genus

$$A(\mathbb{M}) = \det\left(\frac{\Omega}{2\sinh\frac{1}{2}\Omega}\right)^{1/2},$$

and Ω is the Riemannian curvature form defined in local orthonormal frame e_i with dual frame e_i^* by

$$\Omega = \frac{1}{2}\sum_{1\leqslant i,j\leqslant d} R(e_i, e_j)e_i^* \wedge e_j^*,$$

with R, Riemannian curvature.

Before we turn to the proof. Let us first recall some linear algebra constructions as can be found in [8, Chapter 3].

Let V be an oriented d-dimensional Euclidean space. We assume that the dimension d is even. The Clifford algebra $\mathbf{Cl}(V)$ over V is the algebra

$$\mathbf{T}(V) = \mathbb{R} \oplus V \oplus (V \otimes V) \oplus \cdots$$

quotiented by the relations

$$u \otimes v + v \otimes u + 2\langle u, v\rangle 1 = 0. \tag{4.4}$$

Let $e_1, \dots, e_d$ be an oriented basis of V. The family

$$e_{i_1} \dots e_{i_k}, \quad 0 \leqslant k \leqslant d,\ 1 \leqslant i_1 < \cdots < i_k \leqslant d,$$

forms a basis of $\mathbf{Cl}(V)$ that is therefore of dimension 2^d. In $\mathbf{T}(V)$ we can distinguish elements that are even from elements that are odd. This leads to a decomposition:

$$\mathbf{Cl}(V) = \mathbf{Cl}^-(V) \oplus \mathbf{Cl}^+(V),$$

with $V \subset \mathbf{Cl}^-(V)$.

A Clifford module is a vector space E over $\mathbb{R}$ (or $\mathbb{C}$) that is also a $\mathbf{Cl}(V)$-module and that admits a direct sum decomposition

$$E = E^- \oplus E^+$$

with

$$\mathbf{Cl}^-(V) \cdot E^- \subset E^-, \qquad \mathbf{Cl}^+(V) \cdot E^+ \subset E^+.$$

It can be shown that there is a unique Clifford module S, called the spinor module over V such that:

$$\mathbf{End}(S) \simeq \mathbb{C} \otimes \mathbf{Cl}(V).$$

In particular $\dim S = 2^{d/2}$. There is therefore a natural notion of supertrace on $\mathbf{Cl}(V)$ that is given by

$$\mathbf{Str}\, a = \mathbf{Tr}_{S^+} a - \mathbf{Tr}_{S^-} a,$$

where $a \in \mathbf{Cl}(V)$ is seen as an element of $\mathbf{End}(S)$. If

$$a = \sum_{k=0}^{d} \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant d} a_{i_1 \ldots i_k} e_{i_1} \ldots e_{i_k},$$

then we have

$$\mathbf{Str}\, a = \left(\frac{2}{i}\right)^{d/2} a_{1 \ldots d}. \tag{4.5}$$

If $\psi \in \mathfrak{so}(V)$, that is if $\psi : V \to V$ is a skew-symmetric map, we define

$$D\psi = \frac{1}{2} \sum_{1 \leqslant i < j \leqslant d} \langle \psi(e_i), e_j \rangle e_i e_j \in \mathbf{Cl}(V),$$

and observe that $D[\psi_1, \psi_2] = [D\psi_1, D\psi_2]$. The set $\mathbf{Cl}^2(V) = D\mathfrak{so}(V)$ is therefore a Lie algebra. The Lie group $\mathbf{Spin}(V)$ is the group obtained by exponentiating $\mathbf{Cl}^2(V)$ inside the Clifford algebra $\mathbf{Cl}(V)$; it is the two-fold universal covering of the orthogonal group $\mathbf{SO}(V)$. It can also be described as the set of $a \in \mathbf{Cl}(V)$ such that

$$a = v_1 \ldots v_{2k}, \quad 1 \leqslant k \leqslant \frac{d}{2},\ v_i \in V,\ \|v_i\| = 1.$$

We now come back to differential geometry and carry the above construction on the cotangent spaces of a spin manifold.

So, let $\mathbb{M}$ be a compact d-dimensional, Riemannian and oriented manifold. We assume that d is even. We furthermore assume that $\mathbb{M}$ admits a spin structure: that is, there exists a principal bundle on $\mathbb{M}$ with structure group $\mathbf{Spin}(\mathbb{R}^d)$ such that the bundle charts are compatible with the universal covering $\mathbf{Spin}(\mathbb{R}^d) \to \mathbf{SO}(\mathbb{R}^d)$. This bundle will be denoted $\mathcal{SP}(\mathbb{M})$ and π will

denote the canonical surjection. The spin bundle $\mathcal{S}$ over $\mathbb{M}$ is the vector bundle such that for every $x \in \mathbb{M}$, $\mathcal{S}_x$ is the spinor module over the cotangent space $\mathbf{T}_x^*\mathbb{M}$. At each point x, there is therefore a natural action of $\mathbf{Cl}(\mathbf{T}_x^*\mathbb{M}) \simeq \mathbf{End}(\mathcal{S}_x)$; this action will be denoted by $\mathbf{c}$.

On $\mathcal{S}$, there is a canonical elliptic first-order differential operator called the Dirac operator and denoted $\mathbf{D}$. In a local orthonormal frame e_i, with dual frame e_i^*, we have

$$\mathbf{D} = \sum_i c(e_i^*)\nabla_{e_i},$$

where ∇ is the Levi-Civita connection. We have an analogue of Weitzenböck formula which is the celebrated Lichnerowicz formula (see [8, Theorem 3.52]):

$$\mathbf{D}^2 = \Delta + \frac{s}{4},$$

where s is the scalar curvature of $\mathbb{M}$ and Δ is given in a local orthonormal frame e_i by

$$\Delta = -\sum_{i=1}^{d}(\nabla_{e_i}\nabla_{e_i} - \nabla_{\nabla_{e_i}e_i}).$$

After these preliminaries, we can now turn to the proof of the local index theorem for $\mathbf{D}$.

The first crucial step in all heat equations approaches of index theorems is the McKean–Singer type formula (see [21] and [8, Theorem 3.50]):

$$\mathbf{ind}(\mathbf{D}) = \mathbf{Str}(\mathbf{P}_t) = \int_{\mathbb{M}} \mathbf{Str}\, p_t(x,x)\,dx, \quad t > 0,$$

where $\mathbf{P}_t = e^{-\frac{1}{2}t\mathbf{D}^2}$, p_t is the corresponding Schwartz kernel, and dx is the Riemannian volume form.

By using the results of Section 3 we now show that we actually have

$$\lim_{t\to 0} \mathbf{Str}\, p_t(x,x)\,dx = \left(\frac{1}{2i\pi}\right)^{d/2}\big[A(\mathbb{M})\big]_d(x).$$

This last statement is first due to Patodi [22] and Gilkey [14] and implies the index theorem.

Let us fix $x_0 \in \mathbb{M}$ once and for all in what follows. Let e_i be a synchronous local orthonormal frame centered at x_0 with dual frame e_i^*. If needed, with a cut-off function, we extend smoothly the vector field e_i to be zero outside a neigborhood of x_0. At the center x_0 of the frame, we have therefore

$$\Delta = -\sum_{i=1}^{d}\nabla_{e_i}\nabla_{e_i}$$

and

$$[\nabla_{e_i}, \nabla_{e_j}] = R(e_i, e_j),$$

where R is the Riemannian curvature.

Let us denote ∇_0 the multiplication operator by $-\frac{s}{8}$ and $\nabla_i = \nabla_{e_i}$, $1 \leqslant i \leqslant d$.

If $1 \leqslant i < j \leqslant d$, we have at x_0

$$[\nabla_i, \nabla_j] = \mathbf{c}\big(DR(e_i, e_j)\big)$$

with

$$DR(e_i, e_j) = \frac{1}{2} \sum_{1 \leqslant k < l \leqslant d} \big\langle R(e_i, e_j)e_k, e_l\big\rangle e_k^* e_l^* \in \mathbf{Cl}^2\big(\mathbf{T}^*_{x_0}\mathbb{M}\big).$$

Since the Levi-Civita connection is a Clifford connection, if $1 \leqslant i < j < k \leqslant d$, we have at x_0,

$$\begin{aligned}\big[\nabla_i, [\nabla_j, \nabla_k]\big] &= \big[\nabla_i, \nabla_{[e_j, e_k]} + \mathbf{c}\big(DR(e_j, e_k)\big)\big] \\ &= \nabla_{[e_i,[e_j,e_k]]} + \mathbf{c}\big(DR\big(e_i, [e_j, e_k]\big) + \nabla_i DR(e_j, e_k)\big)\end{aligned}$$

and we observe that $DR(e_i, [e_j, e_k]) + \nabla_i DR(e_j, e_k) \in \mathbf{Cl}^2(\mathbf{T}^*_{x_0}\mathbb{M})$.

More generally, a recurrence procedure shows that if $1 \leqslant i_1 < \cdots < i_k \leqslant d$, then at x_0,

$$\nabla_I - \nabla_{e_I} = c(\mathcal{F}_I),$$

where $\mathcal{F}_I \in \mathbf{Cl}^2(\mathbf{T}^*_{x_0}\mathbb{M})$. If $0 \in I$, then it is seen that, at x_0, ∇_I acts by multiplication with a scalar, say $\mathcal{G}_I$. Let us consider

$$\Theta_t(x_0) = \mathbb{E}\Bigg(\mathbf{c}\Bigg(\exp\Bigg(X_t + \sum_{I, |I| \leqslant d, 0 \notin I} t^{|I|/2} \Lambda_I(B)_1 \mathcal{F}_I\Bigg)\Bigg) \Bigg| \sum_{I, |I| \leqslant d, 0 \notin I} t^{|I|/2} \Lambda_I(B)_1 e_I(x_0) = 0\Bigg),$$

where $X_t = \sum_{I, |I| \leqslant d, 0 \in I} t^{|I|/2} \Lambda_I(B)_1 \mathcal{G}_I$ is scalar term such that $X_0 = 0$.

Proposition 4.2.

$$\lim_{t \to 0} \frac{\mathbf{Str}\, \Theta_t(x_0)}{t^{d/2}} = \frac{1}{2^{d/2}(d/2)!} \mathbf{Str}\, \mathbb{E}\Bigg(\Bigg(\sum_{1 \leqslant i < j \leqslant d} DR(e_i, e_j) \int_0^1 B_s^i\, dB_s^j - B_s^j\, dB_s^i\Bigg)^{d/2} \Bigg| B_1 = 0\Bigg).$$

Proof. We have

$$\mathbf{Str}\, \Theta_t(x_0) = \mathbb{E}\Bigg(e^{X_t}\, \mathbf{Str} \exp\Bigg(\sum_{I, |I| \leqslant d, 0 \notin I} t^{|I|/2} \Lambda_I(B)_1 \mathcal{F}_I\Bigg) \Bigg| \sum_{I, |I| \leqslant d, 0 \notin I} t^{|I|/2} \Lambda_I(B)_1 e_I(x_0) = 0\Bigg).$$

Since $\mathcal{F}_I \in \mathbf{Cl}^2(\mathbf{T}^*_{x_0}\mathbb{M})$, according to (4.5), for any $k < \frac{d}{2}$,

$$\mathbf{Str}\Bigg(\sum_{I, |I| \leqslant d, 0 \notin I} t^{|I|/2} \Lambda_I(B)_1 \mathcal{F}_I\Bigg)^k = 0.$$

Therefore

$$\begin{aligned}
&\mathbf{Str}\,\Theta_t(x_0)\\
&\quad= \frac{1}{(d/2)!}\mathbb{E}\left(e^{X_t}\mathbf{Str}\left(\sum_{I,|I|\leqslant d,0\notin I} t^{|I|/2}\Lambda_I(B)_1\mathcal{F}_I\right)^{d/2} \,\Big|\, \sum_{I,|I|\leqslant d,0\notin I} t^{|I|/2}\Lambda_I(B)_1 e_I(x_0)=0\right)\\
&\quad= \frac{1}{(d/2)!}\mathbb{E}\left(e^{X_t}\mathbf{Str}\left(\sum_{I,|I|\leqslant d,0\notin I} t^{|I|/2}\Lambda_I(B)_1\mathcal{F}_I\right)^{d/2} \,\Big|\, \sum_{I,|I|\leqslant d,0\notin I} t^{(|I|-1)/2}\Lambda_I(B)_1 e_I(x_0)=0\right).
\end{aligned}$$

Let us now observe that almost surely,

$$\begin{aligned}
&\left(e^{X_t}\mathbf{Str}\left(\sum_{I,|I|\leqslant d,0\notin I} t^{|I|/2}\Lambda_I(B)_1\mathcal{F}_I\right)^{d/2}, \sum_{I,|I|\leqslant d,0\notin I} t^{(|I|-1)/2}\Lambda_I(B)_1 e_I(x_0)\right)\\
&\quad= \left(t^{d/2}\mathbf{Str}\left(\frac{1}{2}\sum_{1\leqslant i<j\leqslant d} DR(e_i,e_j)\int_0^1 B_s^i\,dB_s^j - B_s^j\,dB_s^i\right)^{d/2}, \sum_{i=1}^d B_1^i e_i\right)\\
&\qquad+ \left(t^{\frac{d+1}{2}}\mathbf{R}_1(t), t^{\frac{1}{2}}\mathbf{R}_2(t)\right),
\end{aligned}$$

where $\mathbf{R}_1$ and $\mathbf{R}_2$ are bounded when $t\to 0$.

Therefore, from the dominated convergence theorem

$$\begin{aligned}
&\lim_{t\to 0}\frac{\mathbf{Str}\,\Theta_t(x_0)}{t^{d/2}}\\
&\quad= \mathbb{E}\left(\mathbf{Str}\frac{1}{(d/2)!}\left(\frac{1}{2}\sum_{1\leqslant i<j\leqslant d} DR(e_i,e_j)\int_0^1 B_s^i\,dB_s^j - B_s^j\,dB_s^i\right)^{d/2} \,\Big|\, B_1=0\right). \qquad \square
\end{aligned}$$

We can now obtain the required limit:

Theorem 4.3.

$$\lim_{t\to 0}\mathbf{Str}\,p_t(x_0,x_0)\,dx_0 = \left(\frac{1}{2i\pi}\right)^{d/2}\big[A(\mathbb{M})\big]_d(x_0).$$

Proof. By Theorem 3.2, Corollary 3.5 and Proposition 4.2, we get

$$\begin{aligned}
&\lim_{t\to 0}\mathbf{Str}\,p_t(x_0,x_0)\,dx_0\\
&\quad= \frac{1}{(4\pi)^{d/2}(d/2)!}\mathbf{Str}\,\mathbb{E}\left(\left(\sum_{1\leqslant i<j\leqslant d} DR(e_i,e_j)\int_0^1 B_s^i\,dB_s^j - B_s^j\,dB_s^i\right)^{d/2} \,\Big|\, B_1=0\right)\\
&\qquad\times e_1^*\wedge\ldots\wedge e_d^*.
\end{aligned}$$

From (4.5) and from the expression of $DR(e_i, e_j)$, the above expression is also equal to the d-form piece of

$$\frac{1}{(2i\pi)^{d/2}(d/2)!}\mathbb{E}\left(\left(\sum_{1\leqslant i<j\leqslant d}\frac{1}{2}\Omega(e_i,e_j)\int_0^1 B_s^i\,dB_s^j - B_s^j\,dB_s^i\right)^{d/2}\,\Big|\, B_1=0\right).$$

This last expression is also the d-form piece of

$$\frac{1}{(2i\pi)^{d/2}}\mathbb{E}\left(\exp\left(\sum_{1\leqslant i<j\leqslant d}\frac{1}{2}\Omega(e_i,e_j)\int_0^1 B_s^i\,dB_s^j - B_s^j\,dB_s^i\right)\,\Big|\, B_1=0\right),$$

which turns out to be the d-form piece of the A-genus

$$\frac{1}{(2i\pi)^{d/2}}A(\mathbb{M}) = \frac{1}{(2i\pi)^{d/2}}\det\left(\frac{\Omega}{2\sinh\frac{1}{2}\Omega}\right)^{1/2},$$

from Lévy's area formula (see for instance [15, Lemma 7.6.6]). □

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The Taylor map on complex path groups ☆

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Abstract

Let $\mathcal{W}(G)$ denote the path group of an arbitrary complex connected Lie group. The existence of a heat kernel measure ν_t on $\mathcal{W}(G)$ has been shown in [M. Cecil, B.K. Driver, Heat kernel measure on loop and path groups, preprint, http://www.math.uconn.edu/~cecil/papers/p2.pdf; Infin. Dimens. Anal. Quantum Probab. Relat. Top., submitted for publication]. The present work establishes an isometric map, the Taylor map, from the space of $L^2(\nu_t)$-holomorphic functions on $\mathcal{W}(G)$ to a subspace of the dual of the universal enveloping algebra of $\mathrm{Lie}(H(G))$, where $H(G)$ is the Lie subgroup of finite energy paths. This map is shown to be surjective in the case where G is a simply connected graded Lie group.

Keywords: Path group; Taylor map; Skeleton theorem

1. Introduction

1.1. Background

A holomorphic function $u : \mathbb{C} \to \mathbb{C}$ is determined by its derivatives at the origin. One can recover values of u by its everywhere convergent Taylor expansion

$$u(z) = \sum_{k=0}^{\infty} \frac{u^{(k)}(0) z^k}{k!}. \tag{1}$$

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Let μ_t denote the Gaussian $\mu_t(z) = \frac{1}{\pi t} e^{\frac{-|z|^2}{t}}$. One can check that powers of z are orthogonal in $L^2(\mu_t)$. Specifically,

$$\int_{\mathbb{C}} z^k \bar{z}^l \mu_t(z)\, dx\, dy = \delta_{kl} t^k k!. \tag{2}$$

It is known that the Taylor expansion of u in Eq. (1) also gives a convergent expansion of u in the Hilbert space $L^2(\mu_t)$ with respect to the orthogonal basis $\{z^k\}_{k=0}^{\infty}$ (see [1,23]). In particular,

$$\|u\|^2_{L^2(\mu_t)} = \sum_{k=0}^{\infty} \frac{|u^{(k)}(0)|^2}{(k!)^2} \|z^k\|^2_{L^2(\mu_t)} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \left|u^{(k)}(0)\right|^2. \tag{3}$$

More generally, if $u : \mathbb{C}^d \to \mathbb{C}$ is holomorphic and $\mu_t(z) = (\frac{1}{\pi t})^d e^{\frac{-|z|^2}{t}}$, then

$$\|u\|^2_{L^2(\mu_t)} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{i_1,\ldots,i_k=1}^{d} \left|(\partial_{e_{i_1}} \partial_{e_{i_2}} \ldots \partial_{e_{i_k}} u)(0)\right|^2, \tag{4}$$

where $\{e_i\}_{i=1}^d$ is the standard basis for $\mathbb{C}^d$.

Let $T(\mathbb{C}^d)$ denote the tensor algebra over $\mathbb{C}^d$, $T(\mathbb{C}^d) := \bigoplus_{k=0}^{\infty} (\mathbb{C}^d)^{\otimes k}$. To every holomorphic $u : \mathbb{C}^d \to \mathbb{C}$ we can associate an element $\alpha_u = \bigoplus_{k=0}^{\infty} \alpha_k \in T(\mathbb{C}^d)$, where $\alpha_k \in (\mathbb{C}^d)^{\otimes k}$ is the symmetric tensor defined by

$$(\alpha_k, z_1 \otimes z_2 \otimes \cdots \otimes z_k)_{(\mathbb{C}^d)^{\otimes k}} = (\partial_{z_1} \partial_{z_2} \ldots \partial_{z_k} u)(0)$$

for every $z_1, z_2, \ldots, z_k \in \mathbb{C}^d$. Here $(,)_{(\mathbb{C}^d)^{\otimes k}}$ denotes the inner product on $(\mathbb{C}^d)^{\otimes k}$ arising from the standard inner product on $\mathbb{C}^d$. If we define a norm $\|\cdot\|_t$ on $T(\mathbb{C}^d)$ by

$$\|\beta\|_t^2 := \sum_{k=0}^{\infty} \frac{t^k}{k!} \|\beta_k\|^2_{(\mathbb{C}^d)^{\otimes k}} \tag{5}$$

for $\beta = \bigoplus_{k=0}^{\infty} \beta_k$ with $\beta_k \in (\mathbb{C}^d)^{\otimes k}$, then Eq. (4) indicates that the map $u \to \alpha_u$ is unitary.

The physicist V.A. Fock introduced this isomorphism in 1932 in [10], and the work was later clarified by Segal and Bargmann in the 1950s and 1960s (see [1,22,23]). The correspondence proves useful in understanding the structure of quantum fields. It is also closely related to the characterization theorem for generalized function in white noise analysis (see, for example, [16, 19,20]).

In [5], Driver and Gross proved a generalization of the above result on a complex connected Lie group G with given Hermitian inner product $(\ ,\ )$ on the Lie algebra $\mathfrak{g} = T_e G$. In this context, μ_t denotes heat kernel measure on G with respect to a right invariant Haar measure dx. Let $T(\mathfrak{g})$ denote the tensor algebra over $\mathfrak{g}$. Then $T(\mathfrak{g})_t^*$ denotes the completion of $T(\mathfrak{g})^*$ with respect to a norm inspired by Eq. (1). Let J denote the ideal in $T(\mathfrak{g})$ generated by $\{\xi \otimes \eta - \eta \otimes \xi -$

$[\xi, \eta]$: $\xi, \eta \in \mathfrak{g}\}$, and $J_t^0 = \{\alpha \in T(\mathfrak{g})_t^*$: $\langle \alpha, v\rangle = 0$ for all $v \in J\}$. To any holomorphic function u on G, and element α_u of J_t^0 is associated by

$$\langle \alpha_u, \xi_1 \otimes \cdots \otimes \xi_k\rangle = (\tilde{\xi}_1 \cdots \tilde{\xi}_k u)(e). \tag{6}$$

Then the main theorem of [5] states that if G is simply connected, then the map $u \in \mathcal{H}L^2(G, \mu_t(x)dx) \to \alpha_u \in J_t^0$ is unitary. Infinite-dimensional analogues have been proven by Gordina in [12,13] on $GL(H)$, the group of invertible operators on a complex Hilbert space H, and groups associated with a *Π_1-factor*. The goal of this work is to establish yet another infinite-dimensional Taylor map, this one on $\mathcal{W}(G)$, the group of paths based at the identity of a simply connected complex Lie group G.

1.2. Statement of results

Let G be an arbitrary complex simply connected Lie group and $\mathfrak{g} = T_e G$ its Lie algebra. Assume there is a given Hermitian inner product $(,)_\mathfrak{g}$ on $\mathfrak{g}$. Let $\langle,\rangle$ denote the real left-invariant Riemannian metric on G determined by

$$\langle \tilde{A}, \tilde{B}\rangle = \mathrm{Re}(A, B)_\mathfrak{g} \quad \forall A, B \in \mathfrak{g}$$

where $\tilde{A}$ denotes the unique left invariant vector field satisfying $\tilde{A}(e) = A \in \mathfrak{g}$. We will use $\langle,\rangle_\mathfrak{g}$ to denote this inner product on $\mathfrak{g}$.

We will also use the notation $L_g x = gx$ for $g, x \in G$ and f_* to denote the differential of a smooth map between two manifolds, $f : M \to N$. With this notation, $\tilde{A}(g) = L_{g*}A$ and hence $\langle \tilde{A}(g), \tilde{B}(g)\rangle = \langle L_{g^{-1}*}A, L_{g^{-1}*}B\rangle_\mathfrak{g}$.

Choose $\mathfrak{X}_\mathbb{C}$ to be an orthonormal basis for the complex inner product space $(\mathfrak{g}, (,)_\mathfrak{g})$. If we denote the complex structure on $\mathfrak{g}$ by $\mathcal{J}$, then $\mathfrak{X}_\mathbb{R} = \{\mathfrak{X}_\mathbb{C}, \mathcal{J}\mathfrak{X}_\mathbb{C}\}$ is an orthonormal basis of the real inner product space $(\mathfrak{g}, \langle,\rangle_\mathfrak{g})$. Define the Laplacian on G by

$$\Delta_G = \sum_{A \in \mathfrak{X}_\mathbb{C}} \tilde{A}^2 + \widetilde{\mathcal{J}A}^2 = \sum_{A \in \mathfrak{X}_\mathbb{R}} \tilde{A}^2. \tag{7}$$

Then Δ_G is a strongly elliptic operator and in the case where G is unimodular, it is the Laplace–Beltrami operator. Let $\mathcal{H}(G)$ denote the space of complex-valued holomorphic functions on G, and let dx denote a fixed right invariant Haar measure.

Define $\mathcal{W}(G)$ to be the based path group on G, i.e. the continuous paths $\sigma : [0, 1] \to G$ such that $\sigma(0) = e$. Similarly, we will let $\mathcal{W}(\mathfrak{g})$ denote the continuous paths $h : [0, 1] \to \mathfrak{g}$ such that $h(0) = 0$. The group operation on $\mathcal{W}(G)$ is given pointwise and is denoted with $\cdot$,

$$(\sigma \cdot \tau)(s) := \sigma(s)\tau(s) \tag{8}$$

for $\sigma, \tau \in \mathcal{W}(G)$, while $\mathcal{W}(\mathfrak{g})$ has a pointwise defined bracket,

$$[h, k](s) := \big[h(s), k(s)\big] \tag{9}$$

for $h, k \in \mathcal{W}(\mathfrak{g})$. We also define a pointwise exponential map $\mathcal{W}(\mathfrak{g}) \to \mathcal{W}(G)$ by

$$\big(e^h\big)(s) := e^{h(s)} \tag{10}$$

for $h \in \mathcal{W}(\mathfrak{g})$. We will use $\underline{e}$ to denote the identity path in $\mathcal{W}(G)$ and $\underline{0}$ to denote the identity path in $\mathcal{W}(\mathfrak{g})$.

Define the *energy* of a path $\sigma \in \mathcal{W}(G)$ by

$$E(\sigma) := \begin{cases} \int_0^1 |L_{\sigma(s)^{-1}*}\sigma'(s)|_{\mathfrak{g}}^2 \, ds, & \text{if } \sigma \text{ is absolutely continuous,} \\ \infty, & \text{otherwise.} \end{cases}$$

The *finite energy subgroup of* $\mathcal{W}(G)$ is then given by

$$H(G) = \{\sigma \in \mathcal{W}(G) \mid E(\sigma) < \infty\}.$$

Similarly, for a $h \in \mathcal{W}(\mathfrak{g})$, let

$$(h,h)_{H(\mathfrak{g})} := \begin{cases} \int_0^1 |h'(s)|_{\mathfrak{g}}^2 \, ds, & \text{if } h \text{ is absolutely continuous,} \\ \infty, & \text{otherwise.} \end{cases}$$

We define the *Cameron–Martin subspace of* $\mathcal{W}(\mathfrak{g})$ as

$$H(\mathfrak{g}) = \{h \in \mathcal{W}(\mathfrak{g}) \mid (h,h)_{H(\mathfrak{g})} < \infty\}.$$

Given $h, k \in H(\mathfrak{g})$, we can define a Hermitian inner product on $H(\mathfrak{g})$ by

$$(h,k)_{H(\mathfrak{g})} = \int_0^1 \big(h'(s), k'(s)\big)_{\mathfrak{g}} \, ds.$$

With this inner product, $H(\mathfrak{g})$ is a Hilbert space. We let $\langle h,k\rangle_{H(\mathfrak{g})} = \operatorname{Re}(h,k)_{H(\mathfrak{g})}$. It is often convenient to think of $H(\mathfrak{g})$ as the "Lie algebra" of $\mathcal{W}(G)$.

Let $S_{\mathbb{C}} \subset H(\mathfrak{g})$ be an orthonormal basis for the complex inner product space $(H(\mathfrak{g}), (,)_{H(\mathfrak{g})})$. The complex structure $\mathcal{J}$ on $H(\mathfrak{g})$ is that on $\mathfrak{g}$ defined pointwise. That is, for $h \in H(\mathfrak{g})$, $\mathcal{J}h \in H(\mathfrak{g})$ is given by $(\mathcal{J}h)(t) = \mathcal{J}(h(t))$ for all $t \in [0,1]$. Then $S_{\mathbb{R}} = \{S_{\mathbb{C}}, \mathcal{J}S_{\mathbb{C}}\}$ is an orthonormal basis for the real inner product space $(H(\mathfrak{g}), \langle,\rangle_{H(\mathfrak{g})})$. In the above definitions, if we consider $\mathfrak{g} = \mathbb{C}$ with standard inner product, then $H(\mathbb{C})$ is the classical complex Wiener space. We will use $S(\mathbb{C})$ to denote an orthonormal basis of $H(\mathbb{C})$.

Given a partition of $[0,1]$, $\mathcal{P} = \{0 = s_0 < s_1 < \cdots < s_n < s_{n+1} = 1\}$, and $g \in \mathcal{W}(G)$, define $\pi_{\mathcal{P}} : \mathcal{W}(G) \to G^n$ by

$$\pi_{\mathcal{P}}(g) = \big(g(s_1), g(s_2), \ldots, g(s_n)\big).$$

Notation 1. We will use the notation $\#(\mathcal{P})$ to denote the number of partition points of $\mathcal{P}$.

Definition 2. A function f is a *smooth cylinder function* on $\mathcal{W}(G)$ if there exists a partition $\mathcal{P}$ and a C^∞ function $F : G^{\#(\mathcal{P})} \to \mathbb{C}$ such that $f = F \circ \pi_{\mathcal{P}}$. f is a *holomorphic cylinder function* if $F : G^{\#(\mathcal{P})} \to \mathbb{C}$ is holomorphic. We will use $\mathcal{F}(\mathcal{W})$ to denote the set of smooth cylinder functions on $\mathcal{W}(G)$ and $\mathcal{HF}(\mathcal{W})$ to denote the set of holomorphic cylinder functions on $\mathcal{W}(G)$,

For $f \in \mathcal{F}(\mathcal{W})$, $g \in \mathcal{W}(G)$, and $h \in H(\mathfrak{g})$, define

$$(\tilde{h} f)(g) := \frac{d}{dt}\bigg|_0 f\big(g \cdot e^{th}\big).$$

Notation 3. Suppose $F \in C^\infty(G^{\#(\mathcal{P})})$. Then for $A \in \mathfrak{g}$ and $i \in \{1, 2, \ldots, n\}$ let

$$\tilde{A}^{(i)} F(x_1, x_2, \ldots, x_n) := \frac{d}{dt}\bigg|_0 F\big(x_1, \ldots, x_i e^{tA}, x_{i+1}, \ldots, x_n\big). \tag{11}$$

Remark 4. Notice if $f = F \circ \pi_{\mathcal{P}} \in \mathcal{F}(\mathcal{W})$, then for $h \in H(\mathfrak{g})$,

$$\tilde{h} f = \sum_{i=1}^{n} \big(\widetilde{h(s_i)}^{(i)} F\big) \circ \pi_{\mathcal{P}}. \tag{12}$$

In particular, note that $\tilde{h} f$ is still a smooth cylinder function based on the same partition $\mathcal{P}$.

In [2], $\mathcal{W}(G)$-valued diffusions are constructed associated to certain second order differential operators on cylinder functions. For an appropriate choice of initial data, this implies the existence of a $\mathcal{W}(G)$-valued Brownian motion. We define ν_t, our heat kernel measure, to be the endpoint distribution of this process. Section 2 is devoted to a summary of the results of [2], many of which are essential for the development of the Taylor map in this setting.

Definition 5. Let $\mathcal{H}_t$ denote the $L^2(\nu_t)$-closure of $\mathcal{H}\mathcal{F}(\mathcal{W}) \cap L^2(\nu_t)$.

$\mathcal{H}_t$ will serve as our Hilbert space of holomorphic functions. In order to state our version of the Taylor map, we must establish a suitable notion of "derivatives at the origin" for a function $f \in \mathcal{H}_t$, which in general need not be continuous. The following theorem is motivated by the results of Sugita and others [24,25] in the setting of an abstract Wiener space. We use $\mathcal{H}(H(G))$ to denote the functions on $H(G)$ which are holomorphic in the sense of Gross and Malliavin [14]. This is elaborated on in Section 3.

Theorem 6 *(Theorem 29). There exists an injective linear map $R : \mathcal{H}_t \to \mathcal{H}(H(G))$ with the following properties*:

(1) *For f a holomorphic cylinder function, $Rf = f|_{H(G)}$.*
(2) *For $g \in H(G)$, $|(Rf)(g)|^2 \leqslant \|f\|^2_{L^2(\nu_t)} e^{\frac{|g|^2_{H(G)}}{t}}$, where $|g|_{H(G)}$ denotes the Riemannian distance between g and the identity path in $H(G)$.*

Denote by $T(H(\mathfrak{g}))$ the tensor algebra over the complex vector space $H(\mathfrak{g})$. For each $t > 0$, define a norm on $T(H(\mathfrak{g}))$ by

$$\|\beta\|_t^2 = \sum_{k=0}^{\infty} \frac{k!}{t^k} |\beta_k|^2, \quad \text{where } \beta = \bigoplus_{k=0}^{\infty} \beta_k,$$

with $\beta_k \in H(\mathfrak{g})^{\otimes k}$ for $k = 0, 1, 2, \ldots,$ where $|\beta_k|$ denotes the cross norm on $H(\mathfrak{g})^{\otimes k}$ arising from the inner product on $H(\mathfrak{g})^{\otimes k}$ determined by the given inner product on $H(\mathfrak{g})$. We will denote the completion of $T(H(\mathfrak{g}))$ with respect to this norm by $T(H(\mathfrak{g}))_t$. Then the Hermitian inner product on $T(H(\mathfrak{g}))_t$ given by polarizing the above turns $T(H(\mathfrak{g}))_t$ into complex Hilbert space.

The topological dual space of $T(H(\mathfrak{g}))_t$ may be identified with the subspace $T(H(\mathfrak{g}))_t^*$ of the algebraic dual $T(H(\mathfrak{g}))'$ of $T(H(\mathfrak{g}))$ consisting of those $\alpha \in T(H(\mathfrak{g}))'$ such that

$$\|\alpha\|_t^2 := \sum_{k=0}^{\infty} \frac{t^k}{k!} |\alpha_k|^2_{(H(\mathfrak{g})^*)^{\otimes k}} < \infty, \tag{13}$$

where $\alpha_k \in (H(\mathfrak{g})^*)^{\otimes k}$ and $|\alpha_k|_{(H(\mathfrak{g})^*)^{\otimes k}}$ denotes the cross norm on $(H(\mathfrak{g})^*)^{\otimes k}$ determined by the Hermitian inner product on $H(\mathfrak{g})^*$ dual to the given Hermitian inner product on $H(\mathfrak{g})$.

For $u \in \mathcal{H}(H(G))$, let $\alpha_u \in T(H(\mathfrak{g}))'$ be defined by

$$\langle \alpha_u, h_1 \otimes h_2 \otimes \cdots \otimes h_n \rangle = (\tilde{h}_1 \tilde{h}_2 \cdots \tilde{h}_n u)(\underline{e}).$$

We will use Driver's suggestive notation $\alpha_u = (1 - D)_{\underline{e}}^{-1} u$. By definition of the Lie bracket on $H(\mathfrak{g})$, α_u annihilates the two-sided ideal

$$J\big(H(\mathfrak{g})\big) := \big\langle \xi \otimes \eta - \eta \otimes \xi - [\xi, \eta] \,\big|\, \xi,\ \eta \in H(\mathfrak{g}) \big\rangle.$$

Let $J^0(H(\mathfrak{g}))$ denote the annihilator of $J(H(\mathfrak{g}))$, that is

$$J^0\big(H(\mathfrak{g})\big) := \big\{ \alpha \in T\big(H(\mathfrak{g})\big)' \,\big|\, |\alpha|_{J(H(\mathfrak{g}))} \equiv 0 \big\}, \tag{14}$$

and let

$$J_t^0\big(H(\mathfrak{g})\big) := J^0\big(H(\mathfrak{g})\big) \cap T\big(H(\mathfrak{g})\big)_t^*. \tag{15}$$

We are now able to define the Taylor map on $\mathcal{H}_t$. Using the above notation, we send $f \in \mathcal{H}_t \to \alpha_{Rf} \in J_t^0(H(\mathfrak{g}))$. That is, the Taylor map is the composition $(1 - D)_{\underline{e}}^{-1} R$. We are able to show the following in Section 4.

Theorem 7 *(Corollary 33). For any complex Lie group G, the Taylor map, $(1 - D)_{\underline{e}}^{-1} R : \mathcal{H}_t \to J_t^0(H(\mathfrak{g}))$, is an isometry.*

In [1,5,12,13,22,23], the analogous Taylor map was also surjective. Section 5 is devoted to proving that our Taylor map is surjective when G is a simply connected graded Lie group. We very briefly sketch the method here. A more thorough outline can be found at the beginning of Section 5.

To every $\alpha \in J_T^0(H(\mathfrak{g}))$ and $\mathcal{P}$ a partition of $[0, 1]$, we associate a holomorphic cylinder function denoted $F_{\mathcal{P}}$ (Definition 45). A sequence of partitions $\{\mathcal{P}_n\}_{n=1}^{\infty}$ will be called a *refining sequence of partitions* if $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ for all n. For simplicity, we will have $\#(\mathcal{P}_n) = n$. We define $\alpha(\mathcal{P}_n)$ to be the set of derivatives of $F_{\mathcal{P}_n}$ at the identity path (Notation 46). We are able to prove the following in Section 5.5.

Theorem 8 *(Theorem 74). Let $\alpha \in J_T^0(H(\mathfrak{g}))$ be of finite rank. For each $n > 0$, let $\alpha(\mathcal{P}_n)$ be given as in Notation 46. Then for all $n > 0$, $\alpha(\mathcal{P}_n) \in J_T^0(H(\mathfrak{g}))$ and*

$$\left\| \alpha - \alpha(\mathcal{P}_n) \right\|_{J_T^0(H(\mathfrak{g}))} \to 0 \quad \text{as } n \to \infty.$$

When G is a simply connected graded Lie group, the finite rank tensors are dense in $J_T^0(H(\mathfrak{g}))$ (see Theorem 41). In this case, surjectivity of the Taylor map follows from the above theorem.

Corollary 9 *(Corollary 75). For all $T > 0$, the Taylor map $(1-D)_{\underline{e}}^{-1} R : \mathcal{H}_T \to J_T^0(H(\mathfrak{g}))$ is surjective for G a complex simply connected graded Lie group.*

2. Preliminaries

For the entirety of this section, we will treat $\mathfrak{g}$ and $H(\mathfrak{g})$ as real spaces, with inner products $\langle , \rangle_{\mathfrak{g}}$ and $\langle , \rangle_{H(\mathfrak{g})}$, respectively.

2.1. Finite-dimensional approximations

A common technique in the sequel will be to approximate our infinite-dimensional path space with natural finite-dimensional spaces arising from a partition of $[0, 1]$. This subsection is primarily a review of techniques used in [4,6,7]. Recall that we set $\mathcal{P} = \{0 = s_0 < s_1 < \cdots < s_n < s_{n+1} = 1\}$.

Let $K : [0, 1]^2 \to \mathbb{R}$ denote the reproducing kernel of $H(\mathbb{R})$. Specifically,

$$K(s, t) := s \wedge t.$$

For a more detailed discussion of K and its properties, see Section 3.1 of [7] or Proposition 66 below.

Definition 10. Define $\langle , \rangle_{\mathcal{P}}$ to be the unique left invariant Riemannian metric on the fibers of $TG^{\#(\mathcal{P})}$ such that for $1 \leqslant i, j \leqslant n$,

$$\left\langle A^{(i)}, B^{(j)} \right\rangle_{\mathcal{P}} = \langle A, B \rangle_{\mathfrak{g}} Q_{ij} \qquad \text{for all } A, B \in \mathfrak{g},$$

where Q is the inverse of the matrix $\{K(s_i, s_j)\}_{i,j=1}^n$ and $A^{(i)}$ and $B^{(j)}$ are defined as in Notation 3.

Given the metrics described above, our goal is to establish an isometry between an appropriate subspace of $H(\mathfrak{g})$ and $\mathfrak{g}^{\#(\mathcal{P})}$. Given a partition, we consider the subspace of paths piecewise linear off of our partition points.

Definition 11. Let $H_{\mathcal{P}}(\mathfrak{g})$ denote the subspace of $H(\mathfrak{g})$ given by

$$H_{\mathcal{P}}(\mathfrak{g}) := \left\{ h \in H(\mathfrak{g}) \cap C^2\big((0, 1)/\mathcal{P}\big) \;\middle|\; h'' = 0 \text{ on } [0, 1]/\mathcal{P} \right\}.$$

We let $P_{\mathcal{P}} : H(\mathfrak{g}) \to H_{\mathcal{P}}(\mathfrak{g})$ denote orthogonal projection.

$H_{\mathcal{P}}(\mathfrak{g})$ is a subspace of $H(\mathfrak{g})$, but it is not a Lie subalgebra with the inherited pointwise commutator.

Proposition 12. *The commutator defined by $[h,k]_{\mathcal{P}} := P_{\mathcal{P}}[h,k]$, for $h,k \in H_{\mathcal{P}}(\mathfrak{g})$ makes $H_{\mathcal{P}}(\mathfrak{g})$ into a Lie algebra.*

Proof. Bilinearity and antisymmetry are trivial to check. For any $h,k \in H_{\mathcal{P}}(\mathfrak{g})$, $[h,k]_{\mathcal{P}}$ is piecewise linear and therefore determined by its values on the partition points. Since for any $s_i \in \mathcal{P}$, $[h,k]_{\mathcal{P}}(s_i) = [h(s_i), k(s_i)]$, the Jacobi identity follows from that for $[,]$ on $\mathfrak{g}$. □

Proposition 13. *Let $\Lambda_{\mathcal{P}} : H(\mathfrak{g}) \to \mathfrak{g}^{\#(\mathcal{P})}$ be given by*

$$\Lambda_{\mathcal{P}}(h) = \big(h(s_1), \ldots, h(s_n)\big).$$

Note that $\Lambda_{\mathcal{P}} = \pi_{\mathcal{P}*\underline{e}}$. Then $Nul(\Lambda_{\mathcal{P}}) = H_{\mathcal{P}}(\mathfrak{g})^{\perp}$.

Proof. First suppose that $h \in Nul(\Lambda_{\mathcal{P}})$, i.e. $h(s_i) = 0$ for all $i = 0, 1, \ldots, n$. Let $k \in H_{\mathcal{P}}(\mathfrak{g})$. Then there exist $A_0, \ldots, A_n \in \mathfrak{g}$ such that

$$k(t) = \sum_{i=0}^{n} A_i (t \wedge s_{i+1} - t \wedge s_i). \tag{16}$$

Notice that a.e.

$$k'(t) = \sum_{i=0}^{n} A_i 1_{(s_{i-1}, s_i)}(t). \tag{17}$$

Then

$$\begin{aligned}
\langle h,k\rangle_{H(\mathfrak{g})} &= \sum_{i=0}^{n} \int_{s_i}^{s_{i+1}} \big\langle h'(t), A_i\big\rangle_{\mathfrak{g}}\, dt \\
&= \sum_{i=0}^{n} \big\langle h(s_{i+1}) - h(s_i), A_i\big\rangle_{\mathfrak{g}} \\
&= \sum_{i=0}^{n} \langle 0, A_i\rangle_{\mathfrak{g}} = 0.
\end{aligned}$$

Therefore, $Nul(\Lambda_{\mathcal{P}}) \subseteq H_{\mathcal{P}}(\mathfrak{g})^{\perp}$. Now suppose that $h \in H_{\mathcal{P}}(\mathfrak{g})^{\perp}$. Let $A_i = h(s_{i+1}) - h(s_i) \in \mathfrak{g}$ for $i = 0, 1, \ldots, n$. Define $k(t)$ by Eq. (16). Then $k \in H_{\mathcal{P}}(\mathfrak{g})$ and so we necessarily have

$$0 = \langle h,k\rangle_{H(\mathfrak{g})} = \sum_{i=0}^{n} \big\langle h(s_{i+1}) - h(s_i), A_i\big\rangle_{\mathfrak{g}} = \sum_{i=0}^{n} \big\| h(s_{i+1}) - h(s_i)\big\|_{\mathfrak{g}}^{2},$$

which clearly implies that $h(s_{i+1}) - h(s_i) = 0$ for $i = 0, 1, \ldots, n$. But since $h(0) = 0$, we have that $h(s_i) = 0$ for all $i = 0, 1, \ldots, n$. Therefore, $h \in Nul(\Lambda_{\mathcal{P}})$, and $H_{\mathcal{P}}(\mathfrak{g})^{\perp} \subseteq Nul(\Lambda_{\mathcal{P}})$. □

Proposition 14. *Consider $H_{\mathcal{P}}(\mathfrak{g})$ as described in Definition* 11 *with inner product $\langle,\rangle_{H(\mathfrak{g})}$ and commutator $[,]_{\mathcal{P}}$, and $\mathfrak{g}^{\#(\mathcal{P})}$ with inner product $\langle,\rangle_{\mathcal{P}}$ and commutator $[,]$. Then the map $\Lambda_{\mathcal{P}}: H_{\mathcal{P}}(\mathfrak{g}) \to \mathfrak{g}^{\#(\mathcal{P})}$, the map described in Proposition* 13 *restricted to $H_{\mathcal{P}}(\mathfrak{g})$, is an isometric Lie algebra isomorphism.*

Proof. To see that $\Lambda_{\mathcal{P}}$ is an isometry, associate to $A = (A_0, \dots, A_n) \in \mathfrak{g}^{\#(\mathcal{P})}$ a path $h_A(t) := \sum_{i=0}^n K(s_i, t)A_i \in H_{\mathcal{P}}(\mathfrak{g})$. Then if $B = (B_0, \dots, B_n) \in \mathfrak{g}^{\#(\mathcal{P})}$,

$$\begin{aligned} \langle h_A, h_B\rangle_{H(\mathfrak{g})} &= \sum_{i,j=0}^n \bigl\langle K(s_i,\cdot), K(s_j,\cdot)\bigr\rangle_{H(\mathbb{R})}\langle A_i, B_j\rangle_{\mathfrak{g}} \\ &= \sum_{i,j=0}^n K(s_i,s_j)\langle A_i, B_j\rangle_{\mathfrak{g}}, \end{aligned} \tag{18}$$

where we have used the reproducing kernel property of K, see [7, Lemma 3.3].

$\{K(s_i, s_j)\}_{i,j=1}^n$ is a positive definite matrix, so setting $B = A$ in Eq. (18) shows that $A \to h_A$ is injective and hence surjective by the rank nullity theorem. By Definition 10,

$$\begin{aligned} \bigl\langle \Lambda_{\mathcal{P}}(h_A), \Lambda_{\mathcal{P}}(h_B)\bigr\rangle_{\mathcal{P}} &= \sum_{k,l=0}^n \bigl\langle h_A(s_k)^{(k)}, h_B(s_l)^{(l)}\bigr\rangle_{\mathcal{P}} \\ &= \sum_{k,l=0}^n Q_{kl}\bigl\langle h_A(s_k), h_B(s_l)\bigr\rangle_{\mathfrak{g}} \\ &= \sum_{i,j,k,l=0}^n Q_{kl}K(s_i,s_k)K(s_j,s_l)\langle A_i, B_j\rangle_{\mathfrak{g}} \\ &= \sum_{i,j=0}^n K(s_i,s_j)\langle A_i, B_j\rangle_{\mathfrak{g}} \\ &= \langle h_A, h_B\rangle_{H(\mathfrak{g})}, \end{aligned}$$

where Eq. (18) was used in the last equality. □

Remark 15. Since $\Lambda_{\mathcal{P}}$ commutes with the complex structures on $H(\mathfrak{g})$ and $\mathfrak{g}^{\#(\mathcal{P})}$, it follows that Proposition 14 holds when $H(\mathfrak{g})$ and $\mathfrak{g}^{\#(\mathcal{P})}$ are considered as complex Lie algebras with inner products $(\,,)_{H(\mathfrak{g})}$ and $(\,,)_{\mathcal{P}}$, respectively, where $(\,,)_{\mathcal{P}}$ is given by the analogous formula to that in Definition 10. Specifically,

$$\bigl(A^{(i)}, B^{(j)}\bigr)_{\mathcal{P}} = (A,B)_{\mathfrak{g}}Q_{ij} \quad \text{for all } A, B \in \mathfrak{g}.$$

Approximating a path $h \in H(\mathfrak{g})$ by $P_{\mathcal{P}}h \in H_{\mathcal{P}}(\mathfrak{g})$ will play an important role in showing surjectivity of the Taylor map in the sequel. We will revisit this subject in detail in Section 5.3.

It is known that $H(G)$ is a Hilbert manifold, and hence has a natural notion of Riemannian distance. We will not develop this point of view beyond the few statements that follow, though we

refer the interested reader to the work of Klingenberg, for example [18]. The following definition is motivated by these considerations.

Definition 16. We define a Riemannian distance on $H(G)$ as follows. For $g \in H(G)$, let

$$|g|_{H(G)} := \inf \int_0^1 \left\| L_{\sigma(t,\cdot)^{-1}*} \frac{d}{dt} \sigma(t,\cdot) \right\|_{H(\mathfrak{g})} dt,$$

where the infimum is taken over all jointly C^1 paths $\sigma : [0,1]^2 \to G$ such that $\sigma(t,\cdot) \in H(G)$ for all t, $\sigma(0,s) = e$, and $\sigma(1,s) = g(s)$ for all s.

Corollary 17. *For any partition $\mathcal{P}$ and any $g \in H(G)$,*

$$|\pi_{\mathcal{P}} g|_{\mathcal{P}} \leqslant |g|_{H(G)},$$

where $|\pi_{\mathcal{P}} g|_{\mathcal{P}}$ denotes the Riemannian distance between $\pi_{\mathcal{P}} g$ and $(e, e, \dots, e) \in G^{\#(\mathcal{P})}$. That is

$$|\pi_{\mathcal{P}} g|_{\mathcal{P}} = \inf \int_0^1 \left| L_{\sigma(s)^{-1}*} \sigma'(s) \right|_{\mathcal{P}} ds,$$

where the infimum is taken over all C^1-paths σ into $G^{\#(\mathcal{P})}$ such that $\sigma(0) = (e, e, \dots, e)$ and $\sigma(1) = \pi_{\mathcal{P}} g$.

Proof. For any jointly C^1 path $\sigma : [0,1]^2 \to G$ such that $\sigma(0,s) = e$, $\sigma(1,s) = g(s)$, we have $\pi_{\mathcal{P}} \sigma(t,\cdot)$ is a $G^{\#(\mathcal{P})}$-valued C^1 path such that $\pi_{\mathcal{P}} \sigma(0,\cdot) = (e, e, \dots, e)$ and $\pi_{\mathcal{P}} \sigma(1,\cdot) = \pi_{\mathcal{P}} g(\cdot)$. By Proposition 14, it follows that

$$\begin{aligned}
\int_0^1 \left\| L_{\sigma(t,\cdot)^{-1}*} \frac{d}{dt} \sigma(t,\cdot) \right\|_{H(\mathfrak{g})} dt &\geqslant \int_0^1 \left\| \Lambda_{\mathcal{P}} \left(L_{\sigma(t,\cdot)^{-1}*} \frac{d}{dt} \sigma(t,\cdot) \right) \right\|_{\mathcal{P}} dt \\
&= \int_0^1 \left\| L_{\pi_{\mathcal{P}} \sigma(t,\cdot)^{-1}*} \frac{d}{dt} \pi_{\mathcal{P}} \sigma(t,\cdot) \right\|_{\mathcal{P}} dt \\
&\geqslant |\pi_{\mathcal{P}} g|_{\mathcal{P}}.
\end{aligned}$$

Therefore the inequality holds as the infimum is taken over all admissible σ. □

2.2. Heat kernel measure

In [2], $\mathcal{W}(G)$-valued diffusions are constructed associated to a linearly independent subset $\Gamma \subset \mathfrak{g}$. It is shown that these diffusions satisfy heat equations with respect to related second order differential operators on cylinder functions. For our purposes, we set $\Gamma = \mathfrak{X}_{\mathbb{R}}$.

For $0 \leqslant s \leqslant 1$, $0 \leqslant t < \infty$, let $\beta(t,s)$ be the $\mathfrak{g}$-valued process given by

$$\beta(t,s) := \sum_{A \in \mathfrak{X}_{\mathbb{R}}} \beta^A(t,s)A,$$

where $\{\beta^A\}_{A \in \mathfrak{X}_{\mathbb{R}}}$ is an independent collection of $\mathbb{R}$-valued Brownian sheets, i.e. for each $A \in \mathfrak{X}_{\mathbb{R}}$, $\{\beta^A(t,s)\colon s \in [0,1],\ t \geqslant 0\}$ is a mean zero continuous Gaussian process such that

$$\mathbb{E}\big[\beta^A(t,s)\beta^A(\tau,\sigma)\big] = \frac{1}{2}K(s,\sigma)(t \wedge \tau). \tag{19}$$

Remark 18. It should be noted that the factor of $\frac{1}{2}$ appearing in Eq. (19) does not appear in the assumptions of [2]. This addition does not change the methods of [2] and has the primary effect of scaling by $\frac{1}{2}$ the generator of the process $\Sigma(t)$ defined below (compare Proposition 23 below with [2, Proposition 1.1]). This step is necessary for the isometry results to follow in Section 4.

Suppose that $(\Omega, \mathcal{F}, P)$ is a complete probability space on which that processes, $\{\beta^A\}_{A \in \mathfrak{X}_{\mathbb{R}}}$, are defined. Let $\{\mathcal{F}_t\}_{t \geqslant 0}$ be the filtration which is the right continuous extension of the filtration $\{\mathcal{F}_t^0\}_{t \geqslant 0}$, the smallest sub-sigma-algebra of $\mathcal{F}$ such that $\beta^A(\tau,s)$ is measurable for all $s \in [0,1]$ and $\tau \in [0,t]$ and $A \in \mathfrak{X}_{\mathbb{R}}$, augmented by all the P null subsets of $\mathcal{F}$. The following is Theorem 1.2 of [2].

Theorem 19. *Let $\sigma_0 \in \mathcal{W}(G)$. Then there exists a continuous adapted $\mathcal{W}(G)$-valued process $\{\Sigma(t)\}_{t \geqslant 0}$ on the filtered probability space $(\mathcal{W}, \{\mathcal{F}_t\}_{t \geqslant 0}, \mathcal{F}, P)$ such that for each $s \in [0,1]$, $\Sigma(\cdot, s)$ solves the stochastic differential equation*

$$\Sigma(\delta t, s) = L_{\Sigma(t,s)*}\beta(\delta t, s) \quad \text{with } \Sigma(0,s) = \sigma_0(s). \tag{20}$$

More precisely,

$$\Sigma(\delta t, s) = \sum_{A \in \mathfrak{X}_{\mathbb{R}}} \tilde{A}\big(\Sigma(t,s)\big)\beta^A(\delta t, s) \quad \text{with } \Sigma(0,s) = \sigma_0(s). \tag{21}$$

Here $\beta^A(\delta t, s)$ denotes the Stratonovich differential of the process $t \to \beta^A(t,s)$.

Definition 20. The measure $\nu_t := Law(\Sigma(t,\cdot))$ is called *the heat kernel measure* on $\mathcal{W}(G)$. For a cylinder function f, let $\nu_t(f) := \mathbb{E}[f(\Sigma(t,\cdot))]$.

Definition 21. Define a $G^{\#(\mathcal{P})}$-valued process

$$\Sigma_{\mathcal{P}}(t) := \pi_{\mathcal{P}} \circ \Sigma(t,\cdot).$$

Let $\nu_t^{\mathcal{P}} := Law(\Sigma_{\mathcal{P}}(t))$, and for a function $F \in C^\infty(G)$, denote $\nu_t^{\mathcal{P}}(F) := \mathbb{E}[F(\Sigma_{\mathcal{P}}(t))]$.

Definition 22. For $f \in \mathcal{F}C^\infty(\mathcal{W})$, define the Laplacian $\Delta_{H(G)}$ by

$$\Delta_{H(G)} f := \sum_{h \in S_{\mathbb{R}}} \tilde{h}^2 f.$$

If $f = F \circ \pi_{\mathcal{P}}$, then by Eq. (12) and [7, Lemma 3.3],

$$\Delta_{H(G)} f = \sum_{h \in S_{\mathbb{R}}} \sum_{i,j=1}^{n} \left(\widetilde{h(s_j)}^{(j)} \widetilde{h(s_i)}^{(i)} F \right) \circ \pi_{\mathcal{P}}$$

$$= \sum_{A \in \mathfrak{X}_{\mathbb{R}}} \sum_{i,j=1}^{n} K(s_j, s_i) \left(\widetilde{A}^{(j)} \widetilde{A}^{(i)} F \right) \circ \pi_{\mathcal{P}}. \tag{22}$$

So if we define an operator $\Delta_{\mathcal{P}}$ on $C^{\infty}(G^{\#(\mathcal{P})})$ by

$$\Delta_{\mathcal{P}} F := \sum_{A \in \mathfrak{X}_{\mathbb{R}}} \sum_{i,j=1}^{n} K(s_j, s_i) \left(\widetilde{A}^{(j)} \widetilde{A}^{(i)} F \right),$$

then we have the relationship

$$\Delta_{H(G)}(F \circ \pi_{\mathcal{P}}) = (\Delta_{\mathcal{P}} F) \circ \pi_{\mathcal{P}}.$$

The heat kernel measures ν_t and $\nu_t^{\mathcal{P}}$ satisfy (in the distributional sense) the following "heat" equations.

Proposition 23. *$\nu_t^{\mathcal{P}}$ and ν_t satisfy the heat equations on $G^{\#(\mathcal{P})}$ and $\mathcal{W}(G)$ in the following weak sense. If $f = F \circ \pi_{\mathcal{P}}$ is a cylinder function, then*

$$\frac{\partial}{\partial t} \nu_t^{\mathcal{P}}(F) = \nu_t^{\mathcal{P}} \left(\frac{1}{4} \Delta_{\mathcal{P}} F \right) \quad \text{with } \lim_{t \downarrow 0} \nu_t^{\mathcal{P}}(F) = F(e, e, \ldots, e), \tag{23}$$

and

$$\frac{\partial}{\partial t} \nu_t(f) = \nu_t \left(\frac{1}{4} \Delta_{H(G)} f \right) \quad \text{with } \lim_{t \downarrow 0} \nu_t(f) = f(\underline{e}). \tag{24}$$

We conclude with a simple isometry.

Proposition 24. *If $f = F \circ \pi_{\mathcal{P}}$, then*

$$\|f\|_{L^2(\nu_t)} = \|F\|_{L^2(\nu_t^{\mathcal{P}})}.$$

Proof.

$$\|f\|_{L^2(\nu_t)}^2 = \nu_t\left(|f|^2\right) = \nu_t^{\mathcal{P}}\left(|F|^2\right) = \|F\|_{L^2(\nu_t^{\mathcal{P}})}^2. \qquad \square$$

3. Skeleton theorem

We will consider $T > 0$ to be fixed throughout this section. The following presentation of the restriction map is similar to that presented by Hall and Sengupta in [15]. The reader should recall that $\mathcal{H}_T$ denotes the $L^2(\nu_T)$-closure of $\mathcal{HF}(\mathcal{W}) \cap L^2(\nu_T)$.

For $g \in H(G)$, define a function $R_g : \mathcal{HF}(\mathcal{W}) \cap L^2(\nu_T) \to \mathbb{C}$ by

$$R_g(f) := f(g).$$

R_g is clearly linear and defined on a dense subset of $\mathcal{H}_T$.

Proposition 25. *For all $g \in H(G)$, R_g can be extended uniquely to a continuous linear functional on all of $\mathcal{H}_T$.*

Proof. Pick $g \in H(G)$, and suppose $f \in \mathcal{HF}(\mathcal{W}) \cap L^2(\nu_T)$ with $f = F \circ \pi_{\mathcal{P}}$ for some partition $\mathcal{P}$ of [0, 1]. Recall that by Definition 21, $\nu_T^{\mathcal{P}}$ is the heat kernel measure with respect to right invariant Haar measure on $G^{\#(\mathcal{P})}$ associated to the operator $\frac{1}{4}\Delta_{\mathcal{P}}$. Applying the finite-dimensional results of Driver and Gross, specifically [5, Remark 5.5], we find that

$$\left|R_g(f)\right|^2 = \left|F\left(\pi_{\mathcal{P}}(g)\right)\right|^2 \leqslant \|F\|^2_{L^2(\nu_T^{\mathcal{P}})} e^{\frac{|\pi_{\mathcal{P}}(g)|^2_{\mathcal{P}}}{T}}.$$

By Corollary 17, $|\pi_{\mathcal{P}}(g)|^2_{\mathcal{P}} \leqslant |g|^2_{H(G)}$, and so using Proposition 24,

$$\left|R_g(f)\right|^2 \leqslant \|f\|^2_{L^2(\nu_T)} e^{\frac{|g|^2_{H(G)}}{T}}. \tag{25}$$

So $\|R_g\|^2_{\mathrm{op}} \leqslant e^{\frac{|g|^2_{H(G)}}{T}}$, where $\|\cdot\|_{\mathrm{op}}$ denotes the operator norm. R_g is therefore continuous. For $f \in \mathcal{H}_T$, pick $\{f_n\}_{n=1}^{\infty} \subset \mathcal{HF}(\mathcal{W}) \cap L^2(\nu_T)$ such that $f_n \to f$. We then define $R_g(f) = \lim_{n\to\infty} R_g(f_n)$. $\square$

Notation 26. In the sequel, R_g will refer to this extension.

Remark 27. Proposition 25 implies that if $f_n \to f$ in $\mathcal{H}_T$, then for any $g \in H(G)$, $R_g f_n \to R_g f$. More precisely, Eq. (25) indicates that the convergence is locally uniform.

We will show that a function $f \in \mathcal{H}_T$ has a holomorphic "skeleton." That is, despite the fact that f is an $L^2(\nu_T)$ equivalence class, "$(f|_{H(G)})(g)$" $:= R_g(f)$ is holomorphic. We make this precise in Theorem 29. We first need an appropriate notion of holomorphic functions on $H(G)$.

Definition 28. We will refer to a function $u : H(G) \to \mathbb{C}$ as holomorphic if it is holomorphic in the sense of Gross and Malliavin [14]. Specifically, we require that for every $g \in H(G)$, the map $h \in H(\mathfrak{g}) \to u(g \cdot e^h)$ is Fréchet differentiable at $h = 0$ and that this Fréchet derivative is complex, linear and continuous in $H(\mathfrak{g})^*$ as a function of g.

Theorem 29 *(Skeleton theorem). There exists a linear map $R : \mathcal{H}_T \to \mathcal{H}(H(G))$ with the following properties*:

(1) *For f a holomorphic cylinder function, $Rf = f|_{H(G)}$.*

(2) *For $g \in H(G)$, $|(Rf)(g)|^2 \leqslant \|f\|^2_{L^2(\nu_T)} e^{\frac{|g|^2_{H(G)}}{T}}$.*

Proof. Given $f \in \mathcal{H}_T$, define Rf by $(Rf)(g) = R_g f$ for all $g \in H(G)$. By the definition of R_g, if $f \in \mathcal{HFC}^\infty(\mathcal{W}) \cap L^2(\nu_T)$, then $(Rf)(g) = f(g)$ for all $g \in H(G)$. So (1) is satisfied. (2) follows from Eq. (25). It remains to show that $Rf \in \mathcal{H}(H(G))$. For this, we follow [15, Theorem 7]. We reproduce the argument for convenience.

We first suppose that $f \in \mathcal{HF}(\mathcal{W}) \cap L^2(\nu_T)$. Then $f = F \circ \pi_{\mathcal{P}}$ for some $F \in \mathcal{H}(G^{\#(\mathcal{P})})$ and some partition $\mathcal{P}$ of [0, 1]. It follows that $h \to f(g \cdot e^h)$ is holomorphic and jointly continuous in g and h. Using results of [17, Chapter III], we can conclude that $f(g \cdot e^h)$ has a complex-linear Fréchet derivative at $h = 0$ and this derivative is continuous with respect to g.

For a general $f \in \mathcal{H}_T$, we fix $g \in H(G)$ and choose $\{f_n\}_{n=1}^\infty \subset \mathcal{HF}(\mathcal{W}) \cap L^2(\nu_T)$ such that $f_n \to f$. Remark 27 indicates that $(Rf_n)(g \cdot e^h) \to (Rf)(g \cdot e^h)$ uniformly for h in some neighborhood of the zero path. Therefore, by [17, Theorem 3.18.1], $h \to (Rf)(g \cdot e^h)$ is holomorphic and jointly continuous in g, h. $\square$

4. The Taylor isometry

Now we are able to define the Taylor map on $\mathcal{H}_T$. We refer the reader to Eq. (6) of Section 1 for the motivating finite-dimensional statement. We will consider $T > 0$ to be fixed throughout this section.

Definition 30. Given $f \in \mathcal{H}_T$, define $\alpha_{Rf} \in T(H(\mathfrak{g}))'$ by

$$\langle \alpha_{Rf}, h_1 \otimes h_2 \otimes \cdots \otimes h_n \rangle = (\tilde{h}_1 \tilde{h}_2 \cdots \tilde{h}_n Rf)(\underline{e}), \tag{26}$$

where $h_1, \ldots, h_n \in H(\mathfrak{g})$, and $\underline{e}$ denotes the identity path in $\mathcal{W}(G)$. Notice that by Theorem 29, $Rf \in \mathcal{H}(H(G))$, so the right-hand side is well defined.

The above map $f \to \alpha_{Rf}$ will be referred to as the *Taylor map*. The reader is referred to Eqs. (13) and (14) for the definition of $J_T^0(H(\mathfrak{g}))$ and $\|\cdot\|_{J_T^0(H(\mathfrak{g}))}$ appearing in the following theorem.

Theorem 31. *Let $f \in \mathcal{HF}(\mathcal{W}) \cap L^2(\nu_T)$ and $\alpha_{Rf} \in T(H(\mathfrak{g}))'$ as given in the above. Then $\alpha_{Rf} \in J_T^0(H(\mathfrak{g}))$ and $\|f\|^2_{L^2(\nu_T)} = \|\alpha_{Rf}\|^2_{J_T^0(H(\mathfrak{g}))}$.*

Proof. Suppose $f \in \mathcal{HF}(\mathcal{W}) \cap L^2(\nu_T)$ with $f = F \circ \pi_{\mathcal{P}}$. Then $Rf = f|_{H(G)}$, and $\|f\|_{L^2(\nu_T)} < \infty$. Let $S_{\mathbb{C}}^{\mathcal{P}}$ be an orthonormal basis for $(H_{\mathcal{P}}(\mathfrak{g}), (\,,\,)_{H(\mathfrak{g})})$. We extend this to an orthonormal basis $S_{\mathbb{C}}$ for $H(\mathfrak{g}) = H_{\mathcal{P}}(\mathfrak{g}) \oplus^\perp Nul(\Lambda_{\mathcal{P}})$. By Proposition 13 and Remark 15, $\mathfrak{X}_{\mathbb{C}}^{\mathcal{P}} := \{\Lambda_{\mathcal{P}}(h) \mid h \in S_{\mathbb{C}}^{\mathcal{P}}\}$ is an orthonormal basis for $(\mathfrak{g}^{\#(\mathcal{P})}, (\,,\,)_{\mathcal{P}})$. Note that for all $h \in H_{\mathcal{P}}(\mathfrak{g})^\perp$,

$$(\tilde{h} f)(\underline{e}) = \sum_{i=1}^n \big(\big(h(s_i)^{(i)} F\big) \circ \pi_{\mathcal{P}}\big)(\underline{e}) = 0$$

since $h|_{\mathcal{P}} \equiv 0$. Then

$$
\begin{aligned}
\|\alpha_F\|^2_{J^0_T(\mathfrak{g}^{\#(\mathcal{P})})} &:= \sum_{k=0}^{\infty} \frac{t^k}{k!} \biggl(\sum_{A_1,\dots,A_k \in \mathfrak{X}^{\mathcal{P}}_{\mathbb{C}}} \bigl|\langle \alpha_F, A_1 \otimes \cdots \otimes A_k\rangle\bigr|^2 \biggr) \\
&= \sum_{k=0}^{\infty} \frac{t^k}{k!} \biggl(\sum_{A_1,\dots,A_k \in \mathfrak{X}^{\mathcal{P}}_{\mathbb{C}}} \bigl|(\tilde{A}_1 \cdots \tilde{A}_k F)(e, e, \dots, e)\bigr|^2 \biggr) \\
&= \sum_{k=0}^{\infty} \frac{t^k}{k!} \biggl(\sum_{h_1,\dots,h_k \in S^{\mathcal{P}}_{\mathbb{C}}} \bigl|(\tilde{h}_1 \cdots \tilde{h}_k f)(\underline{e})\bigr|^2 \biggr) \\
&= \sum_{k=0}^{\infty} \frac{t^k}{k!} \biggl(\sum_{h_1,\dots,h_k \in S_{\mathbb{C}}} \bigl|(\tilde{h}_1 \cdots \tilde{h}_k f)(\underline{e})\bigr|^2 \biggr) \\
&= \|\alpha_f\|^2_{J^0_T(H(\mathfrak{g}))}.
\end{aligned}
$$

Using the isometry of Proposition 24 and the finite-dimensional results found in [5, Theorem 5.1], we have

$$
\|f\|^2_{L^2(\nu_T)} = \|F\|^2_{L^2(\nu^{\mathcal{P}}_T)} = \|\alpha_F\|^2_{J^0_T(\mathfrak{g}^{\#(\mathcal{P})})} = \|\alpha_f\|^2_{J^0_T(H(\mathfrak{g}))} = \|\alpha_{Rf}\|^2_{J^0_T(H(\mathfrak{g}))}. \qquad \square
$$

Before proving Corollary 33, which extends this result to any $f \in \mathcal{H}_T$, we need the following, whose proof follows from the locally uniform convergence of Remark 27.

Proposition 32. *Suppose $\{f_n\}_{n=1}^{\infty} \in \mathcal{HF}(\mathcal{W}) \cap L^2(\nu_T)$ with $f_n \to f \in \mathcal{H}_T$. Then for all $h_1, h_2, \dots, h_k \in H(\mathfrak{g})$ and $g \in H(G)$,*

$$
(\tilde{h}_1\tilde{h}_2 \cdots \tilde{h}_k R f_n)(g) \to (\tilde{h}_1\tilde{h}_2 \cdots \tilde{h}_k R f)(g) \quad \text{as } n \to \infty.
$$

Corollary 33. *The Taylor map described in Definition* 30 *is an isometry, i.e. for all $f \in \mathcal{H}_T$,*

$$
\|f\|^2_{L^2(\nu_T)} = \|\alpha_{Rf}\|^2_{J^0_T(H(\mathfrak{g}))}.
$$

Proof. Let $\{f_n\}_{n=1}^{\infty} \subset \mathcal{HF}(\mathcal{W}) \cap L^2(\nu_T)$ such that $f_n \to f$. By Theorem 31, $\{\alpha_{Rf_n}\}_{n=1}^{\infty} \subset J^0_T(H(\mathfrak{g}))$ is Cauchy, and hence converges to some $\widehat{\alpha} \in J^0_T(H(\mathfrak{g}))$. It remains to show that $\widehat{\alpha} = \alpha_{Rf}$. For any $h_1, h_2, \dots, h_k \in H(\mathfrak{g})$, we have by Proposition 32,

$$
\begin{aligned}
\langle \widehat{\alpha}, h_1 \otimes h_2 \otimes \cdots \otimes h_k\rangle &= \lim_{n\to\infty} \langle \alpha_{Rf_n}, h_1 \otimes h_2 \otimes \cdots \otimes h_k\rangle \\
&= \lim_{n\to\infty} (\tilde{h}_1\tilde{h}_2 \cdots \tilde{h}_k R f_n)(\underline{e}) \\
&= (\tilde{h}_1\tilde{h}_2 \cdots \tilde{h}_k R f)(\underline{e}) \\
&= \langle \alpha_{Rf}, h_1 \otimes h_2 \otimes \cdots \otimes h_k\rangle. \qquad \square
\end{aligned}
$$

Corollary 34. *Since the Taylor map $(1 - D)^{-1}_{\underline{e}} R : \mathcal{H}_T \to J^0_T(H(\mathfrak{g}))$ is isometric, it necessarily follows that $R : \mathcal{H}_T \to \mathcal{H}(H(G))$ is injective.*

5. Surjectivity

This section is devoted to proving that the Taylor map is surjective when G is a simply connected graded Lie group. Section 5.1 is a review of some preliminary results concerning graded Lie algebras and Lie groups and their application to the path space. In particular, we can globally identify $\mathcal{W}(\mathfrak{g})$ and $\mathcal{W}(G)$ via exponential coordinates with multiplication given by the Baker–Campbell–Hausdorff (BCH) series. We will also show that finite rank tensors are dense in $J_T^0(H(\mathfrak{g}))$ and hence it suffices to show that the Taylor map is onto the set of finite rank $\alpha \in J_T^0(H(\mathfrak{g}))$.

In Section 5.2, we will construct a function $u_\alpha \in \mathcal{H}(H(G))$ which has α as its set of derivatives at the identity path. In Section 5.3, we introduce holomorphic cylinder functions associated to u_α and a partition of [0, 1]. We are able to characterize the derivatives of these cylinder functions at the identity path in terms of α and the projection operator $P_{\mathcal{P}}$ introduced in Definition 11. In Section 5.5, these cylinder function will be shown to be elements of $\mathcal{H}_T$. Furthermore, the cylinder functions associated to a sequence of refining partitions, $\mathcal{P}_1 \subset \mathcal{P}_2 \subset \cdots$, will be shown to be Cauchy and convergent to a function $F \in \mathcal{H}_T$ with the property that $(1-D)_{\underline{e}}^{-1} RF = \alpha$. Section 5.4 is devoted to proving estimates on increments of multilinear functions, which will be essential for the analysis in Section 5.5.

5.1. Introduction

Let $\mathfrak{g}$ is a d-dimensional step r complex graded Lie algebra. This means that there is a sequence of nonzero subspaces V_i for $i = 1, \ldots, r$ such that

$$\mathfrak{g} = \bigoplus_{i=1}^{r} V_i, \tag{27}$$

with $[V_i, V_j] \subset V_{i+j}$, with the convention that $V_s = \{0\}$ for $s > r$. We will furthermore assume that the subspaces $\{V_i\}_{i=1}^r$ are orthogonal with respect to our inner product $(,)_{\mathfrak{g}}$. Let G be the simply connected Lie group associated to $\mathfrak{g}$. G is called a graded Lie group. The reader is referred to Goodman [11, Chapter 1] for a thorough introduction to graded Lie algebras and Lie groups.

The decomposition in Eq. (27) gives an orthogonal decomposition of $H(\mathfrak{g})$

$$H(\mathfrak{g}) = \bigoplus_{i=1}^{r} H(V_i),$$

with $[H(V_i), H(V_j)] \subset H(V_{i+j})$ and $H(V_s) = \{0\}$ for $s > r$. Therefore, $H(\mathfrak{g})$ is a step r complex graded Lie algebra as well.

Example 35. The complex Heisenberg Lie algebra is a 3-dimensional step 2 graded Lie algebra. It is generated by the set $\{X, Y, Z\}$, where Z is in the center and $[X, Y] = Z$. Setting $V_1 = \operatorname{span}\{X, Y\}$, and $V_2 = \operatorname{span}\{Z\}$ gives the decomposition of line (27).

The associated graded Lie group is the complex Heisenberg group, which under exponential coordinates is $\mathbb{C}^3$ with the following multiplication rule given by the BCH series:

$$(a, b, c) \cdot (a', b', c') = \left(a + a', b + b', c + c' + \frac{1}{2}(ab' - a'b)\right).$$

As in the above example, elements of $\mathfrak{g}$ and G are identified via the group exponential map and the BCH series [9, p. 30]. For $A, B \in \mathfrak{g}$,

$$\log\left(e^A e^B\right) = A + B + \sum_{k=1}^{\infty} \frac{(-1)^k}{k+1} \sum_{\substack{n_1,\dots,n_k \geqslant 0 \\ n_j+m_j>0 \\ m_1,\dots,m_k \geqslant 0}} \frac{1}{n_1+\cdots+n_k+1} \frac{\mathrm{ad}_A^{n_1} \circ \mathrm{ad}_B^{m_1} \circ \cdots \circ \mathrm{ad}_A^{n_k} \circ \mathrm{ad}_B^{m_k} A}{n_1!m_1!\cdots n_k!m_k!}. \tag{28}$$

Recall that since we are assuming that $\mathfrak{g}$ is step r nilpotent, it follows that

$$\mathrm{ad}_A^{n_1} \circ \mathrm{ad}_B^{m_1} \circ \cdots \circ \mathrm{ad}_A^{n_k} \circ \mathrm{ad}_B^{m_k} A = 0 \quad \text{if } \sum_{j=1}^{k} n_j + m_j \geqslant r.$$

Since G is nilpotent and simply connected, the exponential map is globally bijective [27, Theorem 3.6.2]. We therefore identify $\mathfrak{g}$ and G globally under exponential coordinates. In particular, we can view $\mathfrak{g}$ as both a Lie algebra and a Lie group with group multiplication given globally by the BCH series, Eq. (28). It follows that we can also identify $\mathcal{W}(\mathfrak{g})$ with $\mathcal{W}(G)$ using the pointwise exponential map of Eq. (10), where the path group multiplication is given by pointwise application of Eq. (28).

We will use this identification throughout the sequel, often without comment. It should therefore be understood that if $g \in \mathcal{W}(\mathfrak{g})$, then we will write $e^g = g \in \mathcal{W}(G)$, where it is understood that we are identifying $\mathfrak{g}$ and G under exponential coordinates. Under this identification, it follows that $\underline{e} = \underline{0}$, and $g^{-1} = -g$ for any path $g \in \mathcal{W}(G)$. Proposition 37 below indicates that this identification preserves the important analytic sub-structures of $\mathcal{W}(\mathfrak{g})$ and $\mathcal{W}(G)$.

Remark 36. Before proceeding, we make an observation that will be used throughout the sequel. If $A, B \in \mathfrak{g} = G$ and $z \in \mathbb{C}$, then the BCH series gives

$$A \cdot zB = A + z \sum_{l=0}^{r-1} C_l \,\mathrm{ad}_A^l B + \sum_{l=2}^{r-1} z^l Q_l(A, B), \tag{29}$$

where Q_l is homogeneous of degree l in B. Therefore

$$\tilde{B}(A) = \frac{d}{dt}\bigg|_0 A \cdot tB = B + \sum_{l=1}^{r-1} C_l \,\mathrm{ad}_A^l B \tag{30}$$

for some constants C_l determined by Eq. (28).

Proposition 37. *$g \in H(\mathfrak{g})$ iff $g \in H(G)$.*

Proof. First observe a few important facts. If $g \in H(\mathfrak{g})$, then $\|g'\|_{L^1([0,1])} \leqslant \|g'\|_{L^2([0,1])} = \|g\|_{H(\mathfrak{g})}$, and the following pointwise estimate holds:

$$\|g(s)\|_{\mathfrak{g}} = \left\| \int_0^s g'(r)\,dr \right\|_{\mathfrak{g}} \leqslant \int_0^1 \|g'(r)\|_{\mathfrak{g}}\,dr = \|g'\|_{L^1([0,1])} \leqslant \|g\|_{H(\mathfrak{g})}. \tag{31}$$

For $g \in H(G)$, Eq. (30) gives the Maurer–Cartan form

$$L_{g^{-1}(s)*}g'(s) = g'(s) + \sum_{i=1}^{r-1} C_i \,\mathrm{ad}^i_{g(s)}\, g'(s). \tag{32}$$

Finally, notice that since $\|\mathrm{ad}_A B\|_{\mathfrak{g}} \leqslant C\|A\|_{\mathfrak{g}}\|B\|_{\mathfrak{g}}$ for some constant C, it follows that $\|\mathrm{ad}^i_{g(s)}\, g'(s)\|_{\mathfrak{g}} \leqslant C^i \|g(s)\|^i_{\mathfrak{g}} \|g'(s)\|_{\mathfrak{g}}$.

Now suppose $g \in H(\mathfrak{g})$. Considering g as an element of $\mathcal{W}(G)$, we calculate

$$\begin{aligned}
E(g) &= \int_0^1 \left\| g'(s) + \sum_{i=1}^{r-1} C_i \,\mathrm{ad}^i_{g(s)}\big(g'(s)\big) \right\|^2_{\mathfrak{g}} ds \\
&\leqslant r^2 \left(\|g'\|^2_{L^2([0,1],\mathfrak{g})} + \sum_{i=1}^{r-1} \tilde{C}_i^2 \int_0^1 \|g(s)\|^{2i}_{\mathfrak{g}} \|g'(s)\|^2_{\mathfrak{g}}\,ds \right) \\
&\leqslant r^2 \left(\|g'\|^2_{L^2([0,1],\mathfrak{g})} + \sum_{i=1}^{r-1} \tilde{C}_i^2 \|g'\|^{2i+2}_{L^2([0,1],\mathfrak{g})} \right) \\
&= poly\big(\|g\|_{H(\mathfrak{g})}\big) < \infty,
\end{aligned} \tag{33}$$

where in line (33) we have used Eq. (31). So $g \in H(G)$ as well.

Now suppose $g \in H(G)$. Considering $g \in \mathcal{W}(\mathfrak{g})$, we write $g = (g_1, g_2, \ldots, g_r)$ where $g_i \in \mathcal{W}(V_i)$. Since $g \in H(G)$ and the subspaces $\{V_i\}_{i=1}^r$ are orthogonal,

$$\sum_{i=1}^{r} \big\| \big(L_{g^{-1}(\cdot)*}g'(\cdot)\big)_i \big\|^2_{L^2([0,1],V_i)} < \infty.$$

In particular, $\|(L_{g^{-1}(\cdot)*}g'(\cdot))_i\|^2_{L^2([0,1],V_i)} < \infty$ for all $i = 1, \ldots, r$. We wish to show that for all $i = 1, \ldots, r$, $\|g_i'\|_{L^2([0,1],V_i)} < \infty$. First note that

$$ad^i_{g(s)} g'(s) \in \bigoplus_{j=i+1}^{r} \mathcal{W}(V_j),$$

i.e. it is identically the zero path in the first i coordinates. Then (32) tells us that

$$\|g_1'\|^2_{L^2([0,1],V_1)} = \big\| \big(L_{g^{-1}(\cdot)*}g'(\cdot)\big)_1 \big\|^2_{L^2([0,1],V_1)} < \infty.$$

Now for the second coordinate, we have

$$\left(L_{g^{-1}(s)*}g'(s)\right)_2 = g_2'(s) + C_1 \operatorname{ad}_{g_1(s)} g_1'(s).$$

Therefore,

$$\begin{aligned}
\|g_2'\|^2_{L^2([0,1],V_2)} &= \int_0^1 \left\|\left(L_{g^{-1}(s)*}g'(s)\right)_2 + C_1 \operatorname{ad}_{g_1(s)} g_1'(s)\right\|^2_{\mathfrak{g}}\, ds \\
&\leqslant 4\left(\left\|\left(L_{g^{-1}(\cdot)*}g'(\cdot)\right)_2\right\|^2_{L^2([0,1],V_2)} + \tilde{C}_1^2 \int_0^1 \|g_1(s)\|^2_{\mathfrak{g}} \|g_1'(s)\|^2_{\mathfrak{g}}\, ds\right) \\
&\leqslant 4\left(\left\|\left(L_{g^{-1}(\cdot)*}g'(\cdot)\right)_2\right\|^2_{L^2([0,1],V_2)} + \tilde{C}_1^2 \|g_1'\|^4_{L^2([0,1],V_1)}\right) < \infty,
\end{aligned}$$

where we have used Eq. (31) restricted to the first coordinate. In this manner, we inductively show $\|g_i'\|_{L^2([0,1],V_i)} < \infty$ for all $i = 1, \ldots, r$. $\square$

Graded Lie groups and Lie algebras have a useful dilation structure. For $\lambda \in \mathbb{C}, \lambda \neq 0$, let $\phi_\lambda : \mathfrak{g} \to \mathfrak{g}$ be given by

$$\phi_\lambda(v_1 + v_2 + \cdots + v_r) := \sum_{i=1}^r \lambda^i v_i \quad \text{for } v_i \in V_i \text{ and } i = 1, \ldots, r. \tag{34}$$

This dilation defines an automorphism of the Lie algebra $\mathfrak{g}$ with inverse $\phi_{\lambda^{-1}}$. The dilation on $\mathfrak{g}$ extends to a dilation on $H(\mathfrak{g})$,

$$(\phi_\lambda h)(s) := \phi_\lambda\big(h(s)\big), \tag{35}$$

which is an automorphism of $H(\mathfrak{g})$.

Definition 38. An element $\alpha \in T(H(\mathfrak{g}))'$ is of *finite rank* if there exists an integer $N > 0$ such that

$$\langle \alpha, h_1 \otimes h_2 \otimes \cdots \otimes h_m \rangle = 0,$$

for all $h_1, \ldots, h_m \in H(\mathfrak{g})$ if $m > N$. The smallest such N is the *rank* of α and we say that α *is of rank* N.

Our goal is to show that given $\alpha \in J_T^0(H(\mathfrak{g}))$, there exists a function $\tilde{u}_\alpha \in \mathcal{H}_T$ such that $(1-D)_{\underline{e}}^{-1} R\tilde{u}_\alpha = \alpha$. Of primary importance will be that finite rank tensors are dense in $J_T^0(H(\mathfrak{g}))$ when $\mathfrak{g}$ is graded. The proof of Theorem 41 closely follows the finite-dimensional result of [8, Lemma 4.4]. We first establish a result which will be used in the proof of Theorem 41 and in the sequel.

Notation 39. Given a Hilbert space H and a linear operator $\phi : H \to H$, we denote $\phi^{\otimes k} : H^{\otimes k} \to H^{\otimes k}$ the operator such that for all $h_1, h_2, \ldots, h_k \in H$

$$\phi^{\otimes k}(h_1 \otimes h_2 \otimes \cdots \otimes h_k) = \phi(h_1) \otimes \phi(h_2) \otimes \cdots \otimes \phi(h_k)$$

extended by linearity.

Proposition 40. *Let H be a Hilbert space and $I : H \to H$ the identity operator. Suppose $\phi_n : H \to H$ is a sequence of linear operators such that $\|\phi_n - I\|_{\mathrm{op}} \to 0$ as $n \to \infty$ and $\|\phi_n\|_{\mathrm{op}} \leqslant 1$ for all n. Then for all $k > 0$, $\|\phi_n^{\otimes k} - I^{\otimes k}\|_{\mathrm{op}} \leqslant 2$ and $\|\phi_n^{\otimes k} - I^{\otimes k}\|_{\mathrm{op}} \to 0$ as $n \to \infty$.*

Proof. By the proposition of Reed and Simon [21, p. 299], for any two bounded linear operators A and B on H, $\|A \otimes B\|_{\mathrm{op}} = \|A\|_{\mathrm{op}} \|B\|_{\mathrm{op}}$. It follows that for any $k > 0$,

$$\left\|\phi_n^{\otimes k} - I^{\otimes k}\right\|_{\mathrm{op}} \leqslant \left\|\phi_n^{\otimes k}\right\|_{\mathrm{op}} + \left\|I^{\otimes k}\right\|_{\mathrm{op}} = \|\phi_n\|_{\mathrm{op}}^k + 1 \leqslant 2.$$

Furthermore, by writing

$$\phi_n^{\otimes k} - I^{\otimes k} = \sum_{j=0}^{k-1} \phi_n^{\otimes j} \otimes (\phi_n - I) \otimes I^{\otimes k-1-j}, \tag{36}$$

it follows that

$$\left\|\phi_n^{\otimes k} - I^{\otimes k}\right\|_{\mathrm{op}} = \sum_{j=0}^{k-1} \|\phi_n\|_{\mathrm{op}}^j \|\phi_n - I\|_{\mathrm{op}} \leqslant k \|\phi_n - I\|_{\mathrm{op}}. \tag{37}$$

It then follows that $\|\phi_n^{\otimes k} - I^{\otimes k}\|_{\mathrm{op}} \to 0$ as $n \to \infty$. $\square$

Theorem 41. *Suppose $\mathfrak{g}$ is complex graded Lie algebra. Then the finite rank tensors in $J_t^0(H(\mathfrak{g}))$ are dense in $J_t^0(H(\mathfrak{g}))$ for each $t > 0$.*

Proof. For $\theta \in \mathbb{R}$, let $\Gamma_\theta : T(H(\mathfrak{g})) \to T(H(\mathfrak{g}))$ be the automorphism induced by the automorphism $\phi_{e^{i\theta}}$ on $H(\mathfrak{g})$. Then $\Gamma_\theta 1 = 1$ and $\Gamma_\theta(h_1 \otimes h_2 \otimes \cdots \otimes h_n) = \phi_{e^{i\theta}}^{\otimes n}(h_1 \otimes h_2 \otimes \cdots \otimes h_n)$ for all $h_1, \ldots, h_n \in H(\mathfrak{g})$. For any $h, k \in H(\mathfrak{g})$, we have

$$\Gamma_\theta\big(h \otimes k - k \otimes h - [h,k]\big) = (\phi_{e^{i\theta}} h) \otimes (\phi_{e^{i\theta}} k) - (\phi_{e^{i\theta}} k) \otimes (\phi_{e^{i\theta}} h) - [\phi_{e^{i\theta}} h, \phi_{e^{i\theta}} k],$$

and so Γ_θ takes $J_t(H(\mathfrak{g}))$ into and onto $J_t(H(\mathfrak{g}))$. If we let Γ_θ' denote the transpose, then for any $\alpha \in J_t^0(H(\mathfrak{g}))$ and $v \in J_t(H(\mathfrak{g}))$,

$$0 = \langle \alpha, \Gamma_\theta v \rangle = \big\langle \Gamma_\theta' \alpha, v \big\rangle.$$

Therefore, Γ_θ' takes $J_t^0(H(\mathfrak{g}))$ into and onto itself. If $S = \{h_j\}_{j=1}^\infty$ is any orthonormal basis for $H(\mathfrak{g})$, then so is $S_\theta = \{\phi_{e^{i\theta}} h_j\}_{j=1}^\infty$, and so

$$\begin{aligned}\|\Gamma'_\theta\alpha\|^2_{J_t^0(H(\mathfrak{g}))} &= \sum_{k=0}^\infty \frac{t^k}{k!}\sum_{h_1,\dots,h_k\in S}\bigl|\langle\Gamma'_\theta\alpha, h_1\otimes\cdots\otimes h_k\rangle\bigr|^2\\ &= \sum_{k=0}^\infty \frac{t^k}{k!}\sum_{h_1,\dots,h_k\in S}\bigl|\langle\alpha, (\phi_{e^{i\theta}}h_1)\otimes\cdots\otimes(\phi_{e^{i\theta}}h_k)\rangle\bigr|^2\\ &= \sum_{k=0}^\infty \frac{t^k}{k!}\sum_{h_1,\dots,h_k\in S_\theta}\bigl|\langle\alpha, h_1\otimes\cdots\otimes h_k\rangle\bigr|^2\\ &= \|\alpha\|^2_{J_t^0(H(\mathfrak{g}))}.\end{aligned}$$

Since $\phi_{e^{i\theta}} = \sum_{j=1}^r e^{ij\theta}P_j$ where $P_j : H(\mathfrak{g}) \to H(V_j)$ is orthogonal projection, $\phi_{e^{i\theta}} : H(\mathfrak{g}) \to H(\mathfrak{g})$ is operator norm-continuous in θ. It follows from Proposition 40 that $\phi_{e^{i\theta}}^{\otimes n} : H(\mathfrak{g})^{\otimes n} \to H(\mathfrak{g})^{\otimes n}$ is operator norm-continuous in θ for all n.

Consider $h_1\otimes h_2\otimes\cdots\otimes h_k \in H(\mathfrak{g})^{\otimes k}$, where $h_p \in H(V_{j_p})$ for $p = 1,\dots,k$, where $1 \leqslant j_p \leqslant r$. Then

$$\bigl(\phi_{e^{i\theta}}^{\otimes k} - I^{\otimes k}\bigr)(h_1\otimes h_2\otimes\cdots\otimes h_k) = \bigl(e^{i\theta\sum_{p=1}^k j_p} - 1\bigr)h_1\otimes h_2\otimes\cdots\otimes h_k,$$

where $|e^{i\theta\sum_{p=1}^k j_p} - 1| \leqslant \|\phi_{e^{i\theta}}^{\otimes k} - I^{\otimes k}\|_{\mathrm{op}}$. If we choose an orthonormal basis S of $H(\mathfrak{g})$ which is a union of orthonormal bases for the spaces $H(V_i)$ for $i = 1,\dots,r$, it is evident that

$$\begin{aligned}\|\Gamma'_\theta\alpha - \alpha\|^2_{J_t^0(H(\mathfrak{g}))} &= \sum_{k=0}^\infty \frac{t^k}{k!}\sum_{h_1,\dots,h_k\in S}\bigl|\langle\alpha, \bigl(\phi_{e^{i\theta}}^{\otimes k} - I^{\otimes k}\bigr)(h_1\otimes\cdots\otimes h_k)\rangle\bigr|^2\\ &\leqslant \sum_{k=0}^\infty \frac{t^k}{k!}\|\alpha_k\|^2_{(H(\mathfrak{g})^*)^{\otimes k}}\bigl\|\phi_{e^{i\theta}}^{\otimes k} - I^{\otimes k}\bigr\|^2_{\mathrm{op}}.\end{aligned}\tag{38}$$

Since by Proposition 40, $\|\phi_{e^{i\theta}}^{\otimes k} - I^{\otimes k}\|^2_{\mathrm{op}} \leqslant 4$, the right-hand side of Eq. (38) is bounded above by $4\|\alpha\|^2_{J_t^0(H(\mathfrak{g}))}$. Therefore, the DCT and Proposition 40 give continuity of $\theta \to \Gamma'_\theta\alpha$.

For every $n \in \mathbb{Z}_+$, let

$$F_n(\theta) = \frac{1}{2\pi n}\sum_{k=0}^{n-1}\sum_{l=-k}^{k} e^{il\theta} = \frac{1}{2\pi n}\frac{\sin^2(n\theta/2)}{\sin^2(\theta/2)}$$

denote Fejer's kernel [26, p. 413]. Then

$$\int_{-\pi}^{\pi} F_n(\theta)\,d\theta = 1\tag{39}$$

for all n, and if ϕ is continuous on $[-\pi, \pi]$, then

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} F_n(\theta)\phi(\theta)\, d\theta = \phi(0). \tag{40}$$

In addition, if $m \in \mathbb{Z}_+$ with $m > n$, then one can show that

$$\int_{-\pi}^{\pi} F_n(\theta) e^{im\theta}\, d\theta = 0. \tag{41}$$

Set $\beta = h_1 \otimes h_2 \otimes \cdots \otimes h_k \in H(\mathfrak{g})^{\otimes k}$, where $h_p \in H(V_{j_p})$ for $p = 1, \ldots, k$, where $1 \leqslant j_p \leqslant r$. Then $\Gamma_\theta \beta = (e^{i\theta \sum_{p=1}^k j_p})\beta$. If $k > n$, then $\sum_{p=1}^k j_p > n$ as well, and by Eq. (41),

$$\int_{-\pi}^{\pi} F_n(\theta) \Gamma_\theta \beta\, d\theta = 0.$$

Any element of $H(\mathfrak{g})^{\otimes k}$ can be written as a sum of elements like β, and so in fact

$$\int_{-\pi}^{\pi} F_n(\theta) \Gamma_\theta \beta\, d\theta = 0 \quad \text{for all } \beta \in H(\mathfrak{g})^{\otimes k} \text{ with } k > n.$$

Consequently,

$$\int_{-\pi}^{\pi} F_n(\theta) \Gamma'_\theta \alpha\, d\theta = 0 \quad \text{for all } \alpha \in \big(H(\mathfrak{g})^*\big)^{\otimes k} \text{ with } k > n. \tag{42}$$

Let $\alpha \in J_t^0(H(\mathfrak{g}))$ and define

$$\gamma_n := \int_{-\pi}^{\pi} F_n(\theta) \Gamma'_\theta \alpha\, d\theta.$$

Then $\gamma_n \in J_t^0(H(\mathfrak{g}))$ for all $n > 0$ and by Eq. (42), it is zero in all ranks greater than n. Therefore

$$\begin{aligned}
\limsup_{n\to\infty} \|\gamma_n - \alpha\|_{J_t^0(H(\mathfrak{g}))} &= \limsup_{n\to\infty} \left\| \int_{-\pi}^{\pi} F_n(\theta)(\Gamma'_\theta \alpha - \alpha)\, d\theta \right\|_{J_t^0(H(\mathfrak{g}))} \\
&\leqslant \limsup_{n\to\infty} \int_{-\pi}^{\pi} F_n(\theta) \left\| \Gamma'_\theta \alpha - \alpha \right\|_{J_t^0(H(\mathfrak{g}))} d\theta \\
&= 0
\end{aligned}$$

by Eq. (40). $\square$

The Taylor map will be shown to be onto the set of $\alpha \in J_T^0(H(\mathfrak{g}))$ of finite rank, and the following proposition states that this is sufficient.

Proposition 42. *Let $E \subset J_T^0(H(\mathfrak{g}))$ be a dense subset. If for every $\alpha \in E$ there exists a function $\tilde{u}_\alpha \in \mathcal{H}_T$ such that $(1-D)_{\underline{e}}^{-1} R\tilde{u}_\alpha = \alpha$, then the result holds for all $\alpha \in J_T^0(H(\mathfrak{g}))$.*

Proof. Let $\alpha \in J_T^0(H(\mathfrak{g}))$, and pick a sequence $\{\alpha_n\}_{n=1}^\infty \subset E$ such that $\alpha_n \to \alpha$ as $n \to \infty$. For each α_n, there exists a $\tilde{u}_{\alpha_n} \in \mathcal{H}_T$ such that $(1-D)_{\underline{e}}^{-1} R\tilde{u}_{\alpha_n} = \alpha_n$. Recall by Corollary 33 that the Taylor map $(1-D)_{\underline{e}}^{-1} R : \mathcal{H}_T \to J_T^0(H(\mathfrak{g}))$ is an isometry. $\mathcal{H}_T$ is closed and hence there exists a $\tilde{u}_\alpha \in \mathcal{H}_T$ such that $\tilde{u}_{\alpha_n} \to \tilde{u}_\alpha$. Finally, since the Taylor map is continuous,

$$\alpha = \lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} (1-D)_{\underline{e}}^{-1} R\tilde{u}_{\alpha_n} = (1-D)_{\underline{e}}^{-1} R\tilde{u}_\alpha. \qquad \square$$

Remark 43. For the remainder of this section, it will be assumed that $\alpha \in J_T^0(H(\mathfrak{g}))$ is of finite rank.

5.2. Construction of $u_\alpha \in \mathcal{H}(H(G))$

Given $\alpha \in J_T^0(H(\mathfrak{g}))$, we would like to construct a holomorphic function u_α on $H(G)$ such that $(1-D)_{\underline{e}}^{-1} u_\alpha = \alpha$. Recall from Section 3 that we require that for every $g \in H(G)$, the map $h \in H(\mathfrak{g}) \to u_\alpha(g \cdot e^h)$ is Fréchet differentiable at $h = 0$ and that this Fréchet derivative is complex linear and continuous in $H(\mathfrak{g})^*$ as a function of g.

Remark 36 can be applied pointwise to paths to give, for $h \in H(\mathfrak{g})$, $g \in H(G)$, and $z \in \mathbb{C}$,

$$g \cdot zh = g + z \sum_{l=0}^{r-1} C_l \operatorname{ad}_g^l h + \sum_{l=2}^{r-1} z^l Q_l(g,h), \tag{43}$$

where Q_l is homogeneous of degree l in h. Therefore

$$\tilde{h}(g) = \frac{d}{dt}\bigg|_0 g \cdot th = h + \sum_{l=1}^{r-1} C_l \operatorname{ad}_g^l h \tag{44}$$

and

$$g \cdot h = g + \tilde{h}(g) + \sum_{l=2}^{r-1} Q_l(g,h). \tag{45}$$

The following theorem is motivated by results in [3], specifically by Remark 5.6 and Proposition 6.2.

Theorem 44. *Given $\alpha \in J_T^0(H(\mathfrak{g}))$ of rank $N < \infty$, for every $g \in H(G)$ define*

$$u_\alpha(g) := \sum_{n=0}^{N} \langle \alpha, g^{\otimes n} \rangle / n!.$$

Then u_α is a holomorphic function on $H(G)$ satisfying $(1-D)_{\underline{e}}^{-1} u_\alpha = \alpha$.

Proof. For $0 < n \leqslant N$, define $f_n : H(G) \to \mathbb{C}$ by

$$f_n(g) = \frac{1}{n!}\langle \alpha, g^{\otimes n}\rangle.$$

Since finite sums of holomorphic functions are holomorphic, it suffices to show that f_n is holomorphic.

For $h \in H(\mathfrak{g})$ and $g \in H(G)$ define

$$(df_n)_g h := \frac{1}{n!}\left\langle \alpha, \sum_{k=0}^{n-1} g^{\otimes k} \otimes \tilde{h}(g) \otimes g^{\otimes n-k-1}\right\rangle,$$

where $\tilde{h}(g)$ is given in Eq. (44). Notice that $\tilde{h}(g)$ is complex linear in h and continuous in g.

To see that $(df_n)_g$ is the Fréchet derivative of f_n at $g \in H(G)$, we first observe that, using Eq. (45),

$$\begin{aligned}(g \cdot h)^{\otimes n} &= \left(g + \tilde{h}(g) + \sum_{l=2}^{r-1} Q_l(g,h)\right)^{\otimes n} \\ &= g^{\otimes n} + \sum_{k=0}^{n-1} g^{\otimes k} \otimes \tilde{h}(g) \otimes g^{\otimes n-k-1} + R^n(g,h),\end{aligned}$$

where $R^n(g,h)$ is a sum of tensors of degree two or greater in h. For $\|h\|_{H(\mathfrak{g})} \leqslant 1$, it follows that $\|R^n(g,h)\|_{H(\mathfrak{g})^{\otimes n}} \leqslant C_g \|h\|^2_{H(\mathfrak{g})}$ for an appropriate constant C_g. Therefore for small h,

$$\begin{aligned}&\frac{|f_n(g \cdot h) - f_n(g) - (df_n)_g h|}{\|h\|_{H(\mathfrak{g})}} \\ &\quad \leqslant \frac{\|\alpha\|_{(H(\mathfrak{g})^*)^{\otimes n}} \|(g \cdot h)^{\otimes n} - g^{\otimes n} - \sum_{k=0}^{n-1} g^{\otimes k} \otimes \tilde{h}(g) \otimes g^{\otimes n-k-1}\|_{H(\mathfrak{g})^{\otimes n}}}{n!\|h\|_{H(\mathfrak{g})}} \\ &\quad = \frac{\|\alpha\|_{(H(\mathfrak{g})^*)^{\otimes n}} \|R^n(g,h)\|_{H(\mathfrak{g})^{\otimes n}}}{n!\|h\|_{H(\mathfrak{g})}} \\ &\quad \leqslant \frac{\|\alpha\|_{(H(\mathfrak{g})^*)^{\otimes n}} C_g \|h\|^2_{H(\mathfrak{g})}}{n!\|h\|_{H(\mathfrak{g})}}.\end{aligned}$$

Which tends to zero as $h \to \underline{0}$. This proves that f_n is holomorphic, and therefore so is u_α.

To see that $(1-D)_{\underline{e}}^{-1} u_\alpha = \alpha$, observe that for $h \in H(\mathfrak{g})$,

$$u_\alpha(th) = \sum_{n=0}^{\infty} \frac{t^n}{n!}\langle \alpha, h^{\otimes n}\rangle.$$

Then

$$\begin{aligned}
\overbrace{\tilde{h}\tilde{h}\cdots\tilde{h}}^{k\text{ times}}\, u_\alpha(\underline{e}) &= \frac{d}{dt_1}\bigg|_0 \cdots \frac{d}{dt_k}\bigg|_0 u_\alpha(t_k h \cdot t_{k-1}h \cdots t_1 h)\\
&= \frac{d}{dt_1}\bigg|_0 \cdots \frac{d}{dt_k}\bigg|_0 u_\alpha\big((t_k + t_{k-1} + \cdots + t_1)h\big)\\
&= \frac{d}{dt_1}\bigg|_0 \cdots \frac{d}{dt_k}\bigg|_0 \sum_{n=0}^{\infty} \frac{(t_k + \cdots + t_1)^n}{n!}\big\langle \alpha, h^{\otimes n}\big\rangle\\
&= \big\langle \alpha, h^{\otimes k}\big\rangle.
\end{aligned}$$

Polarization then gives the result for symmetric tensors. The fact that $\alpha \in J_T^0(H(\mathfrak{g}))$ and the Birkhoff–Witt theorem gives that for $h_1, h_2, \ldots, h_k \in H(\mathfrak{g})$,

$$\tilde{h}_1\tilde{h}_2\ldots\tilde{h}_k u_\alpha(\underline{e}) = \langle \alpha, h_1 \otimes h_2 \otimes \cdots \otimes h_k\rangle. \qquad \square$$

5.3. Construction of cylinder functions

Given a $u_\alpha \in \mathcal{H}(H(G))$, we would like to show the existence of a function $\tilde{u}_\alpha \in \mathcal{H}_T$ such that $R\tilde{u}_\alpha$ agrees with u_α on the finite energy subgroup. We do so by constructing a sequence of approximating cylinder functions, the result of evaluating u_α on piecewise-linear approximations of paths. Recall from Section 2.1 that $P_\mathcal{P}$ denotes orthogonal projection from $H(\mathfrak{g})$ onto $H_\mathcal{P}(\mathfrak{g})$, the subspace of piecewise-linear paths subordinate to a partition $\mathcal{P}$. Due to the identifications between $\mathcal{W}(\mathfrak{g})$ and $\mathcal{W}(G)$ outlined in Section 5.1 and Proposition 37, we can further consider $P_\mathcal{P}$ as a map from $\mathcal{W}(G)$ to $H(G)$. It should be noted that $P_\mathcal{P}$, when considered as a map between groups, is not a group homomorphism. Since this piecewise-linear path only depends on the path at its partition points, we have the following natural cylinder functions.

Definition 45. For a partition $\mathcal{P}$ and a function $u_\alpha \in \mathcal{H}(H(G))$, then $F_\mathcal{P} : \mathcal{W}(G) \to \mathbb{C}$ given by

$$F_\mathcal{P} := u_\alpha \circ P_\mathcal{P}$$

defines a cylinder function.

Our goal in this section is to characterize the derivatives of $F_\mathcal{P}$ in terms of our given α. Notice that given $h \in H(\mathfrak{g})$ and $g \in \mathcal{W}(G)$,

$$\begin{aligned}
(\tilde{h}F_\mathcal{P})(g) &= \frac{d}{dt}\bigg|_0 u_\alpha\big(P_\mathcal{P}(g\cdot th)\big)\\
&= \frac{d}{dt}\bigg|_0 u_\alpha\big(P_\mathcal{P}g \cdot \big(-P_\mathcal{P}g \cdot P_\mathcal{P}(g\cdot th)\big)\big). \qquad (46)
\end{aligned}$$

Then setting

$$h_\mathcal{P}(g) := \frac{d}{dt}\bigg|_0 (-P_\mathcal{P}g)\cdot P_\mathcal{P}(g\cdot th) \qquad (47)$$

yields

$$(\tilde{h}F_{\mathcal{P}})(g) = \langle Du_\alpha(P_{\mathcal{P}}g), h_{\mathcal{P}}(g)\rangle, \tag{48}$$

and in particular

$$(\tilde{h}F_{\mathcal{P}})(\underline{e}) = \langle \alpha, h_{\mathcal{P}}(\underline{0})\rangle \tag{49}$$

since $P_{\mathcal{P}}(\underline{e}) = P_{\mathcal{P}}(\underline{0}) = \underline{0}$.

Notation 46. Let $\alpha(\mathcal{P}) := (1-D)_{\underline{e}}^{-1} F_{\mathcal{P}}$, i.e. for $h_1, h_2, \dots, h_k \in H(\mathfrak{g})$,

$$\langle \alpha(\mathcal{P}), h_1 \otimes h_2 \otimes \cdots \otimes h_k\rangle = (\tilde{h}_1\tilde{h}_2 \cdots \tilde{h}_k F_{\mathcal{P}})(\underline{e}).$$

In which case, Eq. (49) can be restated as

$$\langle \alpha(\mathcal{P}), h\rangle = \langle \alpha, h_{\mathcal{P}}(\underline{0})\rangle. \tag{50}$$

Remark 47. If $P_{\mathcal{P}} : \mathcal{W}(G) \to H(G)$ was a Lie group homomorphism, then it would be clear that

$$\langle \alpha(\mathcal{P}), h\rangle = \langle \alpha, P_{\mathcal{P}}h\rangle,$$

and in general

$$\langle \alpha(\mathcal{P}), h_1 \otimes h_2 \otimes \cdots \otimes h_k\rangle = \langle \alpha, P_{\mathcal{P}}h_1 \otimes P_{\mathcal{P}}h_2 \otimes \cdots \otimes P_{\mathcal{P}}h_k\rangle. \tag{51}$$

This is not the case. We will see, however, Eq. (51) is approximately correct as the partition mesh tends to zero.

An essential feature of the linear map $P_{\mathcal{P}}$ is that it does not raise the "index of nilpotency" of a path. It is not difficult to see that if $g \in H(V_i)$, then $P_{\mathcal{P}}g \in H(V_i)$ as well. A more useful property is phrased in terms of the lower central series. Recall that the lower central series of a step r nilpotent Lie algebra $\mathfrak{g}$ is a sequence of ideals

$$\mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_r \supset \{0\},$$

where $\mathfrak{g}_k = [\mathfrak{g}, \mathfrak{g}_{k-1}]$ with $\mathfrak{g}_0 = \mathfrak{g}$.

Defining the sequence of ideals $H(\mathfrak{g})_k$ for $k = 0, \dots, r$ as above, it can be shown that a path $g \in H(\mathfrak{g})_k$ iff $g \in H(\mathfrak{g})$ and $g(t) \in \mathfrak{g}_k$ for all $t \in [0,1]$. Since $P_{\mathcal{P}}g$ is contained in the convex hull of the points $\{g(s_1), g(s_2), \dots, g(s_n)\}$, it follows that if $g \in H(\mathfrak{g})_k$, then $P_{\mathcal{P}}g \in H(\mathfrak{g})_k$. We will commonly use this fact to conclude that if g is a path such that $[h, g] = \underline{0}$ for all $h \in V \subset H(\mathfrak{g})$, then it follows that $[h, P_{\mathcal{P}}g] = \underline{0}$ for all $h \in V$ as well. The following proposition gives a useful expression for $h_{\mathcal{P}}(g)$ (Eq. (54)).

Proposition 48. *Suppose $\mathfrak{g}$ is a step r graded Lie algebra. Suppose $L : \mathfrak{g} \to \mathfrak{g}$ is a linear map which preserves the lower central series, i.e. $L\mathfrak{g}_k \subset \mathfrak{g}_k$. Then for all $X, Y \in \mathfrak{g}$, we have*

$$\frac{d}{dt}\bigg|_0 \big(-L(X)\big) \cdot L(X \cdot tY) = \sum_{m=0}^{r-1} \sum_{l=0}^{m} C_{l,m} \operatorname{ad}_{L(X)}^{l} L\big(\operatorname{ad}_X^{m-l} Y\big), \tag{52}$$

with the property that $C_{0,0} = 1$ and for all $m \geqslant 1$, $\sum_{l=0}^{m} C_{l,m} = 0$.

Proof. Recall that group product is given globally by the BCH series (see Eq. (28))

$$\begin{aligned} A \cdot B &= A + B \\ &\quad + \sum_{k=1}^{\infty} \frac{(-1)^k}{k+1} \sum_{\substack{n_1,\dots,n_k \geqslant 0 \\ m_1,\dots,m_k \geqslant 0 \\ n_j + m_j > 0}} \frac{1}{n_1 + \cdots + n_k + 1} \frac{\operatorname{ad}_A^{n_1} \circ \operatorname{ad}_B^{m_1} \circ \cdots \circ \operatorname{ad}_A^{n_k} \circ \operatorname{ad}_B^{m_k} A}{n_1! m_1! \cdots n_k! m_k!}. \end{aligned}$$

Consider setting $A = -L(X)$ and $B = L(X \cdot tY) = L(X) + t \sum_{l=0}^{r-1} C_l L(\operatorname{ad}_X^l Y) + O(t^2)$, in which case

$$\begin{aligned} &\operatorname{ad}_A^{n_1} \circ \operatorname{ad}_B^{m_1} \circ \cdots \circ \operatorname{ad}_A^{n_k} \circ \operatorname{ad}_B^{m_k} A \\ &\quad = t(-1)^{(\sum_{i=1}^{k} n_i)+2} \left(\sum_{l=0}^{r-1} C_l \operatorname{ad}_{L(X)}^{(\sum_{i=1}^{k} n_i + m_i)} L\big(\operatorname{ad}_X^l Y\big) \right) + O\big(t^2\big). \end{aligned} \tag{53}$$

Since L preserves the lower central series, our series terminates with r or more brackets. We reindex and group together terms based upon the number of nested brackets (m in Eq. (52)). It should then be clear that all terms of order t on the right-hand side of Eq. (52) are of the form $C_{l,m} \operatorname{ad}_{L(X)}^{l} L(\operatorname{ad}_X^{m-l} Y)$ for $0 \leqslant m < r$, $l \leqslant m$, and some constants $C_{l,m}$, which are independent of L.

The coefficients of the BCH series are universal and hence Eq. (53) is valid for any step r graded Lie algebra. We could make L equal to the identity map, in which case Eq. (52) becomes

$$Y = C_{0,0} Y + \sum_{m=1}^{r-1} \left(\sum_{l=0}^{m} C_{l,m} \right) \operatorname{ad}_X^m Y.$$

So if there is a step r graded Lie algebra in which there exist X and Y such that $\{\operatorname{ad}_X^m Y\}_{m=0}^{r-1}$ are linearly independent, then we necessarily have that $C_{0,0} = 1$ and $\sum_{l=0}^{m} C_{l,m} = 0$ for $m > 0$. This is satisfied by the step r graded Lie algebra determined by the basis $\{X_0, \dots, X_r\}$ and the relations $[X_0, X_i] = X_{i+1}$ for $i = 1, \dots, r-1$, with all undefined brackets being zero. Clearly $\{\operatorname{ad}_{X_0}^m X_1\}_{m=0}^{r-1}$ are linearly independent. $\square$

Corollary 49. *Keeping the same notation as in Proposition 48, there exist constants $\tilde{C}_{l,m}$ such that*

$$\frac{d}{dt}\Big|_0 \big(-L(X)\big)\cdot L(X\cdot tY)$$
$$= L(Y) + \sum_{m=1}^{r-1}\sum_{l=0}^{m-1} \tilde{C}_{l,m}\big(\mathrm{ad}^l_{L(X)} L\big(\mathrm{ad}_X^{m-l} Y\big) - \mathrm{ad}^{l+1}_{L(X)} L\big(\mathrm{ad}_X^{m-l+1} Y\big)\big).$$

Proof. This follows from Proposition 48 after setting $\tilde{C}_{l,m} = \sum_{j=0}^{l} C_{j,m}$ and performing a summation by parts. $\square$

We apply the above to the path algebra $\mathcal{W}(\mathfrak{g})$ with $L = P_{\mathcal{P}}$ to get a useful expression for $h_{\mathcal{P}}(g)$.

Definition 50. For $1 \leqslant k < j \leqslant r$ and $h_1, \ldots, h_j \in \mathcal{W}(\mathfrak{g})$, define $R^{\mathcal{P}}_{j,k} : \mathcal{W}(\mathfrak{g})^j \to \mathcal{W}(\mathfrak{g})$ by

$$\begin{aligned} R^{\mathcal{P}}_{j,k}(h_1, \ldots, h_j) := {} & \mathrm{ad}_{P_{\mathcal{P}} h_1} \cdots \mathrm{ad}_{P_{\mathcal{P}} h_{k-1}} \big(P_{\mathcal{P}}(\mathrm{ad}_{h_k} \cdots \mathrm{ad}_{h_{j-1}} h_j)\big) \\ & - \mathrm{ad}_{P_{\mathcal{P}} h_1} \cdots \mathrm{ad}_{P_{\mathcal{P}} h_k} \big(P_{\mathcal{P}}(\mathrm{ad}_{h_{k+1}} \cdots \mathrm{ad}_{h_{j-1}} h_j)\big). \end{aligned}$$

Observe that $R^{\mathcal{P}}_{j,k}$ is a multilinear function and $R^{\mathcal{P}}_{j,k}|_{H(\mathfrak{g})^j} \subset H(\mathfrak{g})$.

Recalling the definition of $h_{\mathcal{P}}(g)$ in Eq. (47), we can use the above to write

$$\begin{aligned} h_{\mathcal{P}}(g) &= P_{\mathcal{P}} h + \sum_{m=1}^{r-1}\sum_{l=0}^{m-1} \tilde{C}_{l,m}\big(\mathrm{ad}^l_{P_{\mathcal{P}}(g)} P_{\mathcal{P}}\big(\mathrm{ad}_g^{m-l} h\big) - \mathrm{ad}^{l+1}_{P_{\mathcal{P}}(g)} P_{\mathcal{P}}\big(\mathrm{ad}_g^{m-l+1} h\big)\big) \\ &= P_{\mathcal{P}} h + \sum_{1\leqslant k<j\leqslant r} \tilde{C}_{j,k} R^{\mathcal{P}}_{j,k}\big(\overbrace{g, \ldots, g}^{j-1 \text{ times}}, h\big). \end{aligned} \tag{54}$$

We now include an example of the above decomposition of $h_{\mathcal{P}}(g)$ when $\mathfrak{g}$ is a step 3 graded Lie algebra. Step 2 graded Lie algebras (for example the Heisenberg Lie algebra) are the first examples in which such calculations are non-trivial. The reader is encouraged to examine the case of the Heisenberg Lie algebra as an exercise. The following example is perhaps more helpful in understanding the general case.

Example 51. Suppose that $\mathfrak{g}$ is a step 3 graded Lie algebra. Then the BCH series gives

$$g\cdot h = g + h + \frac{1}{2}[g,h] + \frac{1}{12}\big(\big[g,[g,h]\big] - \big[h,[g,h]\big]\big)$$

and so

$$P_{\mathcal{P}}(g\cdot th) = P_{\mathcal{P}} g + t P_{\mathcal{P}} h + \frac{t}{2} P_{\mathcal{P}}[g,h] + \frac{t}{12} P_{\mathcal{P}}\big[g,[g,h]\big] + O\big(t^2\big).$$

We therefore arrive at the following expression for $h_{\mathcal{P}}(g)$:

$$\begin{aligned}
&\frac{d}{dt}\Big|_0(-P_{\mathcal{P}}g)\cdot\left(P_{\mathcal{P}}g+tP_{\mathcal{P}}h+\frac{t}{2}P_{\mathcal{P}}[g,h]+\frac{t}{12}P_{\mathcal{P}}\big[g,[g,h]\big]-\frac{t^2}{12}P_{\mathcal{P}}\big[h,[g,h]\big]\right)\\
&\quad=P_{\mathcal{P}}h+\frac{1}{2}P_{\mathcal{P}}[g,h]+\frac{1}{12}P_{\mathcal{P}}\big[g,[g,h]\big]-\frac{1}{2}[P_{\mathcal{P}}g,P_{\mathcal{P}}h]\\
&\qquad-\frac{1}{4}\big[P_{\mathcal{P}}g,P_{\mathcal{P}}[g,h]\big]+\frac{1}{12}\big[P_{\mathcal{P}}g,[P_{\mathcal{P}}g,P_{\mathcal{P}}h]\big]+\frac{1}{12}\big[P_{\mathcal{P}}g,[P_{\mathcal{P}}g,P_{\mathcal{P}}h]\big]\\
&\quad=P_{\mathcal{P}}h+\frac{1}{2}\big(P_{\mathcal{P}}[g,h]-[P_{\mathcal{P}}g,P_{\mathcal{P}}h]\big)+\frac{1}{12}\big(P_{\mathcal{P}}\big[g,[g,h]\big]-\big[P_{\mathcal{P}}g,P_{\mathcal{P}}[g,h]\big]\big)\\
&\qquad-\frac{1}{6}\big(\big[P_{\mathcal{P}}g,P_{\mathcal{P}}[g,h]\big]-\big[P_{\mathcal{P}}g,[P_{\mathcal{P}}g,P_{\mathcal{P}}h]\big]\big)\\
&\quad=P_{\mathcal{P}}h+\frac{1}{2}R^{\mathcal{P}}_{2,1}(g,h)+\frac{1}{12}R^{\mathcal{P}}_{3,1}(g,g,h)-\frac{1}{6}R^{\mathcal{P}}_{3,2}(g,g,h).
\end{aligned}$$

Remark 52. Notice that if any one of $h_1,\dots,h_j$ is equal to $\underline{0}$, then

$$R^{\mathcal{P}}_{j,k}(h_1,\dots,h_j)=\underline{0}.$$

Recall that our goal is to analyze derivatives of $F_{\mathcal{P}}$ at the identity path. Remark 52 and Eq. (49) imply that

$$\begin{aligned}
\big\langle\alpha(\mathcal{P}),h\big\rangle&=\big\langle\alpha,h_{\mathcal{P}}(\underline{0})\big\rangle\\
&=\left\langle\alpha,P_{\mathcal{P}}h+\sum_{1\leqslant k<j\leqslant r}\tilde{C}_{j,k}R^{\mathcal{P}}_{j,k}(\underline{0},\dots,\underline{0},h)\right\rangle\\
&=\langle\alpha,P_{\mathcal{P}}h\rangle.
\end{aligned}\tag{55}$$

We now consider higher order derivatives of $F_{\mathcal{P}}$. Suppose R is a function defined on $H(\mathfrak{g})^j$. For $k\in H(\mathfrak{g})$, we define, in the spirit of Notation 3,

$$\big(\tilde{k}^{(i)}R\big)(g_1,\dots,g_j):=\frac{d}{dt}\Big|_0R(g_1,\dots,g_{i-1},g_i\cdot tk,g_{i+1},\dots,g_j).$$

Note that if R is multilinear, then

$$\big(\tilde{k}^{(i)}R\big)(g_1,\dots,g_j)=R(g_1,\dots,g_{i-1},\tilde{k}(g_i),g_{i+1},\dots,g_j).\tag{56}$$

The notation above allows us to write

$$\tilde{k}h_{\mathcal{P}}(g)=\frac{d}{dt}\Big|_0h_{\mathcal{P}}(g\cdot tk)=\sum_{1\leqslant k<j\leqslant r}\sum_{i=1}^{j-1}\tilde{C}_{j,k}\tilde{k}^{(i)}R^{\mathcal{P}}_{j,k}(g,\dots,g,h),\tag{57}$$

where $\tilde{k}^{(i)}R^{\mathcal{P}}_{j,k}$ is given as in Eq. (56). Therefore

$$
\begin{aligned}
(\tilde{k}\tilde{h}F_{\mathcal{P}})(g) &= \frac{d}{dt}\Big|_0 (\tilde{h}F_{\mathcal{P}})(g\cdot tk) \\
&= \frac{d}{dt}\Big|_0 \big(h_{\mathcal{P}}(g\cdot tk)u_\alpha\big)\big(P_{\mathcal{P}}(g\cdot tk)\big) \\
&= \big(\tilde{k}_{\mathcal{P}}(g)\tilde{h}_{\mathcal{P}}(g)u_\alpha\big)(P_{\mathcal{P}}g) + \big(\widetilde{\tilde{k}h_{\mathcal{P}}(g)}u_\alpha\big)(P_{\mathcal{P}}g) \\
&= \big\langle D^2u_\alpha(P_{\mathcal{P}}g), k_{\mathcal{P}}(g)\otimes h_{\mathcal{P}}(g)\big\rangle \\
&\quad + \sum_{1\leqslant k<j\leqslant r}\sum_{i=1}^{j-1}\tilde{C}_{j,k}\big\langle Du_\alpha(P_{\mathcal{P}}g), \tilde{k}^{(i)}R^{\mathcal{P}}_{j,k}(g,\dots,g,h)\big\rangle.
\end{aligned} \tag{58}
$$

In particular,

$$
\big\langle\alpha(\mathcal{P}), k\otimes h\big\rangle = \big\langle\alpha, P_{\mathcal{P}}k\otimes P_{\mathcal{P}}h + \tilde{C}_{2,1}R^{\mathcal{P}}_{2,1}(k,h)\big\rangle. \tag{59}
$$

We can calculate higher order derivatives of $F_{\mathcal{P}}$ analogously, with repeated use of the product rule as in line (58). This is summarized in the following proposition, whose proof is evident.

Proposition 53. *Let $\mathcal{P}$ a partition of $[0,1]$, and $F_{\mathcal{P}}$ defined as in Definition 45. Then for any $h_1,h_2,\dots,h_k\in H(\mathfrak{g})$,*

$$
(\tilde{h}_1\tilde{h}_2\cdots\tilde{h}_kF_{\mathcal{P}})(g) = \big\langle Du_\alpha(P_{\mathcal{P}}g), V^k_{\mathcal{P}}(h_1,h_2,\dots,h_k)(g)\big\rangle,
$$

where $V^k_{\mathcal{P}}(h_1,h_2,\dots,h_k)(g)$ is defined inductively with $V^0_{\mathcal{P}}(g)=1$, $V^1_{\mathcal{P}}(h)(g)=h_{\mathcal{P}}(g)$, and for $k>1$,

$$
\begin{aligned}
V^k_{\mathcal{P}}(h_1,h_2,\dots,h_k)(g) &= h_{1\mathcal{P}}(g)\otimes V^{k-1}_{\mathcal{P}}(h_2,h_3,\dots,h_k)(g) \\
&\quad + \tilde{h}_1V^{k-1}_{\mathcal{P}}(h_2,h_3,\dots,h_k)(g).
\end{aligned}
$$

It therefore follows that

$$
\big\langle\alpha(\mathcal{P}), h_1\otimes h_2\otimes\cdots\otimes h_k\big\rangle = \big\langle\alpha, V^k_{\mathcal{P}}(h_1,h_2,\dots,h_k)(\underline{e})\big\rangle. \tag{60}
$$

The following continuation of Example 51 should provide insight into the general case.

Example 54. Again, let $\mathfrak{g}$ be a step 3 graded Lie algebra. Recall from Example 51 that

$$
V^1_{\mathcal{P}}(h)(g) = h_{\mathcal{P}}(g) = P_{\mathcal{P}}h + \frac{1}{2}R^{\mathcal{P}}_{2,1}(g,h) + \frac{1}{12}R^{\mathcal{P}}_{3,1}(g,g,h) - \frac{1}{6}R^{\mathcal{P}}_{3,2}(g,g,h).
$$

The result from Eq. (55) follows easily:

$$
\begin{aligned}
\big\langle\alpha(\mathcal{P}), h\big\rangle &= \Big\langle\alpha, P_{\mathcal{P}}h + \frac{1}{2}R^{\mathcal{P}}_{2,1}(\underline{0},h) + \frac{1}{12}R^{\mathcal{P}}_{3,1}(\underline{0},\underline{0},h) - \frac{1}{6}R^{\mathcal{P}}_{3,2}(\underline{0},\underline{0},h)\Big\rangle \\
&= \langle\alpha, P_{\mathcal{P}}h\rangle.
\end{aligned}
$$

Also note that since $\tilde{k}(g) = k + \frac{1}{2}[g,k] + \frac{1}{12}[g,[g,k]]$, it follows by the definition of $V_{\mathcal{P}}^2$ that

$$\begin{aligned}V_{\mathcal{P}}^2(k,h)(g) &= k_{\mathcal{P}}(g) \otimes h_{\mathcal{P}}(g) + \tilde{k}h_{\mathcal{P}}(g)\\ &= k_{\mathcal{P}}(g) \otimes h_{\mathcal{P}}(g) + \frac{1}{2}R_{2,1}^{\mathcal{P}}(k,h) + \frac{1}{2}R_{2,1}^{\mathcal{P}}\big([g,k],h\big) + \frac{1}{12}R_{3,1}^{\mathcal{P}}(k,g,h)\\ &\quad + \frac{1}{12}R_{3,1}^{\mathcal{P}}(g,k,h) - \frac{1}{6}R_{3,2}^{\mathcal{P}}(k,g,h) - \frac{1}{6}R_{3,2}^{\mathcal{P}}(g,k,h).\end{aligned}$$

Therefore, as in Eq. (59),

$$\big\langle \alpha(\mathcal{P}), k \otimes h\big\rangle = \langle \alpha, P_{\mathcal{P}}k \otimes P_{\mathcal{P}}h\rangle + \frac{1}{2}\big\langle \alpha, R_{2,1}^{\mathcal{P}}(k,h)\big\rangle.$$

We get the following expression for $V_{\mathcal{P}}^3(\underline{e})$:

$$\begin{aligned}V_{\mathcal{P}}^3(l,k,h)(\underline{e}) &= P_{\mathcal{P}}l \otimes P_{\mathcal{P}}k \otimes P_{\mathcal{P}}h + P_{\mathcal{P}}k \otimes \frac{1}{2}R_{2,1}^{\mathcal{P}}(l,h) + \frac{1}{2}R_{2,1}^{\mathcal{P}}(l,k) \otimes P_{\mathcal{P}}h\\ &\quad + P_{\mathcal{P}}l \otimes \frac{1}{2}R_{2,1}^{\mathcal{P}}(k,h) + \frac{1}{4}R_{2,1}^{\mathcal{P}}\big([l,k],h\big) + \frac{1}{12}R_{3,1}^{\mathcal{P}}(k,l,h)\\ &\quad + \frac{1}{12}R_{3,1}^{\mathcal{P}}(l,k,h) - \frac{1}{6}R_{3,2}^{\mathcal{P}}(l,k,h) - \frac{1}{6}R_{3,2}^{\mathcal{P}}(k,l,h). \end{aligned}\tag{61}$$

As the example above indicates, the tensors $V_{\mathcal{P}}^k$ are quite complicated in general. It is clear from the definition that $V_{\mathcal{P}}^k(g) \in \bigoplus_{l=1}^k T(H(\mathfrak{g}))^{\otimes l}$. We are able to show below that for k sufficiently large, $V_{\mathcal{P}}^k(g)$ contains no tensors of "small" order (see Proposition 56). Since α is assumed to be of finite rank, this will imply that $\alpha(\mathcal{P})$ is also of finite rank for any partition $\mathcal{P}$ (see Corollary 57 below).

Since we have assumed that $\mathfrak{g}$, and hence $H(\mathfrak{g})$, are step r nilpotent, it follows that if $R : H(\mathfrak{g}) \to H(\mathfrak{g})$ is a linear function, then for any $l_1, l_2, \dots, l_m \in H(\mathfrak{g})$ with $m > r$,

$$\tilde{l}_1\tilde{l}_2\cdots\tilde{l}_m R(g) = \underline{0}$$

for any $g \in H(G)$. This, along with Eq. (56), implies

$$\tilde{l}_1\tilde{l}_2\cdots\tilde{l}_m R_{j,k}^{\mathcal{P}}(g,\dots,g,h) = \underline{0}\tag{62}$$

if $m > (j-1)r$ and, by Eq. (54),

$$\tilde{l}_1\tilde{l}_2\cdots\tilde{l}_m h_{\mathcal{P}}(g) = \underline{0},\tag{63}$$

for $m > r^2 - r$.

For higher order derivatives, it can be seen that the result of evaluating derivatives of $F_{\mathcal{P}}$ at the identity yields α acting on a sum of tensor products of terms like $P_{\mathcal{P}}h$ and $R_{j,k}^{\mathcal{P}}$, where the arguments of the terms $R_{j,k}^{\mathcal{P}}$ could be nested brackets. This is summarized in the following proposition.

Proposition 55. *For all $k > 0$, $g \in \mathcal{W}(G)$ and $h_1, h_2, \dots, h_k \in H(\mathfrak{g})$,*

$$\tilde{l}_1\tilde{l}_2\cdots\tilde{l}_m V_{\mathcal{P}}^k(h_1, h_2, \dots, h_k)(g) = \underline{0} \tag{64}$$

for any $l_1, l_2, \dots, l_m \in H(\mathfrak{g})$ if $m > k(r^2 - r)$.

Proof. We proceed by induction on k. For $k = 1$, the result follows from Eq. (63). Suppose Eq. (64) holds up to order $k - 1$. Then if $m > k(r^2 - r)$, then we have that by the definition of $V_{\mathcal{P}}^k$, for any $h_1, \dots, h_k \in H(\mathfrak{g})$,

$$\begin{aligned}\tilde{l}_1\cdots\tilde{l}_m V_{\mathcal{P}}^k(h_1, \dots, h_k)(g) &= \tilde{l}_1\cdots\tilde{l}_m\big(h_{1\mathcal{P}}(g) \otimes V_{\mathcal{P}}^{k-1}(h_2, h_3, \dots, h_k)(g)\big)\\ &\quad + \tilde{l}_1\cdots\tilde{l}_m\big(\tilde{h}_1 V_{\mathcal{P}}^{k-1}(h_2, h_3, \dots, h_k)(g)\big).\end{aligned}$$

Since $m + 1 > k(r^2 - r) > (k - 1)(r^2 - r)$, the second term above is zero. For the first term, we use the product rule and the pigeon-hole principle. If fewer than or equal to $r^2 - r$ derivatives fall on $h_{1\mathcal{P}}(g)$, this leaves more than $(k - 1)(r^2 - r)$ to fall on the $V_{\mathcal{P}}^{k-1}$ term, which gives zero by the induction hypothesis. If more than $r^2 - r$ derivatives fall on $h_{1\mathcal{P}}(g)$, then we get zero again by Eq. (63). □

Proposition 56. *For all $k > 0$, $g \in H(G)$, and $h_1, \dots, h_k \in H(\mathfrak{g})$,*

$$V_{\mathcal{P}}^m(h_1, h_2, \dots, h_m)(g) \cap H(\mathfrak{g})^{\otimes k} = \underline{0}$$

if $m \geqslant \frac{1}{2}k(k + 1)(r^2 - r)$.

Proof. We start by observing that if l is the smallest integer such that $V_{\mathcal{P}}^m(h_1, h_2, \dots, h_m)(g) \cap H(\mathfrak{g})^{\otimes l} \neq \underline{0}$, then, if $n > l(r^2 - r)$,

$$\begin{aligned}&V_{\mathcal{P}}^{m+n}(l_1, \dots, l_n, h_1, \dots, h_m)(g) \cap H(\mathfrak{g})^{\otimes l}\\ &\quad = \big(\tilde{l}_1\tilde{l}_2\cdots\tilde{l}_n V_{\mathcal{P}}^m(h_1, h_2, \dots, h_m)(g)\big) \cap H(\mathfrak{g})^{\otimes l} = \underline{0}\end{aligned}$$

by Proposition 55.

We now proceed by induction on k. Note that since $V_{\mathcal{P}}^1(h) = h_{\mathcal{P}}(g)$, then if $m > r(r - 1)$, by Eq. (63) it follows that $V_{\mathcal{P}}^m(h_1, h_2, \dots, h_m)(g) \cap H(\mathfrak{g}) = \underline{0}$. So the result holds for $k = 1$. Now suppose that $V_{\mathcal{P}}^{m'}(h_1, h_2, \dots, h_{m'})(g) \cap H(\mathfrak{g})^{\otimes k-1} = \underline{0}$ for all $m' > \frac{1}{2}(k - 1)k(r^2 - r)$. Then by the above observation,

$$V_{\mathcal{P}}^{m'+k(r(r-1)}(l_1, \dots, l_{k(r(r-1))}, h_1, \dots, h_{m'})(g) \cap H(\mathfrak{g})^{\otimes k} = \underline{0}.$$

The result follows, since $m' + k(r^2 - r) > \frac{1}{2}(k - 1)k(r^2 - r) + k(r^2 - r) = \frac{1}{2}k(k + 1)(r^2 - r)$. □

Corollary 57. *Suppose α is of rank N. Then for every partition $\mathcal{P}$, $\alpha(\mathcal{P})$ is of rank less than $\frac{1}{2}N(N + 1)(r^2 - r)$.*

Proof. Suppose $m \geqslant \frac{1}{2}N(N+1)(r^2-r)$ and $h_1,\dots,h_m \in H(\mathfrak{g})$. Then

$$\langle \alpha(\mathcal{P}), h_1 \otimes h_2 \otimes \cdots \otimes h_m \rangle = \langle \alpha, V_{\mathcal{P}}^m(h_1,\dots,h_m)(\underline{e})\rangle = 0$$

since by Proposition 56, $V_{\mathcal{P}}^m(h_1,\dots,h_m)(\underline{e}) \cap H(\mathfrak{g})^{\otimes k} = \underline{0}$ if $k \leqslant N$. $\square$

In light of the previous Corollary, we can write, for all $h_1, h_2, \dots, h_k \in H(\mathfrak{g})$,

$$\begin{aligned}\langle \alpha(\mathcal{P}), h_1 \otimes h_2 \otimes \cdots \otimes h_k \rangle &= \langle \alpha, P_{\mathcal{P}}h_1 \otimes P_{\mathcal{P}}h_2 \otimes \cdots \otimes P_{\mathcal{P}}h_k \rangle \\ &\quad + \langle \alpha, R_k^{\mathcal{P}}(h_1,\dots,h_k)\rangle, \end{aligned} \tag{65}$$

with $R_k^{\mathcal{P}}(h_1,\dots,h_k) \in \bigoplus_{l=m}^{k-1} H(\mathfrak{g})^{\otimes l}$ where m is the largest positive integer such that $k \geqslant \frac{1}{2}m(m+1)(r^2-r)$. The terms $R_k^{\mathcal{P}}$ will sometimes be referred to as *remainder terms*. It is the subject of Section 5.5 to show that these become sufficiently small as our partition mesh tends toward zero. We end this subsection by establishing some useful shorthand.

Notation 58. For $i = 1,\dots,j$, let $B_i : H(\mathfrak{g})^{p_i} \to H(\mathfrak{g})$ be multilinear functions with p_i are positive integers such that $\sum_{i=1}^{j} p_i = p$. Let $\overline{B} : H(\mathfrak{g})^p \to H(\mathfrak{g})^j$ be defined by

$$\overline{B}(h_1,\dots,h_p) = \big(B_1(h_1,\dots,h_{p_1}), B_2(h_{p_1+1},\dots,h_{p_1+p_2}),\dots,B_j(h_{p-p_j+1},\dots,h_p)\big).$$

We will use the above to denote the terms which arise from derivatives of $R_{j,k}^{\mathcal{P}}$.

Example 59. Equation (61) of Example 54 contains the term $R_{2,1}^{\mathcal{P}}([l,k],h)$. Setting $B_1(l,k) = [l,k]$, $B_2(h) = h$, and $\overline{B} = (B_1, B_2)$, then we can write

$$R_{2,1}^{\mathcal{P}}\big([l,k],h\big) = \big(R_{2,1}^{\mathcal{P}} \circ \overline{B}\big)(l,k,h).$$

In the next subsection, we will prove some results regarding terms of the form $R_{j,k}^{\mathcal{P}} \circ \overline{B}$. These will be independent of the specifics of $\overline{B}$, so we will further shorten our notation with setting

$$\big(R_{j,k}^{\mathcal{P}} \circ \overline{B}\big)(\bar{h}) := \big(R_{j,k}^{\mathcal{P}} \circ \overline{B}\big)(h_1,h_2,\dots,h_p). \tag{66}$$

The meaning of $\overline{B}$ and $\bar{h}$ should be clear from the context.

5.4. Increments and multilinear functions on $H(\mathfrak{g})$

As seen in the previous subsection, our cylinder functions are based on approximations of paths in $\mathcal{W}(\mathfrak{g})$ by paths which are piecewise linear subordinate to a partition of $[0,1]$. Thus we develop some results concerning the increments of paths over small time intervals. What follows is primarily notation and some estimates that will be essential for our analysis to follow in Section 5.5. We generalize to arbitrary multilinear functions, though in the sequel will be primarily concerned with applying the following results to the functions $R_{j,k}^{\mathcal{P}}$ given in Definition 50. The reader may find it convenient to skip to Section 5.5 and revisit this section as necessary.

Notation 60. Given a partition $\mathcal{P} = \{0 = s_0 < s_1 < \cdots < s_n < s_{n+1} = 1\}$, we will let δ_i denote the increment function for $i = 0, \ldots, n$. That is, if V is a vector space, and $f : [0, 1] \to V$ is any function, then

$$\delta_i f := f(s_{i+1}) - f(s_i).$$

In the case that f is the identity function on [0, 1], we omit the f in the notation, i.e.

$$\delta_i := s_{i+1} - s_i.$$

The notation above will be used often in the following contexts. If $h \in H(\mathfrak{g})$, then

$$\delta_i h := h(s_{i+1}) - h(s_i). \tag{67}$$

If $h_1, \ldots, h_k \in H(\mathfrak{g})$, then $(h_1, \ldots, h_k) \in H(\mathfrak{g})^k$, and

$$\delta_i(h_1, \ldots, h_k) := \big(h_1(s_{i+1}), \ldots, h_k(s_{i+1})\big) - \big(h_1(s_i), \ldots, h_k(s_i)\big).$$

Finally, if $h_1, \ldots, h_k \in H(\mathfrak{g})$ and T is a multilinear function on $\mathfrak{g}^k$, then

$$\delta_i T(h_1, \ldots, h_k) := T\big(h_1(s_{i+1}), \ldots, h_k(s_{i+1})\big) - T\big(h_1(s_i), \ldots, h_k(s_i)\big). \tag{68}$$

We introduce a discrete version of the product rule (Proposition 64) to express increments of a multilinear function (as in Eq. (68) above) in terms of increments of the arguments. We need the following notation.

Notation 61. For integers $k \geqslant l \geqslant 0$, let Ω_k^l denote the set of subsets of size l of the integers $1, 2, \ldots, k$. That is

$$\Omega_k^l := \big\{\omega \in 2^{\{1,2,\ldots,k\}} \,\big|\, \#(\omega) = l\big\}.$$

Notation 62. Suppose $u_1, \ldots, u_k \in H(\mathbb{C})$ and $\omega \in \Omega_k^l$. Then

$$\delta_i^\omega(u_1 \cdots u_k) := \prod_{j \in \omega} \delta_i u_j \prod_{j \notin \omega} u_j(s_i).$$

Similarly, if $h_1, \ldots, h_k \in H(\mathfrak{g})$, then

$$\delta_i^\omega(h_1, \ldots, h_k) := (\widehat{h}_1, \ldots, \widehat{h}_k)$$

where $\widehat{h}_j = \delta_i h_j$ if $j \in \omega$ and $\widehat{h}_j = h_j(s_i)$, otherwise.

Example 63. Suppose $\omega = \{1, 3, 4\} \in \Omega_5^3$. Then

$$\delta_i^\omega(u_1 \cdots u_5) = (\delta_i u_1)(\delta_i u_3)(\delta_i u_4)\big(u_2(s_i)\big)\big(u_5(s_i)\big)$$

and

$$\delta_i^{\omega}(h_1, \ldots, h_5) = \big(\delta_i h_1, h_2(s_i), \delta_i h_3, \delta_i h_4, h_5(s_i)\big).$$

Observe that if T is a bilinear function on $\mathfrak{g}^2$, then for $h_1, h_2 \in H(\mathfrak{g})$

$$\begin{aligned}
\delta_i T(h_1, h_2) &= T\big(h_1(s_{i+1}), h_2(s_{i+1})\big) - T\big(h_1(s_i), h_2(s_i)\big) \\
&= T\big(h_1(s_{i+1}), h_2(s_i)\big) - T\big(h_1(s_i), h_2(s_i)\big) + T\big(h_1(s_i), h_2(s_{i+1})\big) \\
&\quad - T\big(h_1(s_i), h_2(s_i)\big) + T\big(h_1(s_{i+1}), h_2(s_{i+1})\big) - T\big(h_1(s_{i+1}), h_2(s_i)\big) \\
&\quad - T\big(h_1(s_i), h_2(s_{i+1})\big) + T\big(h_1(s_i), h_2(s_i)\big) \\
&= T\big(h_1(s_{i+1}), h_2(s_i)\big) - T\big(h_1(s_i), h_2(s_i)\big) + T\big(h_1(s_i), h_2(s_{i+1})\big) \\
&\quad - T\big(h_1(s_i), h_2(s_i)\big) + T\big(h_1(s_{i+1}) - h_1(s_i), h_2(s_{i+1}) - h_2(s_i)\big) \\
&= T\big(\delta_i h_1, h_2(s_i)\big) + T\big(h_1(s_i), \delta_i h_2\big) + T(\delta_i h_1, \delta_i h_2) \\
&= \sum_{l=1}^{2} \sum_{\omega \in \Omega_2^l} T\big(\delta_i^{\omega}(h_1, h_2)\big).
\end{aligned}$$

This suggests the following product rule, which can be easily proved by induction on the above.

Proposition 64 *(Product rule). Suppose T be a multilinear function on $\mathfrak{g}^k$. Then*

$$\delta_i T(h_1, \ldots, h_k) = \sum_{l=1}^{k} \sum_{\omega \in \Omega_k^l} T\big(\delta_i^{\omega}(h_1, \ldots, h_k)\big).$$

Remark 65. Letting T be the identity map on $\mathfrak{g}^k$, the above proposition tells us that

$$\delta_i(h_1, \ldots, h_k) = \sum_{l=1}^{k} \sum_{\omega \in \Omega_n^l} \delta_i^{\omega}(h_1, \ldots, h_k).$$

Furthermore, if $T : \mathbb{C}^k \to \mathbb{C}$ by $T(z_1, \ldots, z_k) = z_1 \cdots z_k$, then we get

$$\delta_i(u_1 \cdots u_k) = \sum_{l=1}^{k} \sum_{\omega \in \Omega_k^l} \delta_i^{\omega}(u_1 \cdots u_k)$$

for $u_1, \ldots, u_k \in H(\mathbb{C})$.

In the sequel we will be summing such increments of multilinear functions over an orthonormal basis for $H(\mathfrak{g})$. We are able to calculate such sums explicitly using the reproducing property of $H(\mathbb{C})$. The following proposition is a summary of well-known properties of the reproducing kernel.

Proposition 66. *Suppose $\mathcal{P} = \{0 = s_0 < s_1 < \cdots < s_n < s_{n+1} = 1\}$ is a partition of $[0,1]$. For any $s, t \in [0,1]$ and any $1 \leqslant i, j \leqslant n$,*

(1) $\sum_{u \in S(\mathbb{C})} u(s)\overline{u(t)} = K(s,t) := s \wedge t$,
(2) $\sum_{u \in S(\mathbb{C})} |\delta_i u|^2 = K(\delta_i, \delta_i) = \delta_i$,
(3) $\sum_{u \in S(\mathbb{C})} (\delta_i u)\overline{u(s_j)} = 1_{j>i}\delta_i$,
(4) $\sum_{u \in S(\mathbb{C})} (\delta_i u)\overline{(\delta_j u)} = 1_{ij}\delta_i$,

where 1_{ij} denotes the Kronecker delta and $1_{j>i} = 1$ if $j > i$ and 0 otherwise.

Proof. Properties (2)–(4) are straightforward applications of (1). So we only prove (1). By the Fundamental theorem of calculus,

$$u(s) = \int_0^1 1_{\tau \leqslant s} u'(\tau)\, d\tau. \tag{69}$$

Let $\kappa_s \in H(\mathbb{C})$ be given by $\kappa_s(\cdot) := s \wedge \cdot$. If $u \in H(\mathbb{C})$, then Eq. (69) is equivalent to

$$u(s) = (u, \kappa_s)_{H(\mathbb{C})},$$

and so

$$\begin{aligned}
\sum_{u \in S(\mathbb{C})} u(s)\overline{u(t)} &= \sum_{u \in S(\mathbb{C})} (u, \kappa_s)_{H(\mathbb{C})}(\kappa_t, u)_{H(\mathbb{C})} \\
&= \sum_{u \in S(\mathbb{C})} (\kappa_t, \kappa_s)_{H(\mathbb{C})} \\
&= \int_0^1 1_{[0,t]}(\tau) 1_{[0,s]}(\tau)\, d\tau \\
&= s \wedge t. \qquad \square
\end{aligned}$$

Notation 67. Suppose $\omega \in \Omega_n^l$, and $\theta \in \Omega_n^m$, then we write $\omega = \theta$ if they are equal as sets of positive integers and $l = m$.

The following proposition will be essential for many calculations to follow.

Proposition 68. *Suppose $\omega \in \Omega_n^l$, and $\theta \in \Omega_n^m$, with $n \geqslant 2$ and $l, m \geqslant 1$. Then*

$$\sum_{u_1, \ldots, u_n \in S(\mathbb{C})} \delta_i^{\omega}(u_1 \cdots u_n)\delta_j^{\theta}(\overline{u_1 \cdots u_n}) = 1_{ij} 1_{\omega\theta}(\delta_i)^l (s_i)^{n-l},$$

where $1_{\omega\theta} = 1$ if $\omega = \theta$ in the sense of Notation 67, and 0, otherwise.

Proof. First suppose $\omega \neq \theta$. If $\omega \cap \theta \neq \emptyset$, then w.l.o.g. there exist elements $p \in \omega$ with $p \notin \theta$, and $q \in \omega \cap \theta$. W.l.o.g., say $p = 1$ and $q = 2$. Then

$$\begin{aligned}
&\sum_{u_1,\dots,u_n \in S(\mathbb{C})} \delta_i^{\omega}(u_1 \cdots u_n)\delta_j^{\theta}(\overline{u_1 \cdots u_n}) \\
&= \Biggl(\sum_{u_1 \in S(\mathbb{C})} (\delta_i u_1)\bigl(\overline{u_1(s_j)}\bigr)\Biggr)\Biggl(\sum_{u_2 \in S(\mathbb{C})} (\delta_i u_2)(\overline{\delta_j u_2})\Biggr) \\
&\quad \times \cdots \times \Biggl(\sum_{u_3,\dots,u_n \in S(\mathbb{C})} \delta_i^{\omega \setminus \{1,2\}}(u_3 \cdots u_n)\delta_j^{\theta \setminus \{2\}}(\overline{u_3 \cdots u_n})\Biggr) \\
&= (1_{j>i}\delta_i)(1_{ij}\delta_i)\Biggl(\sum_{u_3,\dots,u_n \in S(\mathbb{C})} \delta_i^{\omega \setminus \{1,2\}}(u_3 \cdots u_n)\delta_j^{\theta \setminus \{2\}}(\overline{u_3 \cdots u_n})\Biggr) \\
&= 0,
\end{aligned}$$

since $1_{j>i}1_{ij} = 0$. If $\omega \cap \theta = \emptyset$, then there exist elements p, q such that $p \in \omega$, $p \notin \theta$, $q \notin \omega$, and $q \in \theta$. W.l.o.g., say $p = 1$ and $q = 2$. Then

$$\begin{aligned}
&\sum_{u_1,\dots,u_n \in S(\mathbb{C})} \delta_i^{\omega}(u_1 \cdots u_n)\delta_j^{\theta}(\overline{u_1 \cdots u_n}) \\
&= \Biggl(\sum_{u_1 \in S(\mathbb{C})} (\delta_i u_1)\bigl(\overline{u_1(s_j)}\bigr)\Biggr)\Biggl(\sum_{u_2 \in S(\mathbb{C})} \bigl(u_2(s_i)\bigr)(\overline{\delta_j u_2})\Biggr) \\
&\quad \times \cdots \times \Biggl(\sum_{u_3,\dots,u_n \in S(\mathbb{C})} \delta_i^{\omega \setminus \{1\}}(u_3 \cdots u_n)\delta_j^{\theta \setminus \{2\}}(\overline{u_3 \cdots u_n})\Biggr) \\
&= (1_{j>i}\delta_i)(1_{i>j}\delta_j)\Biggl(\sum_{u_3,\dots,u_n \in S(\mathbb{C})} \delta_i^{\omega \setminus \{1\}}(u_3 \cdots u_n)\delta_j^{\theta \setminus \{2\}}(\overline{u_3 \cdots u_n})\Biggr) \\
&= 0,
\end{aligned}$$

since $1_{j>i}1_{i>j} = 0$. Now we assume that $\omega = \theta$. Then

$$\begin{aligned}
&\sum_{u_1,\dots,u_n \in S(\mathbb{C})} \delta_i^{\omega}(u_1 \cdots u_n)\delta_j^{\theta}(\overline{u_1 \cdots u_n}) \\
&= \prod_{p \in \omega}\Biggl(\sum_{u_p \in S(\mathbb{C})} (\delta_i u_p)(\overline{\delta_j u_p})\Biggr) \prod_{q \in \omega^c}\Biggl(\sum_{u_q \in S(\mathbb{C})} \bigl(u_q(s_i)\bigr)\bigl(\overline{u_q(s_j)}\bigr)\Biggr) \\
&= 1_{ij}(\delta_i)^l(s_i)^{n-l}. \qquad \square
\end{aligned}$$

The next three corollaries follow from the above and an application of Proposition 64.

Corollary 69. *For all n,*

$$\sum_{u_1,\ldots,u_n\in S(\mathbb{C})} \delta_i(u_1\cdots u_n)\delta_j(\overline{u_1\cdots u_n}) = 1_{ij}\sum_{l=1}^{n}\binom{n}{l}(\delta_i)^l(s_i)^{n-l}.$$

Notice that if $h_j = u_j A_j \in S_{\mathbb{C}}$ for $u_j \in S(\mathbb{C})$ and $A_j \in \mathfrak{X}_{\mathbb{C}}$, then

$$\delta_i^{\omega} T(h_1,\ldots,h_n) = \delta_i^{\omega}(u_1\cdots u_n)T(A_1,\ldots,A_n)$$

and

$$\delta_i T(h_1,\ldots,h_n) = \delta_i(u_1\cdots u_n)T(A_1,\ldots,A_n).$$

Notation 70. If T_1 and T_2 are $\mathfrak{g}$-valued multilinear functions on $\mathfrak{g}^n$, then define $C_n(T_1,T_2)\in \mathfrak{g}\otimes\mathfrak{g}$ by

$$C_n(T_1,T_2) := \sum_{A_1,\ldots,A_n\in\mathfrak{X}_{\mathbb{C}}} T_1(A_1,\ldots,A_n)\otimes\overline{T_2(A_1,\ldots,A_n)}.$$

Corollary 71. *Suppose $\omega\in\Omega_n^l$, and $\theta\in\Omega_n^m$, with $n\geqslant 2$ and $l,m\geqslant 1$. Let T_1 and T_2 be $\mathfrak{g}$-valued multilinear functions on $\mathfrak{g}^n$. Then*

$$\begin{aligned}&\sum_{h_1,\ldots,h_n\in S_{\mathbb{C}}} T_1\big(\delta_i^{\omega}(h_1,\ldots,h_n)\big)\otimes\overline{T_2\big(\delta_j^{\theta}(h_1,\ldots,h_n)\big)}\\ &\quad = 1_{ij}1_{\omega\theta}(\delta_i)^l(s_i)^{n-l}C_n(T_1,T_2).\end{aligned}$$

Corollary 72. *Let T_1 and T_2 be $\mathfrak{g}$-valued multilinear functions on $\mathfrak{g}^n$. Then*

$$\begin{aligned}&\sum_{h_1,\ldots,h_n\in S_{\mathbb{C}}} \delta_i T_1(h_1,\ldots,h_n)\otimes\overline{\delta_j T_2(h_1,\ldots,h_n)}\\ &\quad = 1_{ij}\sum_{l=1}^{n}\binom{n}{l}(\delta_i)^l(s_i)^{n-l}C_n(T_1,T_2).\end{aligned}$$

Remark 73. It follows from Corollary 71 that for $\omega\in\Omega_n^l$ and $\theta\in\Omega_n^m$

$$\begin{aligned}&\sum_{h_1,\ldots,h_n\in S_{\mathbb{C}}} \big(T_1\big(\delta_i^{\omega}(h_1,\ldots,h_n)\big), T_2\big(\delta_j^{\theta}(h_1,\ldots,h_n)\big)\big)_{\mathfrak{g}}\\ &\quad = 1_{ij}1_{\omega\theta}(\delta_i)^l(s_i)^{n-l}\tilde{C}_n(T_1,T_2),\end{aligned}$$

where

$$\tilde{C}_n(T_1,T_2) := \sum_{A_1,\ldots,A_n\in\mathfrak{X}_{\mathbb{C}}} \big(T_1(A_1,\ldots,A_n), T_2(A_1,\ldots,A_n)\big)_{\mathfrak{g}}.$$

In particular,

$$\sum_{h_1,\dots,h_n\in S_{\mathbb{C}}} \left\| T\left(\delta_i^{\omega}(h_1,\dots,h_n)\right)\right\|_{\mathfrak{g}}^2 = (\delta_i)^l (s_i)^{n-l} \sum_{A_1,\dots,A_n\in\mathfrak{X}_{\mathbb{C}}} \left\| T(A_1,\dots,A_n)\right\|_{\mathfrak{g}}^2$$
$$= \tilde{C}_n(T,T)(\delta_i)^l(s_i)^{n-l}.$$

5.5. Remainder estimates

In this section we use the tools developed in Section 5.4 to prove the following theorem and corollary. We will assume throughout this section that $\{\mathcal{P}_n\}_{n=1}^{\infty}$ is a sequence of refining partitions with $\#(\mathcal{P}_n) = n$.

Theorem 74. *Let $\alpha \in J_T^0(H(\mathfrak{g}))$ be of finite rank. For each $n > 0$, let $\alpha(\mathcal{P}_n)$ be given as in Notation 46. Then for all $n > 0$, $\alpha(\mathcal{P}_n) \in J_T^0(H(\mathfrak{g}))$ and*

$$\left\|\alpha - \alpha(\mathcal{P}_n)\right\|_{J_T^0(H(\mathfrak{g}))} \to 0 \quad \text{as } n \to \infty.$$

Corollary 75. *For all $T > 0$, the Taylor map $(1-D)_{\underline{e}}^{-1}R : \mathcal{H}_T \to J_T^0(H(\mathfrak{g}))$ is surjective for G a simply connected graded complex Lie group.*

We begin by giving an explicit formula for $P_{\mathcal{P}}h$ in terms of increments.

Notation 76. For a given partition $\mathcal{P} = \{0 = s_0 < s_1 < \cdots < s_n < s_{n+1} = 1\}$, for $t \in [0,1]$ and $i = 0,\dots,n$, let

$$t_i(t) := 1_{(s_i,s_{i+1}]}(t)\left(\frac{t-s_i}{\delta_i}\right).$$

In the sequel, we will often omit the t in the notation, that is $t_i(t) = t_i$. Notice that $0 \leqslant t_i(t) \leqslant 1$ for all $i = 0,\dots,n$ and $t \in [0,1]$ and

$$\frac{d}{dt}t_i(t) = 1_{(s_i,s_{i+1}]}(t)\frac{1}{\delta_i}.$$

The notation above gives an explicit formula for $P_{\mathcal{P}}h$,

$$(P_{\mathcal{P}}h)(t) = \sum_{i=0}^{n}\left(h(s_i)1_{(s_i,s_{i+1}]}(t) + (\delta_i h)t_i(t)\right), \tag{70}$$

for $t \in [0,1]$. Then $P_{\mathcal{P}}h$ agrees with h at the partition points and is linear on $[0,1]/\mathcal{P}$.

Proposition 77. *For any $g, h \in H(\mathfrak{g})$ and any partition $\mathcal{P}$,*

$$P_{\mathcal{P}}(\mathrm{ad}_g h) - \mathrm{ad}_{P_{\mathcal{P}}g}(P_{\mathcal{P}}h) = \sum_{i=0}^{n}\mathrm{ad}_{\delta_i g}(\delta_i h)\left(t_i - t_i^2\right).$$

Proof. It suffices to show the result on an individual partition increment. On $(s_i, s_{i+1}]$,

$$\begin{aligned}
&P_{\mathcal{P}}(\mathrm{ad}_g h) - \mathrm{ad}_{P_{\mathcal{P}} g} P_{\mathcal{P}} h \\
&\quad = \big[g(s_i), h(s_i)\big] + \delta_i[g,h]t_i - \big[g(s_i) + \delta_i g t_i, h(s_i) + \delta_i h t_i\big] \\
&\quad = \big[g(s_i), h(s_i)\big] + \delta_i[g,h]t_i - \big[g(s_i), h(s_i)\big] \\
&\qquad - \big[\delta_i g, h(s_i)\big]t_i - \big[g(s_i), \delta_i h\big]t_i - [\delta_i g, \delta_i h]t_i^2 \\
&\quad = \big(\delta_i[g,h] - \big[\delta_i g, h(s_i)\big] - \big[g(s_i), \delta_i h\big]\big)t_i - [\delta_i g, \delta_i h]t_i^2 \\
&\quad = [\delta_i g, \delta_i h]\big(t_i - t_i^2\big) \\
&\quad = \mathrm{ad}_{\delta_i g}(\delta_i h)\big(t_i - t_i^2\big). \qquad \square
\end{aligned}$$

Proposition 77 lets us again rewrite our remainder terms:

$$\begin{aligned}
&R^{\mathcal{P}}_{j,k}(h_1, \ldots, h_j) \\
&\quad = \sum_{i=0}^{n} \mathrm{ad}_{P_{\mathcal{P}} h_1} \cdots \mathrm{ad}_{P_{\mathcal{P}} h_{k-1}} \mathrm{ad}_{\delta_i h_k}\big(\delta_i(\mathrm{ad}_{h_{k+1}} \cdots \mathrm{ad}_{h_{j-1}} h_j)\big)\big(t_i - t_i^2\big).
\end{aligned}$$

In general, the arguments of $R^{\mathcal{P}}_{j,k}$ are nested brackets, so using Notation 58,

$$\begin{aligned}
&\big(R^{\mathcal{P}}_{j,k} \circ \overline{B}\big)(\bar{h}) \\
&\quad := \sum_{i=0}^{n} \mathrm{ad}_{P_{\mathcal{P}}(B_1)} \cdots \mathrm{ad}_{P_{\mathcal{P}}(B_{k-1})} \mathrm{ad}_{\delta_i B_k}\big(\delta_i(\mathrm{ad}_{B_{k+1}} \cdots \mathrm{ad}_{B_{j-1}} B_j)\big)(\bar{h})\big(t_i - t_i^2\big).
\end{aligned}$$

Since in the above expression, both B_k and $\mathrm{ad}_{B_{k+1}} \cdots \mathrm{ad}_{B_{j-1}} B_j$ are multilinear functions of the type considered in Section 5.4, we can use Proposition 64 to write

$$\big(R^{\mathcal{P}}_{j,k} \circ \overline{B}\big)(\bar{h}) = \sum_{i=0}^{n} \sum_{l=2}^{p} \sum_{\omega \in \tilde{\Omega}^l_p} \delta_i^{\omega} T(\bar{h}) f_{\omega}(t_i), \tag{71}$$

where $T : \mathfrak{g}^p \to \mathfrak{g}$ is a multilinear function and $f_\omega(t_i)$ is a polynomial, possibly zero, in t_i depending on ω, where t_i is defined in Notation 76. Refer to Eq. (66) for the meaning of the left-hand side, while Notation 62 indicates the meaning of the right-hand side. The following example should clarify Eq. (71).

Example 78. For $r \geqslant 6$, the following remainder term appears in a sixth order derivative of $F_{\mathcal{P}}$ evaluated at the identity path:

$$\begin{aligned}
&R^{\mathcal{P}}_{3,2}\big(\big[h_1, [h_2, h_3]\big], [h_4, h_5], h_6\big) \\
&\quad = \sum_{i=0}^{n} \big[P_{\mathcal{P}}\big[h_1, [h_2, h_3]\big], \big[\delta_i[h_4, h_5], \delta_i h_6\big]\big(t_i - t_i^2\big)\big]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{n} \big[\big[h_1(s_i), \big[h_2(s_i), h_3(s_i)\big]\big], \big[\delta_i[h_4, h_5], \delta_i h_6\big]\big]\big(t_i - t_i^2\big) \\
&\quad + \sum_{i=0}^{n} \big[\delta_i\big[h_1, [h_2, h_3]\big], \big[\delta_i[h_4, h_5], \delta_i h_6\big]\big]\big(t_i^2 - t_i^3\big) \\
&= \sum_{i=0}^{n} \left[\big[h_1(s_i), \big[h_2(s_i), h_3(s_i)\big]\big], \left[\sum_{l=1}^{2} \sum_{\omega\in\Omega_2^l} \delta_i^{\omega}[h_4, h_5], \delta_i h_6\right]\right]\big(t_i - t_i^2\big) \\
&\quad + \sum_{i=0}^{n} \left[\left[\sum_{l=1}^{3} \sum_{\omega\in\Omega_3^l} \delta_i^{\omega}\big[h_1, [h_2, h_3]\big]\right], \left[\sum_{l=1}^{2} \sum_{\omega\in\Omega_2^l} \delta_i^{\omega}[h_4, h_5], \delta_i h_6\right]\right]\big(t_i^2 - t_i^3\big) \\
&= \sum_{i=0}^{n} \sum_{l=2}^{6} \sum_{\omega\in\Omega_6^l} \delta_i^{\omega}\big[\big[h_1, [h_2, h_3]\big], [h_4, h_5], h_6\big] f_{\omega}(t_i),
\end{aligned}
$$

where

$$
f_{\omega}(t_i) = \begin{cases} 0 & \text{if } \omega \cap \{6\} = \emptyset \text{ or } \omega \cap \{4, 5\} = \emptyset, \\ t_i - t_i^2 & \text{if } \omega \cap \{1, 2, 3\} = \emptyset,\ \omega \cap \{6\} \neq \emptyset, \text{ and } \omega \cap \{4, 5\} \neq \emptyset, \\ t_i^2 - t_i^3 & \text{if } \omega \cap \{1, 2, 3\} \neq \emptyset,\ \omega \cap \{6\} \neq \emptyset, \text{ and } \omega \cap \{4, 5\} \neq \emptyset. \end{cases}
$$

We will now prove some limiting properties of the $R_{j,k}^{\mathcal{P}}$ terms as the partition mesh tends to zero.

Proposition 79. *Suppose* $1 \leqslant k < j \leqslant r$, *and for* $1 \leqslant i \leqslant j$, $B_i : \mathfrak{g}^{p_i} \to \mathfrak{g}$ *be multilinear functions such that* $\sum_{i=1}^{j} p_i = p$. *Then*

$$
\lim_{n\to\infty} \sup \sum_{\bar{h}\in(S_{\mathbb{C}})^p} \big\| \big(R_{j,k}^{\mathcal{P}_n} \circ \overline{B}\big)(\bar{h}) \big\|_{H(\mathfrak{g})}^2 < \infty.
$$

Proof. By Eq. (71), we can write

$$
\big(R_{j,k}^{\mathcal{P}_n} \circ \overline{B}\big)(\bar{h}) = \sum_{i=0}^{n} \sum_{l=2}^{p} \sum_{\omega\in\Omega_p^l} \delta_i^{\omega} T(\bar{h}) f_{\omega}(t_i),
$$

where $T : \mathfrak{g}^p \to \mathfrak{g}$ is a multilinear function and, for each $\omega \in \Omega_p^l$, f_{ω} is a polynomial. Notice that both T and f_{ω} are independent of n and

$$
\begin{aligned}
\big\| \big(R_{j,k}^{\mathcal{P}_n} \circ \overline{B}\big)(\bar{h}) \big\|_{H(\mathfrak{g})}^2 &= \left\| \sum_{i=0}^{n} \sum_{l=2}^{p} \sum_{\omega\in\Omega_p^l} \delta_i^{\omega} T(\bar{h}) f_{\omega}(t_i) \right\|_{H(\mathfrak{g})}^2 \\
&= \sum_{i=0}^{n} \left\| \sum_{l=2}^{p} \sum_{\omega\in\Omega_p^l} \delta_i^{\omega} T(\bar{h}) f_{\omega}(t_i) \right\|_{H(\mathfrak{g})}^2
\end{aligned}
$$

$$\leqslant C(p)\sum_{i=0}^{n}\sum_{l=2}^{p}\sum_{\omega\in\Omega_p^l}\left\|\delta_i^{\omega}T(\bar{h})f_{\omega}(t_i)\right\|_{H(\mathfrak{g})}^2$$

$$= C(p)\sum_{i=0}^{n}\sum_{l=2}^{p}\sum_{\omega\in\Omega_p^l}\left\|\delta_i^{\omega}T(\bar{h})\right\|_{\mathfrak{g}}^2\left\|\frac{d}{dt}f_{\omega}(t_i)\right\|_{L^2([0,1],\mathbb{C})}^2.$$

Furthermore,

$$\begin{aligned}\left\|\frac{d}{dt}f_{\omega}(t_i)\right\|_{L^2([0,1],\mathbb{C})}^2 &= \int_{s_i}^{s_{i+1}}\left|\frac{d}{dt}f_{\omega}(t_i)\right|^2 dt\\ &= \frac{1}{\delta_i^2}\int_{s_i}^{s_{i+1}}\left|f'_{\omega}\left(\frac{t-s_i}{\delta_i}\right)\right|^2 dt\\ &= \frac{1}{\delta_i}\int_0^1\left|f'_{\omega}(u)\right|^2 du\\ &= \frac{C(\omega)}{\delta_i},\end{aligned}$$

where $C(\omega)$ is an appropriate constant independent of both i and n.

By Remark 73, $\sum_{\overline{h}\in(S_{\mathbb{C}})^p}\sum_{l=2}^{p}\sum_{\omega\in\Omega_p^l}\|\delta_i^{\omega}T(\overline{h})\|_{\mathfrak{g}}^2$ is $O(\delta_i^2)$. Therefore,

$$\begin{aligned}&\limsup_{n\to\infty}\sum_{\overline{h}\in(S_{\mathbb{C}})^p}\left\|\left(R_{j,k}^{\mathcal{P}_n}\circ\overline{B}\right)(\bar{h})\right\|_{H(\mathfrak{g})}^2\\ &\leqslant \limsup_{n\to\infty}\sum_{\overline{h}\in(S_{\mathbb{C}})^p}C(p)\sum_{i=0}^{n}\sum_{l=2}^{p}\sum_{\omega\in\Omega_p^l}\left\|\delta_i^{\omega}T(\bar{h})\right\|_{\mathfrak{g}}^2\left\|\frac{d}{dt}f_{\omega}(t_i)\right\|_{L^2([0,1],\mathbb{C})}^2\\ &\leqslant \limsup_{n\to\infty}\tilde{C}(p)\sum_{i=0}^{n}\frac{1}{\delta_i}\left(\sum_{\overline{h}\in(S_{\mathbb{C}})^p}\sum_{l=2}^{p}\sum_{\omega\in\Omega_p^l}\left\|\delta_i^{\omega}T(\bar{h})\right\|_{\mathfrak{g}}^2\right)\\ &= \limsup_{n\to\infty}\tilde{C}(p)\sum_{i=0}^{n}\frac{1}{\delta_i}O\left(\delta_i^2\right)\\ &= \limsup_{n\to\infty}\tilde{C}(p)\sum_{i=0}^{n}O(\delta_i)\\ &< \infty. \qquad \square\end{aligned}$$

Proposition 80. *Suppose $1\leqslant k<j\leqslant r$, and for $1\leqslant i\leqslant j$, $B_i : H(\mathfrak{g})^{p_i}\to H(\mathfrak{g})$ be bounded multilinear functions such that $\sum_{i=1}^{j}p_j = p$. Define a function $G_{j,k}^{\mathcal{P}_n}(\overline{B}) : [0,1]^2\to\mathfrak{g}^{\otimes 2}$ by*

$$G_{j,k}^{\mathcal{P}_n}(\overline{B})(u,t) := \sum_{\overline{h}\in(S_{\mathbb{C}})^p} \frac{d}{du}\big(R_{j,k}^{\mathcal{P}_n}\circ\overline{B}\big)(\overline{h})(u)\otimes\frac{d}{dt}\overline{\big(R_{j,k}^{\mathcal{P}_n}\circ\overline{B}\big)(\overline{h})}(t).$$

Then $\|G_{j,k}^{\mathcal{P}_n}(\overline{B})\|_{L^2([0,1]^2;\mathfrak{g}^{\otimes 2})}\to 0$ *as* $n\to 0$.

Proof. Again by Eq. (71), we can write

$$\begin{aligned}\frac{d}{dt}\big(R_{j,k}^{\mathcal{P}_n}\circ\overline{B}\big)(\overline{h})(t) &= \frac{d}{dt}\sum_{i=0}^{n}\sum_{l=2}^{p}\sum_{\omega\in\Omega_p^l}\delta_i^{\omega}T(\overline{h})f_{\omega}\big(t_i(t)\big)\\ &= \sum_{i=0}^{n}\sum_{l=2}^{p}\sum_{\omega\in\Omega_p^l}\delta_i^{\omega}T(\overline{h})\frac{1_{(s_i,s_{i+1}]}(t)}{\delta_i}f'_{\omega}\big(t_i(t)\big).\end{aligned}$$

So if $s_i<u\leqslant s_{i+1}$ and $s_j<t\leqslant s_{j+1}$, then

$$\begin{aligned}G_{j,k}^{\mathcal{P}_n}(\overline{B})(u,t) &= \sum_{\overline{h}\in(S_{\mathbb{C}})^p}\left(\sum_{l=2}^{p}\sum_{\omega\in\Omega_p^l}\delta_i^{\omega}T(\overline{h})\frac{1}{\delta_i}f'_{\omega}\big(t_i(u)\big)\right)\otimes\left(\sum_{l'=2}^{p}\sum_{\theta\in\Omega_p^{l'}}\overline{\delta_j^{\theta}T(\overline{h})}\frac{1}{\delta_j}\overline{f'_{\theta}\big(t_j(t)\big)}\right)\\ &= \left(\sum_{l,l'=2}^{p}\sum_{\substack{\omega\in\Omega_p^l\\ \theta\in\Omega_p^{l'}}}\frac{f'_{\omega}(t_i(u))\overline{f'_{\theta}(t_j(t))}}{\delta_i\delta_j}\right)\left(\sum_{\overline{h}\in(S_{\mathbb{C}})^p}\delta_i^{\omega}T(\overline{h})\otimes\overline{\delta_j^{\theta}T(\overline{h})}\right)\\ &= \left(\sum_{l,l'=2}^{p}\sum_{\substack{\omega\in\Omega_p^l\\ \theta\in\Omega_p^{l'}}}\frac{f'_{\omega}(t_i(u))\overline{f'_{\theta}(t_j(t))}}{\delta_i\delta_j}\right)\big(1_{ij}1_{\omega\theta}(\delta_i)^l(s_i)^{p-l}C_p(T,T)\big)\\ &= 1_{ij}\sum_{l=2}^{p}\sum_{\omega\in\Omega_p^l}f'_{\omega}\big(t_i(u)\big)\overline{f'_{\omega}\big(t_i(t)\big)}(\delta_i)^{l-2}(s_i)^{p-l}C_p(T,T),\end{aligned}$$

where $C_p(T,T)\in\mathfrak{g}\otimes\mathfrak{g}$ is defined as in Notation 70. So $G_{j,k}^{\mathcal{P}_n}(\overline{B})$ has support concentrated near the diagonal, on the set $\{(u,t)\in[0,1]^2 \mid s_i\leqslant u,t\leqslant s_{i+1}$ for some $i=0,\ldots,n-1\}$, which is going to zero in measure as $n\to\infty$. Also note that our functions f_{ω} are polynomials, and hence there exists a constant $\tilde{C}$ such that $|f'_{\omega}(t)|\leqslant\tilde{C}$ for all $t\in[0,1]$ and $\omega\in\Omega_p^l$ for $l=2,\ldots,p$. For any $i=0,\ldots,n-1$ and $s_i\leqslant u,t\leqslant s_{i+1}$, it follows that

$$\begin{aligned}\big\|G_{j,k}^{\mathcal{P}_n}(\overline{B})(u,t)\big\|_{\mathfrak{g}\otimes\mathfrak{g}} &= \left\|\sum_{l=2}^{p}\sum_{\omega\in\Omega_p^l}f'_{\omega}(u_i)\overline{f'_{\omega}(t_i)}(\delta_i)^{l-2}(s_i)^{p-l}C_p(T,T)\right\|_{\mathfrak{g}\otimes\mathfrak{g}}\\ &\leqslant C(p)\tilde{C}^2\big\|(\delta_i)^{l-2}(s_i)^{p-l}C_p(T,T)\big\|_{\mathfrak{g}\otimes\mathfrak{g}}\\ &\leqslant C(p)\tilde{C}^2\big\|C_p(T,T)\big\|_{\mathfrak{g}\otimes\mathfrak{g}}<\infty.\end{aligned}$$

So $G_{j,k}^{\mathcal{P}_n}(\overline{B})$ is pointwise bounded independent of partition with measure of the support going to zero as $n \to \infty$. Therefore, $\|G_{j,k}^{\mathcal{P}_n}(\overline{B})\|_{L^2([0,1]^2)} \to 0$ as $n \to 0$. $\square$

Corollary 81. *Suppose $1 \leqslant k < j \leqslant r$, and for $1 \leqslant i \leqslant j$, $B_i : H(\mathfrak{g})^{p_i} \to H(\mathfrak{g})$ be linear functions such that $\sum_{i=1}^{j} p_i = p$. Given $\alpha \in H(\mathfrak{g})^*$,*

$$\sum_{\overline{h} \in (S_{\mathbb{C}})^p} \left|\left\langle \alpha, \left(R_{j,k}^{\mathcal{P}_n} \circ \overline{B}\right)(\overline{h})\right\rangle\right|^2 \to 0 \quad \textit{as } n \to \infty.$$

Proof. For all $\alpha \in H(\mathfrak{g})^*$, there exists an $\widetilde{\alpha} \in H(\mathfrak{g})$, such that

$$\left\langle \alpha, \left(R_{j,k}^{\mathcal{P}_n} \circ \overline{B}\right)(\overline{h})\right\rangle = \left(\left(R_{j,k}^{\mathcal{P}_n} \circ \overline{B}\right)(\overline{h}), \widetilde{\alpha}\right)_{H(\mathfrak{g})}.$$

In particular,

$$\begin{aligned}
&\left|\left\langle \alpha, \left(R_{j,k}^{\mathcal{P}_n} \circ \overline{B}\right)(\overline{h})\right\rangle\right|^2 \\
&= \left(\int_0^1 \left(\left(R_{j,k}^{\mathcal{P}_n} \circ \overline{B}\right)(\overline{h})'(t), \widetilde{\alpha}'(t)\right)_{\mathfrak{g}} dt\right)\overline{\left(\int_0^1 \left(\left(R_{j,k}^{\mathcal{P}_n} \circ \overline{B}\right)(\overline{h})'(u), \widetilde{\alpha}'(u)\right)_{\mathfrak{g}} du\right)} \\
&= \int_{[0,1]^2} \left(\left(R_{j,k}^{\mathcal{P}_n} \circ \overline{B}\right)(\overline{h})'(t) \otimes \overline{\left(R_{j,k}^{\mathcal{P}_n} \circ \overline{B}\right)(\overline{h})'(u)}, \widetilde{\alpha}'(t) \otimes \overline{\widetilde{\alpha}'(u)}\right)_{\mathfrak{g}^{\otimes 2}} dt \otimes du.
\end{aligned}$$

We sum the above over all possible $\overline{h} \in (S_{\mathbb{C}})^p$ and use Fubini's theorem and Proposition 79 to justify moving the sum inside the integral. Therefore,

$$\sum_{\overline{h} \in (S_{\mathbb{C}})^p} \left|\left\langle \alpha, \left(R_{j,k}^{\mathcal{P}_n} \circ \overline{B}\right)(\overline{h})\right\rangle\right|^2 = \left(G_{j,k}^{\mathcal{P}_n}(\overline{B}), \widetilde{\alpha}' \otimes \overline{\widetilde{\alpha}'}\right)_{L^2([0,1]^2, \mathfrak{g}^{\otimes 2})}.$$

By Cauchy–Schwarz,

$$\sum_{\overline{h} \in (S_{\mathbb{C}})^p} \left|\left\langle \alpha, \left(R_{j,k}^{\mathcal{P}_n} \circ \overline{B}\right)(\overline{h})\right\rangle\right|^2 \leqslant \|\widetilde{\alpha}' \otimes \overline{\widetilde{\alpha}'}\|_{L^2([0,1]^2, \mathfrak{g}^{\otimes 2})} \left\|G_{j,k}^{\mathcal{P}_n}(\overline{B})\right\|_{L^2([0,1]^2, \mathfrak{g}^{\otimes 2})}.$$

We have shown in the above proposition that $\|G_{j,k}^{\mathcal{P}_n}(\overline{B})\|_{L^2([0,1]^2, \mathfrak{g}^{\otimes 2})} \to 0$ as $n \to 0$. The result follows since $\|\widetilde{\alpha}' \otimes \overline{\widetilde{\alpha}'}\|_{L^2([0,1]^2, \mathfrak{g}^{\otimes 2})} = \|\widetilde{\alpha}\|_{H(\mathfrak{g})}^2 < \infty$. $\square$

We wish to extend the result of Corollary 81 to arbitrary tensor products of remainder terms. This is accomplished in Proposition 82 below. We first shorten our notation further. We will use

$$R_{j_1,k_1}^{\mathcal{P}_n}\left(\overline{B}^1\right) \otimes \cdots \otimes R_{j_q,k_q}^{\mathcal{P}_n}\left(\overline{B}^q\right)(\overline{h})$$

to denote

$$R^{\mathcal{P}_n}_{j_1,k_1}(\overline{B}^1)(\overline{h_1}) \otimes \cdots \otimes R^{\mathcal{P}_n}_{j_q,k_q}(\overline{B}^q)(\overline{h_q}),$$

where it is understood that $\overline{h} = (\overline{h_1}, \ldots, \overline{h_q})$. We will typically assume that $\overline{h} \in H(\mathfrak{g})^{\otimes p}$.

Proposition 82. *Given $\alpha \in (H(\mathfrak{g})^{\otimes q})^*$, then*

$$\sum_{\overline{h} \in (S_{\mathbb{C}})^p} \left| \left\langle \alpha, R^{\mathcal{P}_n}_{j_1,k_1}(\overline{B}^1) \otimes \cdots \otimes R^{\mathcal{P}_n}_{j_q,k_q}(\overline{B}^q)(\overline{h}) \right\rangle \right|^2 \to 0 \quad \text{as } n \to \infty.$$

Proof. For $\alpha \in (H(\mathfrak{g})^{\otimes q})^*$, define

$$\phi_{\mathcal{P}}(\alpha) := \sqrt{\sum_{\overline{h} \in (S_{\mathbb{C}})^p} \left| \left\langle \alpha, R^{\mathcal{P}_n}_{j_1,k_1}(\overline{B}^1) \otimes \cdots \otimes R^{\mathcal{P}_n}_{j_q,k_q}(\overline{B}^q)(\overline{h}) \right\rangle \right|^2}.$$

Then $\phi_{\mathcal{P}}$ is a seminorm on $(H(\mathfrak{g})^{\otimes q})^*$. Using Proposition 79, we can say that

$$\begin{aligned}
\phi_{\mathcal{P}}(\alpha)^2 &\leqslant \|\alpha\|^2_{(H(\mathfrak{g})^{\otimes q})^*} \sum_{\overline{h} \in (S_{\mathbb{C}})^p} \left\| R^{\mathcal{P}_n}_{j_1,k_1}(\overline{B}^1) \otimes \cdots \otimes R^{\mathcal{P}_n}_{j_q,k_q}(\overline{B}^q)(\overline{h}) \right\|^2_{H(\mathfrak{g})^{\otimes q}} \\
&\leqslant \|\alpha\|^2_{(H(\mathfrak{g})^{\otimes q})^*} \left(\sum_{\overline{h} \in (S_{\mathbb{C}})^{p^1_{j_1}}} \left\| R^{\mathcal{P}_n}_{j_1,k_1}(\overline{B}^1)(\overline{h}) \right\|^2_{H(\mathfrak{g})} \right) \\
&\quad \times \cdots \times \left(\sum_{\overline{h} \in (S_{\mathbb{C}})^{p^q_{j_q}}} \left\| R^{\mathcal{P}_n}_{j_q,k_q}(\overline{B}^q)(\overline{h}) \right\|^2_{H(\mathfrak{g})} \right) \\
&\leqslant C^2 \|\alpha\|^2_{(H(\mathfrak{g})^{\otimes q})^*},
\end{aligned}$$

for some $C^2 < \infty$. Or equivalently, $\phi_{\mathcal{P}}(\alpha) \leqslant C\|\alpha\|_{(H(\mathfrak{g})^{\otimes q})^*}$. Suppose $\langle \alpha, \cdot \rangle = (\cdot, k_1 \otimes \cdots \otimes k_q)_{H(\mathfrak{g})^{\otimes q}}$ for some $k_1, \ldots, k_q \in H(\mathfrak{g})$. Then

$$\begin{aligned}
\phi_{\mathcal{P}}(\alpha)^2 &= \sum_{\overline{h} \in (S_{\mathbb{C}})^p} \left| \left(R^{\mathcal{P}_n}_{j_1,k_1}(\overline{B}^1) \otimes \cdots \otimes R^{\mathcal{P}_n}_{j_q,k_q}(\overline{B}^q)(\overline{h}), k_1 \otimes \cdots \otimes k_q \right)_{H(\mathfrak{g})^{\otimes q}} \right|^2 \\
&\leqslant \sum_{\overline{h} \in (S_{\mathbb{C}})^p} \left\| \left(R^{\mathcal{P}_n}_{j_1,k_1}(\overline{B}^1), k_1 \right) \right\|^2_{H(\mathfrak{g})} \times \cdots \times \left\| \left(R^{\mathcal{P}_n}_{j_q,k_q}(\overline{B}^q), k_m \right) \right\|^2_{H(\mathfrak{g})} \\
&= \|k_1\|^2_{H(\mathfrak{g})} \left\| G^{\mathcal{P}_n}_{j_1,k_1}(\overline{B}^1) \right\|_{L^2([0,1]^2)} \times \cdots \times \|k_m\|^2_{H(\mathfrak{g})} \left\| G^{\mathcal{P}_n}_{j_q,k_q}(\overline{B}^q) \right\|_{L^2([0,1]^2)}.
\end{aligned}$$

Proposition 80 also tells us that $\|G^{\mathcal{P}_n}_{j_i,k_i}(\overline{B}^i)\|_{L^2([0,1]^2)} \to 0$ as $|\mathcal{P}| \to 0$ for $i = 1, \ldots, q$. Therefore, $\phi_{\mathcal{P}}(\alpha) \to 0$ as $|\mathcal{P}| \to 0$. Furthermore, the same is true for all finite linear combinations of such α. Standard density arguments yield the result for any $\alpha \in (H(\mathfrak{g})^{\otimes q})^*$. $\square$

The typical remainder term is a tensor product of $P_{\mathcal{P}}h$ and $R^{\mathcal{P}}_{j,k}$ terms (see Proposition 53 and Example 54). The next propositions generalize our results to this situation.

Proposition 83. *Given $\alpha \in T(H(\mathfrak{g}))'$ and $h_1, h_2, \ldots, h_k \in S_{\mathbb{C}}$, there exists $\beta_{h_1 h_2 \ldots h_k} \in (H(\mathfrak{g})^*)^{\otimes n}$ which satisfies*

$$\langle \alpha, h_1 \otimes \cdots \otimes h_k \otimes \eta \rangle = \langle \beta_{h_1 h_2 \ldots h_k}, \eta \rangle, \tag{72}$$

for any $\eta \in H(\mathfrak{g})^{\otimes n}$, and furthermore,

$$\sum_{h_1, h_2, \ldots, h_k \in S_{\mathbb{C}}} \|\beta_{h_1 h_2 \ldots h_k}\|^2_{(H(\mathfrak{g})^*)^{\otimes n}} = \|\alpha_{n+k}\|^2_{(H(\mathfrak{g})^*)^{\otimes (n+k)}} < \infty. \tag{73}$$

Proof. Since $\alpha \in T(H(\mathfrak{g}))^*$, we can write

$$\langle \alpha_{n+k}, \cdot \rangle = \sum_{h_1, h_2, \ldots, h_{n+k} \in S_{\mathbb{C}}} a_{h_1 h_2 \ldots h_{n+k}} (\cdot, h_1 \otimes \cdots \otimes h_{n+k})_{H(\mathfrak{g})^{\otimes n+k}},$$

for some square summable $a_{h_1 h_2 \ldots h_{n+k}} \in \mathbb{C}$. The result follows by setting

$$\langle \beta_{h_1 h_2 \ldots h_k}, \cdot \rangle := \sum_{h_{k+1}, \ldots, h_{n+k} \in S_{\mathbb{C}}} a_{h_1 h_2 \ldots h_{n+k}} (\cdot, h_{k+1} \otimes \cdots \otimes h_{n+k})_{H(\mathfrak{g})^{\otimes n}}. \qquad \square$$

It should be noted that the above proposition is true regardless of the ordering of $h_1, h_2, \ldots, h_k, \eta$ in Eq. (72).

Proposition 84. *Given $\alpha \in T(H(\mathfrak{g}))'$ and $q \geqslant 0$, then as $n \to \infty$*

$$\sum_{\substack{h_1, \ldots, h_q \in S_{\mathbb{C}} \\ \bar{h} \in (S_{\mathbb{C}})^p}} \left| \left\langle \alpha, P_{\mathcal{P}_n} h_1 \otimes \cdots \otimes P_{\mathcal{P}_n} h_q \otimes R^{\mathcal{P}_n}_{j_1, k_1}\big(\overline{B}^1\big) \otimes \cdots \otimes R^{\mathcal{P}_n}_{j_q, k_q}\big(\overline{B}^q\big)(\bar{h}) \right\rangle \right|^2 \to 0.$$

Proof. First notice that for any partition $\mathcal{P}$, we can first select a basis $S^{\mathcal{P}}_{\mathbb{C}}$ for $H_{\mathcal{P}}(\mathfrak{g})$, and then extend it to a basis $S_{\mathbb{C}}$ for $H(\mathfrak{g})$. Then it follows that

$$\begin{aligned}
&\lim_{n \to \infty} \sum_{\substack{h_1, \ldots, h_q \in S_{\mathbb{C}} \\ \bar{h} \in (S_{\mathbb{C}})^p}} \left| \left\langle \alpha, P_{\mathcal{P}} h_1 \otimes \cdots \otimes P_{\mathcal{P}} h_q \otimes R^{\mathcal{P}}_{j_1, k_1}(\overline{B}^1) \otimes \cdots \otimes R^{\mathcal{P}}_{j_q, k_q}\big(\overline{B}^q\big)(\bar{h}) \right\rangle \right|^2 \\
&\quad \leqslant \lim_{n \to \infty} \sum_{\substack{h_1, \ldots, h_q \in S_{\mathbb{C}} \\ \bar{h} \in (S_{\mathbb{C}})^p}} \left| \left\langle \alpha, h_1 \otimes \cdots \otimes h_q \otimes R^{\mathcal{P}_n}_{j_1, k_1}\big(\overline{B}^1\big) \otimes \cdots \otimes R^{\mathcal{P}_n}_{j_q, k_q}\big(\overline{B}^q\big)(\bar{h}) \right\rangle \right|^2 \\
&\quad = \lim_{n \to \infty} \sum_{\substack{h_1, \ldots, h_q \in S_{\mathbb{C}} \\ \bar{h} \in (S_{\mathbb{C}})^p}} \left| \left\langle \beta_{h_1 \cdots h_q}, R^{\mathcal{P}_n}_{j_1, k_1}\big(\overline{B}^1\big) \otimes \cdots \otimes R^{\mathcal{P}_n}_{j_q, k_q}\big(\overline{B}^q\big)(\bar{h}) \right\rangle \right|^2 \\
&\quad = 0.
\end{aligned} \tag{74}$$

In Eq. (74) we have used Proposition 83. We are able to move the limit inside by the DCT, which is justified by Eq. (73) and Proposition 79. The case where $q = 0$ is merely a restatement of Proposition 82. $\square$

Remark 85. For all $k > 0$ and any partition $\mathcal{P}$, let $R_k^{\mathcal{P}}$ be defined as in Eq. (65), i.e.

$$\langle \alpha(\mathcal{P}), h_1 \otimes h_2 \otimes \cdots \otimes h_k \rangle = \langle \alpha, P_{\mathcal{P}} h_1 \otimes P_{\mathcal{P}} h_2 \otimes \cdots \otimes P_{\mathcal{P}} h_k + R_k^{\mathcal{P}}(h_1, \ldots, h_k) \rangle.$$

Since we have shown that $R_k^{\mathcal{P}}$ consists of a finite sum of terms like those in Proposition 84, we have shown that for a refining sequence of partitions $\{\mathcal{P}_n\}_{n=1}^{\infty}$,

$$\sum_{h_1,\ldots,h_k \in S_{\mathbb{C}}} \left| \langle \alpha, R_k^{\mathcal{P}_n}(h_1, \ldots, h_k) \rangle \right| \to 0 \quad \text{as } n \to \infty.$$

We leave the proof of the following to the reader.

Lemma 86. *Let $\{\mathcal{P}_n\}_{n=1}^{\infty}$ be a sequence of refining partitions. Then*

(1) $\bigcup_{n=1}^{\infty} H_{\mathcal{P}_n}(\mathfrak{g})$ *is dense in* $H(\mathfrak{g})$.
(2) $H_{\mathcal{P}_n}(\mathfrak{g}) \subset H_{\mathcal{P}_{n+1}}(\mathfrak{g})$ *for all* n.
(3) *For all* $h \in H(\mathfrak{g})$, $\|h - P_{\mathcal{P}_n} h\|_{H(\mathfrak{g})} \to 0$ *as* $n \to \infty$.

Remark 87. Lemma 86 allows us to construct an orthonormal basis for $H(\mathfrak{g})$ adapted to our sequence of partitions in the following sense. After first constructing an orthonormal basis $S_{\mathbb{C}}^{\mathcal{P}_1}$ for $H_{\mathcal{P}_1}(\mathfrak{g})$, extend this basis inductively from $H_{\mathcal{P}_i}(\mathfrak{g})$ to $H_{\mathcal{P}_{i+1}}(\mathfrak{g})$ for $i = 1, 2, \ldots$. Then $S_{\mathbb{C}} = \bigcup_{n=1}^{\infty} S_{\mathbb{C}}^{\mathcal{P}_n}$ is an orthonormal basis for $H(\mathfrak{g})$.

Proposition 88. *Let $\alpha \in T(H(\mathfrak{g}))^*$. Then for any $h_1, \ldots, h_k \in S_{\mathbb{C}}$,*

$$\lim_{n\to\infty} \left| \langle \alpha, h_1 \otimes \cdots \otimes h_k - P_{\mathcal{P}_n} h_1 \otimes \cdots \otimes P_{\mathcal{P}_n} h_k \rangle \right|^2 = 0.$$

Proof. Setting $\phi_n = I - P_{\mathcal{P}_n}$ in Proposition 40 gives that $\|I^{\otimes k} - P_{\mathcal{P}_n}^{\otimes k}\|_{\mathrm{op}} \to 0$ as $n \to \infty$. $\square$

We are now set to prove Theorem 74 and Corollary 75.

Proof of Theorem 74. We assume that α is of rank N. Recall from Corollary 57 that $\alpha(\mathcal{P}_n)$ is of rank $M = \frac{1}{2}N(N+1)(r^2 - r)$ for all n. Choose $S_{\mathbb{C}}$ to be an orthonormal basis for $H(\mathfrak{g})$ adapted to $\{\mathcal{P}_n\}_{n=1}^{\infty}$ as in Remark 87. To see that $\alpha(\mathcal{P}_n) \in J_T^0(H(\mathfrak{g}))$, notice

$$\begin{aligned}
\|\alpha(\mathcal{P}_n)\|^2_{J_T^0(H(\mathfrak{g}))} &= \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{h_1,\ldots,h_k \in S_{\mathbb{C}}} \left| \langle \alpha(\mathcal{P}_n), h_1 \otimes \cdots \otimes h_k \rangle \right|^2 \\
&= \sum_{k=0}^{M} \frac{t^k}{k!} \sum_{h_1,\ldots,h_k \in S_{\mathbb{C}}} \left| \langle \alpha, P_{\mathcal{P}} h_1 \otimes \cdots \otimes P_{\mathcal{P}} h_k \rangle + \langle \alpha, R_k^{\mathcal{P}_n}(h_1, \ldots, h_k) \rangle \right|^2 \\
&\leqslant \sum_{k=0}^{M} \frac{4t^k}{k!} \sum_{h_1,\ldots,h_k \in S_{\mathbb{C}}} \left| \langle \alpha, P_{\mathcal{P}} h_1 \otimes \cdots \otimes P_{\mathcal{P}} h_k \rangle \right|^2 && (75) \\
&\quad + \sum_{k=0}^{M} \frac{4t^k}{k!} \sum_{h_1,\ldots,h_k \in S_{\mathbb{C}}} \left| \langle \alpha, R_k^{\mathcal{P}_n}(h_1, \ldots, h_k) \rangle \right|^2. && (76)
\end{aligned}$$

It follows that the sum in Eq. (75) is less than $4\|\alpha\|^2_{J_T^0(H(\mathfrak{g}))} \leqslant \infty$, while the sum in Eq. (76) is bounded independent of n by Proposition 79. So $\alpha(\mathcal{P}_n) \in J_T^0(H(\mathfrak{g}))$ for all n.

To show convergence in $J_T^0(H(\mathfrak{g}))$, notice that

$$\begin{aligned}
&\lim_{n\to\infty}\big\|\alpha-\alpha(\mathcal{P}_n)\big\|^2_{J_T^0(H(\mathfrak{g}))}\\
&\quad=\lim_{n\to\infty}\sum_{k=0}^{N}\frac{t^k}{k!}\sum_{h_i,\dots,h_k\in S_{\mathbb{C}}}\big|\langle\alpha,h_1\otimes\cdots\otimes h_k-P_{\mathcal{P}_n}h_1\otimes\cdots\otimes P_{\mathcal{P}_n}h_k\rangle-\big\langle\alpha,R_k^{\mathcal{P}_n}(h_i,\dots,h_k)\big\rangle\big|^2\\
&\qquad-\lim_{n\to\infty}\sum_{k=N+1}^{M}\frac{t^k}{k!}\sum_{h_i,\dots,h_k\in S_{\mathbb{C}}}\big|\langle\alpha,P_{\mathcal{P}}h_1\otimes\cdots\otimes P_{\mathcal{P}}h_k\rangle-\big\langle\alpha,R_k^{\mathcal{P}_n}(h_i,\dots,h_k)\big\rangle\big|^2\\
&\quad\leqslant\lim_{n\to\infty}4\sum_{k=0}^{N}\frac{t^k}{k!}\sum_{h_i,\dots,h_k\in S_{\mathbb{C}}}\Big(\big|\langle\alpha,h_1\otimes\cdots\otimes h_k-P_{\mathcal{P}_n}h_1\otimes\cdots\otimes P_{\mathcal{P}_n}h_k\rangle\big|^2\\
&\qquad+\big|\big\langle\alpha,R_k^{\mathcal{P}_n}(h_i,\dots,h_k)\big\rangle\big|^2\Big)+\lim_{n\to\infty}\sum_{k=N+1}^{M}\frac{t^k}{k!}\sum_{h_i,\dots,h_k\in S_{\mathbb{C}}}\big|\big\langle\alpha,R_k^{\mathcal{P}_n}(h_i,\dots,h_k)\big\rangle\big|^2,\\
&\quad=4\sum_{k=0}^{N}\frac{t^k}{k!}\lim_{n\to\infty}\sum_{h_i,\dots,h_k\in S_{\mathbb{C}}}\big|\langle\alpha,h_1\otimes\cdots\otimes h_k-P_{\mathcal{P}_n}h_1\otimes\cdots\otimes P_{\mathcal{P}_n}h_k\rangle\big|^2,
\end{aligned}$$

since we have shown in Remark 85 that for all $k>0$,

$$\lim_{n\to\infty}\sum_{h_i,\dots,h_k\in S_{\mathbb{C}}}\big|\big\langle\alpha,R_k^{\mathcal{P}_n}(h_i,\dots,h_k)\big\rangle\big|^2=0.$$

If we choose a basis $S_{\mathbb{C}}$ adapted to our sequence of partition as in Remark 87, then $|\langle\alpha,P_{\mathcal{P}_n}h_1\otimes\cdots\otimes P_{\mathcal{P}_n}h_k\rangle|$ is either equal to zero or $|\langle\alpha,h_1\otimes\cdots\otimes h_k\rangle|$. Therefore

$$\begin{aligned}
&\sum_{h_i,\dots,h_k\in S_{\mathbb{C}}}\big|\big\langle\alpha,h_1\otimes\cdots\otimes h_k-P_{\mathcal{P}_n}h_1\otimes\cdots\otimes P_{\mathcal{P}_n}h_k\big\rangle\big|^2\\
&\quad\leqslant 2\sum_{h_i,\dots,h_k\in S_{\mathbb{C}}}\big|\langle\alpha,h_1\otimes\cdots\otimes h_k\rangle\big|^2+2\sum_{h_i,\dots,h_k\in S_{\mathbb{C}}}\big|\langle\alpha,P_{\mathcal{P}_n}h_1\otimes\cdots\otimes P_{\mathcal{P}_n}h_k\rangle\big|^2\\
&\quad\leqslant 4\sum_{h_i,\dots,h_k\in S_{\mathbb{C}}}\big|\langle\alpha,h_1\otimes\cdots\otimes h_k\rangle\big|^2\\
&\quad=4\|\alpha_k\|^2_{(H(\mathfrak{g})^*)^{\otimes k}}.
\end{aligned}$$

This justifies the use of the DCT in calculating

$$
\begin{aligned}
&\lim_{n\to\infty}\left\|\alpha-\alpha(\mathcal{P}_n)\right\|^2_{J_T^0(H(\mathfrak{g}))}\\
&\quad\leqslant 4\sum_{k=0}^{N}\lim_{n\to\infty}\sum_{h_i,\dots,h_k\in S_{\mathbb{C}}}\left|\langle\alpha,h_1\otimes\cdots\otimes h_k-P_{\mathcal{P}_n}h_1\otimes\cdots\otimes P_{\mathcal{P}_n}h_k\rangle\right|^2\\
&\quad=4\sum_{k=0}^{N}\sum_{h_i,\dots,h_k\in S_{\mathbb{C}}}\lim_{n\to\infty}\left|\langle\alpha,h_1\otimes\cdots\otimes h_k-P_{\mathcal{P}_n}h_1\otimes\cdots\otimes P_{\mathcal{P}_n}h_k\rangle\right|^2\\
&\quad=0,
\end{aligned}
$$

by Proposition 88. □

Proof of Corollary 75. Theorem 74 implies that $\{\alpha(\mathcal{P}_n)\}_{n=1}^{\infty}$ is Cauchy in $J_T^0(H(\mathfrak{g}))$. Since the Taylor map is an isometry (Corollary 33), it follows that the holomorphic cylinder functions $\{F_{\mathcal{P}_n}\}_{n=1}^{\infty}$ converge to some $\tilde{u}_\alpha\in\mathcal{H}_T$. It then follows by the continuity of the Taylor map that $(1-D)_{\underline{e}}^{-1}R\tilde{u}_\alpha=\alpha$. So the Taylor map is onto the set of finite rank α and therefore onto $J_T^0(H(\mathfrak{g}))$ by Proposition 42. □

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A priori bounds and weak solutions for the nonlinear Schrödinger equation in Sobolev spaces of negative order

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Abstract

Solutions to the Cauchy problem for the one-dimensional cubic nonlinear Schrödinger equation on the real line are studied in Sobolev spaces H^s, for s negative but close to 0. For smooth solutions there is an *a priori* upper bound for the H^s norm of the solution, in terms of the H^s norm of the datum, for arbitrarily large data, for sufficiently short time. Weak solutions are constructed for arbitrary initial data in H^s.

Keywords: Nonlinear Schrödinger equation; Wellposedness

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1. Introduction

The Cauchy problem for the one-dimensional cubic nonlinear Schrödinger equation is

$$\begin{cases} iu_t + u_{xx} + \omega|u|^2 u = 0, \\ u(0,x) = u_0(x). \end{cases} \tag{NLS}$$

Here $u = u(t,x)$ with $(t,x) \in [0,T] \times \mathbb{R}^1$, and $\omega = \pm 1$. As is well known, this Cauchy problem is globally wellposed in H^0 [13]. For all negative s it is illposed in H^s, in the sense that solutions (for smooth initial data) fail to depend *uniformly* continuously on initial data in the H^s norm [5,9]. Moreover, for $s < -\frac{1}{2}$, there is a stronger form of illposedness: the solution operator fails even to be continuous at 0; there exist smooth solutions with arbitrarily small H^s norms at time 0, yet arbitrarily large H^s norms at time ε, for arbitrarily small $\varepsilon > 0$.

Our first result, concerning smooth (or more precisely, H^0) solutions, implies continuity of the solution map at $u_0 = 0$ in the $C^0(H^s)$ norm for negative s sufficiently close to 0, in contrast with the strong illposedness for $s < -\frac{1}{2}$. It asserts an *a priori* upper bound for the H^s norm of an arbitrary smooth solution, in terms of the H^s norm of its datum.

Theorem 1.1 *(A priori bound). Let $s > -\frac{1}{12}$. Then for all $R < \infty$, there exist $R' < \infty$ and $T > 0$ such that for all $u_0 \in H^0$ satisfying $\|u_0\|_{H^s} < R$, the standard solution u of* (NLS) *with initial datum u_0 satisfies* $\max_{t\in[0,T]} \|u(t,\cdot)\|_{H^s} \leqslant R'$.

For large R, T scales like a certain negative power of R.

By the standard solution we mean the unique solution of (NLS) belonging to the function space $X^{0,b}$ for some $b > \frac{1}{2}$, or equivalently to $C^0(H^0) \cap L^4([0,T] \times \mathbb{R})$. Koch and Tataru [10] have obtained the same result in the larger range $s \geqslant -\frac{1}{6}$. It remains an open question whether this type of result is valid over a yet larger range.

Wellposedness of (NLS) has been established by earlier authors in various function spaces which are wider than H^0 [4,7,14] and scale like negative order Sobolev spaces, but do not contain H^s for any $s < 0$. We emphasize that those results have a different character than ours; *uniformly* continuous dependence on the initial datum in the norm in question is established in those works, whereas it certainly fails to hold [5,9] in H^s for $s < 0$.

Our second main result asserts the solvability of the Cauchy problem, in a weak sense, for all initial data in H^s for a range of negative exponents s. The precise statement involves certain function spaces $Y^{s,b}$, which will be specified in Definition 6.1. These are variants of the spaces $X^{s,b}$ commonly employed in connection with this equation. For any $u \in Y^{s,b}$, $|u|^2u$ has a natural interpretation as a distribution for the range of parameters s, b covered by our results, in the sense that when the space–time Fourier transform of $|u|^2u$ is written as an integral expression directly in terms of the space–time Fourier transforms of the factors $u, \bar{u}, u$, the resulting integral is absolutely convergent almost everywhere and defines a tempered locally integrable function; see (7.2). Thus there is a natural notion of a weak solution in $Y^{s,b}$: We say that $u \in Y^{s,b}$ is a weak solution of (NLS) if the equation holds in the sense of distributions, when $|u|^2u$ is interpreted as the inverse Fourier transform of the function defined by this absolutely convergent integral.

Theorem 1.2 *(Existence of weak solutions). Let $s > -\frac{1}{12}$. Then there exists $b > \frac{1}{2}$ such that for each $R < \infty$ there exist $R' < \infty$ and $T > 0$ such that for all $u_0 \in H^s$ satisfying $\|u_0\|_{H^s} < R$,*

there exists a weak solution $u \in C^0([0,T], H^s) \cap Y^{s,b}$ *of* (NLS) *with initial datum* u_0 *which satisfies* $\max_{t\in[0,T]} \|u(t)\|_{H^s} \leqslant R'$.

$Y^{s,b}$ embeds continuously in $C^0(H^s)$ for $b > \frac{1}{2}$, so the $C^0(H^s)$ part of the conclusion is redundant, and is included only for emphasis.

The solutions guaranteed by this theorem are weak limits of smooth solutions with smooth initial data approximating given H^s data. We do not know whether these solutions are unique, that is, independent of the choice of approximating sequence, let alone whether there exists any $s < 0$ for which the mapping from datum to solution is continuous.

Our analysis does not rely on the complete integrability [1] of (NLS). Our arguments would apply, with essentially no changes, to nonintegrable vector-valued generalizations of the one-dimensional cubic nonlinear Schrödinger equation, provided that those systems obey H^0 norm conservation.

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2. Strategy of the analysis

The strategy is as follows. We begin by using the differential equation to (formally, at least) rewrite the increment $\|u(t)\|_{H^s}^2 - \|u_0\|_{H^s}^2$ as a multilinear expression in terms of the space–time Fourier transform of u. Certain cancellations arise, which have no analogues in the corresponding expression for $u(t,x) - u_0(x)$. This leads to an *a priori* inequality of the form $|\|u(t)\|_{H^s}^2 - \|u_0\|_{H^s}^2| \leqslant C\|u\|_{X^{r,b}}^4$, for certain r, s, b with $s < 0$ and $r < s$. It is this initial step which breaks down if u is replaced by the difference of two solutions, preventing us from establishing any continuity of the map $u_0 \mapsto u$.

Thus a bound is required for the $X^{r,b}$ norm, but a loss relative to the $C^0(H^s)$ norm is permitted in the sense that r can be less than s. In Section 6 we introduce certain function spaces $Y^{s,b}$. Their main relevant properties are:

(1) For $s < 0$, $Y^{s,b}$ embeds in $X^{r,b}$, provided that $r < (1+4b)s$.
(2) $Y^{s,b}$ embeds in $C_t^0(H_x^{s-\varepsilon})$ for all[4] $\varepsilon > 0$, provided that $b > \frac{1}{2}$.
(3) If $u, v, w \in Y^{s,b}$ then $u\bar{v}w \in Y^{s,b-1}$, under certain restrictions on s, b.
(4) If $u \in C^0(H^s)$ and $(i\partial_t - \Delta_x)u \in Y^{s,b-1}$ then $u \in Y^{s,b}$.
(5) For solutions of (NLS), there is an *a priori* bound for the $Y^{s,b}$ norm in terms of the $C^0(H^s)$ norm, of the form $\|u\|_{Y^{s,b}} \leqslant C\|u\|_{C^0(H^s)} + C\|u\|_{Y^{s,b}}^3$, valid under certain restrictions on s, b.

Thus one obtains a coupled system of two inequalities relating $\|u\|_{C^0(H^s)}$ and $\|u\|_{Y^{s,b}}$ to $\|u_0\|_{H^s}$. By restricting attention to a short time T and rescaling, one can reduce matters (for $s > -\frac{1}{2}$) to the case where u_0 has small H^s norm. Via a continuity argument, the coupled system then yields a bound for $\|u\|_{C^0(H^s)} + \|u\|_{Y^{s,b}}$ in terms of $\|u_0\|_{H^s}$.

Weak solutions are obtained as limits of smooth solutions. An *a priori* bound in H^s yields compactness in $H^{s-\varepsilon}$ on bounded spatial regions. It follows readily from the machinery developed below that if smooth solutions u_j with uniformly bounded $Y^{s,b}$ norms converge weakly to u, then $|u_j|^2 u_j$ converges weakly to $|u|^2 u$ for some subsequence.

---

[4] We find it convenient to work with Besov-like spaces $Y^{s,b}$ rather than Sobolev-like versions. Their Besov character accounts for the infinitesimal loss of derivatives in the embedding into $C^0(Y^s)$.

An additional argument is needed to place these weak solutions in $C^0(H^s)$, rather than $C^0(H^{s-\varepsilon}) \cap L^\infty(H^s)$. We refine the machinery by replacing the squared H^s norm $\int |\hat{u}(t,\xi)|^2(1+|\xi|^2)^s\,d\xi$ by $\int |\hat{u}(t,\xi)|^2\varphi(\xi)\,d\xi$ for weight functions φ adapted to individual initial data, so that $\varphi(\xi) \gg (1+|\xi|^2)^s$ for very large $|\xi|$, and show that control of $\int |\widehat{u_0}(\xi)|^2\varphi\,d\xi$ extends to control of $\int |\hat{v}(t,\xi)|^2\varphi\,d\xi$ for all solutions v of (NLS) with smooth initial data sufficiently close in H^s norm to u_0. This extra control at high frequencies leads to compactness in $C^0(H^s)$.

3. Bounding the norm

In this section we begin to establish an *a priori* bound for the $C^0(H^s)$ norm of any sufficiently smooth solution of (NLS), in terms of certain other norms. For technical reasons we work with the modified Cauchy problem

$$\begin{cases} iu_t + u_{xx} + \zeta_0(t)\omega|u|^2u = 0, \\ u(0,x) = u_0(x) \end{cases} \tag{NLS*}$$

where ζ_0 is a smooth real-valued function which is $\equiv 1$ on $[0,T]$, and is supported in $(-2T,2T)$. Standard proofs of wellposedness in H^0 (or in H^t for $t \geqslant 0$) apply to this modified equation. One advantage is that u can be extended to a solution defined for all $t \in \mathbb{R}$.

We will study $\zeta_1(t)u(t,x)$, where ζ_1 is another real-valued smooth cutoff function supported in $(-2T,2T)$ which satisfies $\zeta_1\zeta_0 \equiv \zeta_0$. Because the equation is simply the linear Schrödinger equation outside the support of ζ_0, a $C^0(H^s)$ bound holds for $\zeta(t)u(t,x)$ for one real-valued cutoff function in $\zeta \in C_0^\infty(-2T,2T)$ satisfying $\zeta\zeta_0 \equiv \zeta_0$ if and only if such a bound holds for every such function.

Recall [2] the function space $X^{s,b}$, which is defined to be the set of all space–time distributions u whose space–time Fourier transform $\hat{u}$ is such that

$$\|u\|_{X^{s,b}}^2 := \iint\limits_{\mathbb{R}^2} \big|\hat{u}(\xi,\tau)\big|^2 \big\langle \tau - \xi^2 \big\rangle^{2b} \langle \xi \rangle^{2s}\,d\xi\,d\tau < \infty$$

where $\langle x \rangle := (1+|x|^2)^{1/2}$.

One of the two principal inequalities underlying our theorems is as follows. The second is formulated in Proposition 8.1.

Proposition 3.1. *Let $T_0 < \infty$, $T \in [0,T_0]$, $s \in (-\frac{1}{2},0)$, $b \in (\frac{1}{2},1)$. There exists $C < \infty$ such that for any sufficiently smooth solution[5] u of* (NLS*) *with initial datum u_0,*

$$\Big| \|u\|_{C^0([-2T,2T],H^s)}^2 - \|u_0\|_{H^s}^2 \Big| \leqslant C\|\zeta_1 u\|_{X^{r,b}}^4 \tag{3.1}$$

provided that

$$r > -\frac{1}{4} \quad \textit{and} \quad b > \frac{1}{2}. \tag{3.2}$$

[5] For instance, $u_0 \in H^{10}$ would suffice.

For $s > -\frac{1}{4}$, the right-hand side involves a norm which is weaker, in terms of the number of spatial derivatives involved, than the $C^0(H^s)$ norm. The proof of this result is begun below and completed in Section 5, using some of the inequalities established in Section 4.

We will work with both spatial Fourier coefficients

$$\hat{u}(t,\xi) := \int_{\mathbb{R}} e^{-ix\xi} u(t,x)\,dx \tag{3.3}$$

and space–time Fourier coefficients[6]

$$\hat{u}(\xi,\tau) := \int_{\mathbb{R}^2} e^{-ix\xi} e^{-it\tau} u(t,x)\,dx\,dt; \tag{3.4}$$

it will be clear from context and from the names of the variables which of these two is meant in any particular instance. The differential equation (NLS*) is expressed in terms of spatial Fourier coefficients as

$$\frac{d}{dt}\hat{u}(t,\xi) = -i\xi^2\hat{u}(t,\xi) + i\omega\zeta_0(t) \int_{\xi_1-\xi_2+\xi_3=\xi} \hat{u}(t,\xi_1)\overline{\hat{u}(t,\xi_2)}\hat{u}(t,\xi_3)\,d\lambda_\xi \tag{3.5}$$

where λ_ξ is appropriately normalized Lebesgue measure on $\{(\xi_1,\xi_2,\xi_3) \in \mathbb{R}^3\colon\ \xi_1-\xi_2+\xi_3=\xi\}$.

Consider any sufficiently regular solution u of (NLS*). Let $\varphi:\mathbb{R} \to [0,\infty)$ and define the modified mass

$$\Phi_\varphi(t) = \Phi_\varphi(t,u) := \int_{\mathbb{R}} \left|\hat{u}(t,\xi)\right|^2 \varphi(\xi)\,d\xi. \tag{3.6}$$

We will be primarily interested in $\varphi(\xi) = \langle\xi\rangle^{2s}$, but more general weights will be needed to establish the full conclusion of Theorem 1.2.

A short calculation establishes the "almost conservation law"

$$\frac{d\Phi}{dt} = \mathrm{Re}(c\omega\mathcal{I})$$

for Φ, where c is an absolute constant, $\mathcal{I}$ is the multilinear integral

$$\mathcal{I}(t) = \mathcal{I}_\varphi(u,t) := \zeta_0(t) \int_{\Xi} \hat{u}(t,\xi_1)\overline{\hat{u}(t,\xi_2)}\hat{u}(t,\xi_3)\overline{\hat{u}(t,\xi_4)}\,\psi(\vec{\xi})\,d\lambda(\vec{\xi}), \tag{3.7}$$

$\vec{\xi} = (\xi_1,\ldots,\xi_4) \in \mathbb{R}^4$ is a multi-frequency, $\Xi \subset \mathbb{R}^4$ is the hyperplane

$$\Xi := \{\vec{\xi}\colon\ \xi_1-\xi_2+\xi_3-\xi_4=0\}, \tag{3.8}$$

[6] The order of the variables is reversed in our space–time transform; $u(t,x)$ is transformed to $\hat{u}(\xi,\tau)$ where ξ,τ are dual to x,t, respectively.

λ is appropriately normalized Lebesgue measure on Ξ, and

$$\psi(\vec{\xi}) := \varphi(\xi_1) - \varphi(\xi_2) + \varphi(\xi_3) - \varphi(\xi_4). \tag{3.9}$$

Thus[7] $|\Phi(t) - \Phi(0)| \lesssim |\int_0^t \mathcal{I}(r)\,dr|$.

Introduce also

$$\sigma(\xi_1, \ldots, \xi_4) := \xi_1^2 - \xi_2^2 + \xi_3^2 - \xi_4^2. \tag{3.10}$$

σ has the useful alternative expressions

$$\begin{aligned}\sigma(\vec{\xi}) &= 2(\xi_1 - \xi_2)(\xi_1 - \xi_4) = -2(\xi_1 - \xi_2)(\xi_3 - \xi_2)\\ &= -2(\xi_1 - \xi_4)(\xi_3 - \xi_4) \quad \forall \vec{\xi} \in \Xi.\end{aligned} \tag{3.11}$$

We have the following basic cancellation bound (cf. [6]):

Lemma 3.2 *(Double mean value theorem). Let $\vec{\xi} = (\xi_1, \ldots, \xi_4) \in \Xi \subset \mathbb{R}^4$. If $\varphi \in C^2$ and all ξ_j belong to a common interval I then $|\psi(\vec{\xi})| \leqslant |\sigma(\vec{\xi})| \max_{y \in I} |\varphi''(y)|$.*

Proof. $\varphi(\xi_2) - \varphi(\xi_1) = (\xi_2 - \xi_1)\int_0^1 \varphi'(\xi_1 + t(\xi_2 - \xi_1))\,dt$. Writing the corresponding expression for $\varphi(\xi_4) - \varphi(\xi_3)$, and noting that $(\xi_2 - \xi_1) = -(\xi_4 - \xi_3)$ since $\vec{\xi} \in \Xi$, gives

$$\begin{aligned}\psi(\xi) &= (\xi_2 - \xi_1)\int_0^1 \big[\varphi'\big(\xi_1 + t(\xi_2 - \xi_1)\big) - \varphi'\big(\xi_4 + t(\xi_3 - \xi_4)\big)\big]\,dt\\ &= (\xi_2 - \xi_1)(\xi_1 - \xi_4)\iint_{[0,1]^2} \varphi''\big(\xi_1 + t(\xi_2 - \xi_1) + s(\xi_4 - \xi_1)\big)\,ds\,dt. \qquad \square\end{aligned}$$

In order to control the contribution made by the region not close to the diagonal, express each factor $\hat{u}(t, \xi)$ in the integral as the inverse Fourier transform of its Fourier transform with respect to t, to obtain for all $t \in [-2T, 2T]$

$$\left|\int_0^t \mathcal{I}_\varphi(u, r)\,dr\right| \leqslant C \int_\Xi \int_{\mathbb{R}^4} \prod_{j=1}^4 \big|\hat{u}(\xi_j, \tau_j)\big| \langle \tau_1 - \tau_2 + \tau_3 - \tau_4\rangle^{-1} \big|\psi(\vec{\xi})\big|\,d\vec{\tau}\,d\lambda(\vec{\xi}) \tag{3.12}$$

where C depends on T and $\vec{\tau} = (\tau_1, \ldots, \tau_4)$. The notation $\hat{u}$ denotes here the Fourier transform with respect to both spatial and temporal variables.

Write

$$\big|\hat{u}(\xi_j, \tau_j)\big| =: \langle \xi_j\rangle^{-r} \big\langle \tau_j - \xi_j^2\big\rangle^{-b} g_j(\xi_j, \tau_j). \tag{3.13}$$

[7] As usual, we use $X \lesssim Y$ to denote an estimate of the form $X \leqslant CY$ for some constant C, depending only on the exponents r, s and b which will appear later in this paper.

Then $\|g_j\|_{L^2(\mathbb{R})} = \|u\|_{X^{r,b}}$. The right-hand side of (3.12) becomes

$$\int_{\mathbb{R}^4}\int_{\Xi}\prod_{n=1}^{4}\big(g_n(\xi_n,\tau_n)\langle\xi_n\rangle^{-r}\langle\tau_n-\xi_n^2\rangle^{-b}\big)\big|\psi(\vec{\xi})\big|\,d\lambda(\vec{\xi})\langle\tau_1-\tau_2+\tau_3-\tau_4\rangle^{-1}\,d\vec{\tau}. \tag{3.14}$$

In Section 5 we will complete the proof of Proposition 3.1 by showing that for $\varphi(\xi)=\langle\xi\rangle^{2s}$, the integral (3.14) is majorized by $C\prod_{n=1}^{4}\|g_n\|_{L^2(\mathbb{R}^2)}$ provided that s, r, b satisfy the hypotheses of the proposition.

4. Trilinear inequalities of Strichartz type

A prototypical inequality of Strichartz type says that for $h \in L^2(\mathbb{R})$, the solution u of the linear Schrödinger equation with initial datum h belongs to $L^6(\mathbb{R}^2)$. Therefore any three such solutions satisfy $u_1\bar{u}_2u_3 \in L^2$. Rewritten on the Fourier side by means of the Plancherel identity, this becomes

$$\left|\int f\big(\xi_1-\xi_2+\xi_3,\xi_1^2-\xi_2^2+\xi_3^2\big)\prod_{n=1}^{3}g_n(\xi_n)\,d\xi_n\right| \lesssim \prod_{n=1}^{3}\|g_n\|_{L^2(\mathbb{R})}\|f\|_{L^2(\mathbb{R}^2)}. \tag{4.1}$$

One version of the bilinear Strichartz inequality, expressed directly in terms of Fourier variables, states that for any subset $E\subset\mathbb{R}^2$,

$$\begin{aligned}&\left|\int_{\mathbb{R}^2} f\big(\xi_1\pm\xi_2,\xi_1^2\pm\xi_2^2\big)h_1(\xi_1)h_2(\xi_2)\chi_E(\xi_1,\xi_2)\,d\xi_1\,d\xi_2\right|\\ &\quad\lesssim\Big(\min_{(\xi_1,\xi_2)\in E}|\xi_1-\xi_2|\Big)^{-1/2}\|f\|_{L^2(\mathbb{R}^2)}\|h_1\|_{L^2(\mathbb{R}^1)}\|h_2\|_{L^2(\mathbb{R}^1)}\end{aligned} \tag{4.2}$$

where the two $\pm$ signs are either both $+$, or both $-$; this represents the pairing of f with a bilinear operator applied to h_1, h_2. This is implicit in Carleson and Sjölin [3], and is a direct consequence of Cauchy–Schwarz via the substitution $(\xi_1,\xi_2)\mapsto(\xi_1\pm\xi_2,\xi_1^2\pm\xi_2^2)$. Its advantage, in practice, is that it provides a superior bound when $|\xi_1-\xi_2|$ is large.

In this section we establish certain versions of the trilinear inequality (4.1) which incorporate improvements similar to the factor $|\xi_1-\xi_2|^{-1/2}$ in (4.2). These arise naturally in the analysis of the Fourier transform of a threefold product $u\bar{v}w$ of functions in spaces $X^{r,b}$ or $Y^{s,b}$.

4.1. Statements of inequalities

Proposition 4.1. *Consider*

$$\int_{\vec{\xi}\in S\subset\Xi}\int_{\tau_1-\tau_2+\tau_3-\tau_4=0}\prod_{n=1}^{4}g_n(\xi_n,\tau_n)\langle\tau_n-\xi_n^2\rangle^{-\beta_n}\chi_E\big(L(\vec{\xi})\big)\,d\lambda(\vec{\tau})\,d\lambda(\vec{\xi}) \tag{4.3}$$

where each $g_n\geqslant 0$, $i,j\in\{1,2,3,4\}$ are distinct, $E\subset\mathbb{R}^1$ is any measurable set, and $L:\mathbb{R}^4\to\mathbb{R}$ is a linear transformation. Suppose that

- *$\beta_n > \frac{1}{2}$ for all but at most one index n, and $\beta_n > 0$ for all n;*
- *i, j have opposite parity;*
- *L belongs neither to the linear span of $\{\xi_i, \xi_j, \xi_1 - \xi_2 + \xi_3 - \xi_4\}$, nor to the linear span of $\{\xi_k, \xi_l, \xi_1 - \xi_2 + \xi_3 - \xi_4\}$, where $\{i, j, k, l\} = \{1, 2, 3, 4\}$.*

Then there exists $C < \infty$ depending on L such that (4.3) *is majorized by*

$$C|E|^{1/2} \max_{\vec{\xi} \in S} \bigl(|\xi_i - \xi_j|^{-1/2}\bigr) \cdot \max_{\vec{\xi} \in S} \bigl(\langle \sigma(\vec{\xi})\rangle^{-\beta}\bigr) \prod_{n=1}^{4} \|g_n\|_{L^2(\mathbb{R}^2)} \tag{4.4}$$

where $\beta = \min_n \beta_n$.

In our application, L will take the form $L(\vec{\xi}) = \xi_\mu - \xi_\nu$ for some $\mu \neq \nu$. If $\{\mu, \nu\}$ equals neither $\{i, j\}$ nor $\{1, 2, 3, 4\} \setminus \{i, j\}$ then L satisfies the hypothesis.

A variant of this inequality applies to other linear transformations L:

Proposition 4.2. *Consider* (4.3) *with $L(\vec{\xi}) = \xi_k$ for some $k \notin \{i, j\}$. Suppose again that $\beta_n > \frac{1}{2}$ for all but at most one index n, and $\beta_n > 0$ for all n. Suppose that $|E| \lesssim \min_{\vec{\xi} \in S} |\xi_i - \xi_j|$. Let $\beta := \min_n \beta_n$. Then there exists $C < \infty$ such that* (4.3) *is majorized by*

$$|E|^{1/4} \max_{\vec{\xi} \in S} \bigl(|\xi_i - \xi_j|^{-1/4}\bigr) \cdot \max_{\vec{\xi} \in S} \bigl(\langle \sigma(\vec{\xi})\rangle^{-\beta}\bigr) \prod_{n=1}^{4} \|g_n\|_{L^2(\mathbb{R}^2)}. \tag{4.5}$$

Remark 4.1 *(Trilinear Knapp example).* (4.5) is (in practice) weaker than (4.4), because $|E| / \min_S |\xi_i - \xi_j|$ is raised only to the power $\frac{1}{4}$, rather than $\frac{1}{2}$ as in Proposition 4.1. We discuss here the simplified expression

$$\int_{\mathbb{R}^3} G\bigl(\xi_1 + \xi_3 - \xi_4, \xi_1^2 + \xi_3^2 - \xi_4^2\bigr) \prod_{n \neq 2} h_n(\xi_n)\, \chi_{|\xi_4| \leqslant 1} \chi_S(\vec{\xi})\, d\xi_1\, d\xi_3\, d\xi_4, \tag{4.6}$$

which arises in the proof of Proposition 4.2 (see the case $\nu = 2$). The example can be adapted to the situation of the proposition. The analogue of (4.5) for (4.6) is the bound $\min_S |\xi_1 - \xi_2|^{-1/4} \|G\|_{L^2} \prod_{n \neq 2} \|h_n\|_{L^2}$, which is established in the proof of (4.5) below. We show now that the exponent $\frac{1}{4}$ cannot be improved in this bound for (4.6).

Define h_4 to be the characteristic function of the interval $[0, 1]$, h_1 to be the characteristic function of $[N, N + N^{1/2}]$, and h_3 to be the characteristic function of $[N + N^{1/2}, N + 2N^{1/2}]$. Define S to be the set of all $\vec{\xi} = (\xi_1, \xi_2, \xi_3, \xi_4)$ for which $h_1(\xi_1) h_3(\xi_3) h_4(\xi_4) \neq 0$; ξ_2 is always regarded as a function of (ξ_1, ξ_3, ξ_4) via the relation $\xi_1 - \xi_2 + \xi_3 - \xi_4 = 0$. Define $G(x, y)$ to be the characteristic function of the set of all $(x, y) \in \mathbb{R}^2$ satisfying $|x - 2N| \leqslant 3N^{1/2}$ and $|y + 2N^2 - N - 2Nx| \leqslant 4N$. Then a short calculation shows that $G(\xi_1 + \xi_3 - \xi_4, \xi_1^2 + \xi_3^2 - \xi_4^2) \equiv 1$ for all $\vec{\xi} \in S$, and consequently the integral (4.6) is simply $\prod_{n \neq 2} \int_{\mathbb{R}^1} h_n = N^{1/2} \cdot N^{1/2} \cdot 1 = N$.

On the other hand, $\|G\|_{L^2} = CN^{3/4}$, while $\|h_n\|_{L^2} = N^{1/4}$ for $n = 1, 3$ and $= 1$ for $n = 4$. Thus the product of the four L^2 norms has order of magnitude $N^{5/4}$, and consequently the ratio of (4.6) to the product of norms has order of magnitude $N / N^{5/4} = N^{-1/4}$. Since $\xi_2 - \xi_1 = \xi_3 - \xi_4$ has order of magnitude N for all $\vec{\xi} \in S$, this is the ratio claimed.

The analysis of (3.14) is a bit more complicated because the relation $\tau_1 - \tau_2 + \tau_3 - \tau_4 = 0$ is replaced by the slowly decaying factor $\langle \tau_1 - \tau_2 + \tau_3 - \tau_4 \rangle^{-1}$. It requires a third variant:

Proposition 4.3. *Consider*

$$\int\limits_{\vec{\xi} \in S \subset \Xi} \int\limits_{\vec{\tau} \in \mathbb{R}^4} \langle \tau_1 - \tau_2 + \tau_3 - \tau_4 \rangle^{-1} \prod_{n=1}^{4} g_n(\xi_n, \tau_n) \langle \tau_n - \xi_n^2 \rangle^{-\beta_n} \phi(\vec{\xi}) \chi_E\big(L(\vec{\xi})\big)\, d\lambda(\vec{\xi})\, d\vec{\tau} \tag{4.7}$$

where $g_n \geqslant 0$, $\phi \geqslant 0$, and $\beta_n > \frac{1}{2}$ for all n. Let $i \neq j \in \{1, 2, 3, 4\}$ and let $L : \mathbb{R}^4 \to \mathbb{R}$ be a linear functional satisfying the hypotheses of Proposition 4.1. *Then* (4.7) *is majorized by*

$$\begin{aligned} C_\beta \prod_{n=1}^{4} \|g_n\|_{L^2(\mathbb{R}^2)} |E|^{1/2} \cdot \Big[&\max_{\vec{\xi} \in S} \big(\phi(\vec{\xi}) \langle \sigma(\vec{\xi}) \rangle^{-1}\big) \cdot \max_{\vec{\xi} \in S} |\vec{\xi}|^{1/2} \\ &+ \max_{\vec{\xi} \in S} \phi(\vec{\xi}) \cdot \max_{\vec{\xi} \in S} \big(|\xi_i - \xi_j|^{-1/2}\big) \cdot \max_{\vec{\xi} \in S} \big(\langle \sigma(\vec{\xi}) \rangle^{-\beta}\big) \Big] \end{aligned} \tag{4.8}$$

for any $\beta < \min_n \beta_n$.

While (4.7) is formally similar to (4.3), a significant contribution to (4.7) can arise from a region in Fourier space which has no analogue in (4.3). This region contributes an additional term in (4.8). In our application, ϕ will be $|\psi|$.

The factor $|\vec{\xi}|^{1/2}$ in (4.8) can be replaced by $\max_{\vec{\xi} \in S} |\tilde{L}(\vec{\xi})|^{1/2}$ for any linear functional $\tilde{L}$ such that $\{\tilde{L}, L, \xi_1 - \xi_2 + \xi_3 - \xi_4\}$ is linearly independent.

4.2. Proofs of inequalities

The essence of Propositions 4.1–4.3 lies in the following two simpler inequalities.

Lemma 4.4. *Let i, j, k be the three elements of $\{1, 2, 3\}$, written in any order. Let $\ell : \mathbb{R}^3 \mapsto \mathbb{R}^1$ satisfy $\partial \ell / \partial \xi_k \neq 0$. Then for any nonnegative measurable functions G, g_n of two and one real variables, respectively, and for any measurable sets $E \subset \mathbb{R}^1$ and $S \subset \Xi$, the quantity*

$$\int\limits_{\mathbb{R}^3} \prod_{n=1}^{3} g_n(\xi_n) G\big(\xi_1 - \xi_2 + \xi_3, \xi_1^2 - \xi_2^2 + \xi_3^2\big) \chi_S(\vec{\xi}) \chi_E\big(\ell(\vec{\xi})\big)\, d\xi_1\, d\xi_2\, d\xi_3 \tag{4.9}$$

is majorized by

$$\lesssim \|G\|_{L^2} \prod_{n=1}^{3} \|g_n\|_{L^2} |E|^{1/2} \Big(\min_{\vec{\xi} \in S} |\xi_i - \xi_j| \Big)^{-1/2} \tag{4.10}$$

where the implied constant depends on ℓ.

Proof. Consider the case where $\{i,j\}=\{1,2\}$. Apply Cauchy–Schwarz to majorize by

$$\left(\int_{\vec{\xi}\in S} G^2\big(\xi_1-\xi_2+\xi_3,\xi_1^2-\xi_2^2+\xi_3^2\big)g_3^2(\xi_3)\,d(\xi_1,\xi_2,\xi_3)\right)^{1/2}$$
$$\times\left(\int_{\mathbb{R}^3} g_1^2(\xi_1)g_2^2(\xi_2)\chi_E\big(\ell(\vec{\xi})\big)\,d(\xi_1,\xi_2,\xi_3)\right)^{1/2}. \tag{4.11}$$

The left-hand factor is majorized by $\lesssim \|G\|_{L^2}\|g_3\|_{L^2}(\min_S|\xi_1-\xi_2|)^{-1/2}$; this is seen by first fixing ξ_3 and integrating with respect to (ξ_1,ξ_2), making the change of variables $(\xi_1,\xi_2)\mapsto(\xi_1-\xi_2,\xi_1^2-\xi_2^2)$.

To analyze the right-hand factor, first integrate with respect to ξ_3, obtaining a bound of

$$\lesssim |E|^{1/2}\left(\int g_1^2(\xi_1)g_2^2(\xi_2)\,d\xi_1\,d\xi_2\right)^{1/2} \tag{4.12}$$

since $\partial\ell/\partial\xi_3\neq 0$. Then integrate with respect to (ξ_1,ξ_2). Multiplying these bounds for the two factors yields $\lesssim \|G\|_{L^2}\prod_{n=1}^3\|g_n\|_{L^2}|E|^{1/2}(\min_S|\xi_1-\xi_2|)^{-1/2}$.

The same reasoning applies for other $\{i,j\}$; in all cases $|\xi_i-\xi_j|$ arises, rather than $|\xi_i+\xi_j|$. $\square$

Lemma 4.5. *For $m=1,2$ let $L_m:\mathbb{R}^4\to\mathbb{R}$ be linear functionals such that $\{\xi_1-\xi_2+\xi_3-\xi_4,L_1,L_2\}$ is linearly independent. Then all nonnegative measurable functions $g_n\in L^2(\mathbb{R}^1)$ and all measurable sets $E_m\subset\mathbb{R}^1$,*

$$\int_{\Xi}\prod_{n=1}^4 g_n(\xi_n)\prod_{m=1}^2\chi_{E_m}\big(L_m(\vec{\xi})\big)\,d\lambda(\vec{\xi})\lesssim\prod_{m=1}^2|E_m|^{1/2}\prod_{n=1}^4\|g_n\|_{L^2}. \tag{4.13}$$

Proof. Consider the multilinear form

$$T(g_1,\dots,g_6):=\int_{\Xi}\prod_{n=1}^4 g_n(\xi_n)g_5\big(L_1(\vec{\xi})\big)g_6\big(L_2(\vec{\xi})\big)\,d\lambda(\vec{\xi}).$$

By Cauchy–Schwarz,

$$\big|T(g_1,\dots,g_6)\big|\leqslant\left(\int_{\Xi}\prod_{n=1}^3\big|g_n(\xi_n)\big|^2\,d\lambda(\vec{\xi})\right)^{1/2}$$
$$\times\left(\int_{\Xi}\big|g_4(\xi_4)\big|^2\big|g_5\big(L_1(\vec{\xi})\big)\big|^2\big|g_6\big(L_2(\vec{\xi})\big)\big|^2\,d\lambda(\vec{\xi})\right)^{1/2}.$$

The first factor is a constant multiple of $\prod_{j=1}^{3}\|g_j\|_2$. The assumption that $\{\xi_1-\xi_2+\xi_3-\xi_4, L_1, L_2\}$ is linearly independent implies that the second factor is likewise proportional to $\prod_{j=4}^{6}\|g_j\|_2$. $\square$

Proof of Proposition 4.1. Consider the quantity (4.3) given in the statement of the proposition. Define $A=\min_{\vec\xi\in S}\langle\sigma(\vec\xi)\rangle\geqslant 1$. Introduce $\rho_n=\tau_n-\xi_n^2$. Since $\rho_1-\rho_2+\rho_3-\rho_4=\xi_1^2-\xi_2^2+\xi_3^2-\xi_4^2=\sigma(\vec\xi)$, we have $\langle\rho_n\rangle\gtrsim\langle\sigma(\vec\xi)\rangle$ for some n.

Partition the region of integration into four subregions, according to the index n for which $|\rho_n|$ is largest. By symmetry, it suffices to prove the stated bound for one of these subregions. Let $\nu\in\{1,2,3,4\}$ be arbitrary, and consider the subregion consisting of all $\vec\xi$ satisfying $|\rho_\nu(\vec\xi)|=\max_n|\rho_n(\vec\xi)|$. Partition further into subregions, in each of which $\langle\rho_\nu\rangle\sim 2^\kappa A$ for some nonnegative integer κ, and consider the contribution of any one of these subregions.

Suppose first that $\nu\notin\{i,j\}$. Denote by μ the remaining index, so that $\{1,2,3,4\}=\{i,j,\mu,\nu\}$. The contribution of the subregion under examination is

$$\lesssim\int_{(\rho_i,\rho_j,\rho_\mu)\in\mathbb{R}^3}\left(\int_{\mathbb{R}^3} g_\nu\big(\pm\xi_\mu\pm\xi_i\pm\xi_j,\pm\xi_\mu^2\pm\xi_i^2\pm\xi_j^2\pm\rho_\mu\pm\rho_i\pm\rho_j\big)\prod_{n\neq\nu}h_n(\xi_n,\rho_n)\right.$$
$$\left.\times\langle\rho_\nu\rangle^{-\beta_\nu}\chi_S(\vec\xi)\chi_E\big(L(\vec\xi)\big)\,d\xi_i\,d\xi_j\,d\xi_\mu\right)\prod_{n\neq\nu}\langle\rho_n\rangle^{-\beta_n}\,d(\rho_i,\rho_j,\rho_\mu)\tag{4.14}$$

where the $\pm$ sign preceding ξ_n^2 agrees with the sign preceding ξ_n for each $n\in\{\mu,i,j\}$, and where the outer integral extends only over those (ρ_i,ρ_j,ρ_μ) satisfying $|\rho_n|\leqslant|\rho_\nu|$ for all n and $\langle\rho_\nu\rangle\sim 2^\kappa A$. Here $h_n(\xi_n,\rho_n)=g_n(\xi_n,\tau_n)=g_n(\xi_n,\rho_n+\xi_n^2)$, and consequently $\|h_n\|_{L^2}=\|g_n\|_{L^2}$.

Fix (ρ_i,ρ_j,ρ_μ). The linear transformation $\Xi\in\vec\xi\mapsto(\xi_i,\xi_j,\xi_\mu)\in\mathbb{R}^3$ is invertible, so there is a unique linear functional $\tilde L:\mathbb{R}^3\mapsto\mathbb{R}$ satisfying $\tilde L(\xi_i,\xi_j,\xi_\mu)=L(\vec\xi)$. The hypothesis on L ensures that $\partial\tilde L/\partial\xi_\mu\neq 0$. The inner integral thus takes the form discussed in Lemma 4.4, and is consequently majorized by

$$\|g_\nu\|_{L^2(\mathbb{R}^2)}\prod_{n\neq\nu}\big\|g_n(\cdot,\rho_n)\big\|_{L^2(\mathbb{R}^1)}|E|^{1/2}\sup_S\big(|\xi_i-\xi_j|^{-1/2}\big)\big(2^\kappa A\big)^{-\beta_\nu}\tag{4.15}$$

since $\langle\rho_\nu\rangle\gtrsim 2^\kappa A$.

It remains to bound $\prod_{n\neq\nu}\int_{\langle\rho_n\rangle\lesssim 2^\kappa A}\|g_n(\cdot,\rho_n)\|_{L^2(\mathbb{R}^1)}\langle\rho_n\rangle^{-\beta_n}\,d\rho_n$. If $\beta_n>\frac12$ then

$$\int_{\mathbb{R}}\big\|g_n(\cdot,\rho_n)\big\|_{L^2(\mathbb{R}^1)}\langle\rho_n\rangle^{-\beta_n}\,d\rho_n\lesssim\|g_n\|_{L^2(\mathbb{R}^2)}\tag{4.16}$$

by Cauchy–Schwarz. If $\beta_n>\frac12$ for all $n\neq\nu$ then, since A was defined to be $\min_S\langle\sigma\rangle$, the desired bound is obtained from (4.15) by summation over all integers $\kappa\geqslant 0$.

Otherwise there remains exactly one index $m\neq\nu$ such that $\beta_m\leqslant\frac12$. Then $\beta=\beta_m$, and $\beta_\nu\geqslant\beta_m$. Since $2^\kappa A\sim\langle\rho_\nu\rangle\gtrsim\langle\rho_m\rangle$ throughout the region of integration, one has

$$\big(2^\kappa A\big)^{-\beta_\nu}\lesssim\big(2^\kappa A\big)^{-\beta_m}\langle\rho_m\rangle^{\beta_m-\beta_\nu}.$$

This factor of $\langle\rho_m\rangle^{\beta_m-\beta_\nu}$, multiplied by the factor of $\langle\rho_m\rangle^{-\beta_m}$ already present in the integral, becomes $\langle\rho_m\rangle^{-\beta_\nu}$. Since $\beta_\nu > \frac{1}{2}$, the analysis can be completed as above, yielding a bound of

$$\prod_{n=1}^{4} \|g_n\|_{L^2(\mathbb{R}^2)} |E|^{1/2} \sup_S\bigl(|\xi_i-\xi_j|^{-1/2}\bigr)\bigl(2^\kappa A\bigr)^{-\beta_m}.$$

The desired bound again follows by summation over κ, in the case where $\nu \notin \{i,j\}$.

Suppose finally that $\nu \in \{i,j\}$; by symmetry, we may suppose that $\nu = i$. If we write $\{1,2,3,4\} = \{i,j,k,l\}$, then the equation for Ξ, together with the hypothesis that i, j have opposite parity, imply that $|\xi_i-\xi_j| \equiv |\xi_k-\xi_l|$ for all $\vec{\xi} \in \Sigma$. Thus $\min_S|\xi_i-\xi_j| = \min_S|\xi_k-\xi_l|$, so $\{i,j\}$ can be interchanged with $\{k,l\}$. Denote by μ the remaining index, so that $\{1,2,3,4\} = \{\mu,\nu,k,l\}$. The hypothesis on L is formulated so as to be unaffected when $\{i,j\}$ is interchanged with $\{k,l\}$. Therefore the above reasoning again applies. □

Proof of Proposition 4.2. (3.14) is invariant under the permutations $(1,2,3,4) \mapsto (2,1,4,3)$, $(1,2,3,4) \mapsto (3,2,1,4)$, $(1,2,3,4) \mapsto (1,4,3,2)$, and consequently also $(1,2,3,4) \mapsto (3,4,1,2)$ of the indices. Therefore it is no loss of generality to assume that $i=1$, $j=2$, and $k=4$.

We follow the proof of Proposition 4.1. In the case when $\nu \notin \{1,2\}$, because $L(\vec{\xi}) = \xi_4$ does not belong to the span of the three linear transformations ξ_1, ξ_2, and $\xi_1-\xi_2+\xi_3-\xi_4$, that proof applies without alteration and yields the upper bound (4.4). Since $|E|/\min_S|\xi_1-\xi_2| \lesssim 1$ by hypothesis, (4.4) is majorized by a constant multiple of the desired bound (4.5).

Consider next the case where $\nu = 2$. Then because $\xi_1-\xi_2 \equiv -(\xi_3-\xi_4)$, Lemma 4.4 can be applied with the roles of the indices 2, 4 interchanged to obtain a bound

$$\lesssim |E|^{1/2} \max_{\vec{\xi}\in S}\Bigl(|\xi_1-\xi_3|^{-1/2} \cdot \max_{\vec{\xi}\in S}\bigl\langle\sigma(\vec{\xi})\bigr\rangle^{-\beta}\Bigr) \prod_{n=1}^{4} \|g_n\|_{L^2(\mathbb{R}^2)}. \tag{4.17}$$

Another bound is also available. Apply Proposition 4.1 with L replaced by $\tilde{L}(\vec{\xi}) = \xi_1-\xi_3$; $\tilde{L}$ belongs to neither the span of $\{\xi_1,\xi_2,\xi_1-\xi_2+\xi_3-\xi_4\}$ nor the span of $\{\xi_3,\xi_4,\xi_1-\xi_2+\xi_3-\xi_4\}$, so the hypotheses are satisfied. This yields an alternative bound

$$\max_{\vec{\xi}\in S}|\xi_1-\xi_3|^{1/2} \cdot \max_{\vec{\xi}\in S}\bigl(|\xi_1-\xi_2|\bigr)^{-1/2} \cdot \max_{\vec{\xi}\in S}\bigl(\bigl\langle\sigma(\vec{\xi})\bigr\rangle^{-\beta}\bigr) \prod_{n=1}^{4} \|g_n\|_{L^2(\mathbb{R}^2)}. \tag{4.18}$$

If $\max_S|\xi_1-\xi_3|$ is comparable to $\min_S|\xi_1-\xi_3|$, then taking the geometric mean of these two upper bounds yields the desired bound (4.5). Decomposing S into subsets S_κ in which $|\xi_1-\xi_3|$ is comparable to 2^κ for arbitrary $\kappa \in \mathbb{Z}$, invoking whichever of (4.17), (4.18) is more favorable for each κ, and summing over κ yields the same bound in the general case.

Finally, when $\nu = 1$, apply Lemma 4.4 with the roles of the indices 1, 4 interchanged, and repeat the above discussion for the case $\nu = 2$, replacing $\xi_1-\xi_3$ by $\xi_2-\xi_3$ throughout. The reasoning is otherwise unchanged. □

Proof of Proposition 4.3. Substitute $\tau_n = \rho_n + \xi_n^2$ and $g_n(\xi_n, \tau_n) = \tilde{g}_n(\xi_n, \rho_n)$ for all $n \in \{1, 2, 3, 4\}$ to transform the integral into

$$\int_{\mathbb{R}^4} \int_{S \subset \Xi} \langle \rho_1 - \rho_2 + \rho_3 - \rho_4 + \sigma(\vec{\xi}) \rangle^{-1} \prod_{n=1}^{4} \tilde{g}_n(\xi_n, \rho_n) \langle \rho_n \rangle^{-\beta_n} \phi(\vec{\xi}) \chi_E\big(L(\vec{\xi})\big)\, d\lambda(\vec{\xi})\, d\vec{\rho} \tag{4.19}$$

where $\tilde{g}_n$ has the same L^2 norm as g_n.

Begin with the region where $|\rho_n| \leqslant \frac{1}{8} \langle \sigma(\vec{\xi}) \rangle$ for all $n \in \{1, 2, 3, 4\}$, which has no counterpart in Proposition 4.1. Its contribution is comparable to

$$\int_{S \subset \Xi} \int_{\mathbb{R}^4} \prod_{n=1}^{4} \tilde{g}_n(\xi_n, \rho_n) \langle \rho_n \rangle^{-\beta_n} \langle \sigma(\vec{\xi}) \rangle^{-1} \phi(\vec{\xi}) \chi_E\big(L(\vec{\xi})\big)\, d\vec{\rho}\, d\lambda(\vec{\xi}). \tag{4.20}$$

Since all β_n are assumed to be strictly $> \frac{1}{2}$, applying Cauchy–Schwarz to the integral with respect to each variable ρ_n gives an upper bound

$$\lesssim \int_{S \subset \Xi} \prod_{n=1}^{4} h_n(\xi_n) \langle \sigma(\vec{\xi}) \rangle^{-1} \phi(\vec{\xi}) \chi_E\big(L(\vec{\xi})\big)\, d\lambda(\vec{\xi}) \tag{4.21}$$

where $h_n(\xi_n) = \|\tilde{g}_n(\xi_n, \cdot)\|_{L^2(\mathbb{R}^1)} = \|g_n(\xi_n, \cdot)\|_{L^2(\mathbb{R}^1)}$.

According to Lemma 4.5, (4.21) is

$$\lesssim |E|^{1/2} \prod_n \|g_n\|_{L^2(\mathbb{R}^2)} \max_S \big(\phi(\vec{\xi}) \langle \sigma(\vec{\xi}) \rangle^{-1} |\vec{\xi}|^{1/2}\big), \tag{4.22}$$

since the linear functional L does not vanish identically on Ξ.

Consider next the region where $\max_n \langle \rho_n \rangle \geqslant \frac{1}{8} \langle \sigma(\vec{\xi}) \rangle$. By symmetry, it is no loss of generality to restrict attention to the region where $|\rho_4| = \max_n |\rho_n|$. An upper bound for the integral over this region is

$$\begin{aligned} &\int_{\mathbb{R}^3} \int_{S \subset \Xi} \Bigg(\int_{|\rho_4| \geqslant \max_{j \leqslant 3} |\rho_j|} \langle \rho_1 - \rho_2 + \rho_3 - \rho_4 + \sigma(\vec{\xi}) \rangle^{-1} g_4(\xi_4, \rho_4)\, \langle \rho_4 \rangle^{-\beta_4}\, d\rho_4 \Bigg) \\ &\quad \times \phi(\vec{\xi}) \chi_E\big(L(\vec{\xi})\big) \prod_{n=1}^{3} \big(g_n(\xi_n, \rho_n) \langle \rho_n \rangle^{-\beta_n}\big)\, d\lambda(\vec{\xi}) \prod_{n=1}^{3} d\rho_n. \end{aligned} \tag{4.23}$$

Consider the contribution made to (4.23) by the subregion in which $\langle \rho_4 \rangle$ is comparable to an arbitrary constant $\Lambda \geqslant 2$. Since $\langle \rho_4 \rangle = \max_n \langle \rho_n \rangle \gtrsim \langle \sigma(\vec{\xi}) \rangle$, necessarily $\Lambda \gtrsim \langle \sigma(\vec{\xi}) \rangle$, and thus $\langle \rho_1 - \rho_2 + \rho_3 - \rho_4 + \sigma(\vec{\xi}) \rangle \lesssim \Lambda$. Therefore the innermost integral in (4.23) is majorized by the convolution of $\Lambda^{-\beta_4} g(\xi_4, \cdot)$ with $\langle \rho_4 \rangle^{-1} \cdot \chi_{[-C\Lambda, C\Lambda]}(\rho_4)$, evaluated at $\rho_1 - \rho_2 + \rho_3 + \sigma(\vec{\xi})$.

Since $\|\langle \rho_4 \rangle^{-1} \cdot \chi_{[-C\Lambda, C\Lambda]}(\rho_4)\|_{L^1(\mathbb{R})} \lesssim \log \Lambda$, the contribution of the region $\langle \rho_4 \rangle \sim \Lambda$ to (4.23) is majorized by

$$C\Lambda^{-\beta_4}\log\Lambda\int_{\mathbb{R}^3}\int_{\langle\sigma(\vec{\xi})\rangle\lesssim\Lambda}\chi_S(\vec{\xi})G_4\big(\xi_4,\rho_1-\rho_2+\rho_3+\sigma(\vec{\xi})\big)$$

$$\times\prod_{n=1}^{3}g_n(\xi_n,\rho_n)\phi(\vec{\xi})\chi_E\big(L(\vec{\xi})\big)\,d\lambda(\vec{\xi})\prod_{n=1}^{3}\langle\rho_n\rangle^{-\beta_n}\,d\rho_n \tag{4.24}$$

where $\|G_4\|_{L^2}\leqslant C\|g_4\|_{L^2}$. $G_4(\xi_4,\rho_1-\rho_2+\rho_3+\sigma(\vec{\xi}))$ can be reexpressed as $\tilde{G}_4(\xi_4,\rho_1-\rho_2+\rho_3+\xi_1^2-\xi_2^2+\xi_3^2)$ where $\|\tilde{G}_4\|_{L^2}=\|G_4\|_{L^2}$. Summing over dyadic values of Λ yields for (4.23) the upper bound

$$C_\beta\int_{\mathbb{R}^3}\int_{S\subset\Xi}G_4\big(\xi_4,\rho_1-\rho_2+\rho_3+\sigma(\vec{\xi})\big)$$

$$\times\prod_{n=1}^{3}g_n(\xi_n,\rho_n)\phi(\vec{\xi})\big\langle\sigma(\vec{\xi})\big\rangle^{-\beta}\chi_E\big(L(\vec{\xi})\big)\,d\lambda(\vec{\xi})\prod_{n=1}^{3}\langle\rho_n\rangle^{-\beta_n}\,d\rho_n \tag{4.25}$$

for any $\beta<\beta_4$.

This is nearly identical to the expression (4.14) reached in the proof of Proposition 4.1, with the factor $\langle\rho_\nu\rangle^{-\beta_\nu}$ in (4.14) now replaced by $C_\beta\langle\sigma(\vec{\xi})\rangle^{-\beta}$. It suffices to repeat the analysis above of (4.14), with the simplification that here all β_m are $>\frac{1}{2}$. $\square$

5. Conclusion of the proof of Proposition 3.1

Lemma 5.1. *Suppose that $s<0$, $r>-\frac{1}{4}$, and $b>\frac{1}{2}$. Let $\psi(\vec{\xi}):=\sum_{n=1}^{4}(-1)^n\langle\xi_n\rangle^{2s}$. Then*

$$\int_{\mathbb{R}^4}\int_{\Xi}\prod_{n=1}^{4}\big(g_n(\xi_n,\tau_n)\langle\xi_n\rangle^{-r}\big\langle\tau_n-\xi_n^2\big\rangle^{-b}\big)\big|\psi(\vec{\xi})\big|\,d\lambda(\vec{\xi})\,d\vec{\tau}\lesssim\prod_{n=1}^{4}\|g_n\|_{L^2(\mathbb{R}^2)}. \tag{5.1}$$

Proof. It is no loss of generality to assume throughout the proof that $\|g_n\|_{L^2}=1$ for all n. We analyze the integral (3.14) using Proposition 4.3, with $\phi\equiv\psi$. Recall the symmetries discussed in the proof of Proposition 4.2. These will be used to reduce the number of cases that must be discussed in the proof.

Let $N\geqslant 1$, and consider the contribution to the integral made by the subregion S_N of integration in which all $\langle\xi_j\rangle$ are comparable to N. Because of the symmetries listed above, we may restrict attention to the region where $|\xi_1-\xi_2|\leqslant|\xi_1-\xi_4|$. Let $S_{N,A,B}$ be the subregion where $|\xi_1-\xi_2|\sim AN$ and $|\xi_1-\xi_4|\sim B$, for arbitrary $0<A\leqslant B\lesssim 1$. We majorize the contribution of $S_{N,A,B}=S$ via the bound given by Proposition 4.3, with $L(\vec{\xi})=\xi_1-\xi_2$ and $E=[-CAN,CAN]$. Since $|\psi(\vec{\xi})|\lesssim N^{2s-2}|\sigma(\vec{\xi})|\lesssim N^{2s}AB$, this yields the sum of the following two quantities:

$$C|E|^{1/2}\max_S\big|\psi(\vec{\xi})\big|\max_S\big(\big\langle\sigma(\vec{\xi})\big\rangle^{-1}\big)\max_S|\vec{\xi}|^{1/2}$$

$$\lesssim(AN)^{1/2}N^{2s}AB\big\langle ABN^2\big\rangle^{-1}N^{1/2}\leqslant A^{1/2}N^{2s-1}$$

and

$$C|E|^{1/2}\max_S\big(|\xi_1-\xi_4|^{-1/2}\big)\max_S\big|\psi(\vec{\xi})\big|\max_S\big(\langle\sigma(\vec{\xi})\rangle^{-\beta}\big)$$
$$\lesssim (AN)^{1/2}(BN)^{-1/2}\big(N^{2s}AB\big)\big\langle ABN^2\big\rangle^{-\beta}\leqslant AN^{2s-1}$$

since $\beta\geqslant\frac{1}{2}$.

Summing over dyadic values of $A\leqslant B\lesssim 1$ gives a total bound of $\lesssim N^{2s-1}$ for the contribution of S_N. Taking the factors $\langle\xi_n\rangle^{-r}$ into account yields a net bound of $\lesssim N^{2s-4r-1}$ for the contribution of S_N to (3.14). Provided that $-r<\frac{1}{4}-\frac{1}{2}s$, this is $\lesssim N^{-\delta}$ for some $\delta>0$ and hence we can sum over dyadic values of $N\geqslant 1$ to majorize the contribution of the entire region on which all four quantities $\langle\xi_n\rangle$ are mutually comparable. Since $s<0$, this is a less stringent condition on r than the hypothesis $-r<\frac{1}{4}$.

The relation $\xi_1-\xi_2+\xi_3-\xi_4=0$ defining Ξ implies that the largest two of the four quantities $|\xi_n|$ must remain uniformly comparable. Consider next the contribution of a region of integration in which some two variables ξ_n with indices n of opposite parity are large, and at least one of the other two variables is comparatively small. Because of symmetries, it is then no loss of generality to restrict attention to the region where $|\xi_1|,|\xi_2|\sim N_2$, $\langle\xi_3\rangle\sim N_1$, and $\langle\xi_4\rangle\sim N_0$, where the parameters $N_0,N_1,N_2\geqslant 1$ satisfy $N_0\leqslant N_1\leqslant N_2$.

In the subcase in which $N_0\sim N_1$, consider the subregion S_Δ where $|\xi_4-\xi_3|\equiv|\xi_1-\xi_2|$ has some fixed order of magnitude Δ; necessarily $\Delta\lesssim N_1$. There $|\sigma(\vec{\xi})|=|\xi_1-\xi_4|\cdot|\xi_1-\xi_2|\sim\Delta N_2$, and

$$\big|\psi(\vec{\xi})\big|\leqslant\big|\varphi(\xi_3)-\varphi(\xi_4)\big|+\big|\varphi(\xi_1)-\varphi(\xi_2)\big|\lesssim N_0^{2s-1}\Delta+N_2^{2s-1}\Delta\lesssim N_0^{2s-1}\Delta \tag{5.2}$$

since $|\xi_1-\xi_2|=|\xi_3-\xi_4|$.

Apply Proposition 4.3 with $L(\vec{\xi})=\xi_4-\xi_3$ and $\phi(\vec{\xi})=|\psi(\vec{\xi})|$ to the contribution made by the region of integration S_Δ to (3.14):

$$\max_{S_\Delta}\big|L(\vec{\xi})\big|^{1/2}\max_{S_\Delta}\big|\psi(\vec{\xi})\big|\max_{S_\Delta}\big(\langle\sigma(\vec{\xi})\rangle^{-1}\big)\max_{S_\Delta}|\vec{\xi}|^{1/2}$$
$$\lesssim\Delta^{1/2}\cdot N_0^{2s-1}\Delta\cdot\langle\Delta N_2\rangle^{-1}N_2^{1/2}$$
$$\lesssim\Delta^{1/2}N_0^{2s-1}N_2^{-1/2}=(\Delta/N_0)^{1/2}N_0^{2s-\frac{1}{2}}N_2^{-1/2}, \tag{5.3}$$

while

$$\max_{S_\Delta}\big|L(\vec{\xi})\big|^{1/2}\max_{S_\Delta}|\xi_1-\xi_4|^{-1/2}\max_{S_\Delta}\big|\psi(\vec{\xi})\big|\max_{S_\Delta}\big(\langle\sigma(\vec{\xi})\rangle^{-\beta}\big)$$
$$\lesssim\Delta^{1/2}\cdot N_2^{-1/2}\cdot N_0^{2s-1}\Delta\cdot\langle\Delta N_2\rangle^{-1/2}\lesssim(\Delta/N_0)N_0^{2s}N_2^{-1}. \tag{5.4}$$

Since $\Delta\lesssim N_0\lesssim N_2$, the maximum of these two maxima is $\lesssim(\Delta/N_0)^{1/2}N_0^{2s-\frac{1}{2}}N_2^{-1/2}$. Incorporating the factors $\langle\xi_n\rangle^{-r}$ from (3.14) introduces an additional factor of $N_0^{-2r}N_2^{-2r}$, leaving a net bound

$$\lesssim(\Delta/N_0)^{1/2}N_0^{2s-\frac{1}{2}-2r}N_2^{-\frac{1}{2}-2r}.$$

Summing over dyadic values of $\Delta \lesssim N_0$ yields a bound

$$\lesssim N_0^{2s-\frac{1}{2}-2r} N_2^{-\frac{1}{2}-2r}$$

for the original region. This quantity is $\lesssim N_2^{-\delta}$ for some $\delta > 0$ if (and only if) $-r < \frac{1}{4}$. We may then sum over dyadic $N_0 \lesssim N_2$, then over all dyadic N_2.

If on the other hand $N_0 \leqslant \frac{1}{10} N_1$ then $\Delta \sim N_1$ and $|\psi(\vec{\xi})| \lesssim N_0^{-2s}$, so

$$\max\left|L(\vec{\xi})\right|^{1/2} \max\left|\psi(\vec{\xi})\right| \max\left(\left\langle\sigma(\vec{\xi})\right\rangle^{-1}\right) \max|\vec{\xi}|^{1/2} \lesssim N_1^{1/2} \cdot N_0^{2s} \cdot \langle N_1 N_2\rangle^{-1} N_2^{1/2},$$

giving a net bound of $N_0^{2s-r} N_1^{-\frac{1}{2}-r} N_2^{-\frac{1}{2}-2r}$, which again is $\lesssim N_2^{-\delta}$ for some $\delta > 0$ if and only if $-r < \frac{1}{4}$. Likewise

$$\begin{aligned} &\max\left|L(\vec{\xi})\right|^{1/2} \max|\xi_1 - \xi_4|^{-1/2} \max\left|\psi(\vec{\xi})\right| \max\left(\left\langle\sigma(\vec{\xi})\right\rangle^{-\beta}\right) \\ &\quad \lesssim N_1^{1/2} \cdot N_2^{-1/2} \cdot N_0^{2s} \cdot (N_1 N_2)^{-1/2} \lesssim N_0^{2s} N_1^0 N_2^{-1}, \end{aligned} \tag{5.5}$$

leading once again to the less stringent requirement $-r < \frac{1}{4} - \frac{1}{2}s$.

Because the roles of the four variables ξ_n are not completely symmetric, it is necessary to analyze separately the subcase in which again $N_0 \leqslant \frac{1}{10} N_1 \leqslant \frac{1}{10} N_2$, but $\langle\xi_4\rangle \sim N_0$, $\langle\xi_2\rangle \sim N_1$, and $|\xi_1|, |\xi_3| \sim N_2$. Thus $|\sigma(\vec{\xi})|$ is at least as large as in the above analysis. Since it was raised to negative powers above, this new situation is more favorable. Therefore the hypothesis $-r < \frac{1}{4}$ again suffices.

When the various symmetries between the indices $\{1, 2, 3, 4\}$ are taken into account, the above discussion exhausts all possible cases, and the proof is complete. □

Proof of Proposition 3.1. It suffices to bound $\|u(t,\cdot)\|_{H^s}$ for t in the support of ζ_0, since $\mathcal{I}(t) \equiv 0$ for other t. For such t, $u(t,x) \equiv \zeta_1(t)u(t,x)$ and hence $\hat{u}$ can be replaced by $\widehat{\zeta_1(t)u}$ throughout the above discussion. Thus $\|u\|_{X^{r,b}}$ can be replaced by $\|\zeta_1 u\|_{X^{r,b}}$ on the right-hand side of the inequality. □

6. $Y^{s,b}$ norms

The purpose of this section is to introduce certain function spaces $Y^{s,b}$, variants of the spaces $X^{s,b}$ employed by Bourgain [2] and then Kenig, Ponce, and Vega [8] to establish wellposedness of the nonlinear Schrödinger and Korteweg–de Vries equations. An *a priori* bound for $|u|^2 u$ in these spaces, in terms of u, will be proved in the following section.

Proposition 3.1 asserts an *a priori* upper bound for a solution in $C^0(H^s)$ in terms of an $X^{r,b}$ bound. Rather than establishing an $X^{r,b}$ bound directly, we will work with $Y^{s,b}$. Whereas the usual argument establishing an *a priori* $X^{0,b}$ bound for a solution breaks down for $X^{s,b}$ for s strictly negative, it continues to apply for $Y^{s,b}$ when an upper bound in $C^0(H^s)$ is known. $Y^{s,b}$ strictly contains $X^{s,b}$, but embeds in $X^{r,b}$ for certain $r < s$; see Lemma 6.2.

Define the scaling operator

$$T_\lambda u(t,x) := \lambda u\left(\lambda^2 t, \lambda x\right); \tag{6.1}$$

T_λ acts on distributions u defined on $\mathbb{R}^2$. It maps any solution of the cubic nonlinear Schrödinger equation to another solution. We use the same notation for functions of x alone: $T_\lambda f(x) := \lambda f(\lambda x)$.

Define also the (rough) Littlewood–Paley projections

$$\widehat{P_{<N}u}(\xi,\tau) := \begin{cases} \hat{u}(\xi,\tau) & \text{if } |\xi| \leqslant N, \\ 0 & \text{if } |\xi| > N. \end{cases} \tag{6.2}$$

We say that a function f is *M-band-limited* if $\hat{f}(\xi,\tau) = 0$ whenever $|\xi| > M$.

Fix an infinitely differentiable, compactly supported cutoff function $\eta \in C_0^\infty(\mathbb{R}^1)$ satisfying $\eta(0) \neq 0$.

Definition 6.1 *($Y^{s,b}$ norm).* Let $s, b \in \mathbb{R}$ with $s \in [-\frac{1}{2}, 0]$. For any tempered distribution u defined on $\mathbb{R}^2$ whose space–time Fourier transform $\hat{u}(\xi,\tau)$ belongs to $L^2_{\mathrm{loc}}(\mathbb{R}^2)$,

$$\|u\|_{Y^{s,b}} := \sup_{t_0 \in \mathbb{R}} \sup_{N \geqslant 1} \big\| \eta(t-t_0) T_{N^{2s}}(P_{<N}u) \big\|_{X^{0,b}}. \tag{6.3}$$

It would be slightly more natural to form an ℓ^2 norm over a dyadic sequence of values of N, rather than a supremum, but the definition used here is a bit simpler to work with, and is sufficient for our purpose. Observe that if f is N-band-limited, then $T_{N^{2s}} P_{<N} f$ is N^{1+2s}-band-limited.

For functions f supported in any fixed bounded interval with respect to time t,

$$\sup_N \int_{\langle\xi\rangle\sim N} \int_{\tau\in\mathbb{R}} \big|\hat{f}(\xi,\tau)\big|^2 \langle\xi\rangle^{2s} \big\langle N^{4s}\big(\tau - \xi^2\big)\big\rangle^{2b} \, d\xi \, d\tau \lesssim \|f\|^2_{Y^{s,b}}, \tag{6.4}$$

although the reverse inequality does not hold[8]; this inequality can be derived as in the proof of Lemma 6.2 below. Because s is negative and b positive, the factor $\langle N^{4s}(\tau - \xi^2)\rangle^{2b}$ is weaker than the corresponding factor $\langle \tau - \xi^2 \rangle^{2b}$ that appears in the $X^{s,b}$ norm. Thus $X^{s,b}$ embeds continuously in $Y^{s,b}$.

Our first lemma is a simple consequence of the definition; the proof is omitted.

Lemma 6.1 *(Insensitivity to smooth cutoffs).* (i) *If $h : \mathbb{R} \to \mathbb{C}$ is compactly supported and infinitely differentiable then $\|hu\|_{Y^{s,b}} \lesssim \|u\|_{Y^{s,b}}$ for all $u \in Y^{s,b}$.*

(ii) *Changing the cutoff function η in the definition of $Y^{s,b}$ leads to an equivalent norm, provided that $\eta \in C^\infty$ is compactly supported, and not identically zero.*

Remark 6.1. For $s < 0$, the spaces $Y^{s,b}$ are natural from the point of view of the extant H^0 theory. If an initial datum u_0 for (NLS) is N-band-limited in the sense that $\widehat{u_0}(\xi)$ is supported where $|\xi| \sim N$, and if $\|u_0\|_{H^s} \sim 1$, then $u_0 \in H^0$, but with large norm $\|u_0\|_{H^0} \sim N^{-s}$. Hence the Cauchy problem with initial datum u_0 has a solution belonging to $X^{0,b}$. This does not follow from the usual fixed point argument, since u_0 may be quite large in H^0. Instead one can partition

[8] For $r = (1+4b)s$ and $|\xi|$ of some fixed order of magnitude $N \geqslant 1$, the left-hand side of (6.4) is equivalent to the $X^{r,b}$ norm squared in the region where $|\tau - \xi^2| \gtrsim N^{-4s}$; it becomes larger as $|\tau - \xi^2|$ becomes smaller than this threshold.

the interval $[0, t]$ into sufficiently short subintervals that a fixed point argument applies on each, and invoke H^0 norm conservation.

An equivalent way to do the first time step is to solve the Cauchy problem for unit time with rescaled initial datum $T_{\lambda_N} u_0$, where $\lambda_N = N^{2s}$, then to reverse the scaling. The exponent is chosen so that $\|T_{\lambda_N} u_0\|_{H^0} \lesssim 1$ uniformly in $N \geqslant 1$. Successive time steps are done in the same way.

The next simple lemma makes possible the conversion of bounds in $Y^{s,b}$ to the more standard spaces $X^{r,b}$.

Lemma 6.2 *(Y controls X). Let $s < 0$ and $b \geqslant 0$. For any $A < \infty$ and any $r < (1+4b)s$ and all Schwartz class functions $f(t,x)$ supported where $|t| \leqslant A$, we have*

$$\|f\|_{X^{r,b}} \lesssim \|f\|_{Y^{s,b}}. \tag{6.5}$$

The converse inequality is not true; in the region where $|\tau - \xi^2| \ll \langle \xi \rangle^{-4s}$, the $Y^{s,b}$ norm is stronger than the $X^{r,b}$ norm even for $r = (1+4b)s$. We make this conversion both for the sake of conceptual simplicity, and because it simplifies certain calculations later on.

While Lemma 6.2 is needed to control $d\Phi/dt$, a variant will be used in establishing the $Y^{s,b}$ norm bound. For any real number $M \geqslant 1$ define the $X_M^{r,b}$ and $\widehat{X}_M^{r,b}$ norms by

$$\|f\|_{X_M^{r,b}}^2 := \iint_{\mathbb{R}^2} \left|\hat{f}(\xi,\tau)\right|^2 \langle \xi/M \rangle^{2r} \left\langle \tau - \xi^2 \right\rangle^{2b} \, d\tau \, d\xi,$$

$$\|g\|_{\widehat{X}_M^{r,b}}^2 := \iint_{\mathbb{R}^2} \left|g(\xi,\tau)\right|^2 \langle \xi/M \rangle^{2r} \left\langle \tau - \xi^2 \right\rangle^{2b} \, d\tau \, d\xi.$$

Likewise define

$$\|g\|_{\widehat{X}^{r,b}}^2 := \iint_{\mathbb{R}^2} \left|g(\xi,\tau)\right|^2 \langle \xi \rangle^{2r} \left\langle \tau - \xi^2 \right\rangle^{2b} \, d\tau \, d\xi.$$

Lemma 6.3 *(Y controls X, refined). Let $s < 0$, $b \in (\frac{1}{2}, 1)$, and suppose that $\eta \in C^\infty(\mathbb{R})$ has compact support. Let $r < (1+4b)s$. Then there exists $C < \infty$ such that for any $f \in Y^{s,b}$, any $N \geqslant 1$, and any $t_0 \in \mathbb{R}$, the function $g(t,x) = \eta(t-t_0) T_{N^{2s}} f(t,x)$ belongs to $X_{N^{1+2s}}^{r,b}$ with bound*

$$\|g\|_{X_{N^{1+2s}}^{r,b}} \leqslant C \|f\|_{Y^{s,b}}. \tag{6.6}$$

The constant C can be taken to depend only on s, b, r, η.

Choose any smooth, compactly supported function η such that $\sum_{j \in \mathbb{Z}} \eta(t-j) \equiv 1$ for all $t \in \mathbb{R}$.

Lemma 6.4 *(Almost-orthogonality). Let $b \in \mathbb{R}$. Let g be any Schwartz function, and define $g_j = \eta(t-j)g$ so that $g = \sum_{j\in\mathbb{Z}} g_j$. Then the summands g_j are almost orthogonal in $X^{0,b}$ norm, in the sense that*

$$\|g\|_{X^{0,b}} \leqslant C\Bigl(\sum_j \|g_j\|^2_{X^{0,b}}\Bigr)^{1/2} \tag{6.7}$$

where $C<\infty$ depends only on b, η.

Proof. Introduce the spatial Fourier transform $\mathcal{F}g(t,\xi) = \int_{\mathbb{R}} g(t,x)e^{-ix\xi}\,dx$. Let $J(t)$ be the distribution in $\mathcal{S}'(\mathbb{R}^1)$ whose Fourier transform is $\langle\tau\rangle^b$. Then J may be decomposed as $J = J_0 + J_\infty$ where J_0 is compactly supported and J_∞ belongs to the Schwartz class.

Now

$$\|g\|_{X^{0,b}} = \bigl\|\mathcal{F}g * \bigl(e^{i\xi^2 t}J(t)\bigr)\bigr\|_{L^2} \tag{6.8}$$

where $*$ denotes convolution, taken with respect to the t variable alone for each fixed value of ξ. Since J_∞ is a Schwartz function,

$$\bigl\|\mathcal{F}g * \bigl(e^{i\xi^2 t}J_\infty(t)\bigr)\bigr\|_{L^2} \lesssim \Bigl(\sum_j \|g_j\|^2_{L^2}\Bigr)^{1/2}, \tag{6.9}$$

and since $b \geqslant 0$, $\|g_j\|_{L^2} \lesssim \|g_j\|_{X^{0,b}}$.

There exists a finite constant C_0, depending only on η and on the support of J_0, such that no point (t,x) belongs to the support of g_j for more than C_0 integers j. Because the cutoff functions $\eta(t-j)$ are independent of x, the same bounded overlap property holds for their spatial Fourier transforms $\mathcal{F}g_j(t,\xi)$. Because J_0 has compact support, it follows that likewise no point (t,ξ) belongs to the support of $\mathcal{F}g_j * (e^{i\xi^2 t}J_0(t))$ for more than C_0 integers j.

Therefore

$$\begin{aligned}
\bigl\|\mathcal{F}g * \bigl(e^{i\xi^2 t}J_0(t)\bigr)\bigr\|^2_{L^2} &\lesssim \sum_j \bigl\|\mathcal{F}g_j * \bigl(e^{i\xi^2 t}J_0(t)\bigr)\bigr\|^2_{L^2} \\
&\lesssim \sum_j \iint \bigl|\widehat{g_j}(\tau,\xi)\bigr|^2 \bigl|\widehat{J_0}\bigl(\tau-\xi^2\bigr)\bigr|^2\,d\tau\,d\xi \\
&\lesssim \sum_j \|g_j\|^2_{X^{0,b}}
\end{aligned}$$

since $|\widehat{J_0}| = |\widehat{J} - \widehat{J_\infty}| \leqslant |\widehat{J}| + C \lesssim \langle\tau\rangle^b + C \leqslant \langle\tau\rangle^b$ since $b \geqslant 0$. $\square$

Proof of Lemma 6.2. Let f be given. Let $r := (1+4b)s$. It suffices to show that for all $N \geqslant 1$,

$$\int\limits_{\langle\xi\rangle\sim N}\int\limits_{\tau\in\mathbb{R}} \bigl|\hat{f}(\xi,\tau)\bigr|^2\bigl\langle\tau-\xi^2\bigr\rangle^{2b}\langle\xi\rangle^{2r}\,d\xi\,d\tau \lesssim \|f\|^2_{Y^{s,b}}, \tag{6.10}$$

since summing over all $N = 1, 2, 4, 8, \ldots$ then yields the required bound for all r strictly less than $(1 + 4b)s$.

Define $g_j := \eta(t - j) \cdot T_{N^{2s}} P_{<N} f$, and $g := \sum_{j \in \mathbb{Z}} g_j$, as in Lemma 6.4. All but at most CN^{-4s} terms in this decomposition vanish identically, because of the hypothesis restricting the support of f with respect to t. Moreover $\hat{f}(\xi, \tau) = N^{4s} \hat{g}(N^{2s}\xi, N^{4s}\tau)$. Consequently a trivial majorization of the ℓ^2 outer norm in (6.7) gives

$$\|g\|_{X^{0,b}} \lesssim N^{-2s} \max_j \|g_j\|_{X^{0,b}} \lesssim N^{-2s} \|f\|_{Y^{s,b}}. \tag{6.11}$$

Now (since $1 + 2s > 0$)

$$\begin{aligned}
&\int\limits_{\langle\xi\rangle \sim N} \int\limits_{\tau \in \mathbb{R}} \left|\hat{f}(\xi, \tau)\right|^2 \left\langle \tau - \xi^2 \right\rangle^{2b} \langle\xi\rangle^{2r} \, d\xi \, d\tau \\
&= N^{8s} \int\limits_{\langle\xi\rangle \sim N} \int\limits_{\tau \in \mathbb{R}} \left|\hat{g}\left(N^{2s}\xi, N^{4s}\tau\right)\right|^2 \left\langle \tau - \xi^2 \right\rangle^{2b} \langle\xi\rangle^{2r} \, d\xi \, d\tau \\
&= N^{2s} \int\limits_{\langle\xi\rangle \sim N^{1+2s}} \int\limits_{\tau \in \mathbb{R}} \left|\hat{g}(\xi, \tau)\right|^2 \left\langle N^{-4s}\left(\tau - \xi^2\right)\right\rangle^{2b} \left\langle N^{-2s}\xi\right\rangle^{2r} \, d\xi \, d\tau \\
&\sim N^{2s} \int\limits_{\langle\xi\rangle \sim N^{1+2s}} \int\limits_{\tau \in \mathbb{R}} \left|\hat{g}(\xi, \tau)\right|^2 \left\langle N^{-4s}\left(\tau - \xi^2\right)\right\rangle^{2b} N^{2r} \, d\xi \, d\tau \\
&\lesssim N^{2s-8bs+2r} \int\limits_{\langle\xi\rangle \sim N^{1+2s}} \int\limits_{\tau \in \mathbb{R}} \left|\hat{g}(\xi, \tau)\right|^2 \left\langle \tau - \xi^2 \right\rangle^{2b} \, d\xi \, d\tau \\
&\leqslant N^{2s-8bs+2r} \|g\|_{X^{0,b}}^2 \\
&\lesssim N^{-2s-8bs+2r} \|f\|_{Y^{s,b}}^2
\end{aligned}$$

by (6.11). This is $\lesssim \|f\|_{Y^{s,b}}^2$ under the hypothesis that $r \leqslant (1 + 4b)s$. $\square$

Proof of Lemma 6.3. This argument is nearly identical to the proof of Lemma 6.2, except that additional parameters are involved.

Let $f \in Y^{s,b}$ be arbitrary. Let $g(t, x) = \eta(t - t_0) T_{N^{2s}} f(t, x)$ and $M = N^{1+2s}$. Consider $\int_{|\xi/M| \sim \Lambda} \int_{\tau \in \mathbb{R}} |\hat{g}(\xi, \tau)|^2 \langle \xi/M \rangle^{2r} \langle \tau - \xi^2 \rangle^{2b} \, d\tau \, d\xi$ for arbitrary $\Lambda \geqslant 1$. The contribution of the region $|\xi| \leqslant N^{1+2s}$ to this integral is controlled directly by $\|f\|_{Y^{s,b}}^2$, and hence requires no further discussion.

Since $\widehat{T_{N^{2s}} f}(\xi, \tau) = \hat{f}(N^{-2s}\xi, N^{-4s}\tau)$, the above integral can be reexpressed in terms of $\hat{f}(\tilde{\xi}, \tilde{\tau})$ with $|\tilde{\xi}| \sim \Lambda M N^{-2s} = \Lambda N$, thus in terms of $F = \eta(t - t_0) T_{N^{2s}}(P_{<C\Lambda N} f)$. This function F can be naturally decomposed as $F = \sum_j F_j$ where each function $F_j(t, x)$ is supported where $t \in I_j$, each $I_j \subset \mathbb{R}$ is an interval of length Λ^{4s}, no point of $\mathbb{R}$ belongs to more than 2 intervals I_j, the sum extends over at most $C\Lambda^{-4s}$ indices j, and F_j satisfies

$$\iint \left|\widehat{F_j}(\xi, \tau)\right|^2 \left\langle \Lambda^{4s}\left(\tau - \xi^2\right)\right\rangle^{2b} \Lambda^{-4s} \, d\tau \, d\xi \leqslant C\Lambda^{-2s} \|f\|_{Y^{s,b}}^2. \tag{6.12}$$

This decomposition is obtained by a smooth partition of unity in the t variable, which decomposes the portion of $f(t,x)$ with Fourier transform (with respect to x) supported where $|\xi| \leqslant C\Lambda N$ into summands which (as functions of t) are supported on intervals of lengths $(\Lambda N)^{4s}$. The $Y^{s,b}$ norm directly gives a bound for each summand, and substitution via the dilations $T_{N^{2s}}$ yields (6.12).

By dilating time by a factor of $|\Lambda|^{-4s}$, invoking Lemma 6.4, and reversing the dilation, we conclude that $F = \sum_j F_j$ satisfies

$$\iint \big|\widehat{F}(\xi,\tau)\big|^2 \big\langle \Lambda^{4s}\big(\tau - \xi^2\big)\big\rangle^{2b} \Lambda^{-4s}\, d\tau\, d\xi \lesssim \Lambda^{-2s} \cdot \Lambda^{-4s} \|f\|_{Y^{s,b}}^2; \tag{6.13}$$

whereas an application of the triangle inequality would yield a factor of Λ^{-8s} on the right-hand side, the orthogonality expressed by Lemma 6.4 saves a factor of Λ^{4s}. Since $\langle \Lambda^{4s}(\tau - \xi^2)\rangle \gtrsim \Lambda^{4s}\langle \tau - \xi^2\rangle$, it follows that

$$\iint \big|\widehat{F}(\xi,\tau)\big|^2 \big\langle \tau - \xi^2\big\rangle^{2b}\, d\tau\, d\xi \lesssim \Lambda^{-2s(1+4b)} \|f\|_{Y^{s,b}}^2. \tag{6.14}$$

Since $-2s(1+4b) \leqslant -2r$, this yields the desired bound for the contribution made by F to g, that is, the contribution of the region where $|\xi/M| \sim \Lambda$. Since $-2s(1+4b)$ is strictly less than $-2r$, summation over dyadic values of $\Lambda \geqslant 1$ completes the proof. □

Proposition 3.1 together with the embedding of $Y^{s,b}$ in $X^{r,b}$ established in Lemma 6.2 yield

Proposition 6.5. *Let $T_0 < \infty$, $T \in [0, T_0]$, $s \in (-\frac{1}{2}, 0)$, $b \in (\frac{1}{2}, 1)$. For any sufficiently smooth solution u of* (NLS*) *with initial datum u_0,*

$$\|u\|_{C^0([-2T,2T],H^s)}^2 \leqslant \|u_0\|_{H^s}^2 + C\|\zeta_1 u\|_{Y^{s,b}}^4 \tag{6.15}$$

provided that $s < 0$, $b > \frac{1}{2}$, and $-s < \frac{1}{4}(1+4b)^{-1}$.

To use this bound we of course need to control the $Y^{s,b}$ norm of u. This will be accomplished in the next two sections.

7. Bound for $|u|^2 u$

The objective of this section is to prove the following nonlinear estimate.

Proposition 7.1 *(Trilinear estimate in $Y^{s,b}$). Suppose that $s > -\frac{2}{15}$ and $b \in (\frac{1}{2}, 1)$ satisfy*

$$-s < (1+4b)^{-1} \min\left(\frac{1}{10} + \frac{3}{5}(1-b), \frac{1}{12} + \frac{2}{3}(1-b)\right). \tag{7.1}$$

Then for any $u, v, w \in Y^{s,b}$,

$$\|u\bar{v}w\|_{Y^{s,b-1}} \lesssim \|u\|_{Y^{s,b}} \|v\|_{Y^{s,b}} \|w\|_{Y^{s,b}}. \tag{7.2}$$

The product $u\bar{v}w$, by virtue of having a locally integrable space–time Fourier transform, consequently has a natural interpretation as a distribution.

(7.2) is a variant of a well-known inequality in which $Y^{s,c}$ is replaced by $X^{0,c}$ throughout. Here there is a tradeoff: Once the parameter N in the definition of $Y^{s,b-1}$ is fixed, no bound is asserted for $\widehat{u\bar{v}w}(\xi,\tau)$ for $|\xi| \gg N$, but u, v, w are allowed to lie in spaces of mildly negative order.

The right-hand side of (7.1) equals $\frac{2}{15}$ when $b = \frac{1}{2}$. Thus for any $s > -\frac{2}{15}$ there does exist $b \in (\frac{1}{2}, 1)$ satisfying (7.1).

Proof of Proposition 7.1. The definition of the $Y^{s,b}$ norm involves a supremum over $N \geqslant 1$; fix N. Set $M := N^{1+2s}$. Choose r very slightly less than $(1+4b)s$, and recall the $X^{r,b}_{N^{1+2s}}$ bound formulated in Lemma 6.3.

Pair the space–time Fourier transform of $u\bar{v}w$ with $\langle \tau - \xi^2 \rangle^{b-1} g_4(\xi,\tau)$ where $g_4 \in L^2(\mathbb{R}^2)$. Substitute for the Fourier transforms of u, v, w as in (3.13). Matters then reduce to showing that

$$\int_{\vec{\xi}\in\Xi}\int_{\vec{\tau}\in\Xi} \prod_{n=1}^{4} \bigl(g_n(\xi_n,\tau_n)\langle \xi_n/M\rangle^{-r}\langle \tau_n - \xi_n^2\rangle^{-\beta_n}\bigr)\chi_{S_0}(\vec{\xi})\,d\lambda(\vec{\tau})\,d\lambda(\vec{\xi}) \lesssim \prod_{n=1}^{4} \|g_n\|_{L^2(\mathbb{R}^n)}$$

uniformly for all $M \geqslant 1$, where $\beta_n := b$ for $n \leqslant 3$ and $\beta_4 := 1-b$, and $S_0 := \{\vec{\xi}\colon |\xi_4| \lesssim M\}$. Assume with no loss of generality that $\|g_n\|_{L^2(\mathbb{R}^2)} = 1$ for all indices n.

An important special case arises when all $|\xi_n|$ are $\lesssim M = N^{1+2s}$. For this subregion, the desired inequality is nothing more than the well-known $X^{0,b-1}$ bound for $|u|^2u$ in terms of $\|u\|^3_{X^{0,b}}$ (see e.g. [11]).

Consider next the contribution to the integral of the region where $|\xi_n| \sim AM$ for all $n \neq 4$ for some single $A \gg 1$. For all such $\vec{\xi}$, $|\sigma(\vec{\xi})| \sim (AM)^2$, so since $\min(b, 1-b) = 1-b$, an application of Proposition 4.2 with $L(\vec{\xi}) = \xi_4$ yields an upper bound of the form

$$\frac{M^{1/4}}{(AM)^{1/4}}(AM)^{-2(1-b)}A^{-3r} = M^{-2(1-b)}A^{-\frac{1}{4}-2(1-b)-3r} \tag{7.3}$$

and we need both exponents to be negative. The exponent $-2(1-b)$ on M is certainly negative since $b < 1$. Thus we need

$$-r < \frac{1}{12} + \frac{2}{3}(1-b). \tag{7.4}$$

$Y^{s,b}$ embeds in $X^{r,b}_M$ for all $r < (1+4b)s$ uniformly in $M \geqslant 1$, in the sense expressed by Lemma 6.3, so this expression is appropriately controlled by the product of $Y^{s,b}$ norms provided that (7.1) is satisfied.

A more delicate case arises when $|\xi_j| \sim AM$ with $A \gg 1$ for two values of $j \in \{1,2,3\}$, but $|\xi_n| \sim BM$ where $B \leqslant A/10$ for the third index. If $n = 2$, then $\sigma(\vec{\xi}) \sim (AM)^2$, and the above analysis applies; the sole change is that one factor of A^{-r} is now merely $\lesssim B^{-r}$, which is a more favorable bound since $B \leqslant A$ and $r < 0$. Thus it remains only to discuss the case where n is odd; by virtue of the symmetries of the problem, it is then no loss of generality to suppose that $n = 3$.

In the subcase where $B \gtrsim 1$, we have $|\sigma| \gtrsim AMBM$ and Proposition 4.2, again with $L(\vec{\xi}) = \xi_4$, yields the upper bound

$$\frac{M^{1/4}}{(AM)^{1/4}}\left(ABM^2\right)^{-(1-b)}A^{-2r}B^{-r} = M^{-2(1-b)}A^{-\frac{1}{4}-2r-(1-b)}B^{-r-(1-b)}. \tag{7.5}$$

Provided that $-r < 1-b$, the exponent on B is negative, so when $B \gtrsim A^{1/2}$ this is $\lesssim M^{-2(1-b)}A^{-\frac{1}{4}-\frac{5}{2}r-\frac{3}{2}(1-b)}$. In the case $1 \lesssim B \lesssim A^{1/2}$ we invoke instead Proposition 4.1 with $L = \xi_4 - \xi_3$ to obtain an upper bound

$$\begin{aligned}\frac{(BM)^{1/2}}{(AM)^{1/2}}\left(ABM^2\right)^{-(1-b)}A^{-2r}B^{-r} &= M^{-2(1-b)}B^{\frac{1}{2}-(1-b)-r}A^{-\frac{1}{2}-(1-b)-2r}\\ &\lesssim M^{-2(1-b)}A^{-\frac{1}{4}-\frac{3}{2}(1-b)-\frac{5}{2}r}\end{aligned} \tag{7.6}$$

since the exponent $\frac{1}{2}-(1-b)-r$ is positive for $b > \frac{1}{2}$ and $r < 0$, and $B \lesssim A^{1/2}$. This is the same bound as obtained above for $B \gtrsim A^{1/2}$. The exponent on M is negative since $b < 1$, while the exponent on A is negative if

$$-r < \frac{1}{10} + \frac{3}{5}(1-b). \tag{7.7}$$

Under those conditions, this bound is summable over dyadic values of M, A, B.

$1-b > \frac{1}{2} > \min(\frac{1}{10}+\frac{3}{5}(1-b), \frac{1}{12}+\frac{2}{3}(1-b))$ for all $b \in (\frac{1}{2}, 1)$, so the condition that $-r < 1-b$ does not appear in the hypotheses of the proposition.

If $ABM^2 \lesssim 1$ then we use the upper bound $\lesssim 1$ for $\langle\sigma\rangle$ in place of $(ABM^2)^{-(1-b)}$, and obtain the upper bound

$$(BM)^{1/2}(AM)^{-1/2}A^{-2r} = B^{1/2}A^{-\frac{1}{2}-2r} \lesssim \left(A^{-1}M^{-2}\right)^{1/2}A^{-\frac{1}{2}-2r} = A^{-1-2r}M^{-1}. \tag{7.8}$$

Both exponents are negative for all $-r < \frac{1}{2}$, so this is a less stringent requirement than (7.7).

Choosing r to be sufficiently close to $(1+4b)s$ reduces all these restrictions to the stated hypothesis on s. $\square$

8. *A priori* bound in $Y^{s,b}$

The next result is the second main inequality underlying our theorems.

Proposition 8.1. *For any $s > -\frac{2}{15}$ and $b \in (\frac{1}{2}, 1)$ satisfying*

$$-s < (1+4b)^{-1}\min\left(\frac{1}{10}+\frac{3}{5}(1-b), \frac{1}{12}+\frac{2}{3}(1-b)\right) \tag{8.1}$$

any sufficiently smooth solution u of (NLS*) *with initial datum u_0 satisfies*

$$\|u\|_{Y^{s,b}} \lesssim \|u\|_{C^0(H^s)} + \|u\|^3_{Y^{s,b}} \tag{8.2}$$

where $\|\cdot\|_{C^0(H^s)} := \|\cdot\|_{C^0([-2T,2T],H^s)}$.

Proof. Choose $r < (1+4b)s$ sufficiently close to $(1+4b)s$. Let $N \geqslant 1$, let η be a smooth, compactly supported function, and let $t_0 \in \mathbb{R}$. Recall that u may be considered to be defined, and to satisfy the modified equation (NLS*), for all $t \in \mathbb{R}$.

Consider $w(t,x) := \eta(t-t_0)T_{N^{2s}}(u)$, which satisfies the equation

$$iw_t + w_{xx} = \eta'(t-t_0)T_{N^{2s}}u + \eta(t-t_0)\zeta_0\big(N^{4s}(t-t_0)\big)|T_{N^{2s}}u|^2 T_{N^{2s}}u. \tag{8.3}$$

It suffices to bound $\hat{w}(\xi,\tau)$ in the region where $|\tau-\xi^2| \geqslant 1$, for the contribution of the region $|\tau-\xi^2| \leqslant 1$ to the $X^{0,b}$ norm of w is majorized by $\lesssim \|w\|_{L^2(dt\,dx)}$, hence by $\lesssim \|w\|_{C^0(H^0)}$ because as a function of t, $w(t,x)$ is supported in an interval of uniformly bounded length; hence this contribution is majorized by $\lesssim \|u\|_{C^0(H^s)}$.

We may express $\hat{w}(\xi,\tau)$ as a constant times $(\tau-\xi^2)^{-1}$ times the Fourier transform of the right-hand side of (8.3). The contribution of the first term on the right is then easily handled, for $\|\eta'(t-t_0)T_{N^{2s}}u\|_{L^2(dt\,dx)} \leqslant C\|T_{N^{2s}}u\|_{C^0(H^0)} \leqslant C\|u\|_{C^0(H^s)}$. After dividing by $\langle\tau-\xi^2\rangle^{-1}$ we therefore have a quantity whose norm in $X^{0,1}$ is majorized by $\lesssim \|u\|_{C^0(H^s)}$.

The function $\eta(t-t_0)\zeta_0(N^{4s}(t-t_0))$ may be expressed as $\tilde{\eta}^3(t-t_0)$ where $\tilde{\eta} \in C^\infty$ is real-valued, is supported in a bounded interval independent of N, and is bounded in any C^k norm uniformly in N. The second term on the right-hand side of (8.3) thus becomes $|\tilde{\eta}(t-t_0)T_{N^{2s}}u|^2\tilde{\eta}(t-t_0)T_{N^{2s}}u$.

By Lemma 6.3, the norm of $\tilde{\eta}(t-t_0)T_{N^{2s}}u$ in $X^{r,b}_{N^{1+2s}}$ is $\lesssim \|u\|_{Y^{s,b}}$. Proposition 7.1 says that the $X^{0,b}$ norm of the function whose Fourier transform is $(\tau-\xi^2)^{-1}$ times the characteristic function of the region $|\xi| \lesssim N^{1+2s}$ times the space–time Fourier transform of $|\tilde{\eta}(t-t_0)T_{N^{2s}}u|^2\tilde{\eta}(t-t_0)T_{N^{2s}}u$ is majorized by $\lesssim \|u\|^3_{Y^{s,b}}$, provided that $-2+2b \leqslant 1-2b$. $\square$

Proof of Theorem 1.1. For any finite T and $\delta' > 0$, there exists $\delta > 0$ such that the bounds of Propositions 8.1 and 6.5 together imply an *a priori* upper bound $\|u\|_{C^0([0,T],H^s)} \leqslant \delta'$ provided that $\|u_0\|_{H^s} \leqslant \delta$ and $\|u\|_{C^0([0,T],H^s)} \leqslant 2\delta'$.

To prove the theorem, it suffices to show that given any $R < \infty$, there exists $\varepsilon_0 > 0$ such that for any $u_0 \in H^0$ satisfying $\|u_0\|_{H^s} \leqslant R$, if u denotes the solution of (NLS) with initial datum u_0, then $T_{\varepsilon_0}u$ satisfies an *a priori* $C^0([0,1],H^s)$ bound. Because $s > -\frac{1}{2}$, the equation is subcritical in H^s; there exists ε_0 so that $\|\varepsilon u_0(\varepsilon x)\|_{H^s} \leqslant \delta$ whenever $\|u_0\|_{H^s} \leqslant R$ and $0 < \varepsilon \leqslant \varepsilon_0$. We know that $u \in C^0(H^0)$, hence $u \in C^0(H^s)$. For very small ε, depending on $\|u_0\|_{H^0}$, we have $\|T_\varepsilon u\|_{C^0([0,1],H^s)} \leqslant \delta'$.

Now a continuity argument can be applied. If $\varepsilon > 0$ has the property that $\|T_\varepsilon u\|_{C^0([0,1],H^s)} \leqslant \delta'$, then there exists $\varepsilon' > \varepsilon$ such that $\|T_{\varepsilon'}u\|_{C^0([0,1],H^s)} \leqslant 2\delta'$, and provided that $\varepsilon' \leqslant \varepsilon_0$ and ε_0 is chosen to be sufficiently small but depending only on R, this implies that $\|T_{\varepsilon'}u\|_{C^0([0,1],H^s)} \leqslant \delta'$. Standard reasoning shows that this must then hold for $\varepsilon' = \varepsilon_0$. $\square$

9. Existence of weak solutions

We now prove a weakened variant of Theorem 1.2 on the existence of weak solutions, showing merely that weak solutions exist in $L^\infty(H^s) \cap C^0(H^{s'}) \cap Y^{s,b}$ for all $s' < s$. The last detail, existence in $C^0(H^s)$, will be addressed in Section 10.

Lemma 9.1. *Let $s > -\frac{1}{12}$. Let $u_0 \in H^s$, $\varepsilon > 0$, and $M < \infty$ be given. There exist $T' > 0$ and $\delta > 0$ such that for any initial datum $v_0 \in H^0$ satisfying $\|v_0 - u_0\|_{H^s} < \delta$, the standard solution v of* (NLS) *with initial datum v_0 satisfies*

$$\int\limits_{|\xi| \leqslant M} \left|\hat{v}(t_1, \xi) - \hat{v}(t_2, \xi)\right|^2 \langle\xi\rangle^{2s}\, d\xi < \varepsilon \quad \textit{for all } t_1, t_2 \in [0, T'] \textit{ satisfying } |t_1 - t_2| < \delta. \tag{9.1}$$

Proof. Fix any $b > \frac{1}{2}$. For any $\varepsilon' > 0$ there exists $\delta' > 0$ such that any $w \in X^{0,b}$ satisfies $\|w(t_1, \cdot) - w(t_2, \cdot)\|_{L^2} \lesssim |t_1 - t_2|^\gamma \|w\|_{X^{0,b}}$ for all $\gamma < b - \frac{1}{2}$ whenever $|t_1 - t_2| \leqslant 1$, as follows from a standard Cauchy–Schwarz calculation. By rescaling we conclude that

$$\left\| P_{<M} v(t_1, \cdot) - P_{<M} v(t_2, \cdot)\right\|_{H^s} \leqslant C_M |t_1 - t_2|^\gamma \|v\|_{Y^{s,b}} \tag{9.2}$$

whenever $|t_1 - t_2| \lesssim M^{4s}$.

We have already established an *a priori* upper bound for $\|v\|_{Y^{s,b}}$ in terms of $\|v_0\|_{H^s}$, hence in terms of $\|u_0\|_{H^s}$ so long as $\delta \leqslant 1$. Consequently

$$\int\limits_{|\xi| \leqslant M} \left|\hat{v}(t_1, \xi) - \hat{v}(t_2, \xi)\right|^2 \langle\xi\rangle^{2s}\, d\xi \leqslant C'_M {\varepsilon'}^2 \tag{9.3}$$

provided that $|t_1 - t_2| < \delta' M^{4s}$. The claim follows. □

Proof of Theorem 1.2. Let $s \in (-\frac{1}{12}, 0)$, and then let $s' \in (-\frac{1}{12}, s)$ be arbitrary. Consider any initial datum $u_0 \in H^s$. Let $(v_{0,j})$ be any sequence of functions in $H^0(\mathbb{R})$ such that $v_{0,j} \to u_0$ in H^s norm as $j \to \infty$. Let $v^{(j)} \in X^{0,b}$ be the unique standard solution of the Cauchy problem (NLS) with initial datum $v_{0,j}$.

There exist $b > \frac{1}{2}$ and T such that the sequence $v^{(j)}$ is uniformly bounded in $C^0((-2T, 2T), H^s) \cap Y^{s,b}$ norm. Moreover, the mappings $(-2T, 2T) \ni t \mapsto v^{(j)}(t, \cdot) \in H^{s'}$ are equicontinuous, by virtue of Lemma 9.1 and the inequality

$$\int\limits_{|\xi| \geqslant M} \left|\hat{f}(\xi)\right|^2 \langle\xi\rangle^{2s'}\, d\xi \leqslant C M^{2s' - 2s} \|f\|_{H^s}^2. \tag{9.4}$$

For any large N, decompose $v^{(j)}$ as

$$v^{(j)} = v^{(j)}_{N;\,\mathrm{high}} + v^{(j)}_{N;\,\mathrm{low}}$$

where $\widehat{v^{(j)}_{N;\,\mathrm{low}}}(t, \xi) := \widehat{v^{(j)}}(t, \xi)$ when $|\xi| \leqslant N$ and $:= 0$ otherwise. The equicontinuity of the mapping $t \mapsto v^{(n)}(t, \cdot) \in H^{s'}$ implies precompactness of $\{v^{(j)}_{N;\,\mathrm{low}}\}$ in $C^0_t(C^\infty_x)$ for x in every bounded region, for every N. A diagonal argument produces a subsequence, denoted again by $v^{(j)}$, such that for every N, $v^{(j)}_{N;\,\mathrm{low}}$ converges in the $C^0(C^\infty)$ topology in every bounded region. Since $v^{(j)}$ is uniformly bounded in $C^0(H^s)$, there exists a distribution $u \in \mathcal{D}'$ such that $v^{(j)} \to u$ in the topology of $\mathcal{D}'$.

Equicontinuity, the uniform upper bound on $v^{(n)}$ in $C^0(H^s) \cap Y^{s,b}$, and (9.4) together ensure (possibly after passage to the limit of some subsubsequence) that $u \in C^0(H^{s'}) \cap L^\infty(H^s) \cap Y^{s,b}$. It follows likewise that $u(0,\cdot) \equiv u_0(\cdot)$. The proof that the limit of some subsequence actually belongs to $C^0(H^s)$ will be completed in Section 10.

It remains to show that u is a weak solution of the equation. To simplify notation, denote the nonlinearity by $\mathcal{N}(v) := |v|^2 v$. It follows directly from the above convergence that $\mathcal{N}(v^{(j)}_{N;\,\mathrm{low}})$ converges to $\mathcal{N}(u_{N;\,\mathrm{low}})$ in $C^0(C^\infty_{\mathrm{loc}})$ for every N.

For any $\varepsilon > 0$ there exists N such that

$$\big\|\mathcal{N}\big(v^{(j)}\big) - \mathcal{N}\big(v^{(j)}_{N;\,\mathrm{low}}\big)\big\|_{Y^{s',b-1}} \leqslant \varepsilon \quad \text{for all } j. \tag{9.5}$$

This follows from the basic trilinear estimate, Proposition 7.1, since $v^{(j)}_{N;\,\mathrm{high}}$ is arbitrarily small in $Y^{s',b}$ provided N is sufficiently large, while the low part is bounded uniformly in N. Likewise $\mathcal{N}(u) - \mathcal{N}(u_{N;\,\mathrm{low}})$ is $\leqslant \varepsilon$ in $Y^{s',b-1}$ for all j.

These conclusions together imply that $\mathcal{N}(v^{(j)}) \to \mathcal{N}(u)$ in the topology of $\mathcal{D}'$. Since $v^{(j)}$ is a solution of (NLS), it follows that u is likewise a solution. □

10. Continuity in time

Since weak limits cannot be taken directly in spaces $C^0(H^s)$, some additional argument is needed to ensure that the weak limits constructed above do belong to these spaces. In this section we bridge that gap by establishing a certain limited equicontinuity with respect to time.

Recall the expressions $\Phi_\varphi(t,u) = \int_{\mathbb{R}} |\hat{u}(t,\xi)|^2 \varphi(\xi)\,d\xi$. Additional control on the solution u can obtained by analyzing $\Phi_\varphi(t,u)$ for weights φ which are more general than $\langle\xi\rangle^{2s}$, and are specifically adapted to the initial datum u_0 (cf. the "frequency envelopes" used for instance in [12]). We have actually proved the following statement more general than that announced earlier.

Lemma 10.1. *Let $s > -\frac{1}{12}$ and $s' \in (s,0)$. For any nonnegative C^2 weight function φ satisfying*

$$\varphi(\xi) \leqslant \langle\xi\rangle^{2s'}, \qquad \varphi'(\xi) \leqslant \langle\xi\rangle^{2s'-1}, \qquad \varphi''(\xi) \leqslant \langle\xi\rangle^{2s'-2}, \tag{10.1}$$

for any initial datum $u_0 \in H^0$, the standard solution $u(t,x)$ of (NLS) *satisfies*

$$\big|\Phi_\varphi(t,u) - \Phi_\varphi(0,u)\big| \leqslant C\|u\|^4_{Y^{s,b}}. \tag{10.2}$$

From this can be extracted a high-frequency continuity result.

Lemma 10.2. *Let $s > -\frac{1}{12}$. Let $u_0 \in H^s$ and $\varepsilon > 0$ be given. There exist $\delta > 0$ and $N < \infty$ such that for all $v_0 \in H^0$ satisfying $\|v_0 - u_0\|_{H^s} < \delta$, the standard solution v of* (NLS) *with initial datum v_0 satisfies*

$$\int_{|\xi| \geqslant N} \big|\hat{v}(t,\xi)\big|^2 \langle\xi\rangle^{2s}\,d\xi < \varepsilon \tag{10.3}$$

for all $t \in [0,T]$.

Here the timespan $T \in (0, \infty)$ is fixed, and it is assumed that $\|u_0\|_{H^s}$ is sufficiently small that the proof of Theorem 1.1 applies to all smooth solutions with initial data satisfying $\|v_0 - u_0\|_{H^s} \leqslant \delta_0$, where δ_0 depends on T.

Proof. Fix any exponent $s' \in (s, 0)$. Let $\varepsilon > 0$ be given. Choose $M < \infty$ so that $\int_{|\xi| \geqslant M} |\widehat{u_0}(\xi)|^2 \langle \xi \rangle^{2s}\, d\xi < \varepsilon^2$. Then there exist a large parameter $M' \geqslant M$ and a weight function φ satisfying the three inequalities hypothesized in Lemma 10.1, with exponent s', such that

$$\varepsilon^{-1} \langle \xi \rangle^{2s} \geqslant \varphi(\xi) \geqslant \langle \xi \rangle^{2s} \quad \text{for all } \xi, \tag{10.4}$$

$$\varphi(\xi) = \varepsilon^{-1} \langle \xi \rangle^{2s} \quad \text{for all } |\xi| \geqslant M', \tag{10.5}$$

$$\varphi(\xi) = \langle \xi \rangle^{2s} \quad \text{for all } |\xi| \leqslant M. \tag{10.6}$$

M', φ depend on ε and on s'. The conclusion (10.2) of Lemma 10.1 holds with a constant C independent of M, ε.

Thus by (10.2),

$$\begin{aligned} \int\limits_{|\xi| \geqslant M'} \bigl|\widehat{v_0}(\xi)\bigr|^2 \varphi(\xi)\, d\xi &\leqslant 2 \int\limits_{|\xi| \geqslant M'} \bigl|\widehat{u_0}(\xi)\bigr|^2 \varphi(\xi)\, d\xi + 2 \int\limits_{|\xi| \geqslant M'} \bigl|\widehat{v_0}(\xi) - \widehat{u_0}(\xi)\bigr|^2 \varphi(\xi)\, d\xi \\ &\leqslant 2\varepsilon + C_\varphi \|v_0 - u_0\|_{H^s}^2 \end{aligned} \tag{10.7}$$

where C_φ depends on φ, hence ultimately on ε. Therefore there exists $\delta > 0$ such that

$$\int\limits_{|\xi| \geqslant M'} \bigl|\widehat{v_0}(\xi)\bigr|^2 \varphi(\xi)\, d\xi \leqslant 3\varepsilon \tag{10.8}$$

for every $v_0 \in H^0$ satisfying $\|v_0 - u_0\|_{H^s} < \delta$.

For such initial data v_0, the associated solutions v have uniformly bounded $Y^{s,b}$ norms, with a bound independent of ε, provided that δ is sufficiently small. Therefore by Lemma 10.1, $\Phi_\varphi(t, v) = \int_{\mathbb{R}} |\hat{v}(t, \xi)|^2 \varphi(\xi)\, d\xi$ is bounded by a finite constant independent of ε, M, M' uniformly for all $t \in [0, T]$. Therefore since $\langle \xi \rangle^{2s} \leqslant \varepsilon \varphi(\xi)$ for all $|\xi| \geqslant M'$,

$$\int\limits_{|\xi| \geqslant M'} \bigl|\hat{v}(t, \xi)\bigr|^2 \langle \xi \rangle^{2s}\, d\xi \leqslant \varepsilon \int\limits_{|\xi| \geqslant M'} \bigl|\hat{v}(t, \xi)\bigr|^2 \varphi(\xi)\, d\xi \lesssim \varepsilon, \tag{10.9}$$

provided that $t \in [0, T]$ and $\|v_0 - u_0\|_{H^s} < \delta$. $\square$

Thus if $u_0, v_0^{(j)}$ are initial data with $u_0 \in H^s$ and $v_0^{(j)} \in H^0$, and if $v^{(j)} \to u_0$ in the H^s norm, then the corresponding standard solutions $v^{(j)}$ form an equicontinuous family in $C^0(H^s)$. Therefore passage to the limit through an appropriate subsequence produces a solution in $C^0(H^s)$, satisfying the other conclusions of Theorem 1.2.

Remark 10.1. Lemma 10.2 has the following direct consequence. Let $s > -\frac{1}{12}$. If there exists $r > -\infty$ for which the solution mapping from datum to solution of (NLS) is continuous from H^s to $C^0([0, T], H^r)$, then the solution mapping is continuous from H^s to $C^0([0, T], H^s)$.

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On tracial approximation

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Abstract

Let $\mathscr{C}$ be a class of unital C*-algebras. The class $\mathrm{TA}\mathscr{C}$ of C*-algebras which can be tracially approximated (in the Egorov-like sense first considered by Lin) by the C*-algebras in $\mathscr{C}$ is studied (Lin considered the case that $\mathscr{C}$ consists of finite-dimensional C*-algebras or the tensor products of such with $\mathrm{C}([0,1])$). In particular, the question is considered whether, for any simple separable $A \in \mathrm{TA}\mathscr{C}$, there is a C*-algebra B which is a simple inductive limit of certain basic homogeneous C*-algebras together with C*-algebras in $\mathscr{C}$, such that the Elliott invariant of A is isomorphic to the Elliott invariant of B. An interesting case of this question is answered. In the final part of the paper, the question is also considered which properties of C*-algebras are inherited by tracial approximation. (Results of this kind are obtained which are used in the proof of the main theorem of the paper, and also in the proof of the classification theorem of the second author given in [Z. Niu, A classification of tracially approximately splitting tree algebra, in preparation] and [Z. Niu, A classification of certain tracially approximately subhomogeneous C*-algebras, PhD thesis, University of Toronto, 2005]—which also uses the main result of the present paper.)

Keywords: Elliott invariant; Tracial approximation

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1. Introduction

The Elliott program for the classification of amenable C*-algebras might be said to have begun with the K-theoretical classification of AF-algebras [8]. (This was closely related to the Bratteli diagram classification given earlier in [3], but differed in introducing the K-functor.) Since then (but only after a fifteen-year hiatus), many classes of amenable C*-algebras have been found to be classified by their Elliott invariants. Among them, one important class is the class of simple unital inductive limits of homogeneous C*-algebras (AH-algebras for short). In this paper, by a unital homogeneous C*-algebra, we refer to a C*-algebra which is isomorphic to

$$p\mathrm{M}_n\big(\mathrm{C}(X)\big)p$$

for some compact Hausdorff space X, and some projection p in $\mathrm{M}_n(\mathrm{C}(X))$. (Noted that these C*-algebras are exactly the homogeneous C*-algebras with trivial Dixmier–Douady class. See, for example, p. 14 of [1]. In general, a homogeneous C*-algebra may not have this form.) In [16] and [14], Elliott, Gong, and Li showed that C*-algebras in this class can be classified by their Elliott invariant, provided that the dimensions of the base spaces of their building blocks are uniformly bounded. (Such AH-algebras are referred as AH-algebras without dimension growth.) Many naturally arising C*-algebras—for instance, the irrational rotation C*-algebras [12]—are known to be AH-algebras without dimension growth. (Note that AH-algebras not in this class were constructed by Villadsen in [33] and [34].)

A very important axiomatic version of the classification of the AH-algebras without dimension growth was given by Huaxin Lin. Instead of assuming inductive limit structure, he started with a certain abstract approximation property, and showed that C*-algebras with this abstract approximation property and certain additional properties are AH-algebras without dimension growth [18, 20,21,23]. More precisely, Lin introduced the class of tracially approximate interval algebras—TAI-algebras for short. Recall that an interval algebra is a C*-algebra isomorphic to $F\otimes\mathrm{C}([0,1])$ for a finite-dimensional C*-algebra F. Then TAI-algebras are defined by the following.

Definition. A unital C*-algebra A is a TAI-algebra if for any $\varepsilon>0$, any finite subset $\mathcal{F}\subseteq A$, and any non-zero $a\in A^+$, there exist a non-zero projection $p\in A$ and a sub-C*-algebra $I\subseteq A$ such that I is an interval algebra, $1_I=p$, and for all $x\in\mathcal{F}$,

(1) $\|xp-px\|<\varepsilon$,
(2) there exists $x'\in I$ such that $\|pxp-x'\|<\varepsilon$, and
(3) $1-p$ is Murray–von Neumann equivalent to a projection in $\overline{aAa}$.

The classification theorem for TAI-algebras was given by Lin in [23]. (The second author also contributed to the final part of this work; see [26].)

Theorem. *(See [23].) Let A and B be two simple amenable unital TAI-algebras satisfying the UCT. Then A is isomorphic to B if the Elliott invariant of A is isomorphic to the Elliott invariant of B. Moreover, the isomorphism between the algebras can be chosen in such a way that it induces the given isomorphism between the invariants.*

Since AH-algebras without dimension growth are TAI-algebras by the work of Gong [16], and since AH-algebras exhaust the invariant for the class in question (see [32], combined with

Lemma 10.9 of [23]; see also Corollary 4.20 below), the TAI-algebras which are classified by the theorem above must be AH-algebras. Thus the theorem above also provides a method to verify whether a given C*-algebra is an AH-algebras. In this way, the higher-dimensional simple noncommutative tori were shown to be AT-algebras in [22] and [29], and the crossed-product C*-algebras arising from certain minimal homeomorphisms were shown to be AH-algebras in [24].

Motivated by Lin's work on AH-algebras and TAI-algebras, the second author succeeded on using the axiomatic approach to obtain a classification for certain simple inductive limits of subhomogeneous C*-algebras in his PhD thesis [27]. More precisely, C*-algebras which can be tracially approximated by splitting interval algebras—TASI-algebras for short—were introduced and studied in [27]. Under a certain assumption on the value of the invariant arising—to be shown now to be redundant!—these C*-algebras were classified. Recall that a splitting interval algebra is defined as follows.

Definition. Let k be a natural number and $(\bar{k}_i = \{k_{i1}, \ldots, k_{ij_i}\})_{i=0,1}$ be two partitions of k. (All numbers non-zero.) The *splitting interval algebra* $S(\bar{k}_0, \bar{k}_1)$ is defined as follows:

$$S(\bar{k}_0, \bar{k}_1) := \{ f \in \mathrm{M}_k(\mathrm{C}([0,1])); \ f(i) \in \mathrm{M}_{k_{i1}}(\mathbb{C}) \oplus \cdots \oplus \mathrm{M}_{k_{ij_i}}(\mathbb{C}), \ i = 0, 1 \}.$$

These building blocks were introduced by Su in his work [31], in which he classified simple unital inductive limits of splitting interval algebras with real rank zero. General unital simple inductive limits of splitting interval algebras were classified by Jiang and Su in [17]. These authors also pointed out that there exist simple inductive limits of splitting interval algebras with K_0-groups failing to have the Riesz decomposition property, which is one of the properties held by simple AH-algebras without dimension growth. (Such examples were also constructed by the first author in [11].)

The classification theorem of [25] or [27] was as the following.

Theorem. *(See [25,27].) Let A be an amenable simple separable TASI-algebra which satisfies the UCT. If there exists a C*-algebra B which is a simple unital inductive limit of splitting interval algebras together with certain basic homogeneous C*-algebras* (*specified in* [27]; *see also below, where we refer to these algebras as the Gong standard homogeneous algebras*)*, such that*

$$(\mathrm{K}_0(A), \mathrm{K}_0(A)^+, [1_A]_0, \mathrm{K}_1(A), T(A)) \cong (\mathrm{K}_0(B), \mathrm{K}_0(B)^+, [1_B]_0, \mathrm{K}_1(B), T(B)),$$

then $A \cong B$. Moreover, the $*$-isomorphism can be chosen to induce the given isomorphism between the invariants.

As one can see, the above classification theorem covers a priori only a subclass of TASI-algebras, namely, those with the same invariant as an inductive limit C*-algebra of certain kind. The main purpose of the present paper is to show that the assumption of the theorem is satisfied automatically.

Recall that the Gong standard homogeneous C*-algebras consist of

(1) matrix algebras over the C*-algebras of continuous functions on $T_{2,k}$, and
(2) matrix algebras over the C*-algebras of continuous functions on $S^1 \vee \cdots \vee S^1 \vee T_{3,k_i} \vee \cdots \vee T_{3,k_i}$,

where $T_{2,k}$ is the two-dimensional CW complex obtained by attaching a two-dimensional disk D to S^1 via a map S^1 ($\cong \partial D$) $\to S^1$ of degree k, and $T_{3,k}$ is the three-dimensional CW complex obtained by attaching a three-dimensional ball B to S^2 via a map S^2 ($\cong \partial B$) $\to S^2$ of degree k. (See [13,16], and [14].) We shall prove the following theorem in this paper:

Theorem A. *Let $\mathscr{S}$ be a class of splitting tree algebras containing the interval algebras, and let A be a simple separable C*-algebra in the class* $\mathrm{TA}\mathscr{S}$. *There exists a simple inductive limit C*-algebra B of C*-algebras in the class $\mathscr{S}'$ consisting of $\mathscr{S}$ together with the Gong standard homogeneous C*-algebras such that the Elliott invariant of A is isomorphic to the Elliott invariant of B.*

Hence with $\mathscr{S}$ the class of splitting interval algebras, we have that for any TASI-algebra, there is a simple inductive limit C*-algebras of certain building blocks which shares the same invariant. As a consequence, the class of all separable simple amenable TASI-algebras satisfying the UCT is classified by the Elliott invariant.

The proof of this theorem is quite different from the proof for TAI-algebras. For a unital simple TAI-algebra, the K_0-group is weakly unperforated and has the Riesz decomposition property, and the pairing map preserves extreme points. Thus, by the Effros–Handelman–Shen type theorem for simple AH-algebras without dimension growth given by Villadsen in [32], the invariants of such AH-algebras and the invariants of TAI-algebras coincide. However, we do not have an Effros–Handelman–Shen theorem for simple inductive limits of our building blocks (whether splitting tree algebras, or just splitting interval algebras).

The main argument of the decomposition theorem (Theorem A) uses the following local criterion (Lemma 3.1) to determine whether an ordered group is an inductive limit of certain building blocks.

Lemma. *Let $\mathcal{G}$ be a set of ordered groups closed under direct sums and containing the group of integers with the usual order, and assume that the positive cone of every group in $\mathcal{G}$ is finite generated. Then a countable ordered group G is an inductive limit of ordered groups in $\mathcal{G}$ if G has the following lifting property*:

For any $G_1 \in \mathcal{G}$, any ordered group homomorphism $\theta : G_1 \to G$ and any $\alpha \in \ker(\theta)$, there exist $G_2 \in \mathcal{G}$ and ordered homomorphisms $\iota : G_1 \to G_2$ and $\theta' : G_2 \to G$, such that $\iota(\alpha) = 0$ and the following diagram commutes:

$$\begin{tikzcd} G_2 \arrow[dr, "\theta'"] & \\ G_1 \arrow[u, "\iota"] \arrow[r, "\theta"'] & G. \end{tikzcd}$$

Let G be the ordered group $\mathrm{K}_0(A)$, where A is a simple C*-algebra which can be tracially approximated by splitting tree algebras (a class of subhomogeneous C*-algebras which contains the class of splitting interval algebras). We are going to show that the ordered group G satisfies the criterion above for the class $\mathcal{G}$ consisting of the ordered K_0-groups of certain splitting tree algebras, and hence the ordered group G is an inductive limit of ordered groups in $\mathcal{G}$. Furthermore, we show that the groups G_1 and G_2 in the lemma above can be chosen in such a way that a large piece of them comes from a splitting tree algebra sitting inside A, and the restrictions

of the lifting maps to these large pieces are induced by maps between the splitting tree algebras which come from the tracial approximation structure and the semiprojectivity of the splitting tree algebras. Therefore, we get an inductive system (not necessary unital) of certain splitting tree algebras sitting inside large pieces of A, such that the inductive system realizes most of the torsion free part of the K_0-group and the pairing map of A. Then, we shall put suitable Gong standard homogeneous C*-algebras into the inductive system to make it to be a unital inductive system, such that it realizes the K_1-group and the torsion part of K_0-group. Since these new building blocks only sit inside small pieces of the new inductive system, it will not change the pairing map. In this way, we construct an inductive limit C*-algebra using certain special building blocks, such that its Elliott invariant is isomorphic to that of A.

In the final section of this paper, we report some results on the tracial approximation by general C*-algebras, showing that certain properties of C*-algebras in a class $\mathscr{C}$ are possessed by simple C*-algebras which can be tracially approximated by C*-algebras in $\mathscr{C}$. More precisely, we investigate the following properties:

(1) being (stably) finite;
(2) having stable rank one;
(3) having at least one tracial state;
(4) the strict order on projections is determined by traces;
(5) any state on the K_0-group comes from a tracial state of the algebra;
(6) if the restriction of a tracial state to the order-unit K_0-group is the average of two distinct states on K_0, then it is the average of two distinct tracial states;
(7) the canonical map from the unitary group modulo the connected component containing the identity to the K_1-group being injective.

2. TAS-algebras

In this paper, $\mathscr{F}$ denotes the class of finite-dimensional C*-algebras, and $\mathscr{I}$ denotes the class of interval algebras (recall that an interval algebra is a C*-algebra isomorphic to $F \otimes \mathrm{C}([0,1])$ for a finite-dimensional C*-algebra F). If A is a unital C*-algebra, let us denote by $\mathrm{T}(A)$ the simplex of tracial states of A. If a, b are two elements of a C*-algebra A, $a =_\varepsilon b$ means that $\|a-b\| < \varepsilon$. If B is a subset of A, then for any $a \in A$ and $\varepsilon > 0$, the notation $a \in_\varepsilon B$ refers to the relation that there is an element $b \in B$ such that $\|b-a\| < \varepsilon$.

Recall that a splitting tree algebra is a certain subhomogeneous C*-algebra defined as follows.

Definition 2.1. (*See [31].*) Let T be a tree (as a topological space) with finitely many vertices $\{v_i\}_{i=1}^n$, k be a natural number, and $(\bar{k}_i = \{k_{i1}, \dots, k_{ij_i}\})_{i=1}^n$ be n partitions of k (with all numbers non-zero). The *splitting tree algebra* $S(\bar{k}_1, \dots, \bar{k}_n; T)$ is defined as follows:

$$S(\bar{k}_1, \dots, \bar{k}_n; T) := \bigl\{ f \in \mathrm{M}_k\bigl(\mathrm{C}(T)\bigr);\ f(v_i) \in \mathrm{M}_{k_{i1}}(\mathbb{C}) \oplus \cdots \oplus \mathrm{M}_{k_{ij_i}}(\mathbb{C}) \text{ for all } i \bigr\}.$$

Let us call the vertices $\{v_i\}$ the singular points of S. In the case that T consists of only two vertices, let us call S a *splitting interval algebra*.

Following the notion of Lin on the tracial approximation by interval algebras, let us consider tracial approximation by more general C*-algebras. Let $\mathscr{C}$ be a class of unital C*-algebras. Then

the class of C*-algebras which can be tracially approximated by C*-algebras in $\mathscr{C}$, denoted by TA$\mathscr{C}$, is defined as follows:

Definition 2.2. A unital C*-algebra A is said to belong to the class TA$\mathscr{C}$ if for any $\varepsilon > 0$, any finite subset $\mathcal{F} \subseteq A$, and any non-zero $a \in A^+$, there exist a non-zero projection $p \in A$ and a sub-C*-algebra $C \subseteq A$ such that $C \in \mathscr{C}$, $1_C = p$, and for all $x \in \mathcal{F}$,

(1) $\|xp - px\| < \varepsilon$,
(2) $pxp \in_\varepsilon C$, and
(3) $1 - p$ is Murray–von Neumann equivalent to a projection in $\overline{aAa}$.

Lemma 2.3. *If the class $\mathscr{C}$ is closed under tensoring with matrix algebras, or closed under taking unital hereditary sub-C*-algebras, then* TA$\mathscr{C}$ *is also closed under passing to matrix algebras or unital hereditary sub-C*-algebras.*

Proof. Let us only verify the lemma for unital hereditary sub-C*-algebras. The matrix algebras can be verified similarly.

Let $A \in$ TA$\mathscr{C}$, and consider the unital hereditary sub-C*-algebra eAe where e is a projection in A. Let us show that $eAe \in$ TA$\mathscr{C}$.

For any $\varepsilon > 0$ (without loss of generality, let us assume that $\varepsilon < 1/32$), any $a \in (eAe)^+$, and any finite subset $\mathcal{F}$ in the unit ball of eAe, since $A \in$ TA$\mathscr{C}$, there is a sub-C*-algebra C with $p = 1_C$ and $C \in \mathscr{C}$ such that for any $x \in \mathcal{F} \cup \{e\}$,

(1) $\|xp - px\| < \varepsilon$,
(2) $pxp \in_\varepsilon C$, and
(3) $1 - p$ is Murray–von Neumann equivalent to a projection in $\overline{aAa}$.

Therefore, there is a projection $e_2 \in eAe$ such that $\|e_2 - pep\| < 4\varepsilon$. Since $pep \in_\varepsilon C$, there is a projection $e_2' \in C$ such that $\|e_2 - e_2'\| < 8\varepsilon$. Therefore, there is a unitary u with $\|u\| < 16\varepsilon$ such that $e_2 = ue_2'u^*$.

Consider the sub-C*-algebra $C' := uCu^*$. We assert that $e_2C'e_2$ satisfies the lemma. Indeed, since $e_2 \in eAe$, we have that $e_2C'e_2 \in eAe$. For any $x \in \mathcal{F} \subset eAe$,

$$e_2xe_2 =_{4\varepsilon} (pep)x(pep) = (pep)pxp(pep) =_{4\varepsilon} e_2pxpe_2 =_{16\varepsilon} e_2u(pxp)u^*e_2 \in_\varepsilon e_2C'e_2,$$

therefore $e_2xe_2 \in_{33\varepsilon} e_2C'e_2$.

Moreover, for any $x \in \mathcal{F}$, we have

$$xe_2 =_{4\varepsilon} x(pep) =_\varepsilon x(pe) =_\varepsilon pepx =_{4\varepsilon} e_2x,$$

therefore, $\|xe_2 - e_2x\| < 10\varepsilon$. Since $\|(e - e_2) - (1 - p)e(1 - p)\| < 8\varepsilon$, the projection $e - e_2$ is Murray–von Neumann equivalent to a projection in $\overline{aAa}$. Therefore, the C*-algebra eAe is in the class TA$\mathscr{C}$. $\square$

The class TA$\mathscr{F}$ is the class of tracially AF C*-algebras [21], and TA$\mathscr{I}$ is the class of C*-algebras of tracial topological rank one (TAI-algebras) in sense of Lin [23]. If $\mathscr{S}$ denotes the class of finite direct sums of splitting tree algebras, for convenience, we shall refer to the C*-algebras in TA$\mathscr{S}$ as TAS-algebras.

The following lemma provides a criterion for the stable finiteness of simple separable C*-algebras in TA$\mathscr{C}$. (In fact, the lemma only uses the conditions (1) and (2) of Definition 2.2.)

Lemma 2.4. *Let $\mathscr{C}$ be a class of unital C*-algebras. Suppose that for any C*-algebra $C \in \mathscr{C}$, there is a unital $*$-homomorphism from C to a matrix algebra* $\mathrm{M}_k(\mathbb{C})$ (*with* $k > 0$). *Then any separable simple unital C*-algebra satisfying the conditions* (1) *and* (2) *of Definition* 2.2 *with respect to the class $\mathscr{C}$ can be unitally embedded into the asymptotic sequence algebra* $\prod_n \mathrm{M}_{k_n}(\mathbb{C})/\bigoplus_i \mathrm{M}_{k_n}(\mathbb{C})$ *for some sequence of positive integers k_n. In particular, it has at least one tracial state and is stably finite.*

Proof. Let A be a separable simple unital C*-algebra which satisfies the conditions (1) and (2) of Definition 2.2. Choose a countable dense subset $\mathcal{F} = \{a_1, a_2, \ldots\}$ of the unit ball of A with $a_1 = 1$, and set $\varepsilon_n = 1/2^n$ for $n = 1, 2, \ldots$. Applying the conditions (1) and (2) of Definition 2.2 to ε_n and $\mathcal{F}_n = \{a_1, \ldots, a_n\}$, we obtain a sub-C*-algebra $C_n \in \mathscr{C}$ and a projection $p_n = 1_{C_n}$ such that $p_n a_i p_n \in_{\varepsilon_n} C_n$ and $\|p_n a_i - a_i p_n\| < \varepsilon_n$ for all a_i with $1 \leqslant i \leqslant n$. Pick a unital $*$-homomorphism $\phi_n : C_n \to \mathrm{M}_{k_n}(\mathbb{C})$ for some k_n. By Arveson's extension theorem, there exists an extension of ϕ_n to a positive linear contraction from $p_n A p_n$ to the same matrix algebra; denote this still by ϕ_n. Then Φ_n, defined by $\Phi_n(a) = \phi_n(pap)$, $a \in A$, is a positive linear contraction from A to the matrix algebra $\mathrm{M}_{k_n}(\mathbb{C})$ with $\|\Phi_n(a_i a_j) - \Phi_n(a_i)\Phi_n(a_j)\| < 2\varepsilon_n$ and $\|\Phi_n(a_i{}^*) - \Phi_n(a_i)^*\| < 2\varepsilon_n$ for any $1 \leqslant i, j \leqslant n$.

Applying this procedure for each n, we obtain a sequence (Φ_n) of unital positive linear contractions from A to various matrix algebras—the sequence $(\mathrm{M}_{k_n}(\mathbb{C}))$—with the approximation properties as above. The unital map Φ from A to the asymptotic sequence algebra $\prod_n \mathrm{M}_{k_n}(\mathbb{C})/\bigoplus_n \mathrm{M}_{k_n}(\mathbb{C})$, which is induced by the Cartesian product of $(\Phi_1, \Phi_2, \ldots)$, is then a unital $*$-homomorphism. By simplicity of A, the $*$-homomorphism Φ maps A injectively into the asymptotic sequence algebra $\prod_n \mathrm{M}_{k_n}(\mathbb{C})/\bigoplus_n \mathrm{M}_{k_n}(\mathbb{C})$.

Since the asymptotic sequence algebra $\prod_n \mathrm{M}_{k_n}(\mathbb{C})/\bigoplus_n \mathrm{M}_{k_n}(\mathbb{C})$ has tracial states, the C*-algebra A (being a unital sub-C*-algebra) has at least one tracial state. Since A is simple, it follows that A is stably finite. $\square$

Corollary 2.5. *Any separable simple TAS-algebra can be embedded into the asymptotic sequence algebra* $\prod_n \mathrm{M}_{k_n}(\mathbb{C})/\bigoplus_n \mathrm{M}_{k_n}(\mathbb{C})$ *for some positive integers k_n. In particular, such a C*-algebra has at least one tracial state, and hence is stably finite.*

Proof. For any splitting tree algebra S and any point t in its spectrum, there is a unital $*$-homomorphism from S to a non-zero matrix algebra consisting of the evaluation map at the point t. Thus, the corollary follows from Lemma 2.4. $\square$

The following lemma concerns the sizes of the matrix algebras in Lemma 2.4.

Lemma 2.6. *Let A be a C*-algebra of Lemma* 2.4. *If A is not of type I, then the sizes of the matrix algebras* M_{k_n} *are unbounded.*

Proof. Suppose that this were not true. Then A can be embedded into an asymptotic sequence algebra $\prod_n \mathrm{M}_{k_n}(\mathbb{C})/\bigoplus_n \mathrm{M}_{k_n}(\mathbb{C})$ with $\{k_n\}$ uniformly bounded. But $\prod_n \mathrm{M}_{k_n}(\mathbb{C})/\bigoplus_n \mathrm{M}_{k_n}(\mathbb{C})$ is a type I C*-algebra (it is a quotient of the type I C*-algebra $\prod_n \mathrm{M}_{k_n}$). Therefore, A must be of type I, which is in contradiction with the assumption. $\square$

Recall that a C*-algebra has the property (SP) if every non-zero hereditary sub-C*-algebra of A contains a non-zero projection. Let $\mathscr{C}$ be a class of C*-algebras, and let $A \in \mathrm{TA}\mathscr{C}$. If A does not have the property (SP), then there is $a \in A^+$ such that the hereditary sub-C*-algebra $\overline{aAa}$ does not contain non-zero projection. Apply Definition 2.2 to any finite subset $\mathcal{F}$, any ε, and a, we conclude that A can be approximated by C*-algebras in $\mathscr{C}$. This observation will be used several times in Section 4 in which certain properties of C*-algebras in $\mathscr{C}$ are shown to be inherited by C*-algebras in the class $\mathrm{TA}\mathscr{C}$.

However, if only the conditions (1) and (2) of Definition 2.2 are satisfied with respect to splitting tree algebras, then the property (SP) holds automatically. More precisely, we have the following proposition.

Proposition 2.7. *If a separable simple unital C*-algebra A satisfies the conditions* (1) *and* (2) *of Definition* 2.2 *with respect to splitting tree algebras, then A has the property* (SP).

Proof. We must show that for any non-zero positive element a, there is a non-zero projection in $\overline{aAa}$. Given such an element a, without loss of generality, we may assume that $\|a\| = 1$.

For any $\varepsilon > 0$, there exists a sub-C*-algebra S with unit p (not necessarily equal to 1) such that there is an element $b \in S^+$ with $\|pap - b\| < \varepsilon$ by the conditions (1) and (2) of Definition 2.2. We assert that we may choose S and b such that for any irreducible representation π of S, the norm of $\pi(b)$ is greater than $1 - \varepsilon$.

Suppose that this were not true. Then there exists an $\varepsilon_0 > 0$ such that for any S_n and b_n obtained by the conditions (1) and (2) of Definition 2.2, there exists an irreducible representation π_n of S_n such that the $\|\pi_n(b_n)\| < 1 - \varepsilon_0$. By the proof of Lemma 2.4, these irreducible representations induce a unital $*$-homomorphism Φ from A to an asymptotic sequence algebra $\prod_n \mathrm{M}_{k_n} / \bigoplus_n \mathrm{M}_{k_n}$, sending a to the quotient of the Cartesian product $(\pi_1(b_1), \pi_2(b_2), \ldots)$. Since $\|\pi_n(b_n)\| < 1 - \varepsilon_0$ for any n, it follows that $\|\Phi(a)\| \leqslant 1 - \varepsilon_0$. But since A is simple, the unital map Φ must be injective, and thus is isometric. This is a contradiction to the assumption that $\|a\| = 1$.

Therefore, we may assume that for any $\varepsilon > 0$, there exist a sub-C*-algebra S of A with unit p, and $b \in S^+$ such that $\|pap - b\| < \varepsilon$, $\|pa - ap\| < \varepsilon$, and the norm of b is greater than $1 - \varepsilon$ pointwisely on the spectrum of S. By perturbation methods as for interval algebras [10], we may assume that b does not have multiple eigenvalues in each canonical quotient of S. Therefore there exists a projection (a spectral projection) $q \in S$ such that $\|qb - q\| < \varepsilon$, and hence $\|qpap - q\| < 2\varepsilon$. Since $\|pa - ap\| < \varepsilon$, we have that $\|q(pap) - qpa\| \leqslant 3\varepsilon$. Thus we have that $\|q - qpa\| \leqslant 4\varepsilon$, and $\|q - apqpa\| \leqslant 8\varepsilon$. Since q is a projection, the element $apqpa \in \overline{aAa}$ has disconnected spectrum when $\varepsilon < 1/16$. By functional calculus, there is a non-zero projection in $\overline{aAa}$. $\square$

Let (G, G^+, u) be an order-unit group. Denote by $\mathrm{S}(G, G^+, u)$ the convex set of order-unit group homomorphisms from (G, G^+, u) to $(\mathbb{R}, \mathbb{R}^+, 1)$. It is compact with respect to the pointwise convergence topology. In the case that there is no confusion, we write $\mathrm{S}(G)$ for $\mathrm{S}(G, G^+, u)$ in short. Any element in $\mathrm{S}(G, G^+, u)$ is called a *state* of the order-unit group (G, G^+, u). The real-valued affine functions on the convex set $\mathrm{S}(G, G^+, u)$, denoted by $\mathrm{Aff}(\mathrm{S}(G, G^+, u))$, is an order-unit vector space with respect to positive functions as the positive cone and the constant

function $\mathbf{1}$ as the order unit. Note that there is a canonical order-unit homomorphism ρ from (G, G^+, u) to $\mathrm{Aff}(\mathrm{S}(G, G^+, u))$, defined by

$$\rho(g)(s) = s(g)$$

for any $g \in G$ and $s \in \mathrm{S}(G)$.

Let A be a unital stably finite C*-algebra. Then $(\mathrm{K}_0(A), \mathrm{K}_0^+(A), [1]_0)$ is an ordered-unit group (see [2]). Moreover, any tracial state τ of A induces a state of $(\mathrm{K}_0(A), \mathrm{K}_0^+(A), [1]_0)$ by the restriction to projections of matrix algebras of A. This defines an affine map from $\mathrm{T}(A)$ to $\mathrm{S}(\mathrm{K}_0(A))$. We shall refer it as the pairing map (between the simplex of traces and the order-unit K_0-group), and denote it by r_A. Then the *Elliott invariant* of A is defined as the tuple

$$\big((\mathrm{K}_0(A), \mathrm{K}_0^+(A), [1]_0), \mathrm{K}_1(A), \mathrm{T}(A), r_A\big).$$

This invariant has been shown to be the complete invariant for simple unital AH-algebras without dimension growth (the same class as the simple separable unital amenable TAI-algebras satisfying the UCT). (See [14,16], and [23].)

3. The Elliott invariants of TAS-algebras

Let $\mathscr{S}$ denote a class of splitting tree algebras which contains all interval algebras and is closed under taking finite direct sums. For example, $\mathscr{S}$ may denote the class of all splitting interval algebras. Any C*-algebra in the class $\mathscr{S}$ can be generated by a finite subset with respect to stable relations (see [7]). (Explicit generating sets and relations for certain splitting tree algebras are found in [28].)

As shown in Section 4, any simple separable C*-algebra A in $\mathrm{TA}\mathscr{S}$ has stable rank one (see Corollary 4.4), and thus has cancellation property for equivalent class of projections. Moreover, the strict order on projections of A is determined by traces, and thus A has weakly unperforated ordered K_0-group (see Corollary 4.14).

We shall investigate the Elliott invariant of C*-algebras in the class $\mathrm{TA}\mathscr{S}$, showing that for any simple separable C*-algebra $A \in \mathrm{TA}\mathscr{S}$, there exists a simple C*-algebra B, of an inductive limit of $\mathscr{S}'$ contaning $\mathscr{S}$ and the Gong standard homogeneous C*-algebras, such that the Elliott invariant of A is isomorphic to the Elliott invariant of B.

Fix a simple separable C*-algebra $A \in \mathrm{TA}\mathscr{S}$ be. Let $(\mathcal{F}_n)$ be an increasing sequence of finite subsets of A with dense union, and let (ε_n) be a decreasing sequence of positive numbers converging to zero. Since A belongs to the class $\mathrm{TA}\mathscr{S}$, there is a sub-C*-algebra S_1 of A in the class $\mathscr{S}$, such that if p_1 denotes 1_{S_1}, then for any $a \in \mathcal{F}_1$,

(1) $\|p_1 a - a p_1\| < \varepsilon_1$,
(2) $p_1 a p_1 \in_{\varepsilon_1} S_1$, and
(3) $\tau(1_A - p_1) < \varepsilon_1$ for any $\tau \in \mathrm{T}(A)$.

Let S_1' denotes the finite set consisting of b.

We may assume that $\mathcal{F}_2$ is sufficently large such that $\mathcal{F}_2$ contains a generating set of S_1. Since A belongs to the class $\mathrm{TA}\mathscr{S}$, for any $\varepsilon' > 0$, there is a C*-algebra S_2 of A which belongs to the class $\mathscr{S}$, such that if p_2 denotes 1_{S_2}, then for any $a \in \mathcal{F}_2$,

(1) $\|p_2a - ap_2\| < \varepsilon'$,
(2) $p_2ap_2 \in_{\varepsilon'} S_2$, and
(3) $\tau(1_A - p_2) < \varepsilon'$ for any $\tau \in \mathrm{T}(A)$.

We may assume that ε' is sufficiently small such that there is a $*$-homomorphism $\phi_1 : S_1 \to S_2$ satisfying

$$\left\|\phi_1(b) - p_2bp_2\right\| \leqslant \varepsilon_2 \quad \text{for any } b \in S_1'.$$

Repeating this procedure, we obtain an inductive system (not necessarily unital) (S_n, ϕ_n). Denote by S the inductive limit C*-algebra of (S_n, ϕ_n). Since each S_n are unital, S is a stably finite C*-algebra with an approximate unit $\{e_n\}$ consisting of projections. Denote by $\mathrm{T}_{u'}(S)$ the set of traces τ on S satisfying $\sup_n \tau(e_n) = 1$. It is a Choquet simplex. Also denote by $\mathrm{S}_{u'}(\mathrm{K}_0(S))$ the set of the positive homomorphisms s from $\mathrm{K}_0(S)$ to $\mathbb{R}$ such that $\sup_n s([e_n]) = 1$. The same argument as that of Lemma 10.8 of [23] shows that there exist affine continuous isomorphisms $t : \mathrm{T}_{u'}(S) \to \mathrm{T}(A)$ and $s : \mathrm{S}_{u'}(\mathrm{K}_0(S)) \to \mathrm{S}(\mathrm{K}_0(A))$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{S}(\mathrm{K}_0(A)) & \mathrm{T}(A) \arrow[l, "r_A"'] \\
\mathrm{S}_{u'}(\mathrm{K}_0(S)) \arrow[u, "s"] & \mathrm{T}_{u'}(S). \arrow[l, "r_S"] \arrow[u, "t"']
\end{tikzcd}$$

In the following part of this section, we shall construct a new inductive system $(C_n \oplus S_n, \phi_n')$ based on (S_n, ϕ_n) where C_n are the Gong standard homogeneous C*-algebras (see 3.3), such that $B := \varinjlim(C_n \oplus S_n)$ is simple and the K-theory of B is isomorphic to that of A. Moreover, the C*-algebras C_n and the maps ϕ_n' are chosen in such a way that $\mathrm{T}(B)$ is isomorphic to $\mathrm{T}_{u'}(S)$, $\mathrm{S}(\mathrm{K}_0(B))$ is isomorphic to $\mathrm{S}_{u'}(\mathrm{K}_0(S))$ and hence isomorphic to $\mathrm{S}(\mathrm{K}_0(A))$, and the map $r_B : \mathrm{T}(B) \to \mathrm{S}(\mathrm{K}_0(B))$ is compatible with the map r_s. Moreover, the isomorphism between $\mathrm{S}(\mathrm{K}_0(B))$ and $\mathrm{S}(\mathrm{K}_0(A))$ is also compatible with the isomorphism between the K_0-groups. Thus the pairing maps between the simplex of traces and the ordered K_0-group of A is isomorphic to that of B, and the Elliott invariant of A is isomorphic to the Elliott invariant of B.

3.1. *Two ordered group lemmas*

In analogy with Chao-Liang Shen's local criterion for dimension groups, we have a local criterion to determinate whether an ordered group is an inductive limit of certain basic building blocks. Let $\mathcal{G}$ be a set of ordered groups closed under direct sum and containing the group of integers with the usual order. Furthermore, let us assume that the positive cone of every group in $\mathcal{G}$ is finitely generated. Then we have

Lemma 3.1. *A countable ordered group G is an inductive limit of ordered groups in $\mathcal{G}$ if G has the following property*:

For any $G_1 \in \mathcal{G}$, any ordered group homomorphism $\theta : G_1 \to G$ and any $\alpha \in \ker(\theta)$, there exist $G_2 \in \mathcal{G}$ and ordered group homomorphisms $\iota : G_1 \to G_2$ and $\theta' : G_2 \to G$, such that $\iota(\alpha) = 0$ and the following diagram commutes:

$$\begin{tikzcd} G_2 \arrow[dr, "\theta'"] & \\ G_1 \arrow[u, "\iota"] \arrow[r, "\theta"'] & G. \end{tikzcd}$$

Proof. The proof is similar to the case considered by Shen. Since G is countable, we may write $G^+ = \{g_1, g_2, \ldots\}$. To prove G is an inductive limit of $\mathcal{G}$, it is enough to construct the following commutative diagram:

$$\begin{tikzcd} G_1 \arrow[r, "\theta_1"] \arrow[d, "\iota_1"'] & G \arrow[d, "\mathrm{id}"] \\ G_2 \arrow[r, "\theta_2"] \arrow[d, "\iota_2"'] & G \arrow[d, "\mathrm{id}"] \\ G_3 \arrow[r, "\theta_3"] \arrow[d, "\iota_3"'] & G \arrow[d, "\mathrm{id}"] \\ \vdots & \vdots \end{tikzcd}$$

where θ_i and ι_i are positive homomorphisms such that $\ker(\theta_i) \subseteq \ker(\iota_i)$ and $\bigcup_i G_i^+ = G^+$.

Take $G_1 = \mathbb{Z}$ and define $\theta_1 : n \mapsto n g_1$. Now, assume the diagram is well defined for G_n with $\theta_n : G_n \to G$. Then, take $\iota' : G_n \to G_n \oplus \mathbb{Z}$ by $s \mapsto (s, 0)$, and $\kappa : G_n \oplus \mathbb{Z} \to G$ by $(s, m) \mapsto \theta_n(s) + m g_n$. Since $G_n \oplus \mathbb{Z} \in \mathcal{G}$, there are an $G_{n+1} \in \mathcal{G}$ and a positive homomorphisms $\theta_{n+1} : G_{n+1} \to G$, $\iota'' : G_n \oplus \mathbb{Z} \to G_{n+1}$ such that $\ker(\kappa) \subseteq \ker(\iota'')$ and we have the following commutative diagram:

$$\begin{tikzcd} G_n \arrow[rr, "\theta_n"] \arrow[dr, "\iota'"'] \arrow[dd, "\iota_n = \iota'' \circ \iota'"'] & & G \arrow[dd, "\mathrm{id}"] \\ & G_n \oplus \mathbb{Z} \arrow[ur, "\kappa"'] \arrow[dl, "\iota''"] & \\ G_{n+1} \arrow[rr, "\theta_{n+1}"'] & & G. \end{tikzcd}$$

It is easy to verify that $\ker(\theta_n) \subseteq \ker(\iota_n)$. Therefore, the ordered group G is isomorphic to the inductive limit of $\{G_n, \iota_n\}$. $\square$

Recall that if $S(\bar{k}_1, \ldots, \bar{k}_n; T)$ is a splitting tree algebra defined by Definition 2.1, its K_0-group can be described as

$$\left\{ (m_1, \ldots, m_n) \in \bigoplus_{i=1}^{n} \mathbb{Z}^{|\bar{k}_i|};\ \sum_i m_1^{(i)} = \cdots = \sum_i m_n^{(i)} \right\}$$

with the usual order. (See [31].) A map $r:\mathrm{K}_0(S)\to\mathbb{Z}$ is called a *point evaluation map* if it is induced by a point evaluation map $S\to\mathrm{M}_n(\mathbb{C})$ for some n. By the *point evaluation map on a regular point*, we refer to the positive map $\mathrm{K}_0(S)\to\mathbb{Z}$ defined by

$$(m_1,\dots,m_n)\mapsto\sum_{i=1}^{|\bar{k}_1|}m_1^{(i)}.$$

The K_0-groups of splitting tree algebras are not necessarily dimension groups (the Riesz decomposition may fail); however, certain positive homomorphisms factor through dimension groups, provided these maps have a large corner factoring though dimension groups. More precisely, we have the following lemma.

Lemma 3.2. *Let $G=\mathrm{K}_0(S)$ where S is a splitting tree algebra, and let $r:G\to\mathbb{Z}$ be the point evaluation map on a regular point. Then there exist $u\in G^+$ and a natural number m such that, if $\theta:G\to G$ is defined by $g\mapsto r(g)u$, the positive homomorphism $\mathrm{id}+m\theta:G\to G$ factors through $\bigoplus_1^n\mathbb{Z}$ for some n.*

Proof. The group G is torsion free and the positive cone of G is finite generated. Therefore, there is a basis of G (as an abelian group) consisting of the positive elements. Denote it by $\{u_1,\dots,u_n\}$. Then, there is an isomorphism ψ from G to $\bigoplus_n\mathbb{Z}$ (as groups) sending $\{u_i\}$ to the canonical basis of $\bigoplus_n\mathbb{Z}$. This isomorphism may not be a positive homomorphism. But its inverse ψ^{-1} is positive.

Define a positive map $\theta_2:\mathbb{Z}\to G$ by sending 1 to $u:=u_1+\cdots+u_n$, and define $\theta=\theta_2\circ r$. Since $\psi(u)=(1,1,\dots,1)$, we have that for any positive element g in G, there is a natural number m_g such that $\psi(g)+m_g\psi\circ\theta(g)$ is positive in $\bigoplus_n\mathbb{Z}$ (with the usual order). Since the positive cone of G is finite generated, there is a natural number m such that $\psi(g)+m\psi\circ\theta(g)$ is positive for any $g\in G^+$. Consider the positive map $\phi=\psi+m\psi\circ\theta$ from G to $\bigoplus_n\mathbb{Z}$. Then the diagram

$$\begin{tikzcd} G \arrow[rr, "\mathrm{id}+m\theta"] \arrow[dr, "\phi"'] & & G \\ & \bigoplus_n\mathbb{Z} \arrow[ur, "\psi^{-1}"'] & \end{tikzcd}$$

commutes, as desired. □

3.2. The K_0*-groups of C*-algebras in* $\mathrm{TA}\mathscr{S}$

Let A be a simple separable C*-algebra in the class $\mathrm{TA}\mathscr{S}$. We shall prove that $\mathrm{K}_0(A)$ is an inductive limit of certain basic building blocks. Let $\mathcal{KS}$ denote the class of ordered K_0-groups of C*-algebras in $\mathscr{S}$, and let $\mathcal{ZT}$ denote the class of finite direct sums of ordered groups

$$\mathbb{Z}\oplus\mathbb{Z}/k\mathbb{Z},\quad k=1,2,\dots,$$

with respect to the order

$$(m,n) > 0 \quad \text{if and only if} \quad m > 0.$$

Denote by $\mathcal{K}$ the class of finite direct sums of ordered groups in $\mathcal{KS}$ and ordered groups in $\mathcal{ZT}$. We also refer to the ordered groups in $\mathcal{K}$ as basic building blocks (of ordered groups). We shall prove the following proposition in this subsection:

Proposition 3.3. *Let $H = \mathrm{K}_0(A)$ for some simple separable C*-algebra $A \in \mathrm{TA}\mathscr{S}$. Then, for any $G_0 \in \mathcal{K}$, any positive homomorphism $\theta : G_0 \to H$, and any $\alpha \in \ker(\theta)$, there exist $G_1 \in \mathcal{K}$, positive homomorphism $\iota : G_0 \to G_1$, and positive homomorphism $\theta' : G_1 \to H$, such that $\iota(\alpha) = 0$ and the following diagram commutes*:

$$\begin{tikzcd} G_1 \arrow[dr, "\theta'"] & \\ G_0 \arrow[u, "\iota"] \arrow[r, "\theta"'] & H. \end{tikzcd}$$

Moreover, if $G_0 = G_0' \oplus G_0''$, where G_0'' is the K_0-group of a sub-C-algebra S_n of A as described at the beginning of this section and the restriction of θ to G_0'' is induced by the inclusion map, and $u \in G_0$ is a positive element with $\theta(u) = [1_A]$, then, for any natural number N and $\varepsilon > 0$, there is a sub-C*-algebra S_k of A as described at the beginning of this section with k sufficiently large, such that G_1 can be chosen as $G_1' \oplus \mathrm{K}_0(S_k)$, where G_1' is an ordered group in $\mathcal{ZT}$. If denoted by u' the restriction of $\iota(u)$ to G_1' and u'' the restriction of $\iota(u)$ to $\mathrm{K}_0(S_k)$, we have that $u'' = [1_{S_k}]$ and $N\theta'(u') < \theta'(u'')$ in $\mathrm{K}_0(A)$. Furthermore, we may assume that $\tau(1_A - 1_{S_k}) < \varepsilon$ for all $\tau \in \mathrm{T}(A)$ and the restriction of ι to G_0'' and $\mathrm{K}_0(S_k)$ can be chosen to be induced by the $*$-homomorphism $\phi_{n,k}$ from S_n to S_k.*

As a straightforward consequence of the first part of this proposition, a certain inductive limit decomposition of the K_0-groups of a simple separable C*-algebra in $\mathrm{TA}\mathscr{S}$ is obtained.

Corollary 3.4. *Let H be the K_0-group of a simple separable C*-algebra in $\mathrm{TA}\mathscr{S}$. Then H is an inductive limit of K_0-groups of splitting tree algebras in the class $\mathscr{S}$ and ordered groups $\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$, $n = 1, 2, \ldots$.*

Proof. It follows from Lemma 3.1 and the first part of Proposition 3.3. $\square$

The proof of Proposition 3.3 needs several lemmas.

Let H be an ordered group with positive cone H^+. Denote by H_{tor} the subgroup of torsion elements of H. Then, one has that $H^+ \cap H_{\mathrm{tor}} = \{0\}$. It follows that the quotient group H/H_{tor}, with the image of H^+ as the positive cone, is an ordered group.

Lemma 3.5. *Let $G \in \mathcal{KS}$ and $H = \mathrm{K}_0(A)$ for a simple C*-algebra $A \in \mathrm{TA}\mathscr{S}$. Then, for any positive homomorphism $\theta : G \to H/H_{\mathrm{tor}}$ and an $\alpha \in \ker(\theta)$, there exist positive homomorphisms θ_1 and θ_2 from G to H/H_{tor} such that $\theta = \theta_1 + \theta_2$ and the following diagrams commute*:

$$\begin{tikzcd} G_1 \arrow[dr, "\psi_1"] & \\ G \arrow[u, "\phi_1"] \arrow[r, "\theta_1"'] & H/H_{\mathrm{tor}} \end{tikzcd} \qquad \begin{tikzcd} G_2 \arrow[dr, "\psi_2"] & \\ G \arrow[u, "\phi_2"] \arrow[r, "\theta_2"'] & H/H_{\mathrm{tor}} \end{tikzcd}$$

where $G_1 = \bigoplus_k \mathbb{Z}$ for some natural number k, $G_2 \in \mathcal{KS}$ and $\phi_2(\alpha) = 0$. Moreover, the positive homomorphism ψ_2 can be lifted to a positive homomorphism to H.

Furthermore, if $G = G' \oplus G''$, where G'' is the K_0-group of a sub-C-algebra S_n of A as described at the beginning of this section and the restriction of θ to G'' is induced by the inclusion map, and $u \in G$ is a positive element with $\theta(u) = [1_A]$, then, for any natural number N and $\varepsilon > 0$, the group G_2 can be chosen to be the K_0-group of a sub-C*-algebra S_k of A as described at the beginning of this section with $\tau(1_A - 1_{S_k}) < \varepsilon$ for every $\tau \in \mathrm{T}(A)$, and if let $u_1 = \phi_1(u)$ and $u_2 = \phi_2(u)$, we have that $N\psi_1(u_1) < \psi_2(u_2)$ in the ordered group H/H_{tor}. Moreover, the restriction of the map ϕ_2 to G'' can be chosen to be induced by the $*$-homomorphism $\phi_{n,k}$ from S_n to S_k.*

Proof. We may assume that θ always send a non-zero positive element in G to a non-zero element in H/H_{tor}. Otherwise, the positive map θ factors through a quotient group of G which still belongs to $\mathcal{KS}$, and we then can investigate the map from this quotient group to H/H_{tor}. Therefore, we may assume that neither α nor $-\alpha$ is positive.

Step 1. We construct a point evaluation map θ' from G to H/H_{tor} which sends α to 0, and θ' is small enough such that $\theta - \theta'$ is still a positive homomorphism.

Let $r_1, \ldots, r_k$ be the point evaluation maps on regular points of each simple summand of the group G. Denote by m the integer $r_1(\alpha) + \cdots + r_k(\alpha)$. If $m \geqslant 0$, one then chooses a singular point t_0 of α such that the value of α at t_0 is negative (we can do this since $\alpha \notin G^+$). Let s denote the point evaluation at t_0, and let $-n = s(\alpha)$. Since the positive cone of G is finite generated and A has the property (SP), there is $h \in H/H_{\mathrm{tor}}$ such that $(m + n)h$ is less than the images of the generators of G^+. Then, we define the positive homomorphism $\theta' : G \to H/H_{\mathrm{tor}}$ by

$$\theta' : g \mapsto s(g)mh + \sum_{i=1}^{k} r_i(g)nh.$$

It is easy to see $\theta - \theta'$ is a positive homomorphism and $\theta'(\alpha) = 0$. Since $\theta(u) = [1_A]$, we can also choose h sufficiently small such that $4\rho(\theta'(u)) \leqslant \min\{\varepsilon, 1/N\}$ for each state ρ of H/H_{tor}.

If $m < 0$, a construction similar to the above also gives us a desired positive homomorphism θ'.

Step 2. Denote by φ the map $\theta - \theta'$. Since $\theta'(\alpha) = 0$, we have that $\varphi(\alpha) = 0$. Apply Lemma 3.2 to each direct summand of G to obtain an integer m_i and a positive element u_i for each $i = 1, \ldots, k$, such that the positive map $(\mathrm{id} + \sum_{i=1}^{k} m_i r_i u_i)$ factors through a finite direct sum of the group of integers.

Since the positive cone of G is finitely generated, one can denote the images of these generators in H/H_{tor} by $\{p_1, \ldots, p_m\}$. Choose a pre-image q_i in $H = \mathrm{K}_0(A)$ for each p_i. Since A is a TAS-algebra, one has that

$$q_i = q_i' + q_i'', \quad i = 1, \ldots, m,$$

where each q_i'' is in the K_0-group of a splitting tree algebra S_k as described at the beginning of this section. Since the positive cone of the group G is generated by finite relations and the K_0-groups of splitting tree algebras are torsion free, with a suitable choosing of the splitting tree algebra S_k (choose S_k far enough), the map sending each generator of G to the corresponding q'' induces a positive homomorphism from G to H and sends α to 0. Denote this positive homomorphism by ϕ_2, and denote by θ_2 the positive map from G to H/H_{tor} induced by ϕ_2. The map sending the generators of G to corresponding q_i' may not extend to a positive map from G to H, but it induces a positive map to H/H_{tor} by passing to the quotient. Denote this positive map by θ_1'. Therefore, we get a decomposition $\varphi = \theta_1' + \theta_2$, where θ_1' and θ_2 are positive homomorphisms from G to H/H_{tor} which satisfy $\theta_1'(\alpha) = 0$, $\theta_2(\alpha) = 0$, and θ_2 factors through G_2, the K_0-group of the splitting tree algebra S_k. Thus, one has the commutative diagram

$$\begin{tikzcd} G \arrow[rr, "\theta_2"] \arrow[dr, "\phi_2"'] & & H/H_{\mathrm{tor}} \\ & G_2 \arrow[ur, "\psi_2"'] & \end{tikzcd}$$

with $\phi_2(\alpha) = 0$.

If $G = G' \oplus G''$ where G' is nonempty, G'' comes from a sub-C*-algebra S_n of A as described at the beginning of this section and the restriction of θ to G'' is induced by the inclusion map, one can choose the element h in Step 1 sufficiently small such that the map θ' is less that the restriction map of θ to the summand G'. Thus the restriction of the map φ to G'' is still induced by the inclusion map of S_n into A, and then we can choose the sub-C*-algebra S_k such that the restriction of the map ϕ_2 to G'' is induced by the $*$-homomorphism $\phi_{n,k}$ constructed at the beginning of this section.

Note that we may choose q_i' sufficiently small such that $m_i\theta_1'(u_i) < h$ and $4\rho(\theta_1'(u)) \leqslant \min\{\varepsilon, 1/N\}$ for all positive state ρ on the ordered group H/H_{tor}. Thus we have the decomposition

$$\theta = \theta' + \varphi = \big(\theta' + \theta_1'\big) + \theta_2,$$

and for any g in a direct summand of G, the following holds:

$$\begin{aligned} \theta'(g) + \theta_1'(g) &= s(g)mh + \sum_{i=1}^{k} r_i(g)nh + \theta_1'(g) \\ &= s(g)mh + \sum_{i=1}^{k} r_i(g)\big(nh - m_i\theta_1'(u_i)\big) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{k} r_i(g)m_i\theta_1'(u_i)+\theta_1'(g)\\
&= s(g)mh+\sum_{i=1}^{k} r_i(g)\big(nh-m_i\theta_1'(u_i)\big)\\
&+\theta_1'\left(\sum_{i=1}^{k} m_i r_i(g)u_i+g\right).
\end{aligned}
$$

The first two terms of the decomposition are point evaluation maps, and thus factor through a finite direct sum of the group of integers. By Lemma 3.2, the positive homomorphism $(\sum_{i=1}^{k} m_i r_i u_i + \mathrm{id})$ also factors through a finite direct sum of the group of integers. Therefore, the positive homomorphism $\theta' + \theta_1'$ factors through the group $G_1 = \bigoplus_k \mathbb{Z}$ for some k. Set $\theta_1 = \theta' + \theta_1'$. Then the positive homomorphisms θ_1 and θ_2 have the desired factorization property. □

Using a slightly modification of the argument of the lemma above, we have the following decomposition for the positive homomorphisms from an ordered group in $\mathcal{K}$ to $H = \mathrm{K}_0(A)$.

Lemma 3.6. *Let $G \in \mathcal{K}$ and $H = \mathrm{K}_0(A)$ for a simple C*-algebra $A \in \mathrm{TA}\mathscr{S}$. Then, for any positive homomorphism $\theta : G \to H$ and $\alpha \in \ker(\theta)$, there exist positive homomorphisms θ_1 and θ_2 from G to H such that $\theta = \theta_1 + \theta_2$ and the following diagrams commute*:

$$
\begin{tikzcd}
G_1 \arrow[dr, "\psi_1"] & \\
G \arrow[u, "\phi_1"] \arrow[r, "\theta_1"'] & H
\end{tikzcd}
\qquad
\begin{tikzcd}
G_2 \arrow[dr, "\psi_2"] & \\
G \arrow[u, "\phi_2"] \arrow[r, "\theta_2"'] & H
\end{tikzcd}
$$

where $G_1 \in \mathcal{ZT}$, $G_2 \in \mathcal{KS}$, and $\phi_2(\alpha) = 0$.

Furthermore, if $G = G' \oplus G''$, where G'' is the K_0-group of a sub-C-algebra S_n of A as described at the beginning of this section and the restriction of θ to G'' is induced by the inclusion map, and $u \in G$ is a positive element with $\theta(u) = [1_A]$, then, for any natural number N and $\varepsilon > 0$, the group G_2 can be chosen to be the K_0-group of a sub-C*-algebra S_k of A as described at the beginning of this section with $\tau(1_A - 1_{S_k}) < \varepsilon$ for each $\tau \in \mathrm{T}(A)$, and if we let $u_1 = \phi_1(u)$ and $u_2 = \phi_2(u)$, we have that $N\psi_1(u_1) < \psi_2(u_2)$ in the ordered group H. Moreover, the restriction of the map ϕ_2 to G'' can be chosen to be induced by the $*$-homomorphism $\phi_{n,k}$ from S_n to S_k.*

Proof. Write $G = G_{\mathrm{ZT}} \oplus G_{\mathrm{KS}}$ where $G_{\mathrm{ZT}} \in \mathcal{ZT}$ and $G_{\mathrm{KS}} \in \mathcal{KS}$. We consider the restriction of θ to G_{ZT}. Since the positive cone of G_{ZT} is finite generated and A is a simple C*-algebra in the class $\mathrm{TA}\mathscr{S}$, the map $G_{\mathrm{ZT}} \to H$ can be decomposed into the sum of positive maps κ_1 and κ_2, where the κ_2 factors through the K_0-group of a sub-C*-algebra of A in the class $\mathscr{S}$. Since the K_0-groups of splitting tree algebras are torsion free, the map κ_2 factors through a finite direct sum of the group of integers. Moreover, given G, $u \in G^+$, natural number N, and $\varepsilon > 0$ as the

second part of the lemma, we may choose κ_1 and κ_2 such that κ_2 factors through the K_0-group of S_k and $2N\kappa_1(u) < \kappa_2(u)$.

Note that the map θ can be decomposed as the sum of the map κ_1, the map κ_2, and the restriction of θ to G_{KS}. Since the map κ_1 is chosen to be sufficiently small, it is enough to prove the lemma for the sum of the map κ_2 and the map $G_{\mathrm{KS}} \to H$. Since the map κ_2 factors through a finite direct sum of the group of integers, it is enough to prove the lemma for $G \in \mathcal{KS}$. Then, a repeating of the argument of Lemma 3.5 gives a proof of the lemma (even more straightforward, one does not need to find liftings of positive elements as that in Step 2 of the proof of Lemma 3.5). $\square$

By Lemma 3.5 (or Lemma 3.6), the map θ can be lifted to a map $\phi_1 \oplus \phi_2$. However, we only have that $\phi_2(\alpha) = 0$. Thus, in order to determine whether the ordered group H (or H/H_{tor}) is an inductive limit of certain basic building blocks, we have to handle the map ψ_1. Note that the domain of the map ψ_1 is a finite direct sum of the group of integers. We can use certain arguments of the Effros–Handelman–Shen theorem in [6].

Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \bigoplus_n \mathbb{Z}$ for some natural number n. Set

$$m = \max\bigl\{|\alpha_1|, \ldots, |\alpha_n|\bigr\},$$

and l to be the number of $|\alpha_i|$'s equal to m. Then we define the *degree* of α to be the pair (m, l). Denote it by $\deg(\alpha)$. The degrees are well ordered by $(m, l) < (m', l')$ if $m < m'$ or $m = m'$ and $l < l'$. Note that $\deg(\alpha) = 0$ implies $\alpha = 0$.

Lemma 3.7. *Let $G \cong \bigoplus_n \mathbb{Z}$ for some natural number n. Let $H = \mathrm{K}_0(A)$ for a simple C*-algebra $A \in \mathrm{TA}\mathscr{S}$. Then, for any positive homomorphism $\theta : G \to H/H_{\mathrm{tor}}$ and $\alpha \in \ker(\theta)$, there exist an ordered group $(\bigoplus_m \mathbb{Z}) \oplus G_1$ with $G_1 \in \mathcal{KS}$ and positive homomorphisms ϕ_1, ϕ_2 and ψ such that the following diagram commutes*:

$$\begin{tikzcd} (\bigoplus_{\mathrm{m}} \mathbb{Z}) \oplus G_1 \arrow[dr, "\psi"] & \\ G \arrow[u, "\phi_1 \oplus \phi_2"] \arrow[r, "\theta"'] & H/H_{\mathrm{tor}}, \end{tikzcd}$$

such that $\phi_1(\alpha) = \phi_2(\alpha) = 0$. Moreover, G_1 comes from a sub-C-algebra of A which is a splitting tree algebra, and the restriction of ψ on G_1 is induced by the inclusion map.*

Furthermore, if the map θ is the quotient of a positive homomorphism $\theta' : G \to H$, the maps ϕ_1, ϕ_2 and the restrictions of ψ can be chosen to satisfy the following commutative diagrams:

$$\begin{tikzcd} \bigoplus_m \mathbb{Z} \arrow[drr, "\psi_1"] & & \\ G \arrow[u, "\phi_1"] \arrow[r, "\theta'_1"'] & H \arrow[r, "\pi"'] & H/H_{\mathrm{tor}} \end{tikzcd} \qquad \begin{tikzcd} G_1 \arrow[dr, "\psi_2"] & & \\ G \arrow[u, "\phi_2"] \arrow[r, "\theta'_2"'] & H \arrow[r, "\pi"'] & H/H_{\mathrm{tor}} \end{tikzcd}$$

where $\theta'_1 + \theta'_2 = \theta'$.

Proof. The proof follows the same line as the proof of the Effros–Handelman–Shen theorem in [6]. Let

$$\alpha = \sum_{i=1}^{r} m_i e_i - \sum_{j=1}^{s} n_j f_j, \quad m_i, n_j \in \mathbb{N},$$

where e_i $(1 \leqslant i \leqslant r)$, f_j $(1 \leqslant j \leqslant s)$ and g_k $(1 \leqslant k \leqslant t)$ $(r+s+t=n)$ are the standard basis of $\bigoplus_n \mathbb{Z}$.

Let $\theta(e_i) = a_i$ and $\theta(f_j) = b_j$. Assume $\deg(\alpha) = (m, l)$ and $m_1 = m$. Since $\theta(\alpha) = 0$, we have

$$\sum_{i=1}^{r} m_i a_i = \sum_{j=1}^{s} n_j b_j.$$

Therefore,

$$m_1 a_1 \leqslant \sum_{i=1}^{r} m_i a_1 = \sum_{j=1}^{s} n_j b_j \leqslant m_1 \left(\sum_{j=1}^{s} b_j \right).$$

Since $H = \mathrm{K}_0(A)$, the ordered group H/H_{tor} is unperforated. Hence we have

$$a_1 \leqslant \sum_{j=1}^{s} b_j.$$

Case 1. If $a_1 = \sum_{j=1}^{s} b_j$, we have that $r = 1$ and $n_j = m_1$ for each j. Then, we can define a positive homomorphism $\phi : \bigoplus_n \mathbb{Z} \to \bigoplus_{n-1} \mathbb{Z}$ by

$$\begin{aligned} e_1 &\mapsto \sum_{j=1}^{s} f'_j, \\ f_j &\mapsto f'_j \quad (1 \leqslant j \leqslant s), \\ g_k &\mapsto g'_k \quad (1 \leqslant k \leqslant t), \end{aligned}$$

where f'_j $(1 \leqslant j \leqslant s)$, g'_k $(1 \leqslant k \leqslant t)$ are the standard basis of $\bigoplus_{n-1} \mathbb{Z}$. We also define the positive homomorphism $\psi : \bigoplus_{n-1} \mathbb{Z} \to H/H_{\mathrm{tor}}$ by

$$\begin{aligned} f'_j &\mapsto b_j \quad (1 \leqslant j \leqslant s), \\ g'_k &\mapsto \theta(g_k) \quad (1 \leqslant k \leqslant t). \end{aligned}$$

A direct calculation shows that $\theta = \psi \circ \phi$. Moreover, we have

$$\phi(\alpha) = m_1 \sum_{j=1}^{s} f'_j - \sum_{j=1}^{s} n_j f'_j = m_1 \left(\sum_{j=1}^{s} f'_j - \sum_{j=1}^{s} f'_j \right) = 0.$$

Hence we get the desired lifting of θ (with $G_1 = \{0\}$).

Case 2. If $a_1 < \sum_{j=1}^{s} b_j$, we have that $r > 1$ or $n_j < m$ for some j. Let a_i''s and b_j''s be pre-images of a_i's and b_j's, respectively, in H. Let θ' be the lifting of θ according to a_i''s and b_j''s. We assert that there is a factorization of θ' as follows:

$$\begin{tikzcd} G_1 \oplus (\bigoplus_k \mathbb{Z}) \arrow[dr, "\psi"] & \\ G \arrow[u, "\phi_1 \oplus \iota"] \arrow[r, "\theta'"'] & H, \end{tikzcd}$$

where $G_1 \in \mathcal{KS}$ comes from a sub-C*-algebra of A, $\phi_1(\alpha) = 0$. Moreover, $\deg(\iota(\alpha)) < \deg(\alpha)$.

To show this, we shall use the same technique of Lemma 3.5. Let $d_1 = \sum_{i=1}^{r} m_i$ and $d_2 = \sum_{j=1}^{s} n_j$. Define the positive homomorphism $r : G \cong \bigoplus_n \mathbb{Z} \to \mathbb{Z}$ by

$$(c_1, \ldots, c_n) \mapsto d_2\left(\sum_{i=1}^{r} c_i\right) + d_1\left(\sum_{i=r+1}^{r+s} c_i\right) + \sum_{i=r+s+1}^{n} c_i.$$

Since A has the property (SP), there is an element $h \in H$ sufficiently small such that $g \mapsto \theta'(g) - r(g)h$ is a positive homomorphism. To save notation, let us still use r to denote the map $g \mapsto r(g)h$. Note that $r(\alpha) = 0$ and r is faithful on the positive cone of G. We have

$$\sum_{i=1}^{r} m_i r(e_i) = \sum_{j=1}^{s} n_j r(f_j).$$

Therefore, we get

$$mr(e_1) < m\left(\sum_{j=1}^{s} r(f_j)\right).$$

In the other words, $r((\sum_{j=1}^{s} f_j) - e_1) > 0$.

Since A is a TAS-algebra and the positive homomorphism $\theta' - r$ is positive, there is a decomposition $\theta' - r = \phi_1' + \phi_1$ such that $\phi_1'(\alpha) = \phi_1(\alpha) = 0$ and ϕ_1 factors through G_1 which is the K_0-group of a splitting tree algebra inside A. Moreover, we may assume that $s\phi_1'(e_1) < r((\sum_{j=1}^{s} f_j) - e_1)$.

Therefore, we have that $\theta' = r + \phi_1' + \phi_1$. We wish to show that the map $r + \phi_1'$ factors through a dimension group and decreases the degree of α strictly. Since $r((\sum_{j=1}^{s} f_j) - e_1) > 0$, we have

$$r(e_1) < \sum_{j=1}^{s} r(f_j).$$

Moreover, the map r factors through $\mathbb{Z}$ by construction. Therefore, there exist $a_{11}', \ldots, a_{1s}'$ such that

$$r(e_1) = \sum_{j=1}^{s} a'_{1j}$$

and $a'_{1j} < r(f_j)$ for any $1 \leqslant j \leqslant s$.

Among the $\{a_{ij}\}$, let us show there exists a_{1j_0} such that $\phi'_1(e_1) < r(f_{j_0}) - a'_{1j_0}$. Suppose this were not true. Then for any $0 \leqslant j \leqslant s$, $r(f_j) - a'_{1j} \leqslant \phi'_1(e_1)$. Therefore, one has

$$\sum_{j=1}^{s} r(f_j) - \sum_{j=1}^{s} a'_{1j} \leqslant s\phi'_1(e_1),$$

and $\sum_{j=1}^{s} r(f_j) - r(e_1) \leqslant s\phi'_1(e_1)$ which is a contradiction to the choice of ϕ'_1.

Set $a_{1j} = a'_{1j}$ for any $1 \leqslant j \leqslant s$, $j \neq j_0$ and $a_{1j_0} = a'_{1j_0} + \phi'(e_1)$. Since $\phi'_1(e_1) < r(f_{j_0}) - a'_{1j_0}$, we have

$$a_{1j} \leqslant r(f_j) \leqslant r(f_j) + \phi'_1(f_j) \quad \text{for any } 1 \leqslant j \leqslant s,$$

and

$$r(e_1) + \phi'_1(e_1) = \left(\sum_{j=1}^{s} a'_{1j}\right) + \phi'_1(e_1) = \sum_{j=1}^{s} a_{1j}.$$

As in the proof of Effros–Handelman–Shen theorem in [6], let e'_{1j} $(0 \leqslant j \leqslant s)$, e'_i $(2 \leqslant i \leqslant r)$, f_j $(1 \leqslant j \leqslant s)$ and g_k $(1 \leqslant k \leqslant t)$ be a standard basis for $\mathbb{Z}^k$, where $k = r + 2s + t - 1$. Define the positive homomorphisms $\iota : G \to \mathbb{Z}^k$ by

$$\begin{aligned} e_1 &\mapsto \sum_j e'_{1j}, \\ e_i &\mapsto e'_i \quad (2 \leqslant i \leqslant r), \\ f_j &\mapsto f'_j + e'_{1j}, \\ g_k &\mapsto g'_k, \end{aligned}$$

and $\psi_1 : \mathbb{Z}^k \to H$ by

$$\begin{aligned} e'_{1j} &\mapsto a_{1j}, \\ e'_i &\mapsto r(e_i) + \phi'_1(e_i) \quad (i \geqslant 2), \\ f'_j &\mapsto r(f_j) + \phi'_1(f_j) - a_{1j}, \\ g'_k &\mapsto r(g_k) - \phi'_1(g_k). \end{aligned}$$

A direct calculation shows that $\psi_1 \circ \iota = r + \phi'_1$. Moreover,

$$\iota(\alpha) = \iota\left(\sum_{i=1}^{r} m_i e_i - \sum_{j=1}^{s} n_j f_j\right)$$

$$= \sum_j (m_1 - n_j) e'_{1j} + \sum_{i=2}^{r} m_i e'_i - \sum_j n_j f'_j,$$

where $n_j > 0$ for any j. Thus, $0 \neq \deg(\iota(\alpha)) < \deg(\alpha)$ as desired.

Applying the factorization above to the restriction of ψ to $\bigoplus_k \mathbb{Z}$ iteratively, after finite steps, we get the following commutative diagram:

$$\begin{tikzcd} G_1 \oplus (\bigoplus_{k'} \mathbb{Z}) \arrow[dr, "\psi"] & \\ G \arrow[u, "\phi_1 \oplus \iota"] \arrow[r, "\theta'"'] & H \end{tikzcd}$$

for some $G_1 \in \mathcal{KS}$ which comes from a sub-C*-algebra of A, such that $\phi_1(\alpha) = 0$ and the element $\iota(\alpha)$ has the form $m' e_1 - \sum m' f_j$ where e_1 and f_j from the standard basis of $\bigoplus_{k'} \mathbb{Z}$.

Therefore, after passing to the quotient H/H_{tor}, in order to prove the lemma, we shall find a suitable factorization of the restriction of ψ to $\bigoplus_{k'} \mathbb{Z}$. Since $\iota(\alpha)$ has the special form, it reduces the lemma to Case 1. Thus, the first part of the lemma holds.

If the positive map θ can be lifted to $\theta' : G \to H$, we can choose this positive homomorphism as the lifting of θ as what we used in the proof of Case 2. The commutative diagrams of the second part of the lemma follows directly from the constructions in Case 2. □

From Lemmas 3.5 and 3.7, we have the following corollary:

Corollary 3.8. *Let $G \cong (\bigoplus_n \mathbb{Z}) \oplus G_0$ with $G_0 \in \mathcal{KS}$. Let $H = \mathrm{K}_0(A)$ for a simple C*-algebra $A \in \mathrm{TA}\mathscr{S}$. Then, for any positive homomorphism $\theta : G \to H/H_{\mathrm{tor}}$, there are an ordered group $(\bigoplus_m \mathbb{Z}) \oplus G_1$ where $G_1 \in \mathcal{KS}$ and positive homomorphisms ϕ_1, ϕ_2 and ψ such that the following diagram commutes*:

$$\begin{tikzcd} (\bigoplus_m \mathbb{Z}) \oplus G_1 \arrow[dr, "\psi"] & \\ G \arrow[u, "\phi_1 \oplus \phi_2"] \arrow[r, "\theta"'] & H/H_{\mathrm{tor}}, \end{tikzcd}$$

such that $\ker(\phi_1) = \ker(\phi_2) = \ker(\theta)$. *Moreover, the group G_1 can be chosen to be the* K_0*-group of a sub-C*-algebra of A which is a splitting tree algebra, and the restriction of ψ on G_1 is induced by the inclusion map.*

Proof. Let $\{\alpha_1, \ldots, \alpha_n\}$ be a set of generators of $\ker(\theta)$. By Lemma 3.5, there is a factorization

$$\begin{tikzcd} (\bigoplus_{n_1'} \mathbb{Z}) \oplus G_1' \arrow[dr, "\psi"] & \\ G \arrow[u, "\phi_1' \oplus \phi_2'"] \arrow[r, "\theta"'] & H/H_{\mathrm{tor}} \end{tikzcd}$$

with some $G_1' \in \mathcal{KS}$, such that $\phi_2'(\alpha_1) = 0$ and the positive homomorphisms in the commutative diagram satisfy the second part of the corollary. Applying Lemma 3.7 to the restriction of ψ' to $\bigoplus_{n_1'} \mathbb{Z}$, we obtain the factorization of θ

$$\begin{tikzcd} (\bigoplus_{n_1} \mathbb{Z}) \oplus G_1 \arrow[dr, "\psi"] & \\ G \arrow[u, "\phi_1 \oplus \phi_2"] \arrow[r, "\theta"'] & H/H_{\mathrm{tor}} \end{tikzcd}$$

with $\phi_1(\alpha_1) = \phi_2(\alpha_1) = 0$.

Then, one can consider the image of α_2 in $(\bigoplus_{n_1} \mathbb{Z}) \oplus G_1$ by $\phi_1 \oplus \phi_2$. If it is not zero, by the same argument above, one gets a factorization of the positive homomorphism ψ, and sends $\phi_1(\alpha_2) \oplus \phi_2(\alpha_2)$ to zero simultaneously. Repeat this procedure for each α_i, we obtain the desired factorization of θ. □

Corollary 3.9. *Let $H = \mathrm{K}_0(A)$ for a simple TAS-algebra A. Then H/H_{tor} is an inductive limit of the* K_0*-groups of splitting tree algebras.*

Proof. This follows directly from the first part of Corollary 3.8 and Lemma 3.1. □

Before proving Proposition 3.3, we need the following lemma about the K_0-groups of simple TAS-algebras.

Lemma 3.10. *Let $H = \mathrm{K}_0(A)$ for some simple separable TAS-algebra A. Then, for any $a \in H^+$ and $b \in H_{\mathrm{tor}}$, b is majorised by a.*

Proof. Since A is a simple separable TAS-algebra, A has the cancellation property for equivalence classes of projections. Since any matrix algebra over a TAS-algebra is a TAS-algebra again, we may assume

$$a = [e], \qquad b = [p] - [q]$$

for projections e, p and q in A with $e \perp q$, $e \perp p$. Since $b \in H_{\mathrm{tor}}$, we have

$$\tau(p) = \tau(q) \quad \text{for any } \tau \in T(A).$$

Hence

$$\tau(e+q)=\tau(e)+\tau(q)>\tau(p) \quad \text{for any } \tau\in T(A).$$

Since A is a simple TAS-algebra, by Theorem 4.12, the strict order on projections is determined by traces. Hence,

$$[p]\preccurlyeq[e+q],$$

which implies $b<a$. $\square$

Proof of Proposition 3.3. Let α be an element in the kernel of the map $\theta: G_0\to H$. Let us first show that it is sufficient to prove the proposition for positive maps from basic building blocks in $\mathcal{ZT}$—i.e., the basic building blocks $\mathbb{Z}\oplus$ (finitecyclic)—to the ordered group H. It follows from Lemma 3.6 that the map $G_0\to H$ factors through $G_1\oplus G_2$ with $G_1\in\mathcal{ZT}$ and $G_2\in\mathcal{KS}$ such that the map $G_0\to G_2$ sends α to 0, and the groups G_1 and G_2 satisfy the second part of the proposition. Thus, to prove the proposition, it is sufficient to show that the map $G_1\to H$ has a lifting in $\mathcal{K}$ which sends the image of α in G_1 to 0.

Therefore, let us assume that $G_0\in\mathcal{ZT}$, and follows an argument similar to that of Theorem 3.2 of [9].

Step 1. Let us show that it is sufficient to verify Lemma 3.1 with the basic building blocks $\mathbb{Z}\oplus$ (finite cyclic) replaced by the ordered groups $\mathbb{Z}\oplus$ (finite) (with the order determined by the first coordinate). To see this, it is enough to show that a map $G_1'\to H$ with $G_1'=\mathbb{Z}\oplus$ (finite) can be factorised as $G_1'\to G_1\to H$ with $G_1\in\mathcal{ZT}$. With the finite part of G_1' be expressed as $F_n\oplus\cdots\oplus F_n$, denote the images of their generator by $t_1,\dots,t_n\in H_{\mathrm{tor}}$, and denote the image in H of the positive generator of $\mathbb{Z}\subseteq G_1'$ by a. Since $H=\mathrm{K}_0(A)$ and A has property (SP), one has that $a=a_1+\cdots+a_n$ with some $a_1,\dots,a_n\in H^+$. By Lemma 3.10, we have $t_i<a_i$ for each $1\leqslant i\leqslant n$. With G_1 to be the direct sum of the basic building blocks $\mathbb{Z}\oplus F_1,\ \dots,\ \mathbb{Z}\oplus F_n$, let us define the map $G_1'\to G_1$ by mapping $F_1\oplus\cdots\oplus F_n\subseteq G_1'$ identically onto $F_1\oplus\cdots\oplus F_n\subseteq G_1$, and $1\in\mathbb{Z}\subseteq G_1'$ into $(1,\dots,1)\in\mathbb{Z}^n\subseteq G_1$. We also define the map $G_1\to H$ by sending the fixed generators of F_i to $t_i\in H_{\mathrm{tor}}$ and the positive generator of the ith summand of G_1 into $a_i\in H$. We then have the factorization $G_1'\to G_1\to H$ as desired.

Step 2. In order to verify the local criterion for the generized building blocks of Step 1, it is enough to consider the case that the restriction of the map $G_0\to H$ to the subgroup $(G_0)_{\mathrm{free}}$ generated by the free direct summands of the generalized basic building blocks $\mathbb{Z}\oplus$ (finite) of G_0 factorises through the direct sum of $(G_1)_{\mathrm{free}}$ a torsion free basic building blocks in $\mathcal{ZT}$ and $(G_1)_{\mathrm{KS}}$ a basic building blocks in $\mathcal{KS}$, in such a way that the kernel of $(G_0)_{\mathrm{free}}\to(G_1)_{\mathrm{free}}$ is equal not only to the kernel of $(G_0)_{\mathrm{free}}\to H$, but also to the kernel of $(G_0)_{\mathrm{free}}\to H/H_{\mathrm{tor}}$.

Let $G_0\to H$ be a map of ordered groups with G_0 a finite direct sum of generalized basic building blocks, and consider the map $(G_0)_{\mathrm{free}}\to H/H_{\mathrm{tor}}$ obtained by restricting $G_0\to H$ to the (noncanonical) subgroup $(G_0)_{\mathrm{free}}$ and composing $(G_0)_{\mathrm{free}}\to H$ with the canonical map $H\to H/H_{\mathrm{tor}}$. By Lemma 3.7, there exists an ordered group $(G_1)_{\mathrm{free}}\oplus(G_1)_{\mathrm{KS}}$, with $(G_1)_{\mathrm{free}}$ the finite direct sum of the ordered groups $\mathbb{Z}$, $(G_1)_{\mathrm{KS}}$ the K_0-group of a splitting tree algebra, and a factorization $(G_0)_{\mathrm{free}}\to(G_1)_{\mathrm{free}}\oplus(G_1)_{\mathrm{KS}}\to H/H_{\mathrm{tor}}$ such that

$$\ker\big((G_0)_{\mathrm{free}}\to H/H_{\mathrm{tor}}\big)=\ker\big((G_0)_{\mathrm{free}}\to\big((G_1)_{\mathrm{free}}\oplus(G_1)_{\mathrm{KS}}\big)\big).$$

Moreover, the restriction of the map $(G_1)_{\mathrm{free}} \oplus (G_1)_{\mathrm{KS}} \to H/H_{\mathrm{tor}}$ to $(G_1)_{\mathrm{KS}}$ has a lifting to H by the second part of Lemma 3.7. Choose a positive map $(G_1)_{\mathrm{free}} \to H$ lifting the restriction of the map $(G_1)_{\mathrm{free}} \oplus (G_1)_{\mathrm{KS}} \to H/H_{\mathrm{tor}}$ to $(G_1)_{\mathrm{free}}$. Then, we get a map $(G_1)_{\mathrm{free}} \oplus (G_1)_{\mathrm{KS}} \to H$ lifting the map $(G_1)_{\mathrm{free}} \oplus (G_1)_{\mathrm{KS}} \to H/H_{\mathrm{tor}}$.

The map $(G_0)_{\mathrm{free}} \to H$ may not equal to the combined map $(G_0)_{\mathrm{free}} \to (G_1)_{\mathrm{free}} \oplus (G_1)_{\mathrm{KS}} \to H$. By the second part of Lemma 3.7, we only need to fix the map which factors through $(G_1)_{\mathrm{free}}$. This can be done by the same argument as that in the proof of Theorem 3.2 of [9].

Step 3. Let us show that the local criterion holds for generalized basic building blocks. Let $G_0 \to H$ be a map of ordered groups with G_0 a finite sum of generalized basic building blocks. By Step 2, we may assume there is a factorization $(G_0)_{\mathrm{free}} \to (G_1)_{\mathrm{free}} \oplus (G_1)_{\mathrm{KS}} \to H$ of the map $(G_0)_{\mathrm{free}} \to H$, where $(G_0)_{\mathrm{free}}$ denotes the direct sum of the free parts of the generalized basic building blocks $\mathbb{Z} \oplus$ (finite) of G_0, and $(G_1)_{\mathrm{free}}$ is some finite ordered group direct sum of copies of $\mathbb{Z}$, such that

$$\ker\big((G_0)_{\mathrm{free}} \to H/H_{\mathrm{tor}}\big) = \ker\big((G_0)_{\mathrm{free}} \to (G_1)_{\mathrm{free}} \oplus (G_1)_{\mathrm{KS}}\big).$$

Let us construct G_1 a finite direct sum of generalized basic building blocks and ordered groups in $\mathcal{KS}$, with torsion free part equal to $(G_1)_{\mathrm{free}} \oplus (G_1)_{\mathrm{KS}}$ as given above, and a factorization of $G_0 \to G_1 \to H$ of $G_0 \to H$ such that

$$\ker(G_0 \to H) = \ker(G_0 \to G_1).$$

Denote by F the image of the finite part of G_0 in H. By Lemma 3.10, the subgroup F is majorised by any positive element in H. Pick a minimal direct summand $\mathbb{Z}$ of $(G_1)_{\mathrm{free}}$ which has non-zero image in H, and append F to this summand. This direct sum can be ordered (in a unique way) so that it is a generalized basic building block. Thus, the group $G_1 = ((G_1)_{\mathrm{free}} \oplus (G_1)_{\mathrm{KS}}) \oplus F$ became an ordered group direct sum of basic building blocks and ordered group in $\mathcal{KS}$. The extension of $(G_0)_{\mathrm{free}} \to (G_1)_{\mathrm{free}} \oplus (G_1)_{\mathrm{KS}} \subseteq G_1$ to G_0 by factoring $(G_0)_{\mathrm{tor}}$ through the maps $(G_0)_{\mathrm{tor}} \to F \subseteq H$ and $F \to (G_1)_{\mathrm{tor}} \subseteq G_1$, and the obvious extension of $(G_1)_{\mathrm{free}} \oplus (G_1)_{\mathrm{KS}} \to H$ to $G_1 = ((G_1)_{\mathrm{free}} \oplus (G_1)_{\mathrm{KS}}) \oplus F$, mapping $F \subseteq G_1$ identically onto $F \subseteq H$, factorise $G_0 \to H$ obviously, and fulfill the condition

$$\ker(G_0 \to H) = \ker(G_0 \to G_1).$$

Positivity of $G_0 \to G_1$ follows directly from the construction and simplicity of the group H/H_{tor}: The finite part of a generalized basic building block of G_0 is majorised by the positive generators of the free part in G_0. By simplicity of the ordered group H/H_{tor}, for each positive element of direct summand of $(G_0)_{\mathrm{free}}$ which does not vanish in H, we may assume that its image in each simple summand of $(G_1)_{\mathrm{free}}$ under the map $(G_0)_{\mathrm{free}} \to (G_1)_{\mathrm{free}} \oplus (G_1)_{\mathrm{KS}}$ is a positive element. Therefore, positive elements of G_0 are sent to positive element of G_1 by the map $G_0 \to G_1$.

Positivity of the map $G_1 \to H$ follows directly from Lemma 3.10: By Lemma 3.10, the image of F is majorized by any positive element of H. Thus, the map $G_1 \to H$ sends positive elements of G_1 to positive elements of H.

Step 4. The proposition follows from Steps 1 and 3. $\square$

Remark 3.11. In fact, in order to prove Proposition 3.3, one only needs Lemmas 3.6, 3.7 and 3.10.

3.3. The K_1*-groups and the pairing maps of C*-algebras in* TA$\mathscr{S}$

If A is a separable simple C*-algebra in the class TA$\mathscr{S}$, we have shown that the K_0-group of A can be realized as an inductive limit of K_0-groups of the C*-algebras in $\mathscr{S}$ and certain ordered groups in the class $\mathcal{ZT}$. We shall go further to construct a C*-algebra B, a simple inductive limit of algebras in $\mathscr{S}$ together with certain homogeneous C*-algebras, such that A and B not only have the same K_0-group, but also have the same K_1-group and pairing.

The basic building blocks of C*-algebras we are going to use are

(1) splitting tree algebras in the class $\mathscr{S}$,
(2) matrix algebras over the C*-algebras of continuous functions on $T_{2,k}$, and
(3) matrix algebras over the C*-algebras of continuous functions on $S^1 \vee \cdots \vee S^1 \vee T_{3,k_i} \vee \cdots \vee T_{3,k_i}$,

where $T_{2,k}$ is the two-dimensional CW complex obtained by attaching a two-dimensional disk D to S^1 via a map S^1 $(\cong \partial D) \to S^1$ of degree k, and $T_{3,k}$ is the three-dimensional CW complex obtained by attaching a three-dimensional ball B to S^2 via a map S^2 $(\cong \partial B) \to S^2$ of degree k. Building blocks in (1) provide the torsion free part of the K_0-group, building blocks in (2) provide the torsion part of the K_0-group, and building blocks in (3) provide the K_1-group in the construction.

The homogeneous C*-algebras of (2) and (3) are called *the Gong standard homogeneous C*-algebras* (see [13,16], and [14]). Denote by $\mathscr{S}'$ the class of C*-algebras containing $\mathscr{S}$ and the Gong standard homogeneous C*-algebras.

Let A be a simple separable C*-algebra in the class TA$\mathscr{S}$. Since $K_1 := \mathrm{K}_1(A)$ is a countable abelian group, we can write K_1 as an inductive limit (in the category of abelian groups) of finitely generated abelian groups with injective maps:

$$K_1^{(1)} \xhookrightarrow{\eta_{12}} K_1^{(2)} \xhookrightarrow{\eta_{23}} \cdots \hookrightarrow \varinjlim K_1^{(i)} = K_1.$$

We may assume

$$K_1^{(i)} \cong \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{m_i} \oplus \big(\mathbb{Z}/n_1^{(i)}\mathbb{Z}\big) \oplus \cdots \oplus \big(\mathbb{Z}/n_{k_i}^{(i)}\mathbb{Z}\big).$$

Then the compact topological space

$$E_i = \underbrace{S^1 \vee \cdots \vee S^1}_{m_i} \vee T_{3,n_1^{(i)}} \vee \cdots \vee T_{3,n_{k_i}^{(i)}}$$

has $K_1^{(i)}$ as its K_1-group and has $(\mathbb{Z}, \mathbb{Z}^+)$ as its ordered K_0-group, i.e., $\mathrm{K}_1(\mathrm{C}(E_i)) = K_1^{(i)}$ and $(\mathrm{K}_0(\mathrm{C}(E_i)), \mathrm{K}_0^+(\mathrm{C}(E_i))) = (\mathbb{Z}, \mathbb{Z}^+)$. Moreover, for any group homomorphism

$$\eta_{ij} : K_1^{(i)} \to K_1^{(j)},$$

there is a $*$-homomorphism

$$\phi_{ij}:\mathrm{C}(E_i)\to \mathrm{M}_{12}\big(\mathrm{C}(E_j)\big)$$

such that

$$[\phi_{ij}]_1=\eta_{ij}\quad \text{and}\quad [\phi_{ij}]_0=\mathrm{id},$$

where $[\phi_{ij}]_*$ denotes the homomorphism induced by ϕ_{ij} on K_*-groups.

We have shown that $\mathrm{K}_0(A)$ is an inductive limit (as ordered groups) of basic building blocks. Moreover, by Proposition 3.3 and the argument of Lemma 3.1, the inductive limit decomposition of $\mathrm{K}_0(A)$ can be chosen to have special forms: let $\{\varepsilon_n\}$ be a sequence of positive numbers which converges to 0. The ordered group $\mathrm{K}_0(A)$ can be realized as an inductive limit of $(G_i = G'_i\oplus G''_i, \iota_i)$ where G''_i comes from a sub-C*-algebra S_i of A satisfying $\tau(1_A-1_{S_i})<\varepsilon_i$ as being described at the beginning of this section, and the restriction of ι_i to $G''_i\to G''_{i+1}$ is induced by the homomorphism (not necessarily unital) $\phi_i:S_i\to S_{i+1}$.

Set the positive map θ_1 in the proof of Lemma 3.1 to be $n\mapsto n[1_A]$. Since A belongs to the class $\mathrm{TA}\mathscr{S}$, the map θ_1 can be decomposed into $\theta'_1+\theta''_1$ such that θ'_1 has the factorization $\mathbb{Z}\xrightarrow{\mathrm{id}}\mathbb{Z}\to\mathrm{K}_0(A)$ and θ''_1 has a factorization $\mathbb{Z}\to G''_1\to\mathrm{K}_0(A)$, where G''_1 is the K_0-group of sub-C*-algebra S_1 of A with $\tau(1_A-1_{S_1})<\varepsilon_1$. Now suppose that the inductive system is constructed up to the nth level; that is, we have groups $G_i=G'_i\oplus G''_i$ and maps θ_i and ι_i such that the following diagram commutes:

$$\begin{tikzcd}
G_1 \arrow[r, "\theta_1"] \arrow[d, "\iota_1"'] & \mathrm{K}_0(A) \arrow[d, "\mathrm{id}"] \\
G_2 \arrow[r, "\theta_2"] \arrow[d, "\iota_2"'] & \mathrm{K}_0(A) \arrow[d, "\mathrm{id}"] \\
\vdots \arrow[d, "\iota_{n-1}"'] & \vdots \arrow[d, "\mathrm{id}"] \\
G_n \arrow[r, "\theta_n"] & \mathrm{K}_0(A),
\end{tikzcd}$$

and moreover, the group G''_i is the K_0-group of sub-C*-algebra S_i, the restriction of θ_i to G''_i is induced by the inclusion map, and the restriction of ι_i to $G''_i\to G''_{i+1}$ is induced by $\phi_{i,i+1}$.

We shall construct an ordered group $G_{n+1}=G'_{n+1}\oplus G''_{n+1}$ and certain positive maps. As in the proof of Lemma 3.1, define the positive maps $\iota':G_n\to G_n\oplus\mathbb{Z}$ by $s\mapsto(s,0)$, and $\kappa:G_n\oplus\mathbb{Z}\to\mathrm{K}_0(A)$ by $(s,m)\mapsto\theta_n(s)+mg_n$. Since A belongs to the class of $\mathrm{TA}\mathscr{C}$, the map κ has a factorization $G_n\oplus\mathbb{Z}\to G_n\oplus\mathbb{Z}\oplus D\to\mathrm{K}_0(A)$, where D is the K_0-group of a sub-C*-algebra S_{n+1} of A as being described at the beginning of this section (we may pass to a subsequence of the original sequence (S_n), and assume that S_{n+1} is far enough from S_n). Moreover, since the restriction of κ to G''_n is induced by the inclusion map, the factorization can be

chosen such that the map $G_n'' \to D$ is induced by the $*$-homomorphism $\phi_n : S_n \to S_{n+1}$. We then apply Proposition 3.3 to the map $(G_n \oplus \mathbb{Z}) \oplus D \to \mathrm{K}_0(A)$ to get a factorization

$$(G_n \oplus \mathbb{Z}) \oplus D \to \left(G_{n+1} = G'_{n+1} \oplus G''_{n+1}\right) \to \mathrm{K}_0(A),$$

such that $\ker((G_n \oplus \mathbb{Z}) \oplus D \to \mathrm{K}_0(A))$ and $\ker((G_n \oplus \mathbb{Z}) \oplus D \to G_{n+1})$ are the same. Moreover, the ordered group G''_{n+1} is the K_0-group of a sub-C*-algebra S_{n+2} of A as being described at the beginning of this section, and the map $D \to G''_{n+1}$ is induced by the $*$-homomorphism $\phi_{n+1} : S_{n+1} \to S_{n+2}$. Set the map $G_n \to G_{n+1}$ to be the composition of $G_n \to (G_n \oplus \mathbb{Z}) \oplus D$ and $(G_n \oplus \mathbb{Z}) \oplus D \to G_{n+1}$, and it is the desired map. We may pass to the subsequence of the inductive system (S_n, ϕ_n) described at the beginning of this section, and still denote the sub-C*-algebra of A associated with G_{n+1} by S_{n+1}. This procedure can be illustrated by the following diagram:

$$\begin{tikzcd}
G_n \arrow[rrr, "\theta_n"] \arrow[dr, "\iota'"] \arrow[dd, "\iota_n=\iota'''\circ\iota''\circ\iota'"'] & & & \mathrm{K}_0(A) \arrow[dd, "\mathrm{id}"] \\
& G_n \oplus \mathbb{Z} \arrow[r, "\iota''"'] \arrow[urr, "\kappa"] & (G_n \oplus \mathbb{Z}) \oplus D \arrow[ur] \arrow[dll, "\iota'''"] & \\
G_{n+1} \arrow[rrr, "\theta_{n+1}"'] & & & G.
\end{tikzcd}$$

Thus, we have an inductive limit decomposition of the ordered group $\mathrm{K}_0(A)$

$$G_1' \oplus G_1'' \to G_2' \oplus G_2'' \to \cdots \to \mathrm{K}_0(A)$$

such that the ordered groups G_n'' are sub-C*-algebras S_n of A, the maps $G_n'' \to G''_{n+1}$ are induced by the $*$-homomorphisms $\phi_n : S_n \to S_{n+1}$, and the maps $G_n'' \to \mathrm{K}_0(A)$ are induced by the inclusion maps.

Since each simple direct summand of the ordered group G_n' belongs to the class $\mathcal{K}$, we can choose a certain C*-algebra to have this direct summand as its ordered K_0-group: if the simple direct summand is $\mathbb{Z}$, we choose the algebra of complex number; if the simple direct summand is $\mathbb{Z} \oplus \mathbb{Z}/k\mathbb{Z}$, we choose the algebra of the continuous function over $T_{2,k}$; if the simple direct summand is the K_0-group of a C*-algebra in $\mathscr{S}$, we choose the corresponding splitting tree algebra. Therefore, there is a C*-algebra C_n of the finite direct sum of basic building blocks such that $\mathrm{K}_0(C_n) = G_n'$. For each G_n', choose $u_n' \in G_n'^{+}$ such that the image of (u_n', u_n'') is $[1_A]$ in $\mathrm{K}_0(A)$. By taking matrix algebras or their cut-down over C_n, there is a C*-algebra of the finite direct sum of the basic building blocks such that its order-unit K_0-group is (G_n', u_n'). Still denote this C*-algebra by C_n.

Set $B_n = C_n \oplus S_n$. It is clear that $(\mathrm{K}_0(B_n), [1_{B_n}]) = (G_n' \oplus G_n'', (u_n', u_n''))$. For any positive homomorphism $\iota_n : G_n \to G_{n+1}$, since any order-unit K_0-map between basic building block C*-algebras can be lifted to a $*$-homomorphism (see [14] for homogeneous building blocks,

and see [27] for splitting tree algebras), and the restriction of ι_n to $G''_n \to G''_{n+1}$ is induced by a $*$-homomorphism $\phi_n : S_n \to S_{n+1}$, there is a $*$-homomorphism $\psi_n : B_n \to B_{n+1}$ in the form

$$\begin{pmatrix} * & * \\ * & \phi_n \end{pmatrix}$$

such that $[\psi_n]_0 = \iota_n$. Moreover, we can choose the maps ψ_n in such a way that $\varinjlim B_n$ is simple. It is clear that the inductive limit of the system (B_n, ψ_n) has the same order-unit K_0-group as that of A. However, the C*-algebra B has trivial K_1-group.

To get the desired K_1-group, we shall replace one direct summand of each B_n by a certain basic building block C*-algebra with nontrivial K_1-groups, and modify the connection homomorphisms without changing the K_0-group. Since at least one of the simple direct summands of G'_n is $\mathbb{Z}$, there is one of the direct summand of C_n which is a matrix algebra, say $\mathrm{M}_k(\mathbb{C})$, and it does not vanish in the inductive limit. We then replace this direct summand by $\mathrm{M}_k(\mathrm{C}(E_n))$, and still denote the new building block by B_n. We then see that $\mathrm{K}_1(B_n) = K_1^{(n)}$, and the K_0-group of B_n remains same. Set the map between two such building blocks $\mathrm{C}(E_n)$ and $\mathrm{M}_k(\mathrm{C}(E_{n+1}))$ by

$$f \mapsto \operatorname{diag}\{\phi_n, f(x_{k'-12}), \ldots, f(x_{k'})\},$$

where each x_i is a point in E_n, ϕ_n is a $*$-homomorphism from $\mathrm{C}(E_n)$ to $\mathrm{M}_{12}(\mathrm{C}(E_{n+1}))$ which induces $\eta_{n,n+1}$ as K_1-map, and k' is the multiplicity of the K_0-map between the two matrix algebras being replaced. Moreover, since the K-group maps do not depend on the choices of the points $\{x_i\}$, we again can choose suitable points such that the inductive limit C*-algebra is simple. Thus, there is a $*$-homomorphism $\psi_n : B_n \to B_{n+1}$ in the form

$$\begin{pmatrix} * & * \\ * & \phi_n \end{pmatrix}$$

such that $[\psi_n]_0 = \iota_n$ and $[\psi_n]_1 = \eta_n$. Denote the inductive limit of (B_n, η_n) by B. We have

$$\mathrm{K}_0(B) = \varinjlim(G_n, \iota_n) = \mathrm{K}_0(A)$$

and

$$\mathrm{K}_1(B) = \varinjlim\bigl(K_1^{(n)}, \eta_n\bigr) = \mathrm{K}_1(A).$$

Thus A and B have the same K-groups.

We assert that A and B have the same pairing map between the simplex of traces and the ordered K_0-group. At the beginning of this section, we have that the map $r_A : \mathrm{T}(A) \to \mathrm{S}(\mathrm{K}_0(A))$ is isomorphic to the map $r_S : \mathrm{T}_{u'}(S) \to \mathrm{S}_{u'}(\mathrm{K}_0(S))$ where S is the inductive limit of (S_n, ϕ_n). We shall first show that $r_B : \mathrm{T}(B) \to \mathrm{S}(\mathrm{K}_0(B))$ is also isomorphic to the above maps.

Since B is a simple inductive limit of splitting tree algebra and homogeneous C*-algebras with bounded dimensional spectra, the strict order on the projections in B is determined by traces. By the construction, there are homomorphisms $\lambda_n : S_n \to B$ by sending S_n to the corresponding direct summand of B_n. Note that we have commutative relations $\lambda_n = \lambda_{n+1} \circ \phi_n$. Since A is simple, the system (S_n, ϕ_n) is injective (by an asymptotic argument as that of Lemma 2.4). Therefore, the $*$-homomorphism $\lambda_n : S_n \to B$ is one-to-one, and we may consider the C*-algebra S and S_n as sub-C*-algebras of B.

Note that for any $\varepsilon > 0$, we have that $\rho([1_B]_0 - [\phi_{n,\infty}(1_{S_n})]) < \varepsilon$ for any $\rho \in \mathrm{S}(\mathrm{K}_0(B))$. Since B is a unital nuclear C*-algebra, any state on $\mathrm{K}_0(B)$ comes from a tracial state on B; we conclude that $\tau(1_B - 1_{S'_n}) < \varepsilon$ for any tracial state $\tau \in \mathrm{T}(B)$. Thus, let $\{\mathcal{F}_1, \ldots, \mathcal{F}_n, \ldots\}$ be an increasing sequence of finite subsets of B with dense union, and $\{\varepsilon_n\}$ be a decreasing sequence of positive number converging to zero. It is easy to see that the sub-C*-algebra S_n of B (we may pass to a subsequence of the system (B_n, ϕ_n)) satisfies the following: let $p_n = [1_{S_n}]_0$, then, for any $b \in \mathcal{F}_n$, we have

(1) $\|pb - bp\| < \varepsilon_n$,
(2) $pbp \in_{\varepsilon_n} S_n$, and
(3) $\tau(1_B - p_n) < \varepsilon_n$ for any $\tau \in \mathrm{T}(B)$.

Using the same argument as that of Lemma 10.8 of [23], we conclude that the map $r_B : \mathrm{T}(B) \to \mathrm{S}(\mathrm{K}_0(B))$ is isomorphic to $r_S : \mathrm{T}_{u'}(S) \to \mathrm{S}_{u'}(\mathrm{K}_0(S))$. In particular, it is isomorphic to the map $r_A : \mathrm{T}(A) \to \mathrm{S}(\mathrm{K}_0(A))$.

In order to prove that the pairing of A is isomorphic to the pairing of B, we must show that the isomorphism between the K_0-groups and the isomorphisms between $\mathrm{S}(\mathrm{K}_0(A))$ and $\mathrm{S}(\mathrm{K}_0(B))$ are compatible; that is, if ψ denotes the isomorphism $\mathrm{K}_0(B) \to \mathrm{K}_0(A)$ and ϱ denote the isomorphism $\mathrm{S}(\mathrm{K}_0(A)) \to \mathrm{S}(\mathrm{K}_0(A))$, one has that

$$s\big(\psi(p)\big) = \varrho(s)(p) \quad \text{for any } p \in \mathrm{K}_0(B),\ s \in \mathrm{S}\big(\mathrm{K}_0(A)\big).$$

Denote by ϱ_A the isomorphism $\mathrm{S}_{u'}(\mathrm{K}_0(S)) \to \mathrm{S}(\mathrm{K}_0(A))$ and denote by ϱ_B the isomorphism $\mathrm{S}_{u'}(\mathrm{K}_0(S)) \to \mathrm{S}(\mathrm{K}_0(B))$. To prove the compatibility, it is sufficient to show that for any $s \in \mathrm{S}_{u'}(\mathrm{K}_0(C))$ and any projection p in B, the equality

$$\varrho_B(s)\big([p]\big) = \varrho_A(s)\big(\psi\big([p]\big)\big)$$

holds. By (iv) of Lemma 10.8 of [23], if the projection q stands for the K_0-element $\psi([p])$, one has that

$$\varrho_A(s)\big(\psi\big([p]\big)\big) = \lim_{n\to\infty} \tau_s\big(\phi_{n,\infty}\big(L_n^{(A)}(q)\big)\big)$$

and

$$\varrho_B(s)\big([p]\big) = \lim_{n\to\infty} \tau_s\big(\phi_{n,\infty}\big(L_n^{(B)}(p)\big)\big),$$

where τ_s is a trace on S which induces s, and $\{L_n^{(A)}\}$ (or $\{L_n^{(B)}\}$) are certain completely positive linear maps from A (or B) to S_n. Let $\varepsilon > 0$. By the construction of B, one may assume that $p \in B_n$ and $[p] = [p'] \oplus [p''] \in G'_n \oplus G''_n$. By the construction of the isomorphism ψ (induced by the positive homomorphisms $\{\theta_n\}$), one has that $\psi([p]) = \theta_n([p])$. Thus, there is a projection $q \in A$ such that $[q] = \psi([p])$ and $q = q' + q''$ where $q'' \in S_{n+1}$. Moreover, since the restriction of θ_n to G''_n is induced by the inclusion map of S_n, the projection q can be chosen such that

$$\big|\tau\big(\phi_n(p'') - q''\big)\big| \leqslant \varepsilon$$

for any $\tau \in \mathrm{T}(A)$. By (iii) of Lemma 10.8 of [23], this implies

$$\left|\tau'\left(\phi_n(p'') - q''\right)\right| \leqslant \varepsilon$$

for any $\tau' \in \mathrm{T}_{u'}(S)$. Since $L_n^{(A)}(q) = q''$ and $L_n^{(B)}(p) = p''$, one has

$$\left|\tau_s\left(L_n^{(A)}(q) - L_n^{(B)}(p)\right)\right| \leqslant \varepsilon$$

when n is sufficiently large. Thus, the equality

$$\varrho_B(s)\left([p]\right) = \varrho_A(s)\left(\psi\left([p]\right)\right)$$

holds, and hence the pairing of the simplex of traces and the K_0-groups of A is isomorphic to that of B. Thus, we proved Theorem A.

Theorem A. *Let A be a simple separable C*-algebra in the class* $\mathrm{TA}\mathscr{S}$. *There exists a simple inductive limit C*-algebra B of C*-algebras in the class $\mathscr{S}'$ such that the Elliott invariant of A is isomorphic to the Elliott invariant of B.*

Remark 3.12. Denote by $\mathscr{S}$ the class of splitting interval algebras. Theorem A concludes that for any simple separable C*-algebra in the class $\mathrm{TA}\mathscr{S}$, the model algebra in the sense of [27] exists. Therefore, by the classification theorem of [25] or [27], the class of simple separable amenable C*-algebras in $\mathrm{TA}\mathscr{S}$ which satisfy the UCT can be classified by the Elliott invariant.

4. Certain properties preserved by tracial approximation

The question of the behaviour of C*-algebra properties under passage from a class $\mathscr{C}$ to the class $\mathrm{TA}\mathscr{C}$ is interesting and sometimes important. In fact, the property of having tracial states, the property of being of stable rank one, and the property that the strict order on projections is determined by traces were used in the proof of Theorem A and also in the proof of the classification theorem of [27].

In this section, we shall show that the following properties of C*-algebras in a class $\mathscr{C}$ are inherited by simple C*-algebras in the class $\mathrm{TA}\mathscr{C}$:

(1) being (stably) finite;
(2) having stable rank one;
(3) having at least one tracial state;
(4) the strict order on projections is determined by traces (if this property holds for every matrix algebra over the given C*-algebra, then it in particular implies that the K_0-group of the given C*-algebra is weakly unperforated);
(5) any state of the order-unit K_0-group comes from a tracial state of the algebra;
(6) if the restriction of a tracial state to the order-unit K_0-group is the average of two distinct states on the K_0-group, then it is the average of two distinct tracial states (in particular, the restriction map preserves extreme points);
(7) the canonical map from the unitary group modulo the connected component containing the identity to the K_1-group being injective.

Theorem 4.1. *Let $\mathscr{C}$ be a class of finite unital C*-algebras. Then any simple C*-algebra in the class* $\mathrm{TA}\mathscr{C}$ *is finite. Moreover, if C*-algebras in $\mathscr{C}$ are stably finite, then simple C*-algebras in the class* $\mathrm{TA}\mathscr{C}$ *are also stably finite.*

Proof. Let $A \in \mathrm{TA}\mathscr{C}$. For any finite subset $\mathcal{F}$ of the unit ball of A, there is a sub-C*-algebra $C_{\mathcal{F}}$ of A with unit $p_{\mathcal{F}}$ such that for any $a \in \mathcal{F}$

(1) $\|p_{\mathcal{F}}a - ap_{\mathcal{F}}\| \leqslant \frac{1}{|\mathcal{F}|}$, and
(2) there is an element $a_{\mathcal{F}} \in C_{\mathcal{F}}$ such that $\|a_{\mathcal{F}} - p_{\mathcal{F}}ap_{\mathcal{F}}\| \leqslant \frac{1}{|\mathcal{F}|}$,

where $|\mathcal{F}|$ denotes the cardinality of $\mathcal{F}$. Let $\Phi_{\mathcal{F}}$ be a unital map (not necessarily a linear map, just a set theoretical map) from A to $C_{\mathcal{F}}$ such that $\Phi_{\mathcal{F}}$ sends a to $a_{\mathcal{F}}$.

Note that the collection of all the finite subsets of the unit ball of A forms an upward directed family. Define the map (which is not a linear map a priori)

$$\Phi : A \to \prod_{\mathcal{F}} C_{\mathcal{F}} \Big/ \bigoplus_{\mathcal{F}} C_{\mathcal{F}} \quad \text{by } a \mapsto \prod_{\mathcal{F}} \Phi_{\mathcal{F}},$$

where $\mathcal{F}$ runs over all the finite subsets of A.

We show that the map Φ is in fact a $*$-homomorphism. Fix a, b in the unit ball of A. Let $\mathcal{F}$ be an element of an upward chain of finite subsets containing $\{a, b, a^*, b^*, a+b, ab\}$. Then, we have that

$$\bigl\|\Phi_{\mathcal{F}}(a) - p_{\mathcal{F}}ap_{\mathcal{F}}\bigr\| = \|a_{\mathcal{F}} - p_{\mathcal{F}}ap_{\mathcal{F}}\| \leqslant \frac{1}{|\mathcal{F}|}.$$

The same argument shows that

$$\bigl\|\Phi_{\mathcal{F}}(b) - p_{\mathcal{F}}bp_{\mathcal{F}}\bigr\| \leqslant \frac{1}{|\mathcal{F}|}$$

and

$$\bigl\|\Phi_{\mathcal{F}}(a+b) - p_{\mathcal{F}}(a+b)p_{\mathcal{F}}\bigr\| \leqslant \frac{1}{|\mathcal{F}|}.$$

Therefore, one has that

$$\bigl\|\Phi_{\mathcal{F}}(a+b) - \bigl(\Phi_{\mathcal{F}}(a) + \Phi_{\mathcal{F}}(b)\bigr)\bigr\| \leqslant 3\frac{1}{|\mathcal{F}|}.$$

A similar argument also shows that

$$\bigl\|\Phi_{\mathcal{F}}(a^*) - \bigl(\Phi_{\mathcal{F}}(a)\bigr)^*\bigr\| \leqslant 2\frac{1}{|\mathcal{F}|}$$

and

$$\bigl\|\Phi_{\mathcal{F}}(ab) - \Phi_{\mathcal{F}}(a)\Phi_{\mathcal{F}}(b)\bigr\| \leqslant 5\frac{1}{|\mathcal{F}|}.$$

Therefore, the families

$$\big(\Phi_{\mathcal{F}}(a+b)-\big(\Phi_{\mathcal{F}}(a)+\Phi_{\mathcal{F}}(b)\big)\big)_{\mathcal{F}},$$
$$\big(\Phi_{\mathcal{F}}(a^*)-\big(\Phi_{\mathcal{F}}(a)\big)^*\big)_{\mathcal{F}}$$

and

$$\big(\Phi_{\mathcal{F}}(ab)-\Phi_{\mathcal{F}}(a)\Phi_{\mathcal{F}}(b)\big)_{\mathcal{F}}$$

are in the ideal $\bigoplus_{\mathcal{F}} C_{\mathcal{F}}$. Note that Φ is also unital, hence the map Φ is a $*$-homomorphism.

Since the C*-algebra A is simple, the map Φ is injective, and the C*-algebra A can be considered as a unital sub-C*-algebra of $\prod_{\mathcal{F}} C_{\mathcal{F}}/\bigoplus_{\mathcal{F}} C_{\mathcal{F}}$. We shall show that the C*-algebra $\prod_{\mathcal{F}} C_{\mathcal{F}}/\bigoplus_{\mathcal{F}} C_{\mathcal{F}}$ is finite if the C*-algebras $C_{\mathcal{F}}$ are finite, and hence the C*-algebra A is finite.

Let v be a partial isometry in $\prod_{\mathcal{F}} C_{\mathcal{F}}/\bigoplus_{\mathcal{F}} C_{\mathcal{F}}$ such that

$$vv^*=1 \quad \text{and} \quad v^*v=p.$$

Note that any partial isometry in $\prod_{\mathcal{F}} C_{\mathcal{F}}/\bigoplus_{\mathcal{F}} C_{\mathcal{F}}$ can be lifted to a partial isometry in $\prod_{\mathcal{F}} C_{\mathcal{F}}$. Pick a lifting and denote it by $(v_{\mathcal{F}})_{\mathcal{F}}$ where $v_{\mathcal{F}}$ is a partial isometry in $C_{\mathcal{F}}$. Therefore, we have that

$$\lim_{\mathcal{F}\to\infty}\big\|v_{\mathcal{F}}v_{\mathcal{F}}^*-1_{C_{\mathcal{F}}}\big\|=0.$$

In particular, $v_{\mathcal{F}}v_{\mathcal{F}}^*=1_{C_{\mathcal{F}}}$ if $\mathcal{F}$ is sufficiently large. Since $C_{\mathcal{F}}$ is finite, we conclude that $v_{\mathcal{F}}^*v_{\mathcal{F}}=1_{C_{\mathcal{F}}}$ if $\mathcal{F}$ is sufficiently large, and hence $p=v^*v=1$. Thus the C*-algebra $\prod_{\mathcal{F}} C_{\mathcal{F}}/\bigoplus_{\mathcal{F}} C_{\mathcal{F}}$ is finite, as desired.

If C*-algebras in $\mathscr{C}$ are stably finite, the argument above applies to matrix algebras of A, and shows that A is stably finite. $\square$

Remark 4.2. In the proof of the theorem above, we only use conditions (1) and (2) of Definition 2.2. In other words, only a piece of A is required to be approximated by C*-algebras in $\mathscr{C}$ without assuming this piece to be large.

Recall that a unital C*-algebra is said to have *stable rank one* if the invertible elements are dense.

Theorem 4.3. *Let $\mathscr{C}$ be a class of unital C*-algebras with stable rank one. Then any simple C*-algebra in the class* $\mathrm{TA}\mathscr{C}$ *has stable rank one.*

Proof. Let A be a C*-algebra in $\mathrm{TA}\mathscr{C}$. Let us show that it is enough to assume that A has the property (SP). If A does not have the property (SP), then there is a positive element $a\in A$ such that the only projection in $\overline{aAa}$ is the zero projection. Since $A\in\mathrm{TA}\mathscr{C}$, by applying Definition 2.2 to a given finite subset $\mathcal{F}\subseteq A$, $\varepsilon>0$, and a, we conclude that $\mathcal{F}$ can be approximated by a C*-algebra in the class $\mathscr{C}$ (which has stable rank one) to within ε. In particular, this implies that the C*-algebra A has stable rank one. Hence it remains to consider the case that A has the property (SP).

We must show that the invertible elements are dense in A. Let a be an element of A which is not invertible. Let ε be any positive number. Note that any C*-algebra with stable rank one is stably finite. By Theorem 4.1, A is stably finite, and thus a is not one-sided invertible. By [30], there exists a zero-divisor a' such that $\|a - a'\| < \varepsilon/2$. Thus, in order to prove the proposition, it is enough to prove that a' can be approximated by invertible elements.

Since A has the property (SP), there is a projection e which is orthogonal to a'. Recall that stable rank one is preserved by unital hereditary sub-C*-algebras. Hence, we may assume that the class $\mathscr{C}$ is closed under passing to unital hereditary sub-C*-algebras, and therefore we may assume that the C*-algebra pAp belongs to the class $\mathrm{TA}\mathscr{C}$ for any projection $p \in A$. Since A has the property (SP), eAe also has the property (SP). Since A is simple, we have that $e = e_1 + e_2$ with $e_2 \preccurlyeq e_1$ by Lemma 4.9 below. Since a' is orthogonal to e_1, we have that $a' \in (1 - e_1)A(1 - e_1)$. Furthermore, since $(1 - e_1)A(1 - e_1)$ is also a C*-algebra in $\mathrm{TA}\mathscr{C}$, there is a projection $p \in (1 - e_1)A(1 - e_1)$, and a sub-C*-algebra $C \in \mathscr{C}$ with $1_C = p$ such that a' is close to $pa'p - (1 - e_1 - p)a'(1 - e_1 - p)$ up to $\varepsilon/4$, $pa'p$ is almost inside C up to $\varepsilon/4$, and $1 - e_1 - p \preccurlyeq e_2$. Since C has stable rank one, there exists an invertible element $b \in C$ which is close to $pa'p$ up to $\varepsilon/2$. In the unital hereditary sub-C*-algebra $(1 - p)A(1 - p)$, if we denote by μ the partial isometry with $\mu\mu^* = 1 - e_1 - p$ and $\mu^*\mu \leqslant e_1$, the element $(1 - e_1 - p)a(1 - e_1 - p) + (\varepsilon/2)\mu + (\varepsilon/2)\mu^* + (\varepsilon/2)(e_1 - \mu^*\mu)$, with the matrix form

$$\begin{pmatrix} (\varepsilon/2)(e_1 - \mu\mu^*) & 0 & 0 \\ 0 & 0 & (\varepsilon/2)\mu^* \\ 0 & (\varepsilon/2)\mu & (1 - e_1 - p)a(1 - e_1 - p) \end{pmatrix},$$

is invertible, and is close to $(1 - e_1 - p)a'(1 - e_1 - p)$ up to $\varepsilon/2$. Thus, the element a' can be approximated by the invertible elements of A. This shows that A has stable rank one. □

Corollary 4.4. *Any simple TAS-algebra has stable rank one.*

Proof. This follows from Theorem 4.3 and Corollary 2.5. □

Remark 4.5. The authors thank the referee for informing us that the same statement as that of Theorem 4.3 has appeared in Qingzhai Fan's PhD thesis, and a weaker version appeared in [15].

Let $\mathscr{T}$ be a class of unital C*-algebras which have tracial states.

Theorem 4.6. *Any C*-algebra in the class* $\mathrm{TA}\mathscr{T}$ *has at least one tracial state.*

Proof. Let A be a C*-algebra in $\mathrm{TA}\mathscr{T}$ and $\mathcal{F}$ be a finite subset of A. Then there is a sub-C*-algebra $B_{\mathcal{F}}$ (with unit p) of A such that for any $x \in \mathcal{F}$,

$$\big\|x - \big(pxp + (1 - p)x(1 - p)\big)\big\| < \frac{1}{|\mathcal{F}|},$$

and there is $b \in B_{\mathcal{F}}$

$$\|pxp - b\| < \frac{1}{|\mathcal{F}|},$$

where $|\mathcal{F}|$ is the cardinality of $\mathcal{F}$. Choose a tracial state τ_B of $B_{\mathcal{F}}$ and extend it to a state of pAp; still denoted by τ_B. Define a state $\tau_{\mathcal{F}}$ on A by

$$\tau_{\mathcal{F}} : a \mapsto \tau_B(pap).$$

The finite subsets of A form an upward directed collection with respect to inclusion. Therefore, all the states $\{\tau_{\mathcal{F}}\}$ on A form an upward directed family. Since the state space is compact, there is a state τ on A such that τ is an accumulation point of $\{\tau_{\mathcal{F}}\}$.

As expected, τ is a trace. To verify this, fix $a, b \in A$, and let us show that $\tau(ab) = \tau(ba)$. For any $\varepsilon > 0$, by definition, there exists a finite subset $\mathcal{F}$ of A containing $\{a, b, ab, ba\}$ and $|\mathcal{F}| > 1/\varepsilon$, such that

$$\left|\tau(ab) - \tau_{\mathcal{F}}(ab)\right| < \varepsilon \quad \text{and} \quad \left|\tau(ba) - \tau_{\mathcal{F}}(ba)\right| < \varepsilon.$$

(One can take $\mathcal{F}_1 = \{a, b, ab, ba\}$; then take an increasing sequence of finite subsets containing $\mathcal{F}_1$.) Then we have

$$\begin{aligned} \tau(ab) &=_\varepsilon \tau_{\mathcal{F}}(ab) = \tau_B(pabp) =_\varepsilon \tau_B(pappbp) \\ &=_\varepsilon \tau_B(a'b') \quad \text{where } a', b' \in B_{\mathcal{F}} \\ &= \tau_B(b'a') \\ &=_{2\varepsilon} \tau_{\mathcal{F}}(ba) \\ &=_\varepsilon \tau(ba), \end{aligned}$$

where $a =_\varepsilon b$ refers to $|a - b| < \varepsilon$. Thus we have

$$\left\|\tau(ab) - \tau(ba)\right\| < 6\varepsilon.$$

Since ε is arbitrary, $\tau(ab) = \tau(ba)$. $\square$

An immediately consequence of the theorem stated above is

Corollary 4.7. *Any simple C*-algebra in the class* $\mathrm{TA}\mathscr{I}$ *is stably finite.*

Remark 4.8. In the proof of Theorem 4.6 and corollary above, we only need the conditions (1) and (2) of Definition 2.2. In other words, one does not need the unknown piece in tracial approximation to be small.

Note that with A a C*-algebra in the class $\mathrm{TA}\mathscr{C}$ for some class $\mathscr{C}$, any element $a \in A$ can be approximated by

$$pap + (1 - p)a(1 - p),$$

where pap is approximately inside a sub-C*-algebra of A which is in the class $\mathscr{C}$. We shall see in the following that if A is an infinite-dimensional simple C*-algebra with the property (SP), we can make the piece $(1 - p)a(1 - p)$ to be uniformly small with respect to tracial states. First, we have the following well-known lemma (for example, see Lemma 3.5.6(b) of [19]):

Lemma 4.9. *Let A be a simple C*-algebra with the property* (SP). *Then, for any finite set of projections $\{p_1, \dots, p_n\}$, there is a non-zero subprojection e of p_1 such that e is Murray–von Neumann equivalent to a subprojection of p_i for all $1 \leqslant i \leqslant n$.*

Combining the above lemma with the fact that any infinite-dimensional unital simple C*-algebra contains a positive element with infinite points in its spectrum, we have the following observation: for any unital simple infinite-dimensional C*-algebra A with the property (SP), and for any $n > 0$, there exist n mutually orthogonal projections $\{q_1, \dots, q_n\}$ in A which are Murray–von Neumann equivalent to each other. Applying this observation to any unital hereditary sub-C*-algebra of A, we have the following lemma.

Lemma 4.10. *Let A be a infinite-dimensional simple unital C*-algebra with the property* (SP). *Then, for any natural number N and any non-zero projection p, there is a non-zero projection $q \in A$ such that $N[q]_0 \leqslant [p]$. Hence, the projection q is less that $1/N$ with respect to any tracial state of A* (*if tracial states exist*).

In particular, this lemma holds for $p = 1$. Therefore, if $A \in \mathrm{TA}\mathscr{C}$ for some class $\mathscr{C}$, by applying tracial approximation to any finite subset $\mathcal{F} \subseteq A$, any $\varepsilon > 0$, and q, we get the following lemma.

Lemma 4.11. *Let $A \in \mathrm{TA}\mathscr{C}$ for some class $\mathscr{C}$ of C*-algebras. If A is simple, infinite-dimensional, and has the property* (SP)*, then, for any natural number N and any finite subset $\mathcal{F} \subseteq A$, any $\varepsilon > 0$ and any $a \in A^+$, there is $C \in \mathscr{C}$ with $I_C = p$ satisfying the approximation with respect to $\mathcal{F}$, ε and a, such that*

$$N[1 - p]_0 \leqslant [1]_0.$$

Recall that a unital C*-algebra A has *the Blackadar comparison property* if $\mathrm{T}(A)$ is non-empty, and for any projections $p, q \in A$, if $\tau(p) < \tau(q)$ holds for any $\tau \in \mathrm{T}(A)$, then p is Murray–von Neumann equivalent to a subprojection of q.

Theorem 4.12. *Let $\mathscr{C}$ be a class of unital C*-algebras with the Blackadar comparison property. Then any simple C*-algebra $A \in \mathrm{TA}\mathscr{C}$ has the Blackadar comparison property.*

Proof. By Theorem 4.6, the C*-algebra A has tracial state. Let p and q be two projections in A with $\tau(p) < \tau(q)$ for any $\tau \in \mathrm{T}(A)$. We shall show that $p \precsim q$.

Let us assume that A has the (SP) property. Since the simplex of tracial states is compact under the pointwise convergence topology, there exists $\delta > 0$ such that $\tau(p) < \tau(q) - \delta$ for any $\tau \in \mathrm{T}(A)$. Since A is simple and has the (SP) property, by Lemma 4.10, there is a subprojection q' of q such that $\tau(q') < \delta/2$ for any $\tau \in \mathrm{T}(A)$.

Applying tracial approximation (Definition 2.2) to $\mathcal{F} = \{p, q - q'\}$, $\varepsilon = 1/16$ and $a = q'$, we obtain a sub-C*-algebra $C \in \mathscr{C}$ with unit e such that for $f = p, q - q'$,

(1) $\|fe - ef\| < 1/16$,
(2) $efe \in_{1/16} C$, and
(3) $1 - e$ is Murray–von Neumann equivalent to a subprojection of q'.

Therefore, we get

$$p \sim p_1 + p_2 \quad \text{and} \quad q - q' \sim q_1 + q_2,$$

where p_1, q_1 are subprojections of $1 - e$ and p_2, q_2 are projections in S.

We assert that we can choose such a sub-C*-algebra C such that $\tau(p_2) < \tau(q_2)$ for any tracial state τ of C. Indeed, if this were not true, then for any sub-C*-algebra C stated above, there would exist a trace τ on C such that $\tau(p_2) \geqslant \tau(q_2)$. This trace could be extended to a positive linear functional τ with norm 1 on eAe. Let us still denote it by τ. Then the map $\phi : a \mapsto \tau(eae)$ would be a positive linear functional on A with norm 1. Note that $\phi(p) \geqslant \phi(q)$. As the argument of Theorem 4.6, we could apply this construction with any finite subset of A, to obtain an upward directed family of states (ϕ_λ) inside the unit ball of the dual space of A. Choose an accumulation point τ_0, and it is easy to verify that τ_0 is a trace of A. But then we have that

$$\tau_0(p) = \lim \phi_\lambda\big(p_2^{(\lambda)}\big) \geqslant \lim \phi_\lambda\big(q_2^{(\lambda)}\big) = \tau_0(q),$$

which is in contradiction with the assumption on p and q.

Therefore, we may assume that $\tau(p_2) < \tau(q_2)$ for every trace τ of S. Then p_2 is Murray–von Neumann equivalent to a subprojection of q_2. Note that p_1 is a subprojection of $1 - e$ and $1 - e$ is Murray–von Neumann equivalent to a subprojection of q', so that p_1 is Murray–von Neumann equivalent to a subprojection of q'. Therefore we have

$$p \sim p_1 + p_2 \preccurlyeq q' + q_2 \leqslant q' + q - q' = q.$$

If A does not have the (SP) property, there is a positive element $a \in A$ such that the only projection in $\overline{aAa}$ is 0. Apply Definition 2.2 to any finite subset $\mathcal{F}$, any ε, and a, we conclude that there is a unital sub-C*-algebra C of A such that $\mathcal{F} \subseteq_\varepsilon C$. With an argument same as that used above, we conclude that the unital sub-C*-algebra C can be chosen such that there are projections p' and q' in C with $\|p - p'\| < 1/2$, $\|q - q'\| < 1/2$, and $\tau(p') < \tau(q')$ for any $\tau \in \mathrm{T}(C)$. Since C has the Blackadar comparison property, one has that

$$p \sim p' \preccurlyeq q' \sim q,$$

as desired. □

Remark 4.13. The proof above is similar to the proof of Theorem 3.7.2 of [19], which states that any simple TAI-algebra has the Blackadar comparison property.

Corollary 4.14. *Let A be a simple TAS-algebra. The ordered group $\mathrm{K}_0(A)$ is weakly unperforated*; *that is, if $g \in \mathrm{K}_0(A)$ and $ng \in \mathrm{K}_0(A)^+ \setminus \{0\}$ for any $n \in \mathbb{Z}^+$, then $g \in \mathrm{K}_0(A)^+$.*

Proof. We must show that for any two projections p and q in a matrix algebra $\mathrm{M}_k(A)$, if $\mathrm{diag}\{\underbrace{p, \ldots, p}_{n}\}$ is Murray–von Neumann equivalent to a proper subprojection of $\mathrm{diag}\{\underbrace{q, \ldots, q}_{n}\}$ in $\mathrm{M}_{nk}(A)$ for some $n \in \mathbb{Z}^+$, then p is Murray–von Neumann equivalent to a subprojection of q.

Since the projection $\mathrm{diag}\{\underbrace{p,\ldots,p}_{n}\}$ is Murray–von Neumann equivalent to a proper subprojection of $\mathrm{diag}\{\underbrace{q,\ldots,q}_{n}\}$, one has that $n\tau(p) < n\tau(q)$ for any $\tau \in \mathrm{T}(A)$, and hence $\tau(p) < \tau(q)$ for any τ of $\mathrm{T}(A)$. Since $\mathrm{M}_k(A)$ is also a simple TAS-algebra, by Theorem 4.12, the projection p is Murray–von Neumann equivalent to a subprojection of q, and hence the ordered group $\mathrm{K}_0(A)$ is weakly unperforated. □

Let $\mathscr{E}$ be a class of unital stably finite unital C*-algebras, such that any state on the order-unit K_0-group comes from a tracial state. It is well known that any stably finite unital exact C*-algebra has this property. We shall show that this property still holds for C*-algebras in $\mathrm{TA}\mathscr{E}$.

Theorem 4.15. *The map $r : T(A) \to S(\mathrm{K}_0(A), \mathrm{K}_0^+(A), [1])$ is always a surjective map for any simple C*-algebra $A \in \mathrm{TA}\mathscr{E}$, where r is induced by the canonical restriction of tracial states to projections in matrix algebras of A.*

Proof. Let ρ be a positive state over $(\mathrm{K}_0(A), \mathrm{K}_0^+(A), [1])$, and let $\mathcal{F}$ be a finite subset of A with $|\mathcal{F}| > 2$ where $|\mathcal{F}|$ is the cardinality of $\mathcal{F}$.

Let us assume that A has the property (SP). Since A is a simple C*-algebra in $\mathrm{TA}\mathscr{E}$, there is a sub-C*-algebra E of A with $p = 1_E$ such that $E \in \mathscr{E}$, and for any $a \in \mathcal{F}$:

(1) $\|pa - ap\| \leqslant 1/|\mathcal{F}|$,
(2) $pap \in_{1/|\mathcal{F}|} E$, and
(3) $M[1-p] \preccurlyeq [1]$ for a natural number $M \geqslant |\mathcal{F}|$.

Then the map $\phi : a \mapsto pap$ is $\mathcal{F} - \frac{2}{|\mathcal{F}|}$ multiplicative, and $\rho([1-p]) \leqslant \frac{1}{M} \leqslant \frac{1}{|\mathcal{F}|}$.

Define the map $\rho_1 : \mathrm{K}_0(S) \to \mathbb{R}$ to be $\rho_1([q]) = \frac{1}{\rho([p])}\rho([q])$ for any projection q in a matrix algebra of E (since E is a sub-C*-algebra of A). It is clear that ρ_1 is a positive state of $\mathrm{K}_0(E)$. Since E is in the class $\mathscr{E}$, ρ_1 arises from a tracial state of S, denote it by τ_1'. One can extend τ_1' to a state on pAp; we still denote it by τ_1'. Then define a positive linear contraction τ_1 on A by $\tau_1 = \tau_1' \circ \phi$. It is clear that

$$\bigl|\tau_1(ab) - \tau_1(ba)\bigr| \leqslant 4/|\mathcal{F}| \quad \text{for any } a, b \in \mathcal{F}.$$

Moreover, for any projection $q \in \mathcal{F}$, the projection q will be unitarily equivalent to a sum of two orthogonal projections q' and q'' where $q' \leqslant 1-p$, $q'' \in E$, and $\|q'' - \phi(q)\| \leqslant 1/|\mathcal{F}|$. Then for any projection $q \in \mathcal{F}$, we have that

$$\begin{aligned}\bigl|\rho([q]) - \tau_1(q)\bigr| &= \bigl|\rho([q'] + [q'']) - \tau_1(q)\bigr| \\ &\leqslant \frac{1}{|\mathcal{F}|} + \bigl|\rho([q'']) - \tau_1' \circ \phi(q)\bigr| \\ &\leqslant \frac{2}{|\mathcal{F}|} + \bigl|\rho([q'']) - \tau_1'(q'')\bigr|\end{aligned}$$

$$\leqslant \frac{2}{|\mathcal{F}|} + \left|\rho\big([q'']\big) - \rho_1\big([q'']\big)\right|$$
$$\leqslant \frac{4}{|\mathcal{F}|}.$$

Repeating this construction for all finite subsets $(\mathcal{F}_\lambda)$, we obtain an upward directed family of positive linear contractions (τ_λ). Pick an accumulation point τ of (τ_λ) in the unit ball of the dual space of A (with respect to the weak topology). It is a state of A. Moreover, for any a and b in the unit ball of A, we have that

$$\tau(ab) - \tau(ba) = \lim_{\lambda\to\infty}\big(\tau_\lambda(ab) - \tau_\lambda(ba)\big) = 0.$$

Therefore, τ is a tracial state of A.

For any projection $q \in A$, there is a finite subset $\mathcal{F}$ containing q with sufficiently large cardinality. Therefore, we have

$$\tau(q) = \lim_{\lambda\to\infty} \tau_\lambda(q) = \rho\big([q]_0\big).$$

Set T_1 to be the set of all the traces on A which have the property $\tau(f) = \rho([f]_0)$ for any projection f in A. T_1 is a compact set with respect to the weak topology. By the argument above, T_1 is not empty. Since the n by n matrix algebra over a C*-algebra in $\mathrm{TA}\mathscr{E}$ still belongs to the class $\mathrm{TA}\mathscr{E}$, we can apply the construction above to $\mathrm{M}_n(A)$, and get a nonempty compact set of traces on A such that $\tau(f) = \rho([f]_0)$ for any projection f in $\mathrm{M}_n(A)$. Denote by T_n all such tracial states of $\mathrm{M}_n(A)$. Since $T_n \supset T_{n+1}$ and all T_n are compact, there exists a tracial state τ in all of T_n. Therefore the state on $(\mathrm{K}_0(A), \mathrm{K}_0^+(A), [1])$ induced by τ is exactly ρ. And the map $r : \mathrm{T}(A) \to \mathrm{S}(\mathrm{K}_0(A), \mathrm{K}_0^+(A), [1])$ is surjective.

If A does not have the property (SP)—say, there is no nontrivial projection in the hereditary sub-C*-algebra generated by $a \in A$—one can apply the tracial approximation property (Definition 2.2) to any finite subset, any ε, and a. One then concludes that A can be locally approximated by C*-algebras in the class $\mathscr{E}$. The same arguments as above (even more directly) show that the map $r : \mathrm{T}(A) \to \mathrm{S}(\mathrm{K}_0(A), \mathrm{K}_0^+(A), [1])$ is surjective. $\square$

Unital stably finite exact C*-algebras are in the class $\mathscr{E}$. However, not all simple C*-algebras in the class $\mathrm{TA}\mathscr{E}$ are exact. There even exist non-exact separable simple C*-algebras in the class $\mathrm{TA}\mathscr{F}$. Here is an example constructed by M. Dadarlat in [5]:

Example 4.16. Let B be a non-exact unital separable C*-algebra which has a separating sequence of finite-dimensional representations $\{\pi_n\}$. (For instance, we can take B to be the full C*-algebra of the group $\mathbb{F}_2$, the free group with two generators. See [4,35].) Denote by d_n the dimension of the representation π_n.

We are going to construct A as an inductive limit. Set $A_1 = B \oplus \mathrm{M}_{d_1}(\mathbb{C})$. Assume that we already have $A_n = \mathrm{M}_{i_n}(B) \oplus \mathrm{M}_{j_n}(\mathbb{C})$, we set $i_{n+1} = i_n + i_n j_n$ and $j_{n+1} = d_n i_n + j_n$. We then set

$$A_{n+1} = \mathrm{M}_{i_{n+1}}(B) \oplus \mathrm{M}_{j_{n+1}}(\mathbb{C}),$$

and set a map $\varphi_n : A_n \to A_{n+1}$ by

$$A_n \ni (b, m) \mapsto \bigl(\mathrm{diag}\{b, \underbrace{m, \dots, m}_{i_n}\}, \mathrm{diag}\bigl\{\pi_n(b), m\bigr\}\bigr) \in A_{n+1}.$$

Denote by A the inductive limit C*-algebra of the system (A_n, φ_n). We claim that A is a simple C*-algebra in the class TA$\mathscr{F}$. But A contains B, a non-exact C*-algebra, as a sub-C*-algebra; thus A cannot be an exact C*-algebra.

The simplicity of A follows directly from the separability of $\{\pi_n\}$ and the construction of the maps $\{\varphi_n\}$. We wish to show that A satisfies Definition 2.2 for the class of C*-algebras $\mathscr{F}$. For any finite subset $\mathcal{F} \subseteq A$ and $\varepsilon > 0$, we may assume $\mathcal{F} \subseteq A_n$ for some n. Thus, A satisfies conditions (1) and (2) of Definition 2.2; by Proposition 2.7, the C*-algebra A has the property (SP). Therefore, in order to verify the condition (3) of Definition 2.2, it is enough to verify it for any unital hereditary sub-C*-algebra of A. In other words, to see A belongs to the class TA$\mathscr{F}$, it is enough to verify Definition 2.2 for any finite subset $\mathcal{F} \subseteq A$, any $\varepsilon > 0$, and any projection q. We then may assume $\mathcal{F}$, and q are in a building block of A, say A_n. Since $A_n = \mathrm{M}_{i_n}(B) \oplus \mathrm{M}_{j_n}(\mathbb{C})$, we even may assume that $q \in \mathrm{M}_{j_n}(\mathbb{C})$.

Denote by p' the unit of $\mathrm{M}_{i_n}(B)$ in A_n. Passing to $A_{n+1} = \mathrm{M}_{i_{n+1}}(B) \oplus \mathrm{M}_{j_{n+1}}(\mathbb{C})$, the image of p' in $\mathrm{M}_{i_{n+1}}(B)$ by the map φ_n has the form of $(1, 0, \dots, 0)$. By the construction of φ_n, restricting to the sub-C*-algebra $\mathrm{M}_{i_{n+1}}(B)$, the projection $(1, 0, \dots, 0)$ is Murray–von Neumann equivalent to a subprojection of any image of minimal projections of $\mathrm{M}_{j_n}(\mathbb{C})$ by the map φ_n; hence it is Murray–von Neumann equivalent to a subprojection of q. Therefore, we may take the sub-C*-algebra F to be the direct sum of the sub-C*-algebra $\mathrm{M}_{j_{n+1}}(\mathbb{C})$ and the image of $\mathrm{M}_{j_n}(\mathbb{C})$ in $\mathrm{M}_{i_{n+1}}(B)$. It is easy to see that this sub-C*-algebra satisfies Definition 2.2 for the finite subset $\mathcal{F}$ and projection q. Thus, A is a simple separable C*-algebra in the class TA$\mathscr{F}$ which is not an exact C*-algebra.

Let A be unital stably finite C*-algebra. Denote by r the canonical affine map

$$\mathrm{T}(A) \to \mathrm{S}\bigl(\mathrm{K}_0(A), \mathrm{K}_0^+(A), [1_A]_0\bigr)$$

induced by restricting traces to projections in matrix algebras.

Definition 4.17. A unital stably finite C*-algebra A is said to have the property (M) if for any tracial state τ of A satisfying

$$r(\tau) = \lambda_1 \rho_1 + \lambda_2 \rho_2$$

for two states ρ_1, ρ_2 on $\mathrm{K}_0(A)$ where $\lambda_1, \lambda_2 \geqslant 0$ with $\lambda_1 + \lambda_2 = 1$, there exist two tracial states τ_1 and τ_2 of A such that $r(\tau_1) = \rho_1$ and $r(\tau_2) = \rho_2$ respectively, and

$$\tau = \lambda_1 \tau_1 + \lambda_2 \tau_2.$$

Remark 4.18. Let A be a C*-algebra with the property (M). Then the map r is surjective (indeed, one can set one of the λ_i's to be zero). If a tracial state τ is sent to a midpoint of $\mathrm{S}(\mathrm{K}_0(A))$ by the map r, then τ must be a midpoint of $\mathrm{T}(A)$. In particular, the map r preserves extreme points.

Theorem 4.19. *Let $\mathscr{M}$ be a class of unital stably finite C*-algebras with the property* (M). *Then any simple C*-algebras in the class* $\mathrm{TA}\mathscr{M}$ *has the property* (M).

Proof. Let A be a C*-algebra in $\mathrm{TA}\mathscr{M}$. Let τ be a tracial state of A satisfying

$$r(\tau)=\lambda_1\rho_1+\lambda_2\rho_2$$

for some states ρ_1, ρ_2 on $\mathrm{K}_0(A)$ where $\lambda_1, \lambda_2 \geqslant 0$ with $\lambda_1+\lambda_2=1$.

Suppose that A has the property (SP). For any subset $\mathcal{F}$ of the unit ball of A, since $A \in \mathrm{TA}\mathscr{M}$, there is a sub-C*-algebra M with unit p such that p commutes with the elements of $\mathcal{F}$ to within $1/|\mathcal{F}|$, and pap belongs to M up to $1/|\mathcal{F}|$ for any $a \in \mathcal{F}$, where $|\mathcal{F}|$ is the cardinality of $\mathcal{F}$. Therefore, if we define the map $\phi : A \to pAp$ by $a \mapsto pap$, ϕ is unital and $\mathcal{F}-\frac{1}{|\mathcal{F}|}$ multiplicative, and the map ϕ sends $\mathcal{F}$ to M to within $1/|\mathcal{F}|$. Moreover, we may assume that $\tau'(1-p)<1/|\mathcal{F}|$ for any tracial state τ' of A.

Consider the restriction of τ to M and the restrictions of ρ_1 and ρ_2 to $\mathrm{K}_0(M)$, one then has

$$r_M(\tau|_M)=\lambda_1\rho_1|_{\mathrm{K}_0(M)}+\lambda_2\rho_2|_{\mathrm{K}_0(M)}.$$

Since M has the property (M), there exist tracial states τ_1' and τ_2' of M such that $r_M(\tau_1')=\rho_1|_{\mathrm{K}_0(M)}$, $r_M(\tau_2')=\rho_2|_{\mathrm{K}_0(M)}$ and

$$\tau|_M(b)=\lambda_1\tau_1'(b)+\lambda_2\tau_2'(b) \quad \text{for any } b\in M.$$

We then extend these two tracial states to states on pAp; still denote them by τ_1' and τ_2' respectively.

For any $a \in \mathcal{F}$, we have

$$\begin{aligned}
&\bigl|\tau(a)-\bigl(\lambda_1\tau_1'\bigl(\phi(a)\bigr)+\lambda_2\tau_2'\bigl(\phi(a)\bigr)\bigr)\bigr| \\
&\quad\leqslant \bigl|\tau\bigl((1-p)a(1-p)+pap\bigr)-\bigl(\lambda_1\tau_1'\bigl(\phi(a)\bigr)+\lambda_2\tau_2'\bigl(\phi(a)\bigr)\bigr)\bigr|+1/|\mathcal{F}| \\
&\quad\leqslant 2/|\mathcal{F}|+\bigl|\tau\bigl(\phi(a)\bigr)-\bigl(\lambda_1\tau_1'\bigl(\phi(a)\bigr)+\lambda_2\tau_2'\bigl(\phi(a)\bigr)\bigr)\bigr| \\
&\quad\leqslant 5/|\mathcal{F}|.
\end{aligned}$$

Denote by $\tau_i^{\mathcal{F}}$ the state $\tau_i'\circ\phi$ of A for $i=1,2$.

Thus, for any finite subset of the unit ball of A, on can construct two states $\tau_1^{\mathcal{F}}$ and $\tau_2^{\mathcal{F}}$ of A such that for any $a\in\mathcal{F}$,

$$\bigl|\tau(a)-\bigl(\lambda_1\tau_1^{\mathcal{F}}(a)-\lambda_2\tau_2^{\mathcal{F}}(a)\bigr)\bigr|\leqslant 1/|\mathcal{F}|,$$

and for any projection $q\in\mathcal{F}$,

$$\bigl|\tau_1(q)-\rho_1\bigl([q]\bigr)\bigr|\leqslant 1/|\mathcal{F}| \quad \text{and} \quad \bigl|\tau_2(q)-\rho_2\bigl([q]\bigr)\bigr|\leqslant 1/|\mathcal{F}|.$$

Moreover, for any $a,b\in\mathcal{F}$, one has that $|\tau_i^{\mathcal{F}}(ab)-\tau_i^{\mathcal{F}}(ba)|<4/|\mathcal{F}|$ for $i=1,2$.

Since the finite subsets of the unital ball of A form an upward directed collection, there are accumulate points τ_1^∞ and τ_2^∞ for the upward directed families $\{\tau_1^{\mathcal{F}}\}$ and $\{\tau_2^{\mathcal{F}}\}$ in the unital ball

of the dual space of A. A simple argument shows that τ_1^∞ and τ_2^∞ are tracial states of A, and moreover,

$$\tau = \lambda_1\tau_1^\infty + \lambda_2\tau_2^\infty,$$

and

$$\tau_1^\infty(q) = \rho_1\big([q]\big) \quad \text{and} \quad \tau_2^\infty(q) = \rho_2\big([q]\big)$$

for any projection q in A. Using the same trick as that of the argument of Theorem 4.15, we can find two traces τ_1 and τ_2 such that $\tau_1(q) = \rho_1([q])$ and $\tau_2(q) = \rho_2([q])$ for any projection q in matrix algebras of A, and $\tau = \lambda_1\tau_1 + \lambda_2\tau_2$. Therefore, we have

$$r(\tau_1) = \rho_1, \qquad r(\tau_2) = \rho_2$$

and

$$\tau = \lambda_1\tau_1 + \lambda_2\tau_2,$$

as desired.

If A does not have the property (SP), one has that A can be locally approximated by C*-algebras in the class $\mathscr{M}$ by the argument of Theorem 4.15. Using an argument similar with above (even more directly), one can show that A has the property (M). □

Note that all unital homogeneous C*-algebras have the property (M). Therefore, we have the following corollary.

Corollary 4.20. *Let A be a separable simple C*-algebras in the class* TA$\mathscr{H}$*, where $\mathscr{H}$ is the class of unital homogeneous C*-algebras. Then the C*-algebra A has the property* (M)*. In particular, the map r preserves extreme points.*

Remark 4.21. A direct consequence of the above corollary is that any separable simple TAI-algebra has this property. This is important for TAI classification. (See [23,26]. Note that this fact is included in [23], as Lemma 10.9, but was erroneously just assumed in [26].) (The present proof is different from that of Lemma 10.9 of [23].)

Let A be a unital C*-algebra. Denote by $\mathrm{U}(A)$ and $\mathrm{U}_0(A)$ the unitary group of A and the path connected component of 1_A, respectively. Then the group $\mathrm{U}_0(A)$ is a normal subgroup of $\mathrm{U}(A)$, and there is a canonical map π from $\mathrm{U}(A)/\mathrm{U}_0(A)$ to $\mathrm{K}_1(A)$.

Let $\mathscr{C}$ be a class of unital C*-algebras such that the map π is injective for any member of this class. We shall show that for any simple C*-algebra $A \in \mathrm{TA}\mathscr{C}$ with the cancellation property for projections, the map π is still injective.

It is well known that if two unitaries u, v satisfy $\|u - v\| < 2$, then u is path connected to v. There is a little improvement of this fact.

Lemma 4.22. *Let u, v be two unitaries in a C*-algebra A. If $\|u - v\| < 2$ and there is a projection $p \in A$ with $[p, u] = [p, v] = 0$, then there exists a path $w(t)$ in* $\mathrm{U}(A)$ *such that $w(0) = v$,*

$w(1) = u$ *and* $[p, w(t)] = 0$, $\forall t \in [0, 1]$. *Moreover, if* pup *and* pvp *are in a unital sub-C*-algebra* B *of* pAp, *then the path* $w(t)$ *can be chosen such that* $pw(t)p$ *is a path of unitaries in the C*-algebra* B.

Proof. Since $\|u - v\| < 2$, the point -1 is not in the spectrum of v^*u. Therefore, there is a real-valued continuous function ψ on $\mathrm{sp}(v^*u)$ such that $z = \exp(i\psi(z))$. Set $h = \psi(v^*u) \in A^{s.a}$, and one has $v^*u = \exp(ih)$. Set $w'(t) = \exp(ith)$. It is a path in the unitary group of A such that $w'(0) = 1_A$ and $w'(1) = v^*u$. Moreover, since $[p, u] = 0$ and $[p, v] = 0$, one has $[p, h] = 0$. Hence, $[p, w'(t)] = 0$. Set $w(t) = vw'(t)$. It is clear that $w(0) = v$, $w(1) = u$ and $[p, w(t)] = 0$ for all $t \in [0, 1]$.

If pup and pvp are in a unital sub-C*-algebra B of pAp, then one has that

$$php = \psi\big((pv^*p)(pup)\big) \in B,$$

and hence for any $t \in [0, 1]$,

$$pw(t)p = pvw'(t)p = (pvp)\big(\exp\big(it(php)\big)\big) \in B,$$

as desired. □

Theorem 4.23. *Let* A *be a simple C*-algebra in* $\mathrm{TA}\mathscr{C}$ *with the cancellation property for projections. Let* $\mathrm{U}(A)$ *and* $\mathrm{U}_0(A)$ *denote the unitary group of* A *and the path connected component containing the identity* 1, *respectively. Then the canonical map* $\mathrm{U}(A)/\mathrm{U}_0(A) \to \mathrm{K}_1(A)$ *is injective.*

Proof. To show the injectivity of the map $\mathrm{U}(A)/\mathrm{U}_0(A) \to \mathrm{K}_1(A)$, it is enough to show for any unitary $u \in A$, if

$$\mathrm{diag}\{\underbrace{u, 1, \ldots, 1}_{n+1}\}$$

is path connected to

$$\mathrm{diag}\{\underbrace{1, 1, \ldots, 1}_{n+1}\}$$

in the unitary group of $\mathrm{M}_{n+1}(A)$ for some n, then u is path connected to 1 in $\mathrm{U}(A)$.

Suppose that A has the property (SP). Let $\mathrm{diag}\{u, 1, \ldots, 1\}$ be a unitary in $\mathrm{M}_{n+1}(A)$ which is path connected to $\mathrm{diag}\{1, 1, \ldots, 1\}$ by a path $W(t)$. Then there is a partition $0 = t_1 < t_2 < \cdots < t_s = 1$ such that

$$\big\|W(t_k) - W(t_{k+1})\big\| < 2, \quad 0 \leqslant k \leqslant s - 1.$$

Since A is simple and has the property (SP), there are $n + 1$ mutually orthogonal projections $\{q_1, \ldots, q_{n+1}\}$ in A which is Murray–von Neumann equivalent to each other. Since $A \in \mathrm{TA}\mathscr{C}$, one may assume that there is a sub-C*-algebra $C \in \mathrm{TA}\mathscr{C}$ with $0 \neq p \in 1_C$ such that

$$W(t_k)_{i,j} = W'(t_k)_{i,j} + W''(t_k)_{i,j}$$

with $W'(t_k)_{i,j} \in (1-p)A(1-p)$, $W''(t_k)_{i,j} \in C$ for each $0 \leqslant k \leqslant s$ and $1 \leqslant i, j \leqslant n+1$, and $(1-p) \preccurlyeq q_1$. Moreover, one can assume $(W'(t_k)_{i,j})$ is a unitary in $\mathrm{M}_{n+1}((1-p)A(1-p))$ and $(W''(t_k)_{i,j})$ is a unitary in $\mathrm{M}_{n+1}(C)$. By Lemma 4.22, there is a path of unitaries $w_k(t)$ between $W(t_k)$ and $W(t_{k+1})$ for each $0 \leqslant k \leqslant s-1$ and w_k commutes with $Q = \mathrm{diag}\{1-p, 1-p, \ldots, 1-p\}$. Therefore, $Qw_k(t)Q$ is a path of unitaries in $\mathrm{M}_{n+1}((1-p)A(1-p))$ which connects $(W'(t_k)_{i,j})$ and $(W'(t_{k+1})_{i,j})$. Moreover, $(1-Q)w_k(t)(1-Q)$ is a path of unitaries in $\mathrm{M}_{n+1}(C)$ which connects $(W''(t_k)_{i,j})$ and $(W''(t_{k+1})_{i,j})$. Then, there is a path of unitaries $Qw(t)Q$ in $\mathrm{M}_{n+1}((1-p)A(1-p))$ which connects $(W'(t_0)_{i,j})$ and $(W'(t_s)_{i,j}) = Q$, and there is a path of unitaries $(1-Q)w(t)(1-Q)$ in $\mathrm{M}_{n+1}(C)$ which connects the unitary

$$\bigl(W''(t_0)_{i,j}\bigr) = \mathrm{diag}\bigl\{W''(t_0)_{1,1}, p, \ldots, p\bigr\}$$

and the unitary

$$\bigl(W''(t_s)_{i,j}\bigr) = 1 - Q = \mathrm{diag}\{p, \ldots, p\}.$$

Therefore, the unitary $W''(t_0)_{1,1}$ has trivial K_1 class. Since the canonical map

$$\pi : \mathrm{U}(C)/\mathrm{U}_0(C) \to \mathrm{K}_1(C)$$

is injective, one has that $W''(t_0)_{1,1}$ is path connected to $p = 1_C$ in the unitary group of C.

Note that $(1-p) \preccurlyeq q_1$ and the mutually orthogonal projections $q_1, \ldots, q_{n+1}$ are Murray–von Neumann equivalent to each other. Since A has the cancellation property for projections, there are partial isometries $\{v_1 = 1-p, v_2, \ldots, v_{n+1}\}$ such that the source projections are $1-p$ and the range projections are mutually orthogonal. Set

$$V = \{v_1, v_2, \ldots, v_{n+1}\} \in \mathrm{M}_{1,n+1}(A).$$

It is easy to verify that $V^*V = Q$. Then

$$c(t) = VQw(t)QV^* + (1 - VV^*)$$

is a path of unitaries in A. One has

$$\begin{aligned} c(0) &= V\,\mathrm{diag}\bigl\{(1-p)u(1-p), 1-p, \ldots, 1-p\bigr\}V^* + (1 - VV^*) \\ &= (1-p)u(1-p) + p \\ &= W'(t_0)_{1,1} + p \end{aligned}$$

and

$$\begin{aligned} c(1) &= V\,\mathrm{diag}\{1-p, 1-p, \ldots, 1-p\}V^* + (1 - VV^*) \\ &= (1-p) + p \\ &= 1. \end{aligned}$$

Therefore $W'(t_0)_{1,1} + p$ is path connected to 1 in the unitary group of A.

Note that

$$u = W(t_0)_{1,1} = W'(t_0)_{1,1} + W''(t_0)_{1,1}.$$

Since $W''(t_0)_{1,1}$ is path connected to $p = 1_C$ in the unitary group of C, one has that u is path connected to $c(0) = W'(t_0)_{1,1} + p$ in the unitary group of A, and hence u is path connected to $c(1) = 1$ as desired.

If A does not have the property (SP), A can be locally approximated by C*-algebras in the class $\mathscr{C}$. Therefore, using an argument same as above, we may assume that the unitary

$$\operatorname{diag}\{\underbrace{u, 1, \ldots, 1}_{n+1}\}$$

is path connected to

$$\operatorname{diag}\{\underbrace{1, 1, \ldots, 1}_{n+1}\}$$

in a unitary group of the matrix algebra $\mathrm{M}_{n+1}(C)$ for a unital sub-C*-algebra C of A which is in the class $\mathscr{C}$. Hence, the unitary u is path connected to the identity in the unitary group of C, and in particular, u is path connected to the identity in the unitary group of A. $\square$

Remark 4.24. If A is a simple TAS-algebra, then A has stable rank one by Corollary 4.4, and therefore, the natural map $\mathrm{U}(A)/\mathrm{U}_0(A) \to \mathrm{K}_1(A)$ is an isomorphism.

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Nonexistence of self-similar singularities in the viscous magnetohydrodynamics with zero resistivity

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Abstract

We are concerned on the possibility of finite time singularity in a partially viscous magnetohydrodynamic equations in $\mathbb{R}^n$, $n = 2, 3$, namely the MHD with positive viscosity and zero resistivity. In the special case of zero magnetic field the system reduces to the Navier–Stokes equations in $\mathbb{R}^n$. In this paper we exclude the scenario of finite time singularity in the form of self-similarity, under suitable integrability conditions on the velocity and the magnetic field. We also prove the nonexistence of asymptotically self-similar singularity. This provides us information on the behavior of solutions near possible singularity of general type as described in Corollary 1.1.

Keywords: Viscous MHD; Self-similar singularity; Asymptotically self-similar singularity

1. Introduction

The equations of magnetohydrodynamics (MHD) with zero resistivity in $\mathbb{R}^n$, $n = 2, 3$, are the following:

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v = \nu \Delta v - \nabla\left(p + \frac{1}{2}|b|^2\right) + (b \cdot \nabla)b, \tag{1.1}$$

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$$\frac{\partial b}{\partial t} + (v \cdot \nabla) b = (b \cdot \nabla) v, \tag{1.2}$$

$$\operatorname{div} v = \operatorname{div} b = 0, \tag{1.3}$$

$$v(x,0) = v_0(x), \quad b(x,0) = b_0(x), \tag{1.4}$$

where $v = (v_1, \dots, v_n)$, $v_j = v_j(x,t)$, $j = 1, \dots, n$, is the velocity of the flow, $b = (b_1, \dots, b_n)$, $b_j = b_j(x,t)$, is the magnetic field, $p = p(x,t)$ is the scalar pressure, $\nu > 0$ is the viscosity of the fluid, and v_0, b_0 are the given initial velocity and magnetic fields, satisfying $\operatorname{div} v_0 = \operatorname{div} b_0 = 0$, respectively. The system (1.1)–(1.4) describes the macroscopic behavior of electrically conducting incompressible fluids with extremely high conductivity. In the original (fully viscous) equations of magnetohydrodynamics, besides the viscosity term, $\nu \Delta v$, in (1.1) we have the resistivity term, $\eta \Delta b$, in the right-hand side of (1.2), where η is the resistivity constant, which is inversely proportional to the electrical conductivity constant, σ. In the extremely high electrical conductivity cases, which occur frequently in the cosmical and geophysical problems we ignore the resistivity term to have our system (1.1)–(1.4) (see e.g. [4]). We are concerned here the mathematical question of the global well-posedness/finite time singularity of the system (1.1)–(1.4). The proof of local well-posedness of the Cauchy problem is rather standard, following argument in [12] (actually the necessary essential estimates are derived in the proof of Lemma 2.1), and similar to the case of fully viscous MHD, which is done in [17]. The question of spontaneous apparition of singularity from a local classical solution is a challenging open problem in the mathematical fluid mechanics. The situation is similar to the both of the cases of ideal MHD and fully viscous MHD. We just refer some of the studies on the finite time blow-up problem in the ideal MHD ([1,6–8,10,16] and references therein). In order to discuss the self-similar singularity of the system (1.1)–(1.4) we first observe that it has the following scaling property: if (v, b, p) is a solution of (1.1)–(1.4) corresponding to the initial data (v_0, b_0), then for any $\lambda > 0$ the functions

$$v^\lambda(x,t) = \lambda v\big(\lambda x, \lambda^2 t\big), \qquad b^\lambda(x,t) = \lambda b\big(\lambda x, \lambda^2 t\big),$$

and

$$p^\lambda(x,t) = \lambda^2 p\big(\lambda x, \lambda^2 t\big)$$

are also solutions with the initial data $v_0^\lambda(x) = \lambda v_0(\lambda x)$, $b_0^\lambda(x) = \lambda b_0(\lambda x)$. In view of the above scaling property the self-similar blowing up solution $(v(x,t), b(x,t))$ of the system (1.1)–(1.4), if it exists, should be of the form,

$$v(x,t) = \frac{1}{\sqrt{T_* - t}} V\left(\frac{x}{\sqrt{T_* - t}}\right), \tag{1.5}$$

$$b(x,t) = \frac{1}{\sqrt{T_* - t}} B\left(\frac{x}{\sqrt{T_* - t}}\right), \tag{1.6}$$

$$p(x,t) = \frac{1}{T_* - t} P\left(\frac{x}{\sqrt{T_* - t}}\right) \tag{1.7}$$

for t close to the possible blow-up time T_*. If we substitute (1.5)–(1.7) into (1.1)–(1.4), then we find that (V, B, P) should be a solution of the stationary system:

$$\frac{1}{2}V + \frac{1}{2}(y\cdot\nabla)V + (V\cdot\nabla)V = \nu\Delta V + (B\cdot\nabla)B - \nabla\left(P + \frac{1}{2}|B|^2\right), \tag{1.8}$$

$$\frac{1}{2}B + \frac{1}{2}(y\cdot\nabla)B + (V\cdot\nabla)B = (B\cdot\nabla)V, \tag{1.9}$$

$$\operatorname{div} V = \operatorname{div} B = 0. \tag{1.10}$$

Conversely, if (V, B, P) is a smooth solution of the system (1.8)–(1.10), then the triple of functions (v, b, p) defined by (1.5)–(1.7) is a smooth solution of (1.1)–(1.4), which blows up at $t = T_*$. The search for self-similar singularities of the form, (1.5)–(1.7) was suggested first by Leray for the 3D Navier–Stokes equations in [14], and its nonexistence was first proved by Nečas, Ružička and Šverák in [15] under the condition of $V \in L^3(\mathbb{R}^3) \cap H^1_{\mathrm{loc}}(\mathbb{R}^3)$, the result of which was generalized later by Tsai to the case $L^p(\mathbb{R}^3) \cap H^1_{\mathrm{loc}}(\mathbb{R}^3)$ with $p > 3$ in [19]. Their proofs crucially depend on the maximum principle of the Leray system,

$$\frac{1}{2}V + \frac{1}{2}(y\cdot\nabla)V + (V\cdot\nabla)V = -\nabla P + \nu\Delta V, \qquad \operatorname{div} V = 0,$$

which corresponds to a special case($B = 0$) in (1.8)–(1.10). The corresponding maximum principle for (1.8)–(1.10), however, cannot be obtained by applying similar method used in [15,19] (the situation is similar even if we have 'special' resistivity term $\nu\Delta B$ to the right-hand side of (1.9)). Due to this fact there are difficulties in extending the nonexistence results for the self-similar singularity of the 3D Navier–Stokes system to our system (1.1)–(1.4). Recently, the author of this paper developed new method to prove nonexistence of the self-similar singularity of the 3D Euler system under suitable integrability condition on the vorticity [2]. Here we first combine the argument in [2] together with the results by [15,19] to obtain the nonexistence of self-similar blowing up solutions, the precise statement of which is in the following theorem.

Theorem 1.1. *Suppose there exists $T_* > 0$ such that we have a representation of a solution (v, b) to* (1.1)–(1.4) *by* (1.5)–(1.6) *for all $t \in (0, T_*)$ with (V, B) satisfying the following conditions*:

(i) $(V, B) \in [C^1(\mathbb{R}^n)]^2$, $\nabla V \in L^\infty(\mathbb{R}^n)$, *and* $\operatorname{div} V = \operatorname{div} B = 0$.
(ii) *In the case $n = 3$, there exists $q_1 \in [3, \infty)$ such that $V \in L^{q_1}(\mathbb{R}^3)$.*
In the case $n = 2$, $V \in L^2(\mathbb{R}^2)$.
(iii) *There exists $q_2 > 0$ such that $B \in L^q(\mathbb{R}^n)$ for all $q \in (0, q_2)$.*

Then, $V = B = 0$.

Remark 1.1. In order to illustrate the integrability condition for B in (iii) above we make the following observations. If a function $f(x)$ on $\mathbb{R}^n$ satisfies

$$\sup_{x\in\mathbb{R}^n}\left(1 + |x|^k\right)\left|f(x)\right| < \infty \quad \forall k \in \mathbb{N},$$

then $f \in L^p(\mathbb{R}^n)$ for all $p \in (0,\infty)$. Indeed, given $p \in (0,\infty)$, we choose $k = [\frac{n+1}{p}]$. Then, we have

$$\int_{\mathbb{R}^n} |f(x)|^p\,dx \leqslant \int_{\mathbb{R}^n} \left(\frac{C}{1+|x|^k}\right)^p dx \leqslant C(n,p)\int_0^\infty \frac{r^{n-1}}{(1+r^{n+1})}\,dr < \infty.$$

Under different type of decay conditions on (V,B) from the above theorem, we could also have similar nonexistence result as follows.

Theorem 1.2. *Suppose there exists $T_* > 0$ such that we have a representation of a solution (v,b) to* (1.1)–(1.4) *by* (1.5)–(1.6) *for all $t \in (0,T_*)$ with (V,B) satisfying the following conditions*:

(i) $(V,B) \in [H^m(\mathbb{R}^n)]^2$, $m > n/2+1$.
(ii) $\|\nabla V\|_{L^\infty} + \|\nabla B\|_{L^\infty} < \eta$, *where η is a sufficiently small constant to be determined in Lemma* 2.1 *in the next section.*

Then, $V = B = 0$.

Remark 1.2. The above theorem implies the '*stability of the null solution*' of the stationary system (1.8)–(1.10). Namely, there exists $\eta > 0$ such that if (V,B) is a solution to (1.8)–(1.10) and belongs to a ball $B(0,\eta) = \{X = (V,B) \in H^m(\mathbb{R}^n) \mid \|\nabla X\|_{L^\infty} < \eta\}$, where $m > n/2+1$, then $(V,B) = (0,0)$.

Next, we consider more refined scenario of 'asymptotically self-similar singularity,' which means that the local in time smooth solution evolves into a self-similar profile as the possible singularity time is approached. A similar notion was considered previously by Giga and Kohn in the context of the nonlinear scalar heat equation in [9]. Recently, the author of this paper [3] considered it in the context of 3D Euler and the 3D Navier–Stokes equations (see also [11]), and excluded its scenario. We apply the idea developed in [3] to exclude asymptotically self-similar singularity of our system (1.1)–(1.4).

Theorem 1.3. *Let $(v,b) \in [C([0,T);H^m(\mathbb{R}^n))]^2$, $m > n/2+1$, be a classical solutions to* (1.1)–(1.4). *Suppose there exist functions $\bar V$, $\bar B$ satisfying the conditions* (i)–(iii) *for V, B in Theorem* 1.1 *such that the following boundedness and the convergence hold true*:

$$\begin{aligned}&\sup_{0<t<T}(T-t)^{\frac{1-n}{2}}\left\|v(\cdot,t) - \frac{1}{\sqrt{T-t}}\bar V\left(\frac{\cdot}{\sqrt{T-t}}\right)\right\|_{L^1}\\ &\quad+\sup_{0<t<T}(T-t)^{\frac{1-n}{2}}\left\|b(\cdot,t) - \frac{1}{\sqrt{T-t}}\bar B\left(\frac{\cdot}{\sqrt{T-t}}\right)\right\|_{L^1} < \infty,\end{aligned} \tag{1.11}$$

$$\begin{aligned}&\lim_{t\nearrow T}(T-t)\left\|\nabla v(\cdot,t) - \frac{1}{\sqrt{T-t}}\nabla\bar V\left(\frac{\cdot}{\sqrt{T-t}}\right)\right\|_{L^\infty}\\ &\quad+\lim_{t\nearrow T}(T-t)\left\|\nabla b(\cdot,t) - \frac{1}{T-t}\nabla\bar B\left(\frac{\cdot}{\sqrt{T-t}}\right)\right\|_{L^\infty} = 0.\end{aligned} \tag{1.12}$$

Then, $\bar{V} = \bar{B} = 0$, *and* (v, b) *can be extended to a solution of* (1.1)–(1.4) *in* $[0, T + \delta] \times \mathbb{R}^n$, *and belongs to* $C([0, T + \delta]; H^m(\mathbb{R}^n))$ *for some* $\delta > 0$.

Remark 1.3. Unlike to the cases of the Euler equations [3], the convergence of (1.12) is not in the critical Besov space norms for the quantities of vorticities and current densities, but in the Lipschitz norm for the gradients of velocities and magnetic fields. Actually due to the non-symmetry of the viscosity terms (the term $\nu \Delta v$ for the velocity evolution equations (1.1), and zero for the magnetic field evolution equations (1.2)) we cannot obtain critical Besov space type of norm estimates in the procedure of proof of the above theorem (see the proof in the next section).

As an immediate corollary of Theorem 1.3 we have the following information of the behaviors of solution near possible singularity, which is not necessarily of the self-similar type.

Corollary 1.1. *Let* $(v, b) \in [C([0, T_*); H^m(\mathbb{R}^n))]^2$, $m > n/2 + 1$, *be a classical solutions to* (1.1)–(1.4), *which blows up at* T. *We expand the solution of the form*:

$$v(x, t) = \frac{1}{\sqrt{T - t}} \bar{V}\left(\frac{x}{\sqrt{T - t}}\right) + \bar{v}(x, t), \tag{1.13}$$

$$b(x, t) = \frac{1}{\sqrt{T - t}} \bar{B}\left(\frac{x}{\sqrt{T - t}}\right) + \bar{b}(x, t), \tag{1.14}$$

where $(\bar{V}, \bar{B})$ *satisfies the conditions* (i)–(iii) *for* (V, B) *in Theorem* 1.1. *Then, either*

$$\limsup_{t \nearrow T} \left[(T - t)^{\frac{1-n}{2}} \left(\left\|\bar{v}(t)\right\|_{L^1} + \left\|\bar{b}(t)\right\|_{L^1}\right)\right] = \infty, \tag{1.15}$$

or there exists $\varepsilon_0 > 0$ *such that*

$$\limsup_{t \nearrow T} \left[(T - t)\left(\left\|\nabla \bar{v}(t)\right\|_{L^\infty} + \left\|\nabla \bar{b}(t)\right\|_{L^\infty}\right)\right] > \varepsilon_0. \tag{1.16}$$

2. Proof of the theorems

Proof of Theorem 1.1. We assume classical solution (v, b) of the form (1.5)–(1.6). We will show that this assumption leads to $V = B = 0$. By consistency with the initial condition, $b_0(x) = \frac{1}{\sqrt{T_*}} B(\frac{x}{\sqrt{T_*}})$, we can rewrite the representation (1.6) in the form,

$$b(x, t) = \left(1 - \frac{t}{T_*}\right)^{-\frac{1}{2}} b_0\left(\left(1 - \frac{t}{T_*}\right)^{-\frac{1}{2}} x\right) \quad \forall t \in [0, T_*). \tag{2.1}$$

Let $a \mapsto X(a, t)$ be the particle trajectory mapping, defined by the ordinary differential equations,

$$\frac{\partial X(a, t)}{\partial t} = v\big(X(a, t), t\big); \qquad X(a, 0) = a.$$

We set $A(x,t) := X^{-1}(x,t)$, which is called the back-to-label map, satisfying

$$A\big(X(a,t),t\big) = a, \qquad X\big(A(x,t),t\big) = x. \tag{2.2}$$

We note that for our smoothness condition (i) decay condition on the velocity (ii) the existence of $A(\cdot,t)$ is guaranteed at lease for t close to T_* (see [5]), which is enough for our purpose in the proof. Taking dot product (1.2) by b, we obtain

$$\frac{\partial |b|}{\partial t} + (v\cdot\nabla)|b| = \alpha|b|, \tag{2.3}$$

where $\alpha(x,t)$ is defined as

$$\alpha(x,t) = \begin{cases} \sum_{i,j=1}^n S_{ij}(x,t)\xi_i(x,t)\xi_j(x,t) & \text{if } b(x,t)\neq 0,\\ 0 & \text{if } b(x,t)=0,\end{cases}$$

with

$$S_{ij} = \frac{1}{2}\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right) \quad \text{and} \quad \xi(x,t) = \frac{b(x,t)}{|b(x,t)|}.$$

In terms of the particle trajectory mapping we can rewrite (2.3) as

$$\frac{\partial}{\partial t}\big|b\big(X(a,t),t\big)\big| = \alpha\big(X(a,t),t\big)\big|b\big(X(a,t),t\big)\big|. \tag{2.4}$$

Integrating (2.4) along the particle trajectories $\{X(a,t)\}$, we have

$$\big|b\big(X(a,t),t\big)\big| = \big|b_0(a)\big|\exp\left[\int_0^t \alpha\big(X(a,s),s\big)\,ds\right]. \tag{2.5}$$

Taking into account the simple estimates

$$-\big\|\nabla v(\cdot,t)\big\|_{L^\infty} \leqslant \alpha(x,t) \quad \forall x\in\mathbb{R}^n,$$

we obtain from (2.5) that

$$\big|b_0(a)\big|\exp\left[-\int_0^t \big\|\nabla v(\cdot,s)\big\|_{L^\infty}\,ds\right] \leqslant \big|b\big(X(a,t),t\big)\big|,$$

which, in terms of the back-to-label map, can be rewritten as

$$\big|b_0\big(A(x,t)\big)\big|\exp\left[-\int_0^t \big\|\nabla v(\cdot,s)\big\|_{L^\infty}\,ds\right] \leqslant \big|b(x,t)\big|. \tag{2.6}$$

Combining this with the self-similar representation formula in (2.1), we have

$$\left|b_0\big(A(x,t)\big)\right| \exp\left[-\int_0^t \left\|\nabla v(\cdot,s)\right\|_{L^\infty} ds\right] \leqslant \left(1-\frac{t}{T_*}\right)^{-\frac{1}{2}} \left|b_0\left(\left(1-\frac{t}{T_*}\right)^{-\frac{1}{2}} x\right)\right|. \tag{2.7}$$

Given $q \in (0, q_2)$, computing $L^q(\mathbb{R}^n)$ norm of the each side of (2.7), we obtain

$$\|b_0\|_{L^q} \exp\left[-\int_0^t \left\|\nabla v(\cdot,s)\right\|_{L^\infty} ds\right] \leqslant \|b_0\|_{L^q} \left(1-\frac{t}{T_*}\right)^{\frac{n}{2q}-\frac{1}{2}} \tag{2.8}$$

where we used the fact $\det(\nabla A(x,t)) \equiv 1$. Now, suppose $B \neq 0$, which is equivalent to assuming that $b_0 \neq 0$, then we divide (2.8) by $\|b_0\|_{L^q}$ to have

$$\exp\left[-\int_0^t \left\|\nabla v(\cdot,s)\right\|_{L^\infty} ds\right] \leqslant \left(1-\frac{t}{T_*}\right)^{\frac{n}{2q}-\frac{1}{2}}. \tag{2.9}$$

Passing $q \searrow 0$ in (2.9), we deduce that

$$\int_0^t \left\|\nabla v(\cdot,s)\right\|_{L^\infty} ds = \infty \quad \forall t \in (0, T_*).$$

This contradicts with the assumption that the flow is smooth on $(0, T_*)$, i.e.

$$v \in C^1\big([0, T_*); C^1\big(\mathbb{R}^n\big) \cap W^{1,\infty}\big(\mathbb{R}^n\big)\big),$$

which is implied by the by the explicit representation formula (1.5)–(1.6), combined with the assumption (i). Hence we need to have $B = 0$. Setting $B = 0$ in the system (1.1)–(1.4), it reduces to the incompressible Navier–Stokes system in $\mathbb{R}^n$. When $n = 3$ we apply Nečas–Ružička–Šverák's result in [15] for $q_1 = 3$ and Tsai's result in [19] for $q_1 \in (3, \infty)$, respectively. Then, we obtain $V = 0$. In the case $n = 2$ we recall that in the 2D Navier–Stokes equations for the initial data $v_0(\cdot) = \frac{1}{\sqrt{T_*}} V(\frac{\cdot}{\sqrt{T_*}}) \in L^2(\mathbb{R}^2)$ the solution v belongs to $C^\infty((0,\infty) \times \mathbb{R}^2)$ (see e.g. [18]), and hence we need to have $V = 0$. $\square$

In order to prove Theorems 1.2 and 1.3 we establish the following continuation principle for local classical solution of (1.1)–(1.4).

Lemma 2.1. *Let $(v, b) \in [C([0,T]; H^m(\mathbb{R}^n))]^2$, $m > n/2 + 1$, be a classical solution to* (1.1)–(1.4)*. There exists an absolute constant $\eta > 0$ such that if*

$$\sup_{0 \leqslant t < T} (T-t)\left\{\left\|\nabla v(t)\right\|_{L^\infty} + \left\|\nabla b(t)\right\|_{L^\infty}\right\} < \eta, \tag{2.10}$$

then the solution $(v(x,t), b(x,t))$ can be extended to be functions on $[0, T+\delta] \times \mathbb{R}^n$, and belongs to $C([0, T+\delta]; H^m(\mathbb{R}^n))$ for some $\delta > 0$.

Proof. Let $\alpha=(\alpha_1,\dots,\alpha_n)\in(\mathbb{N}\cup\{0\})^n$ be a standard multi-index with $|\alpha|=\alpha_1+\cdots+\alpha_n$. We take operation $D^\alpha=\partial_1^{\alpha_1}\dots\partial_n^{\alpha_n}$ on (1.1), and take $L^2(\mathbb{R}^n)$ inner product it with D^α, summing over $|\alpha|\leqslant m$ after integration by parts. Then, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v\|_{H^m}^2+\nu\|\nabla v\|_{H^m}^2={}&\sum_{|\alpha|\leqslant m}\big(D^\alpha(v\cdot\nabla)v-(v\cdot\nabla)D^\alpha v,D^\alpha v\big)_{L^2}\\&+\sum_{|\alpha|\leqslant m}\big(D^\alpha(b\cdot\nabla)b-(b\cdot\nabla)D^\alpha b,D^\alpha v\big)_{L^2}\\&+\sum_{|\alpha|\leqslant m}\big((b\cdot\nabla)D^\alpha b,D^\alpha v\big)_{L^2},\end{aligned}\tag{2.11}$$

where we used the facts,

$$\big((v\cdot\nabla)D^\alpha v,D^\alpha v\big)_{L^2}=\frac{1}{2}\int_{\mathbb{R}^n}(v\cdot\nabla)\big|D^\alpha v\big|^2\,dx=-\frac{1}{2}\int_{\mathbb{R}^n}(\operatorname{div}v)\big|D^\alpha v\big|^2\,dx=0,$$

and

$$\left(D^\alpha v,D^\alpha\nabla\left(p+\frac{1}{2}|v|^2\right)\right)_{L^2}=-\left(D^\alpha(\operatorname{div}v),D^\alpha\left(p+\frac{1}{2}|v|^2\right)\right)_{L^2}=0.$$

Applying the well-known commutator estimate [13],

$$\sum_{|\alpha|\leqslant m}\big\|D^\alpha(fg)-fD^\alpha g\big\|_{L^2}\leqslant C\big(\|\nabla f\|_{H^{m-1}}+\|f\|_{H^m}\|g\|_{L^\infty}\big),$$

to the terms of the right-hand side of (2.11), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|v\|_{H^m}^2+\nu\|\nabla v\|_{H^m}^2\\&\quad\leqslant C\|\nabla v\|_{L^\infty}\|v\|_{H^m}^2+C\|\nabla b\|_{L^\infty}\|b\|_{H^m}\|v\|_{H^m}+\sum_{|\alpha|\leqslant m}\big((b\cdot\nabla)D^\alpha b,D^\alpha v\big)_{L^2}.\end{aligned}\tag{2.12}$$

Similarly, starting from (1.2), we can deduce

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|b\|_{H^m}^2={}&-\sum_{|\alpha|\leqslant m}\big(D^\alpha(v\cdot\nabla)b-(v\cdot\nabla)D^\alpha b,D^\alpha b\big)_{L^2}\\&+\sum_{|\alpha|\leqslant m}\big(D^\alpha(b\cdot\nabla)v-(v\cdot\nabla)D^\alpha v,D^\alpha b\big)_{L^2}+\sum_{|\alpha|\leqslant m}\big((b\cdot\nabla)D^\alpha v,D^\alpha b\big)_{L^2}\\&\leqslant C\|\nabla v\|_{L^\infty}\|b\|_{H^m}^2+C\|\nabla b\|_{L^\infty}\|v\|_{H^m}\|b\|_{H^m}+\sum_{|\alpha|\leqslant m}\big((b\cdot\nabla)D^\alpha v,D^\alpha b\big)_{L^2}.\end{aligned}\tag{2.13}$$

We observe that

$$\sum_{|\alpha|\leqslant m}\big((b\cdot\nabla)D^\alpha b, D^\alpha v\big)_{L^2} = -\sum_{|\alpha|\leqslant m}\big((b\cdot\nabla)D^\alpha v, D^\alpha b\big)_{L^2},$$

which is obvious by the integration by part. Thus, adding (2.12) to (2.13), we obtain

$$\frac{1}{2}\frac{d}{dt}\big(\|v\|_{H^m}^2+\|b\|_{H^m}^2\big)+\nu\|\nabla v\|_{H^m}^2 \leqslant C\big(\|\nabla v\|_{L^\infty}+\|\nabla b\|_{L^\infty}\big)\big(\|v\|_{H^m}^2+\|b\|_{H^m}^2\big), \quad (2.14)$$

where we used the inequality, $ab\leqslant \frac{1}{2}(a^2+b^2)$. From (2.14) we first derive the inequality

$$\begin{aligned}
&\big\|v(t)\big\|_{H^m}^2+\big\|b(t)\big\|_{H^m}^2+\nu\int_{t_0}^{t}\big\|\nabla v(s)\big\|_{H^m}^2\,ds\\
&\quad\leqslant\big(\big\|v(t_0)\big\|_{H^m}^2+\big\|b(t_0)\big\|_{H^m}^2\big)\exp\left[\int_{t_0}^{t}\big(\big\|\nabla v(s)\big\|_{L^\infty}+\big\|\nabla b(s)\big\|_{L^\infty}\big)\,ds\right] \qquad (2.15)
\end{aligned}$$

for all $0\leqslant t_0<t$, which implies the continuation principle that if

$$\int_{t_0}^{T}\big(\big\|\nabla v(s)\big\|_{L^\infty}+\big\|\nabla b(s)\big\|_{L^\infty}\big)\,ds<\infty,$$

then $\|v(T)\|_{H^m}+\|b(T)\|_{H^m}<\infty$, and we can continue our classical solution $(v(t),b(t))\in[H^m(\mathbb{R}^n)]^2$ up to $[t_0,T+\delta]$ so that $(v,b)\in[C([0,T+\delta];H^m(\mathbb{R}^n))]^2$ for some $\delta>0$. Next, using the estimate (2.14), we derive

$$\begin{aligned}
&\frac{d}{dt}\big\{(T-t)\big(\|v\|_{H^m}^2+\|b\|_{H^m}^2\big)\big\}+2\nu(T-t)\|\nabla v\|_{H^m}^2+\big(\|v\|_{H^m}^2+\|b\|_{H^m}^2\big)\\
&\quad\leqslant C_0(T-t)\big(\|\nabla v\|_{L^\infty}+\|\nabla b\|_{L^\infty}\big)\big(\|v\|_{H^m}^2+\|b\|_{H^m}^2\big) \qquad (2.16)
\end{aligned}$$

for a constant $C_0=C_0(m,n)$. We suppose

$$\sup_{0<t<T}\big\{(T-t)\big(\big\|\nabla v(t)\big\|_{L^\infty}+\big\|\nabla b(t)\big\|_{L^\infty}\big)\big\}<\frac{1}{2C_0}.$$

Then,

$$\frac{d}{dt}\big\{(T-t)\big(\|v\|_{H^m}^2+\|b\|_{H^m}^2\big)\big\}+2\nu(T-t)\|\nabla v\|_{H^m}^2+\frac{1}{2}\big(\|v\|_{H^m}^2+\|b\|_{H^m}^2\big)\leqslant 0,$$

and integrating this over $[t_0,T]$, we have

$$\sup_{t_0<t<T}(T-t)\big(\|v\|_{H^m}^2+\|b\|_{H^m}^2\big)+2\nu\int_{t_0}^{T}(T-t)\big\|\nabla v(t)\big\|_{H^m}^2\,dt$$
$$+\frac{1}{2}\int_{t_0}^{T}\big(\big\|v(t)\big\|_{H^m}^2+\big\|b(t)\big\|_{H^m}^2\big)\,dt$$
$$\leqslant (T-t_0)\big(\big\|v(t_0)\big\|_{H^m}^2+\big\|b(t_0)\big\|_{H^m}^2\big).$$

In particular,

$$\int_{t_0}^{T}\big(\big\|v(t)\big\|_{H^m}^2+\big\|b(t)\big\|_{H^m}^2\big)\,dt\leqslant 2(T-t_0)\big(\big\|v(t_0)\big\|_{H^m}^2+\big\|b(t_0)\big\|_{H^m}^2\big)<\infty. \tag{2.17}$$

Since $H^m(\mathbb{R}^n)\hookrightarrow Lip(\mathbb{R}^n)$ for $m>n/2+1$, the estimate (2.17) implies

$$\int_{t_0}^{T}\big(\big\|\nabla v(t)\big\|_{L^\infty}+\big\|\nabla b(t)\big\|_{L^\infty}\big)\,dt\leqslant C\int_{t_0}^{T}\big(\big\|v(t)\big\|_{H^m}+\big\|b(t)\big\|_{H^m}\big)\,dt$$
$$\leqslant C\sqrt{T-t_0}\left[\int_{t_0}^{T}\big(\big\|v(t)\big\|_{H^m}^2+\big\|b(t)\big\|_{H^m}^2\big)\,dt\right]^{\frac{1}{2}}<\infty.$$

Applying the continuation principle derived above, we can continue our local solution as described in the theorem. □

Proof of Theorem 1.2. We just observe that

$$(T-t)\big\|\nabla v(t)\big\|_{L^\infty}=\|V\|_{L^\infty},\qquad (T-t)\big\|\nabla b(t)\big\|_{L^\infty}=\|\nabla B\|_{L^\infty}$$

for all $t\in(t_0,T)$. Hence, our smallness condition, $\|\nabla V\|_{L^\infty}+\|\nabla B\|_{L^\infty}<\eta$, leads to

$$\sup_{t_0<t<T}(T-t)\big\{\big\|\nabla v(t)\big\|_{L^\infty}+\big\|\nabla b(t)\big\|_{L^\infty}\big\}<\eta.$$

Applying Lemma 2.1, for initial time at $t=t_0$, we conclude that $(v,b)\in[C([t_0,T);H^m(\mathbb{R}^n))]^2$ cannot have singularity at $t=t_0$, hence we need to have $V=B=0$. □

Proof of Theorem 1.3. We change variables from the physical ones $(x,t)\in\mathbb{R}^n\times[0,T)$ to the 'self-similar variables' $(y,s)\in\mathbb{R}^n\times[0,\infty)$ as follows:

$$y=\frac{x}{\sqrt{T-t}},\qquad s=\frac{1}{2}\log\left(\frac{T}{T-t}\right).$$

Based on this change of variables, we transform the functions $(v,p)\mapsto(V,P)$ according to

$$v(x,t)=\frac{1}{\sqrt{T-t}}V(y,s), \tag{2.18}$$

$$b(x,t)=\frac{1}{\sqrt{T-t}}B(y,s), \tag{2.19}$$

$$p(x,t)=\frac{1}{\sqrt{T-t}}P(y,s). \tag{2.20}$$

Substituting (v,b,p) into (1.1)–(1.4), we obtain the following equivalent evolution equations for (V,P),

$$\begin{cases} \frac{1}{2}V_s+\frac{1}{2}V+\frac{1}{2}(y\cdot\nabla)V+(V\cdot\nabla)V=\nu\Delta V+(B\cdot\nabla)B-\nabla\left(P+\frac{1}{2}|V|^2\right),\\ \frac{1}{2}B_s+\frac{1}{2}B+\frac{1}{2}(y\cdot\nabla)B+(V\cdot\nabla)B=(B\cdot\nabla)V, \qquad \operatorname{div}V=\operatorname{div}B=0,\\ V(y,0)=V_0(y)=\sqrt{T}v_0(\sqrt{T}y), \qquad B(y,0)=B_0(y)=\sqrt{T}b_0(\sqrt{T}y). \end{cases} \tag{2.21}$$

In terms of (V,B) the conditions (1.11) and (1.12) are translated into

$$\sup_{0<s<\infty}\left(\left\|V(\cdot,s)-\bar{V}(\cdot)\right\|_{L^1}+\left\|B(\cdot,s)-\bar{B}(\cdot)\right\|_{L^1}\right)<\infty$$

and

$$\lim_{s\to\infty}\left\|\nabla V(\cdot,s)-\nabla\bar{V}(\cdot)\right\|_{L^\infty}=\lim_{s\to\infty}\left\|\nabla B(\cdot,s)-\nabla\bar{B}(\cdot)\right\|_{L^\infty}=0,$$

respectively, from which, thanks to the standard interpolation, we can have

$$\lim_{s\to\infty}\left\|V(\cdot,s)-\bar{V}(\cdot)\right\|_{H^1(B_R)}=\lim_{s\to\infty}\left\|B(\cdot,s)-\bar{B}(\cdot)\right\|_{H^1(B_R)}=0 \tag{2.22}$$

for all $0<R<\infty$, where $B_R=\{x\in\mathbb{R}^n \mid |x|<R\}$. Similarly to [11], we consider scalar test functions $\xi\in C^1_{\mathrm{c}}(0,1)$ with $\int_0^1\xi(s)\,ds\neq 0$, $\psi\in C^1_{\mathrm{c}}(\mathbb{R}^n)$ and the vector test function $\phi=(\phi_1,\dots,\phi_n)\in C^1_{\mathrm{c}}(\mathbb{R}^n)$ with $\operatorname{div}\phi=0$. We multiply the first equation of (2.21) by $\xi(s-k)\phi(y)$, and integrate it over $\mathbb{R}^n\times[k,k+1]$, and then we integrate by part for the terms including the time derivative and the pressure term to obtain

$$-\int_0^1\int_{\mathbb{R}^n}\xi_s(s)\phi(y)\cdot V(y,s+k)\,dy\,ds$$
$$+\int_0^1\int_{\mathbb{R}^n}\xi(s)\phi(y)\cdot\big[V+(y\cdot\nabla)V+2(V\cdot\nabla)V\big](y,s+k)\,dy\,ds$$

$$- 2\int_0^1 \int_{\mathbb{R}^n} \xi(s)\phi(y) \cdot \big[(B \cdot \nabla)B\big](y, s+k)\,dy\,ds$$

$$+ 2\nu \int_0^1 \int_{\mathbb{R}^n} \xi(s)\nabla\phi(y) \cdot \nabla V(y, s+k)\,dy\,ds = 0 \tag{2.23}$$

and

$$- \int_0^1 \int_{\mathbb{R}^n} \xi_s(s)\psi(y)B(y, s+k)\,dy\,ds$$

$$+ \int_0^1 \int_{\mathbb{R}^n} \xi(s)\psi(y)\big[B + (y \cdot \nabla)B + 2(V \cdot \nabla)B\big](y, s+k)\,dy\,ds$$

$$- 2\int_0^1 \int_{\mathbb{R}^n} \xi(s)\psi(y)\big[(B \cdot \nabla)V\big](y, s+k)\,dy\,ds = 0. \tag{2.24}$$

Passing to the limit $k \to \infty$ in (2.23)–(2.24), using the convergence (2.22), $\int_0^1 \xi_s(s)\,ds = 0$ and $\int_0^1 \xi(s)\,ds \neq 0$, we find that $\bar{V}, \bar{B} \in C^1(\mathbb{R}^n)$ satisfies

$$\int_{\mathbb{R}^n} \big[\bar{V} + (y \cdot \nabla)\bar{V} + 2(\bar{V} \cdot \nabla)\bar{V} - 2(\bar{B} \cdot \nabla)\bar{B}\big] \cdot \phi\,dy + 2\nu \int_{\mathbb{R}^n} \nabla\bar{V} \cdot \nabla\phi\,dy = 0,$$

$$\int_{\mathbb{R}^n} \big[\bar{B} + (y \cdot \nabla)\bar{B} + 2(\bar{V} \cdot \nabla)\bar{B} - 2(\bar{B} \cdot \nabla)\bar{V}\big]\psi\,dy = 0,$$

for all vector test function $\phi \in C_c^1(\mathbb{R}^n)$ with $\operatorname{div}\phi = 0$, and scalar test function $\psi \in C_c^1(\mathbb{R}^n)$. Hence, there exists a scalar function $\bar{P}'$, which can be written without loss of generality as $\bar{P}' = \bar{P} + \frac{1}{2}|\bar{B}|^2$ for another scalar function $\bar{P}$, such that

$$\bar{V} + (y \cdot \nabla)\bar{V} + 2(\bar{V} \cdot \nabla)\bar{V} = 2\nu\Delta\bar{V} + 2(\bar{B} \cdot \nabla)\bar{B} - 2\nabla\left(\bar{P} + \frac{1}{2}|\bar{B}|^2\right) \tag{2.25}$$

and

$$\bar{B} + (y \cdot \nabla)\bar{B} + 2(\bar{V} \cdot \nabla)\bar{B} = 2(\bar{B} \cdot \nabla)\bar{V}. \tag{2.26}$$

On the other hand, we can pass $s \to \infty$ directly in the incompressibility equations for V and B in (2.21) to have

$$\operatorname{div}\bar{V} = \operatorname{div}\bar{B} = 0. \tag{2.27}$$

Equations (2.25)–(2.27) show that $(\bar{V}, \bar{B})$ is a classical solution of (1.8)–(1.10). Since, by hypothesis, $(\bar{V}, \bar{B})$ satisfies the condition (i)–(iii) of Theorem 1.1, we can deduce $\bar{V} = \bar{B} = 0$ by that theorem. Hence, the equations above (2.22) lead to

$$\lim_{s\to\infty} \|\nabla V(s)\|_{L^\infty} = \lim_{s\to\infty} \|\nabla B(s)\|_{L^\infty} = 0.$$

Thus, for $\eta > 0$ given in Lemma 2.1, there exists $s_1 > 0$ such that

$$\|\nabla V(s_1)\|_{L^\infty} + \|\nabla B(s_1)\|_{L^\infty} < \eta.$$

Let us set $t_1 = T[1 - e^{2s_1}]$. Going back to the original physical variables, we have

$$(T - t_1)\|\nabla v(t_1)\|_{L^\infty} + (T - t_1)\|\nabla b(t_1)\|_{L^\infty} < \eta.$$

Applying Lemma 2.1, we conclude the proof. $\square$

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Sharp estimates for large coupling convergence with applications to Dirichlet operators

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Abstract

Let H be a nonnegative selfadjoint operator, $\mathcal{E}$ the closed quadratic form associated with H, and P a nonnegative quadratic form such that $\mathcal{E} + P$ is closed and $D(P) \supset D(H)$. For every $\beta > 0$ let H_β be the selfadjoint operator associated with $\mathcal{E} + \beta P$. The pairs (H, P) satisfying

$$L(H, P) := \liminf_{\beta \to \infty} \beta \left\| (H_\beta + 1)^{-1} - \lim_{\beta' \to \infty} (H_{\beta'} + 1)^{-1} \right\| < \infty$$

are characterized. A sufficient condition for convergence of the operators $(H_\beta + 1)^{-1}$ within a Schatten–von Neumann class of finite order is derived. It is shown that $L(H, P) = 1$, if $\mathcal{E}$ is a regular conservative Dirichlet form with the strong local property and P the killing form corresponding to the equilibrium measure of a closed set with finite capacity and nonempty interior. An example is given where $L(H, P)$ is finite, H is a regular Dirichlet operator and P the killing form corresponding to a measure which has infinite mass and a support with infinite capacity.

Keywords: Schatten class; Time changed Dirichlet form; Equilibrium measure; Killing measure

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1. Introduction

Let H be a nonnegative selfadjoint operator in a Hilbert $\mathcal{H}$ and $\mathcal{E}$ the closed quadratic form associated to H in the sense of Kato's representation theorem. Let P be a nonnegative quadratic form in $\mathcal{H}$ and $(H_\beta)_{\beta>0}$ be a family of nonnegative selfadjoint operators in $\mathcal{H}$ satisfying

$$\begin{aligned} &D(\sqrt{H_\beta}) = D(P) \cap D(\mathcal{E}), \\ &\|\sqrt{H_\beta} f\|^2 = \mathcal{E}(f,f) + \beta P(f,f) \quad \forall f \in D(P) \cap D(\mathcal{E}), \end{aligned} \tag{1}$$

for all $\beta > 0$. By Kato's monotone convergence theorem (cf. [8]), the operators $(H_\beta + 1)^{-1}$ converge strongly, as β tends to infinity. Both in quantum hard core scattering and in semiclassical analysis one is interested in the question about how fast these resolvents converge, cf. [4–6] and references given therein.

The operators $(H_\beta + 1)^{-1}$ may even converge locally uniformly. Under the additional assumption, that P is not identically equal to zero on $D(P) \cap D(\mathcal{E})$ and

$$D(H) \subset D(P), \tag{2}$$

Brasche and Demuth [3] have shown that the convergence in operator norm cannot be faster than c/β for some positive constant c. In fact, let

$$L(H,P) := \liminf_{\beta\to\infty} \beta \left\| (H_\beta + 1)^{-1} - \lim_{\beta'\to\infty} (H_{\beta'} + 1)^{-1} \right\|. \tag{3}$$

Then [3, Proposition 1(ii)][1]

$$L(H,P) > 0. \tag{4}$$

In [3], one has given both examples where $L(H,P) < \infty$, i.e. where the convergence in operator norm is as fast as c/β for some finite constant c, and examples where $L(H,P) = \infty$. In this paper we shall derive a condition which is necessary and sufficient in order that $L(H,P) < \infty$, i.e. in order that the operators $(H_\beta + 1)^{-1}$ converge in operator norm with maximal rate of convergence, as β tends to infinity. We shall also show that the limit inferior in the definition (3) is a limit and describe a method how to compute it.

It is possible that the resolvents $(H_\beta + 1)^{-1}$ do not only converge with respect to the operator norm but even within some Schatten–von Neumann ideal. We shall derive a condition which is sufficient in order that the operator

$$(H + 1)^{-1} - \lim_{\beta'\to\infty} (H_{\beta'} + 1)^{-1}$$

belongs to a Schatten–von Neumann class of finite order and give an upper bound for the norm of the operator within this ideal. The condition also guarantees convergence of the resolvents $(H_\beta + 1)^{-1}$ with respect to the corresponding Schatten–von Neumann norm.

[1] The formulation in [3] is different, but equivalent, cf. Appendix A.

On the other hand, it is also possible that the resolvents $(H_\beta + 1)^{-1}$ do not even converge with respect to the operator norm, cf. Example 13 below. We shall give a condition which is sufficient in order that locally uniform convergence takes place. In addition, for $0 < r < 1$ we shall present conditions which guarantee that the convergence in operator norm is at least as fast as $O(1/\beta^r)$.

Both in quantum hard core scattering and in semiclassical analysis one is strongly interested in the special case, when H is a regular Dirichlet operator and P a killing term, i.e. $\mathcal{H} = L^2(X, m)$ for some locally compact separable metric space and some positive Radon measure m on X whose support is X, $\mathcal{E}$ is a regular Dirichlet form in $L^2(X, m)$, and $P = P_\mu$ for some positive Radon measure μ on X charging no set with capacity (with respect to $\mathcal{E}$) zero, where

$$D(P_\mu) := \left\{ f \in D(\mathcal{E})\colon \int |f|^2 \, d\mu < \infty \right\},$$
$$P_\mu(f, f) := \int |f|^2 \, d\mu \quad \forall f \in D(P_\mu). \tag{5}$$

Here and in what follows we use the fact that every $f \in D(\mathcal{E})$ admits a version which is quasi-continuous (short q.c.) and two different q.c. versions coincide quasi-everywhere (short q.e.), and we always assume that any q.c. version of f has been chosen. We refer to the standard book [7] for notations and results from the theory of Dirichlet forms used in this paper.

For closed subsets Γ of X with finite capacity let μ_Γ be the (1-)equilibrium measure of Γ. Brasche and Demuth have shown that

$$L(H, P_{\mu_\Gamma}) \leqslant 1 \tag{6}$$

provided that the resolvent $(H + 1)^{-1}$ is conservative and possesses a Green kernel. The question whether there exist also measures μ with $L(H, P_\mu) < \infty$ which are not proportional to an equilibrium measure has been left open. In this paper we shall use our general criterion for the finiteness of $L(H, P)$ in order to give an example where $L(H, P_\mu) < \infty$, the measure μ has infinite mass and the capacity of the support of μ is infinite; in particular, the measure μ is not proportional to any equilibrium measure. We shall also show that (6) holds, provided that the resolvent $(H + 1)^{-1}$ is conservative, i.e. the hypothesis that $(H + 1)^{-1}$ possesses a Green kernel can be omitted.

As mentioned we describe a general method how to compute the number $L(H, P)$. If $(H + 1)^{-1}$ is conservative and the Dirichlet form $\mathcal{E}$ possesses the strong local property, then this method can be used in order to show that

$$L(H, P_{\mu_\Gamma}) = 1, \tag{7}$$

provided Γ is a closed subset of X with finite capacity and nonempty interior. Thus in this case the inequality (6) even becomes an equality.

We shall apply our mentioned general results on Schatten–von Neumann norms in the special case when H equals the Laplacian or a fractional Laplacian in $L^2(\mathbb{R}^d, dx)$ and P the killing term corresponding to the equilibrium measure of a closed subset Γ of $\mathbb{R}^d$ with finite capacity. In this case we shall derive upper bounds for the Schatten–von Neumann norms in terms of the capacity of Γ.

2. General results

Let $\mathcal{H}$ and $\mathcal{H}_{\mathrm{aux}} \neq \{0\}$ be Hilbert spaces with scalar product $(\cdot,\cdot)$ and $(\cdot,\cdot)_{\mathrm{aux}}$ and corresponding norm $\|\cdot\|$ and $\|\cdot\|_{\mathrm{aux}}$, respectively. Let H be a nonnegative selfadjoint operator in $\mathcal{H}$ and put

$$(f,g)_{\sqrt{H}} := (\sqrt{H}f, \sqrt{H}g) + (f,g) \quad \forall f,g \in D(\sqrt{H}).$$

Let J be a closed operator from the Hilbert space $(D(\sqrt{H}), (\cdot,\cdot)_{\sqrt{H}})$ to $\mathcal{H}_{\mathrm{aux}}$. Suppose that the range $\operatorname{ran}(J)$ of J is dense in $\mathcal{H}_{\mathrm{aux}}$ and the domain $D(J)$ of J contains $D(H)$. Let

$$\begin{aligned} D(P) &:= D(J), \\ P(f,f) &:= \|Jf\|^2_{\mathrm{aux}}. \end{aligned} \tag{8}$$

For every $\beta > 0$ there exists a unique nonnegative selfadjoint operator H_β in $\mathcal{H}$ such that

$$\begin{aligned} D(\sqrt{H_\beta}) &= D(P), \\ \|\sqrt{H_\beta}f\|^2 &= \|\sqrt{H}f\|^2 + \beta P(f,f) \quad \forall f \in D(\sqrt{H_\beta}). \end{aligned} \tag{9}$$

In fact, since J is closed, the right-hand side of (9) defines a nonnegative closed quadratic form on $\mathcal{H}$, as it is easily verified. Since the domain of J contains the domain $D(H)$ of H, it is dense in $(D(\sqrt{H}), (\cdot,\cdot)_{\sqrt{H}})$ and hence in $\mathcal{H}$. Thus the existence and uniqueness of the nonnegative selfadjoint operator H_β satisfying (9) follows from Kato's representation theorem.

Since the range of J is dense in $\mathcal{H}_{\mathrm{aux}}$, the operator JJ^* is invertible. Since J is a densely defined closed operator from $(D(\sqrt{H}), (\cdot,\cdot)_{\sqrt{H}})$ to $\mathcal{H}_{\mathrm{aux}}$, the operator JJ^* in $\mathcal{H}_{\mathrm{aux}}$ is nonnegative and selfadjoint. Thus its inverse

$$\check{H} := \bigl(JJ^*\bigr)^{-1} \tag{10}$$

is an invertible nonnegative selfadjoint operator in $\mathcal{H}_{\mathrm{aux}}$. For every $h \in \mathcal{H}_{\mathrm{aux}}$ let μ_h be the spectral measure of h with respect to $\check{H}$, i.e. the unique finite positive Radon measure on $\mathbb{R}$ such that, with $(E_{\check{H}}(\lambda))_{\lambda\in\mathbb{R}}$ being the spectral family of $\check{H}$,

$$\mu_h\bigl((-\infty,\lambda]\bigr) = \bigl\|E_{\check{H}}(\lambda)h\bigr\|^2_{\mathrm{aux}} \quad \forall \lambda \in \mathbb{R}. \tag{11}$$

Since $\check{H}$ is nonnegative and invertible, we have

$$\mu_h\bigl((-\infty,0]\bigr) = 0 \quad \forall h \in \mathcal{H}_{\mathrm{aux}}. \tag{12}$$

We put

$$K_1 := (H+1)^{-1}.$$

Since J is closed and K_1 is bounded and everywhere defined, the operator JK_1 is also closed. Since $\operatorname{ran}(K_1) = D(H) \subset D(J)$, the closed operator JK_1 is also everywhere defined and hence

it is bounded. If, in addition, $\operatorname{ran}(JK_1) \subset D(\check{H})$, then the same reasoning shows that the operator $\check{H}JK_1$ is bounded, too.

By [2, Lemma 3(ii)],

$$(H+1)^{-1} - (H_\beta + 1)^{-1} = (JK_1)^* \left[\frac{1}{\beta} + JJ^*\right]^{-1} JK_1. \tag{13}$$

Let $h \in D(J^*)$. Since

$$\big(J^*h, f\big) = \big(J^*h, K_1 f\big)_{\sqrt{H}} = (h, JK_1 f)_{\mathrm{aux}} = \big((JK_1)^*h, f\big)$$

for every $f \in \mathcal{H}$, we have

$$J^*h = (JK_1)^*h.$$

Note that J is regarded as an operator from $(D(\sqrt{H}), (\cdot,\cdot)_{\sqrt{H}})$ to $\mathcal{H}_{\mathrm{aux}}$ and JK_1 as an operator from $(\mathcal{H}, (\cdot,\cdot))$ to $\mathcal{H}_{\mathrm{aux}}$.

By Kato's monotone convergence for quadratic forms, the strong limit

$$\lim_{\beta' \to \infty} (H_{\beta'} + 1)^{-1}$$

exists. Often these resolvents even converge with respect to the operator norm $\|\cdot\|$. It has been shown in [3] that the convergence in operator norm cannot be faster than c/β for some constant $c > 0$. In the theorem below we shall give a condition which is necessary and sufficient in order that the convergence in operator norm is as fast as $L(H,P)/\beta$ for some finite constant $L(H,P)$ and, in addition, we shall determine the constant $L(H,P)$.

It is convenient to introduce some notation. We put

$$D_\beta := (H+1)^{-1} - (H_\beta + 1)^{-1} \quad \forall \beta > 0, \quad \text{and} \quad D_\infty := \lim_{\beta' \to \infty} D_{\beta'}. \tag{14}$$

Note that with this notation

$$(H_\beta + 1)^{-1} - \lim_{\beta' \to \infty} (H_{\beta'} + 1)^{-1} = D_\infty - D_\beta. \tag{15}$$

Formula (17) in the following theorem is known [3, Proposition 1(ii)]; for convenience of the reader we shall show that it also easily follows from the assertions (ii) and (iii) in the next theorem.

Theorem 1.

(i) *The limit*

$$L(H,P) := \lim_{\beta \to \infty} \beta \left\| (H_\beta + 1)^{-1} - \lim_{\beta' \to \infty} (H_{\beta'} + 1)^{-1} \right\| \quad (\leqslant \infty) \tag{16}$$

exists and

$$L(H,P) > 0. \tag{17}$$

(ii) *The limit* $L(H,P)$ *is finite if and only if the range* $\operatorname{ran}(JJ^*)$ *of* JJ^* *contains the range* $\operatorname{ran}(JK_1)$ *of the operator* $J(H+1)^{-1}$.
(iii) *Let* $\operatorname{ran}(JJ^*) \supset \operatorname{ran}(JK_1)$. *Then*

$$L(H,P) = \left\| \left(JJ^*\right)^{-1} J(H+1)^{-1} \right\|^2. \tag{18}$$

Proof. Let $f \in \mathcal{H}$, $\beta > 0$ and put $h := JK_1 f$. Then

$$\begin{aligned}
\big((D_{\beta'} - D_\beta)f, f\big) &= \left((JK_1)^*\left(\left[\frac{1}{\beta'} + \check{H}^{-1}\right]^{-1} - \left[\frac{1}{\beta} + \check{H}^{-1}\right]^{-1}\right)JK_1 f, f\right) \\
&= \left(\left(\left[\frac{1}{\beta'} + \check{H}^{-1}\right]^{-1} - \left[\frac{1}{\beta} + \check{H}^{-1}\right]^{-1}\right)h, h\right)_{\mathrm{aux}} \\
&= \int \left(\frac{1}{1/\beta' + 1/\lambda} - \frac{1}{1/\beta + 1/\lambda}\right) \mu_h(d\lambda)
\end{aligned}$$

for every $\beta' > 0$. By (12) and the monotone convergence theorem, this implies that

$$\lim_{\beta' \to \infty} \big((D_{\beta'} - D_\beta)f, f\big) = \int \frac{\lambda^2}{\lambda + \beta} \mu_h(d\lambda). \tag{19}$$

By this equality in conjunction with (14), we get

$$\big((D_\infty - D_\beta)f, f\big) = \int \frac{\lambda^2}{\lambda + \beta} \mu_h(d\lambda). \tag{20}$$

By the monotone convergence theorem in conjunction with (12), this implies

$$\lim_{\beta \to \infty} \beta\big((D_\infty - D_\beta)f, f\big) = \int \lambda^2 \mu_h(d\lambda).$$

Since $h = JK_1 f$ and μ_h is the spectral measure with respect to the selfadjoint operator $\check{H}$, it follows that

$$\lim_{\beta \to \infty} \beta\big((D_\infty - D_\beta)f, f\big) = \|\check{H} JK_1 f\|_{\mathrm{aux}}^2, \quad \text{if } JK_1 f \in D(\check{H}), \tag{21}$$

$$\lim_{\beta \to \infty} \beta\big((D_\infty - D_\beta)f, f\big) = \infty, \quad \text{if } JK_1 f \notin D(\check{H}). \tag{22}$$

By (22),

$$\liminf_{\beta \to \infty} \beta \|D_\infty - D_\beta\| = \infty, \tag{23}$$

if there exists an $f \in \mathcal{H}$ such that $JK_1 f \notin D(\check{H})$.

Suppose now that $\operatorname{ran}(JK_1) \subset D(\check{H}) = \operatorname{ran}(JJ^*)$. Then, by (21),

$$\liminf_{\beta\to\infty} \beta \|D_\infty - D_\beta\| \geqslant \|\check{H} J K_1 f\|_{\mathrm{aux}}^2,$$

if $\|f\| = 1$, and hence

$$\liminf_{\beta\to\infty} \beta \|D_\infty - D_\beta\| \geqslant \|\check{H} J K_1\|^2. \tag{24}$$

By (20) in conjunction with (12), $D_\infty - D_\beta$ is a nonnegative selfadjoint operator in $\mathcal{H}$. Thus

$$\|D_\infty - D_\beta\| = \sup_{\|f\|=1} \big((D_\infty - D_\beta) f, f\big). \tag{25}$$

(20) in conjunction with (12) also implies that for every normalized $f \in \mathcal{H}$

$$\beta\big((D_\infty - D_\beta) f, f\big) \leqslant \int \lambda^2 \, \mu_h(d\lambda) \leqslant \|\check{H} J K_1\|^2.$$

In conjunction with (25), this implies that

$$\beta \|D_\infty - D_\beta\| \leqslant \|\check{H} J K_1\|^2 \quad \forall \beta > 0. \tag{26}$$

By (23), (24), (26), the assertions (ii) and (iii) are proved as well as the existence of the limit in the assertion (i). If $L(H, P)$ would be equal to zero, then, by the assertions (ii) and (iii), we would have $JK_1 = 0$. Thus the kernel of J would contain $\operatorname{ran}(K_1) = D(H)$ and hence it would be dense in $(D(\sqrt{H}), (\cdot,\cdot)_{\sqrt{H}})$. Since the kernel of a closed operator is closed it would follow that $J = 0$, which contradicts the fact that the range of J is dense in $\mathcal{H}_{\mathrm{aux}}$. Thus $L(H, P) > 0$. $\square$

Under weaker conditions we still get norm convergence of the resolvents $(H_\beta + 1)^{-1}$ at least as fast as $O(\beta^{-r})$ for some $r \in (0, 1)$:

Proposition 2. *Let $0 < r < 1$. If $\check{H}^{1/2+r/2} J K_1$ is a bounded everywhere defined operator from $\mathcal{H}$ to $\mathcal{H}_{\mathrm{aux}}$, then*

$$\begin{aligned} &\Big\|(H_\beta + 1)^{-1} - \lim_{\beta'\to\infty} (H_{\beta'} + 1)^{-1}\Big\| \\ &\quad \leqslant (1-r)^{1-r} r^r \Big\|\check{H}^{1/2+r/2} J K_1\Big\|^2 \frac{1}{\beta^r} \quad \forall \beta > 0. \end{aligned} \tag{27}$$

Proof. By (15), (20) and (25),

$$\Big\|(H_\beta + 1)^{-1} - \lim_{\beta'\to\infty} (H_{\beta'} + 1)^{-1}\Big\| = \sup_{\|f\|=1} \int \frac{\lambda^2}{\lambda + \beta} \, \mu_h(d\lambda),$$

where f and h are related via $h = JK_1 f$.

$$\int \frac{\lambda^2}{\lambda + \beta} \, \mu_h(d\lambda) \leqslant \max_{\lambda\in(0,\infty)} \frac{\lambda^{1-r}}{\lambda + \beta} \int \big|\lambda^{1/2+r/2}\big|^2 \, \mu_h(d\lambda).$$

By elementary calculus,

$$\max_{\lambda\in(0,\infty)} \frac{\lambda^{1-r}}{\lambda+\beta} = \frac{(1-r)^{1-r}r^r}{\beta^r}.$$

By the spectral calculus,

$$\int \left|\lambda^{1/2+r/2}\right|^2 \mu_h(d\lambda) = \left\|\check{H}^{1/2+r/2}h\right\|_{\mathrm{aux}}^2.$$

If $h = JK_1 f$ and $\|f\| = 1$, then

$$\left\|\check{H}^{1/2+r/2}h\right\|_{\mathrm{aux}} \leqslant \left\|\check{H}^{1/2+r/2}JK_1\right\|,$$

and the proposition is proven. □

The hypothesis that $\check{H}^{1/2+r/2}JK_1$ is an everywhere defined bounded operator from $\mathcal{H}$ to $\mathcal{H}_{\mathrm{aux}}$ can be violated for every $r > 0$, cf. Example 13 below. However, it is always satisfied for $r = 0$ and we have the following results on the representation of the operator D_∞:

Lemma 3.

(i) *We have*

$$\operatorname{ran}(JK_1) \subset D\big(\check{H}^{1/2}\big) \quad \textit{and} \quad D_\infty = \big(\check{H}^{1/2}JK_1\big)^*\check{H}^{1/2}JK_1. \tag{28}$$

In particular, D_∞ is compact if and only if $\check{H}^{1/2}JK_1$ is compact.

(ii) *If* $\operatorname{ran}(JK_1) \subset D(\check{H})$, *then*

$$D_\infty = (JK_1)^*\check{H}JK_1. \tag{29}$$

Proof. (i) Let $f \in \mathcal{H}$ and $h := JK_1 f$. By (13) and (14),

$$(D_\infty f, f) = \lim_{\beta\to\infty}(D_\beta f, f) = \int \frac{1}{1/\beta + 1/\lambda}\,\mu_h(d\lambda) = \int \lambda\,\mu_h(d\lambda).$$

In the last step we have used the monotone convergence theorem and (12). By the spectral calculus, it follows that $h = JK_1 f \in D(\check{H}^{1/2})$ and $\|\check{H}^{1/2}JK_1 f\|_{\mathrm{aux}}^2 = (D_\infty f, f)$. Thus

$$\left\|\check{H}^{1/2}JK_1\right\|^2 = \|D_\infty\|. \tag{30}$$

Since $JK_1 f \in D(\check{H}^{1/2})$ for every $f \in \mathcal{H}$, the spectral calculus yields

$$\left[\frac{1}{\beta} + \check{H}^{-1}\right]^{-1/2} JK_1 \to \check{H}^{1/2}JK_1 \quad \text{strongly, as } \beta \to \infty,$$

and hence

$$\left(\left[\frac{1}{\beta}+\check{H}^{-1}\right]^{-1/2} JK_1\right)^*\left[\frac{1}{\beta}+\check{H}^{-1}\right]^{-1/2} JK_1 \to \left(\check{H}^{1/2}JK_1\right)^*\check{H}^{1/2}JK_1 \tag{31}$$

weakly, as β tends to infinity. By (13) and (14), the operators on the left-hand side of (31) also converge to D_∞ and (28) is proven.

(ii) (29) follows from (28) and the fact that $(JK_1)^*\check{H}^{1/2} \subset (\check{H}^{1/2}JK_1)^*$. $\square$

That $\check{H}^{1/2}JK_1$ is a bounded everywhere defined operator from $\mathcal{H}$ to $\mathcal{H}_{\mathrm{aux}}$ does not guarantee that the resolvents $(H+\beta)^{-1}$ converge locally uniformly, cf. Example 13 below. The stronger requirement that $\check{H}^{1/2}JK_1$ is compact implies convergence of the operators $(H_\beta+1)^{-1}$ with respect to the operator norm:

Theorem 4. *If $\check{H}^{1/2}JK_1$ is a compact operator from $\mathcal{H}$ to $\mathcal{H}_{\mathrm{aux}}$, then the operators $(H_\beta+1)^{-1}$ converge locally uniformly.*

Proof. By (15), we only need to prove that $D_\infty - D_\beta$ converge to zero with respect to the operator norm, as β tends to infinity. By (13) and (14), D_β is a nonnegative bounded selfadjoint operator in $\mathcal{H}$. By (12) and (20), $D_\infty - D_\beta$ is a nonnegative bounded selfadjoint operator in $\mathcal{H}$, too. By the definition (14), $D_\infty - D_\beta$ converge to zero strongly, as β tends to infinity. By (28), along with $\check{H}^{1/2}JK_1$ also D_∞ is a compact operator.

The remaining part of the proof follows now from an idea due to Stollmann [9, p. 34]: the operators $D_\infty - D_\beta$ are nonnegative selfadjoint operators dominated by the compact selfadjoint operator D_∞, and they converge to 0 strongly, as β tends to infinity. Hence $\lim_{\beta\to\infty}\|D_\infty - D_\beta\| = 0$. $\square$

Let $p \in [1,\infty)$. Let $\mathcal{H}_i$ be Hilbert spaces with scalar products $(\cdot,\cdot)_i$, $i = 0, 1, 2, \ldots$. Let C be a compact operator from $\mathcal{H}_1$ to $\mathcal{H}_2$. Then $\mathcal{H}_2$ has an orthonormal basis $\{e_i\}_{i\in I}$ such that, with $|C| := \sqrt{CC^*}$,

$$|C|e_i = \lambda_i e_i \quad \forall i \in I$$

for some suitably chosen family $\{\lambda_i\}_{i\in I}$ in $[0,\infty)$ which is unique up to permutations. One puts

$$\|C\|_{S_p} := \left(\sum_{i\in I}\lambda_i^p\right)^{1/p}.$$

$S_p(\mathcal{H}_1,\mathcal{H}_2)$ (short S_p) denotes the set of compact operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ such that $\|C\|_{S_p} < \infty$ and is called the Schatten–von Neumann class of order p. S_p is a linear space and $\|\cdot\|_{S_p}$ a norm on it. If $C : \mathcal{H}_1 \to \mathcal{H}_2$ belongs to the class $S_p(\mathcal{H}_1,\mathcal{H}_2)$ and $A : \mathcal{H}_0 \to \mathcal{H}_1$ and $B : \mathcal{H}_2 \to \mathcal{H}_3$ are linear and bounded, then $CA \in S_p(\mathcal{H}_0,\mathcal{H}_2)$ and $BC \in S_p(\mathcal{H}_1,\mathcal{H}_3)$ and

$$\|CA\|_{S_p} \leqslant \|C\|_{S_p}\|A\|, \qquad \|BC\|_{S_p} \leqslant \|C\|_{S_p}\|B\|. \tag{32}$$

Moreover

$$\|C\|_{S_p} = \left\|C^*\right\|_{S_p} = \left\||C|\right\|_{S_p} \tag{33}$$

for every compact operator C.

Let $B : \mathcal{H}_1 \to \mathcal{H}_2$ be linear and bounded, Q_1 an orthogonal projection in $\mathcal{H}_1$ and Q_2 an orthogonal projection in $\mathcal{H}_2$ such that the dimension N of the range of Q_2 is finite. Then $|Q_2 B Q_1|^2 = Q_2 B Q_1 B^* Q_2$ and hence $|Q_2 B Q_1|$ is compact and

$$\big\| |Q_2 B Q_1| \big\|_{S_p} = \big\| |Q_2 B Q_1| \lceil \operatorname{ran}(Q_2) \big\|_{S_p}. \tag{34}$$

Since $|Q_2 B Q_1| \lceil \operatorname{ran}(Q_2)$ belongs to the finite-dimensional space of all linear mappings from $\operatorname{ran}(Q_2)$ into itself and all norms on a finite-dimensional space are equivalent, there exists a finite constant c, depending only on p and N such that

$$\big\| |Q_2 B Q_1| \lceil \operatorname{ran}(Q_2) \big\|_{S_p} \leqslant c \big\| |Q_2 B Q_1| \lceil \operatorname{ran}(Q_2) \big\| \leqslant c \|B\|. \tag{35}$$

By (33)–(35),

$$\|Q_2 B Q_1\|_{S_p} \leqslant c \|B\| \tag{36}$$

for some finite constant c, depending only on p and $N < \infty$, provided the range of Q_1 or the range of Q_2 is N-dimensional.

If A is a nonnegative bounded selfadjoint operator and dominated by the compact selfadjoint operator B, then A and $B - A$ are also compact and it follows easily from the min-max principle for compact operators, that

$$\|A\|_{S_p} \leqslant \|B\|_{S_p} \quad \text{and} \quad \|B - A\|_{S_p} \leqslant \|B\|_{S_p}. \tag{37}$$

In the proof of Theorem 4 we have used that strong convergence of nonnegative selfadjoint operators dominated by a compact operator implies operator norm convergence. Similarly, operator norm convergence of nonnegative selfadjoint operators dominated by an operator in S_p implies convergence in S_p:

Lemma 5. *Let $\{A_n\}_{n \in \mathbb{N}}$ be a sequence of nonnegative bounded selfadjoint operators dominated by the bounded nonnegative selfadjoint operator A. Let $1 \leqslant p < \infty$. If $A \in S_p$ and $\lim_{n\to\infty} \|A - A_n\| = 0$, then*

$$\lim_{n\to\infty} \|A - A_n\|_{S_p} = 0. \tag{38}$$

Proof. A admits the representation

$$A = \sum_{i \in I} \lambda_i (e_i, \cdot) e_i$$

for some orthonormal system $\{e_i\}_{i \in I}$ and some family $\{\lambda_i\}_{i \in I}$ of nonnegative real numbers satisfying

$$\sum_{i \in I} \lambda_i^p = \|A\|_{S_p}^p.$$

Let $\varepsilon > 0$. We choose a finite subset I_0 of I such that

$$\sum_{i\in I\setminus I_0} \lambda_i^p \leqslant \varepsilon^p$$

and denote by Q the orthogonal projection onto the orthogonal complement of the finite-dimensional space spanned by $(\{e_i\colon\ i \in I_0\})$.

$$QAQ = \sum_{i\in I\setminus I_0} \lambda_i (e_i, \cdot) e_i$$

and, in particular,

$$\|QAQ\|_{S_p}^p = \sum_{i\in I\setminus I_0} \lambda_i^p \leqslant \varepsilon^p.$$

Since $Q(A - A_n)Q$ is dominated by QAQ, it follows that

$$\bigl\|Q(A - A_n)Q\bigr\|_{S_p} \leqslant \varepsilon \quad \forall n \in \mathbb{N}. \tag{39}$$

Since the range of the orthogonal projection $1 - Q$ is finite-dimensional and $\lim_{n\to\infty}\|A - A_n\| = 0$, it follows from (36), that

$$\begin{aligned}\lim_{n\to\infty}\bigl\|(1-Q)(A-A_n)Q\bigr\|_{S_p} &= \lim_{n\to\infty}\bigl\|(1-Q)(A-A_n)(1-Q)\bigr\|_{S_p} \\ &= \lim_{n\to\infty}\bigl\|Q(A-A_n)(1-Q)\bigr\|_{S_p} = 0.\end{aligned}$$

Since $A - A_n = Q(A - A_n)Q + (1 - Q)(A - A_n)Q + Q(A - A_n)(1 - Q) + (1 - Q)(A - A_n)(1 - Q)$, this implies in conjunction with (39), that

$$\limsup_{n\to\infty}\|A - A_n\|_{S_p} \leqslant \varepsilon,$$

and the lemma is proven. □

The following corollary gives a sufficient condition in order that the operator D_∞ belongs to a Schatten–von Neumann ideal of finite order and gives an upper bound for the corresponding Neumann–von Schatten norm. We refer to Example 14 for an application of the corollary.

Corollary 6. *Let $L(H, P) < \infty$.*

(i) *Let $1 \leqslant p < \infty$. If $JK_1 \in S_p(\mathcal{H}, \mathcal{H}_{\mathrm{aux}})$, then $D_\beta \in S_p(\mathcal{H}, \mathcal{H})$ and*

$$\|D_\infty - D_\beta\|_{S_p} \leqslant \|D_\infty\|_{S_p} \leqslant \sqrt{L(H,P)}\,\|JK_1\|_{S_p} \tag{40}$$

for every $\beta \in (0, \infty)$. Moreover

$$\lim_{\beta\to\infty}\|D_\infty - D_\beta\|_{S_p} = 0. \tag{41}$$

(ii) *Let* $u \in (3/2, \infty)$. *If* JJ^* *is bounded and* JK_1^u *belongs to the Hilbert–Schmidt class* $S_2(\mathcal{H}, \mathcal{H}_{\text{aux}})$, *then* $D_\beta \in S_{4u-2}(\mathcal{H}, \mathcal{H})$ *and*

$$\begin{aligned} \|D_\infty - D_\beta\|_{S_{4u-2}} &\leqslant \|D_\infty\|_{S_{4u-2}} \\ &\leqslant \sqrt{L(H,P)}\left(\|JJ^*\|^{2u-2}\|JK_1^u\|_{S_2}^2\right)^{\frac{1}{4u-2}} \end{aligned} \tag{42}$$

for every $\beta \in (0, \infty)$. *Moreover*

$$\lim_{\beta\to\infty}\|D_\infty - D_\beta\|_{S_{4u-2}} = 0. \tag{43}$$

Proof. By Theorem 1 and since $L(H, P) < \infty$, we have that $\operatorname{ran}(JK_1) \subset D(\check{H})$, $\|\check{H}JK_1\| = \sqrt{L(H,P)}$ and $\lim_{\beta\to\infty}\|D_\infty - D_\beta\| = 0$. By Lemma 3(ii), this implies that

$$D_\infty = (JK_1)^*\check{H}JK_1,$$

hence (40) follows from (32), (33) and (37).

Suppose, in addition, that JJ^* is bounded. For all $h \in \mathcal{H}_{\text{aux}}$ and $f \in D(\mathcal{E})$

$$\big(f, (JK_1)^*h\big) = (JK_1f, h)_{\text{aux}} = \mathcal{E}_1\big(K_1f, J^*h\big) = \big(f, J^*h\big).$$

Thus $J^*h = (JK_1)^*h$ for all $h \in \mathcal{H}_{\text{aux}}$. Thus $JJ^* = JK_1^{1/2}(JK_1^{1/2})^*$ and hence

$$\|JJ^*\| = \|JK_1^{1/2}\|^2.$$

In conjunction with the hypothesis $JK_1^u \in S_2$ this implies, by [2, Lemma 2], that

$$\|JK_1\|_{S_{4u-2}}^{4u-2} \leqslant \|JJ^*\|^{2u-2}\|JK_1^u\|_{S_2}^2,$$

hence (42) follows now from (40).

By Lemma 5 and since $\lim_{\beta\to\infty}\|D_\infty - D_\beta\| = 0$, (41) and (43) follow from (40) and (42), respectively. □

3. Dirichlet operators

Throughout this section we consider the important case when H is a regular Dirichlet operator and P a killing term. Thus $\mathcal{H} = L^2(X, m)$ for some locally compact separable metric space and some positive Radon measure m on X whose support is X, $\mathcal{E}$ is a regular Dirichlet form in $L^2(X, m)$, H is the selfadjoint operator associated to $\mathcal{E}$, $\mathcal{H}_{\text{aux}} = L^2(X, \mu)$ for some positive Radon measure μ on X charging no set with capacity (with respect to $\mathcal{E}$) zero and J equals the operator I_μ from $(D(\mathcal{E}), \mathcal{E}_1)$ to $L^2(X, \mu)$ defined as follows:

$$\begin{aligned} D(I_\mu) &:= \left\{f \in D(\mathcal{E}):\ \int |f|^2\,d\mu < \infty\right\}, \\ I_\mu f &:= f \quad \forall f \in D(I_\mu); \end{aligned} \tag{44}$$

we recall that every $f \in D(\mathcal{E})$ admits a version which is quasi-continuous (short q.c.) and two different q.c. versions coincide quasi-everywhere (short q.e.), and we always assume that any q.c. version of f has been chosen.

We shall always assume that

$$\int |f|^2 \, d\mu < \infty \quad \forall f \in D(H). \tag{45}$$

Then the general hypothesis in the previous section is satisfied and the operators denoted by H_β in the previous section become the selfadjoint operators associated to the Dirichlet form $\mathcal{E}^{\beta\mu}$ satisfying

$$\begin{aligned} D\big(\mathcal{E}^{\beta\mu}\big) &:= D(I_\mu), \\ \mathcal{E}^{\beta\mu}(f,f) &:= \mathcal{E}(f,f) + \beta \int |f|^2 \, d\mu \quad \forall f \in D\big(\mathcal{E}^{\beta\mu}\big). \end{aligned} \tag{46}$$

We shall denote the operator associated to $\mathcal{E}^{\beta\mu}$ by $H + \beta\mu$. With our special choice of H and P we get

$$D_\beta = (H+1)^{-1} - (H+\beta\mu+1)^{-1}, \qquad D_\infty = \lim_{\beta\to\infty} D_\beta, \qquad \check{H} = \big(I_\mu I_\mu^*\big)^{-1}. \tag{47}$$

Here and in what follows we do not indicate the dependence of D_β and $\check{H}$ on the special choice of the measure μ; it will always be clear from the context which measure we refer to.

For a Borel subset B of X we define the form $\mathcal{E}_{X\setminus B}$ as follows:

$$\begin{aligned} D(\mathcal{E}_{X\setminus B}) &:= \big\{ f \in D(\mathcal{E}) \colon\ f = 0 \text{ q.e. on } B \big\}, \\ \mathcal{E}_{X\setminus B}(f,f) &:= \mathcal{E}(f,f) \quad \forall f \in D(\mathcal{E}_{X\setminus B}). \end{aligned} \tag{48}$$

$\mathcal{E}_{X\setminus B}$ is a Dirichlet form in $L^2(X \setminus B, m)$ and we denote by $H_{X\setminus B}$ the selfadjoint operator in $L^2(X \setminus B, m)$ associated to it.

Fairly different measures μ may lead to the same limit $\lim_{\beta\to\infty}(H+\beta\mu+1)^{-1}$; actually, if one passes to the limit, then only the quasi-supports play a role: we have

$$\lim_{\beta\to\infty} (H+\beta\mu+1)^{-1} = (H_{X\setminus\tilde{F}}+1)^{-1} \oplus 0, \tag{49}$$

provided $\tilde{F}$ is a quasi-support of the measure μ. As an abbreviation we shall change the notation and simply write $(H_{X\setminus\tilde{F}}+1)^{-1}$ instead of $(H_{X\setminus\tilde{F}}+1)^{-1} \oplus 0$.

Note that for every $h \in L^2(X,\mu)$ with $h \geqslant 0$ μ-a.e. the following holds: $h\mu$ is a measure with finite energy integral (with respect to $\mathcal{E}$) if and only if $h \in D(I_\mu^*)$. In this case $I_\mu^* h$ equals the (1-)potential $U_1(h\mu)$ of $h\mu$ (with respect to $\mathcal{E}$), i.e.

$$I_\mu^* h = U_1(h\mu) \geqslant 0 \quad m\text{-a.e. } \forall h \in D\big(I_\mu^*\big) \text{ with } h \geqslant 0 \ \mu\text{-a.e.} \tag{50}$$

Let Γ be a closed subset of X such that the (1-)capacity $\operatorname{cap}(\Gamma)$ of Γ is finite. There exists a unique $e_\Gamma \in D(\mathcal{E})$ satisfying

$$e_\Gamma = 1 \quad \text{q.e. on } \Gamma \quad \text{and} \quad \mathcal{E}_1(e_\Gamma, v) \geqslant 0 \quad \forall v \in D(\mathcal{E}) \text{ with } v \geqslant 0 \text{ q.e. on } \Gamma. \tag{51}$$

Moreover there exists a unique positive Radon measure μ_Γ on X such that μ_Γ has finite energy integral,

$$\mu_\Gamma(\Gamma) = \mu_\Gamma(X) = \operatorname{cap}(\Gamma) \quad \text{and} \quad e_\Gamma = U_1\mu_\Gamma. \tag{52}$$

Thus $1 \in D(I_{\mu_\Gamma}^*)$ and

$$I_{\mu_\Gamma} I_{\mu_\Gamma}^* 1 = 1 \quad \text{q.e. on } \Gamma. \tag{53}$$

The (1-)equilibrium potential e_Γ of Γ satisfies, in addition,

$$0 \leqslant e_\Gamma \leqslant 1 \quad m\text{-a.e.} \tag{54}$$

We also recall some basic facts about the trace of the (transient) Dirichlet form $\mathcal{E}$ on the support of μ relative to μ. To this end we denote by $\mathcal{F}$ the extended Dirichlet space of $\mathcal{E}$ (see [7, p. 35]).

Let A be the positive continuous functional whose Revuz measure is μ. We denote by F the support of μ and by $\tilde{F}$ the support of A. It is known [7, p. 265] that $\tilde{F}$ is a quasi-support for μ, $\mu(F \setminus \tilde{F}) = 0$ and that, by [7, Theorem 4.6.2], if two elements from $\mathcal{F}$ coincide μ-a.e. then they coincide q.e. as well.

We introduce the subspaces

$$\mathcal{F}_{X-\tilde{F}} := \{f \in \mathcal{F}\colon\ f = 0 \text{ q.e. on } \tilde{F}\} = \{f \in \mathcal{F}\colon\ f = 0\ \mu\text{-a.e. on } F\},$$

and $\mathcal{H}^{\tilde{F}}$, the $\mathcal{E}$-orthogonal complement of $\mathcal{F}_{X-\tilde{F}}$ in the space $\mathcal{F}$, so that the following decomposition holds true [7, p. 265]:

$$\mathcal{F} = \mathcal{F}_{X-\tilde{F}} \oplus \mathcal{H}^{\tilde{F}}. \tag{55}$$

Let P be the orthogonal projection onto $\mathcal{H}^{\tilde{F}}$. We define the trace of $\mathcal{E}$ on the subset F relative to the measure μ as follows (see [7, p. 266 and Theorem 4.6.5]):

$$D(\check{\mathcal{E}}) := \big\{f \in L^2(F,\mu)\colon\ f = u\ \mu\text{-a.e. on } F, \text{ for some } u \in \mathcal{F}\big\},$$
$$\check{\mathcal{E}}(f,f) = \mathcal{E}(Pu, Pu), \quad f = u \quad \mu\text{-a.e. on } F.$$

It is known that $\check{\mathcal{E}}$ is a regular Dirichlet form in $L^2(F,\mu)$ [7, p. 266] and the Dirichlet space $(\check{\mathcal{E}}, D(\check{\mathcal{E}}))$ is called the *time changed Dirichlet space* or the *trace* of the space $(\mathcal{E}, D(\mathcal{E}))$ on F relative to μ.

In the following, instead of $\mathcal{E}$ we consider $\mathcal{E}_1$, which is a transient Dirichlet form, and put

$$\check{\mathcal{E}}_1 := (\mathcal{E}_1)\check{}.$$

It is known that the operator $\check{H}$, defined by (47), is the selfadjoint operator associated with $\check{\mathcal{E}}_1$ in the sense of Kato's representation theorem, cf. [1, Eq. (4.5)]. We put

$$\check{K} := \check{H}^{-1} = I_\mu I_\mu^* \quad \text{and} \quad \check{K}_\alpha := (\check{H} + \alpha)^{-1} \quad \forall \alpha > 0. \tag{56}$$

Equality (53) can be used in order to prove that $I_{\mu_\Gamma} I_{\mu_\Gamma}^*$ is a bounded operator with norm one. We prepare the proof by the following lemma.

Lemma 7. *Let G be a symmetric Markovian kernel and put*

$$Tf(x) := \int f(y)\, G(x, dy)$$

whenever the expression on the right-hand side is defined. Then

$$\|Tf\| \leqslant \big(\|T1\|_\infty\big)^{1/2} \|f\| \quad \forall f \in L^2(X, m) \cap L^\infty(X, m)$$

and hence T extends to a bounded operator on $L^2(X, m)$ with

$$\|T\| \leqslant \big(\|T1\|_\infty\big)^{1/2}. \tag{57}$$

Proof. Let $f \in L^2(X, m) \cap L^\infty(X, m)$. By Hölder's inequality,

$$|Tf|^2 \leqslant T1 \int_X f^2(y)\, G(\cdot, dy) \leqslant \|T1\|_\infty \int_X f^2(y)\, G(\cdot, dy), \tag{58}$$

which yields, by the Markov property and symmetry of G, that $\|Tf\|^2 \leqslant \|T1\|_\infty \|f\|^2$. $\square$

Corollary 8. *Let Γ be a closed subset of X such that*

$$0 < \operatorname{cap}(\Gamma) < \infty.$$

Then

$$\big\| I_{\mu_\Gamma} I_{\mu_\Gamma}^* \big\| = 1. \tag{59}$$

Proof. By the first resolvent equality and since the operators $\check{K}_\alpha$ are positivity preserving, the sequence $(\check{K}_{1/n} f)_{n=1}^\infty$ is pointwise nondecreasing μ_Γ-a.e. for every $f \in L^2(X, \mu_\Gamma)$ with $f \geqslant 0$ μ_Γ-a.e.

By (56) and (53), $1 \in D(\check{K})$ and $\check{K}1 = 1$ μ_Γ-a.e. and hence $\|\check{K}\| \geqslant 1$. By the spectral calculus

$$\|\check{K}_{1/n} f - \check{K} f\|_{L^2(X, \mu_\Gamma)} \to 0, \quad \text{as } n \to \infty,\ \forall f \in D(\check{K}). \tag{60}$$

Since the sequence $(\check{K}_{1/n} 1)_{n=1}^\infty$ is nondecreasing μ_Γ-a.e., it follows that it converges to 1 μ_Γ-a.e. and, in particular, $\check{K}_{1/n} 1 \leqslant 1$ μ_Γ-a.e. for all $n \in \mathbb{N}$, $n \geqslant 1$. By Lemma 7, this implies that

$$\|\check{K}_{1/n}\| \leqslant 1, \quad n = 1, 2, 3, \ldots.$$

By (60), it follows that $\|\check{K}\| \leqslant 1$. $\square$

We proceed to improve, a bit, the result of Brasche and Demuth [3, Theorem 4]. There the authors proved that the rate of convergence is optimal if the measure μ is the equilibrium measure of a closed subset having finite capacity, provided the operator $(H+1)^{-1}$ is *conservative* and possesses a *Green kernel*. We shall show that the condition that $(H+1)^{-1}$ is an integral operator is not needed. We prepare the proof of our generalization via the following

Lemma 9. *Let μ be a positive Radon measure on X with finite energy integral and satisfying* (45). *Let $\tilde{F}$ be a quasi-support of μ. Then for all $\beta \geqslant 0$ the operator*

$$(H+\beta\mu+1)^{-1}-(H_{X\setminus\tilde{F}}+1)^{-1} \tag{61}$$

is positivity preserving.

Proof. Let $f \in L^2(X,m)$ satisfying $f \geqslant 0$ m-a.e. and $\beta \geqslant 0$ be fixed. For every integer $n > \beta$, set

$$f_n := (H+\beta\mu+1)^{-1}f-(H+n\mu+1)^{-1}f,$$
$$f_\beta := (H+\beta\mu+1)^{-1}f-(H_{X\setminus\tilde{F}}+1)^{-1}f.$$

By [3, Lemma 1], the operator $(H+\beta\mu+1)^{-1}-(H+n\mu+1)^{-1}$ is positivity preserving and thereby $f_n \geqslant 0$ m-a.e. for every $n > \beta$. Further, since $(H+n\mu+1)^{-1}$ converges strongly to $(H_{X\setminus\tilde{F}}+1)^{-1}$, we get

$$\lim_{n\to\infty}\|f_n-f_\beta\|=0. \tag{62}$$

Thus, a suitably chosen subsequence of (f_n) converges to f_β m-a.e. Since the f_n's are nonnegative m-a.e., we conclude that f_β also is nonnegative m-a.e, which yields the result. $\square$

Remark 10. The operators $(H+\beta\mu+1)^{-1}$ and $(H_{X\setminus\tilde{F}}+1)^{-1}$ possess symmetric Markovian kernels, since they are the resolvents of symmetric regular Dirichlet operators. Thus the difference of these operators possesses a symmetric kernel, too. By the latter lemma, the symmetric kernel of the difference $(H+\beta\mu+1)^{-1}-(H_{X\setminus\tilde{F}}+1)^{-1}$ is also Markovian.

Theorem 11. *Let Γ be a closed subset of X with finite capacity and μ_Γ the equilibrium measure of Γ. Let F be the support of μ_Γ. Assume that $(H+1)^{-1}$ is conservative, i.e.,*

$$(H+1)^{-1}1=1 \quad q.e. \tag{63}$$

Then

$$\left\|(H+\beta\mu_\Gamma+1)^{-1}-(H_{X\setminus\tilde{F}}+1)^{-1}\right\| \leqslant \frac{1}{1+\beta} \quad \forall\beta>0. \tag{64}$$

Proof. Making use of Remark 10 and Lemma 7, it suffices to prove that

$$\left\|(H+\beta\mu_\Gamma+1)^{-1}1-(H_{X\setminus\tilde{F}}+1)^{-1}1\right\|_\infty \leqslant \frac{1}{1+\beta} \quad \forall\beta>0. \tag{65}$$

Let $\beta > 0$ and $(f_k) \subset C_c(X)$ such that $f_k \uparrow 1$ everywhere on X. Using the representation of K_1 in term of its Markovian kernel, we obtain, by applying the monotone convergence theorem, that

$$I_{\mu_\Gamma} K_1 f_k \to 1 \quad \text{in } L^2(X, \mu_\Gamma). \tag{66}$$

Thus observing that, by (53), $\check{H}(\check{H} + \beta)^{-1}1 = \frac{1}{1+\beta}$ we get

$$D_\beta f_k = \beta (I_{\mu_\Gamma} K_1)^* \check{H}(\check{H} + \beta)^{-1} I_{\mu_\Gamma} K_1 f_k \to \frac{\beta}{1+\beta} (I_{\mu_\Gamma} K_1)^* 1. \tag{67}$$

By monotone convergence, another time, we get that $D_\beta f_k \uparrow D_\beta 1$ a.e. Thus, by the latter identity and since

$$\frac{\beta}{1+\beta} (I_{\mu_\Gamma} K_1)^* 1 = \frac{\beta}{1+\beta} U_1 \mu_\Gamma,$$

we achieve $D_\beta 1 = \frac{\beta}{1+\beta} U_1 \mu_\Gamma$ for every $0 < \beta < \infty$. Since the operators D_β converge to D_∞ strongly, this implies that $D_\infty 1 = U_1 \mu_\Gamma$. Thus

$$\left\| (H + \beta \mu_\Gamma + 1)^{-1} 1 - (H_{X \setminus \tilde{F}} + 1)^{-1} 1 \right\|_\infty \leqslant \frac{\|U_1 \mu_\Gamma\|_\infty}{1+\beta} \quad \forall \beta > 0. \tag{68}$$

Finally the result follows from (52) and (54). $\square$

Theorem 12. *Suppose that the regular Dirichlet form $\mathcal{E}$ associated to the nonnegative selfadjoint operator H in $L^2(X, m)$ has the strong local property. Let Γ be a closed subset of X with finite capacity. If the interior Γ° of Γ is not empty, then*

$$L(H, P_{\mu_\Gamma}) \geqslant 1. \tag{69}$$

If, in addition, the operator $(H + 1)^{-1}$ is conservative, then

$$L(H, P_{\mu_\Gamma}) = 1. \tag{70}$$

Proof. (70) follows from (69) and Theorem 11. Thus we need only to prove (69).

Since $U_1 \mu_\Gamma = 1$ q.e. on Γ and by the strong locality of $\mathcal{E}$,

$$\int u \, dm = (U_1 \mu_\Gamma, u) = \mathcal{E}_1(U_1 \mu_\Gamma, u) = \int u \, d\mu_\Gamma$$

for all $u \in C_c(\Gamma^\circ) \cap D(\mathcal{E})$. Since $C_c(\Gamma^\circ) \cap D(\mathcal{E})$ is dense in $C_c(\Gamma^\circ)$ with respect to the supremum norm, it follows that

$$\mu_\Gamma = m \quad \text{on } \mathcal{B}(\Gamma^\circ) \tag{71}$$

($\mathcal{B}(D)$ denotes the Borel algebra of D).

Choose $u \in C_c(\Gamma^\circ) \cap D(\mathcal{E})$ such that $\|u\| = 1$. For all $f \in D(I_{\mu\Gamma})$

$$\mathcal{E}_1(f, K_1 u) = (f, u) = (I_{\mu\Gamma} f, u)_{L^2(\mu\Gamma)} = \mathcal{E}_1(f, I^*_{\mu\Gamma} u)$$

(in the second step we have used (71)). Thus $K_1 u = I^*_{\mu\Gamma} u$ and hence $\check{H} I_{\mu\Gamma} K_1 u = u$. Thus

$$\|\check{H} I_{\mu\Gamma} K_1\| \geqslant \|u\|_{L^2(\mu\Gamma)} = \|u\| = 1$$

(again we have used (71) in the second step). By Theorem 1(iii), this implies (69). $\square$

Example 13. Let $\mathcal{E}$ be the classical Dirichlet form in $L^2(\mathbb{R}, dx)$, i.e. the domain of $\mathcal{E}$ equals the Sobolev space $H^1(\mathbb{R})$ of order one and

$$\mathcal{E}(u, u) = \int |u'|^2\, dx \quad \forall u \in D(\mathcal{E}).$$

Denote by δ_a the Dirac measure with mass at a and put

$$\mu := \sum_{n \in \mathbb{Z}} 2^{-|n|} \delta_n.$$

In this case $K_1^{1/2}$ is an integral operator with a square integrable convolution kernel $g(x - y)$ and $\int g(\cdot - y) f(y)\, dy$ is continuous for every $f \in L^2(\mathbb{R}, dx)$ and hence $I_\mu K^{1/2}$ a Hilbert–Schmidt operator. By (13), this implies that D_β even belongs to the trace class for every $\beta \in (0, \infty)$. Thus the essential spectrum $\sigma_{\mathrm{ess}}(-\Delta + \beta\mu)$ of $-\Delta + \beta\mu$ equals the essential spectrum of $-\Delta$ and hence

$$\sigma_{\mathrm{ess}}(-\Delta + \beta\mu) = [0, \infty). \tag{72}$$

On the other hand, as $\beta \to \infty$ the operators $-\Delta + \beta\mu$ converge in the strong resolvent sense to an operator which is unitarily equivalent to the infinite orthogonal sum

$$\bigoplus_{n \in \mathbb{Z}} -\Delta^{\mathrm{D}}_{(n,n+1)}.$$

Here $-\Delta^{\mathrm{D}}_I$ denotes the Dirichlet Laplacian on I. Thus there exists $\lambda \in [0, \infty)$ which does not belong to the spectrum of $\lim_{\beta\to\infty} -\Delta + \beta\mu$. By (72), this would be impossible if the operators $-\Delta + \beta\mu$ would converge in the norm resolvent sense, as β tends to infinity. Thus the operators D_β do not converge to D_∞ with respect to the operator norm. By Proposition 2 and Theorem 4, this implies that for every $r > 0$ the operator $\check{H}^{1/2+r/2} I_\mu K_1$ is not an everywhere defined bounded operator from $L^2(\mathbb{R}, dx)$ to $L^2(\mathbb{R}, \mu)$ and that $\check{H}^{1/2} I_\mu K_1$ is not compact.

As the example shows, even in the classical case the operator D_∞ need not be compact. On the other hand, if the killing measure μ is an equilibrium measure, then in the classical case D_∞ even belongs to a Schatten–von Neumann class of finite order:

Example 14. Let $H = (-\Delta)^\alpha$ for some $\alpha \in (0,1]$ where $-\Delta$ denotes the selfadjoint operator in $L^2(\mathbb{R}^d, dx)$ associated to the classical Dirichlet form. Let Γ be a closed subset of $\mathbb{R}^d$ with finite capacity and μ_Γ the equilibrium measure of Γ ("capacity," "equilibrium measure," etc. refer to the Dirichlet form associated to $(-\Delta)^\alpha$). Let $\tilde{F}$ be a quasi-support of μ_Γ.

Let $4\alpha u > d$. The operator $((-\Delta)^\alpha + 1)^{-u}$ is an integral operator with a kernel $g_{\alpha,u}(x-y)$ satisfying

$$\widehat{g_{\alpha,u}}(k) = \frac{1}{(1+k^{2\alpha})^u} \quad dk\text{-a.e.},$$

$g_{\alpha,u}$ is square-integrable and for every $f \in L^2(\mathbb{R}^d, dx)$ the function $\int g_{\alpha,u}(\cdot, y) f(y)\, dy$ is continuous. Thus $I_{\mu_\Gamma} K_1^u$ is the integral operator with kernel $g_{\alpha,u}(x-y)$, belongs to the Hilbert–Schmidt class $S_2(L^2(\mathbb{R}^d, dx), L^2(\mathbb{R}^d, \mu_\Gamma))$ and

$$\|I_{\mu_\Gamma} K_1^u\|_{S_2}^2 = \iint |g_{\alpha,u}(x-y)|^2\, dy\, \mu_\Gamma(dx) = (2\pi)^{-d} \int \frac{1}{|1+k^{2\alpha}|^{2u}}\, dk\, \mathrm{cap}(\Gamma). \tag{73}$$

By Corollary 8 and Theorem 11,

$$\|I_{\mu_\Gamma} I_{\mu_\Gamma}^*\| = 1 \quad \text{and} \quad L\big((-\Delta)^\alpha, P_{\mu_\Gamma}\big) \leqslant 1. \tag{74}$$

By Corollary 6 and (73) and (74), the following assertions hold true:

(i) Let $4\alpha > d$. Then the operator $((-\Delta)^\alpha + 1)^{-1} - (((-\Delta)^\alpha)_{\mathbb{R}^d \setminus \tilde{F}} + 1)^{-1}$ belongs to the Hilbert–Schmidt class $S_2(L^2(\mathbb{R}^d, dx), L^2(\mathbb{R}^d, dx))$ and we have the following upper bound for its Hilbert–Schmidt norm:

$$\begin{aligned} &\big\|\big((-\Delta)^\alpha + 1\big)^{-1} - \big(((-\Delta)^\alpha)_{\mathbb{R}^d \setminus \tilde{F}} + 1\big)^{-1}\big\|_{S_2}^2 \\ &\quad \leqslant (2\pi)^{-d} \int \frac{1}{|1+k^{2\alpha}|^2}\, dk\, \mathrm{cap}(\Gamma). \end{aligned} \tag{75}$$

(ii) Let $4\alpha u > d$ and $u > 3/2$. Then the operator $((-\Delta)^\alpha + 1)^{-1} - (((-\Delta)^\alpha)_{\mathbb{R}^d \setminus \tilde{F}} + 1)^{-1}$ belongs to the Schatten–von Neumann class $S_{4u-2}(L^2(\mathbb{R}^d, dx), L^2(\mathbb{R}^d, dx))$ and we have the following upper bound for its norm with respect to this class:

$$\begin{aligned} &\big\|\big((-\Delta)^\alpha + 1\big)^{-1} - \big(((-\Delta)^\alpha)_{\mathbb{R}^d \setminus \tilde{F}} + 1\big)^{-1}\big\|_{S_{4u-2}}^{4u-2} \\ &\quad \leqslant (2\pi)^{-d} \int \frac{1}{|1+k^{2\alpha}|^2}\, dk\, \mathrm{cap}(\Gamma). \end{aligned} \tag{76}$$

We recall that (75) and (76) imply, that the operators $((-\Delta)^\alpha + \beta\mu_\Gamma + 1)^{-1}$ converge to $(((-\Delta)^\alpha)_{\mathbb{R}^d \setminus \tilde{F}} + 1)^{-1}$ with respect to the Hilbert–Schmidt norm and the norm in S_{4u-2}, respectively.

Let us give a concrete example where $L(H, P_\mu)$ is finite despite the fact that the measure μ has infinite mass and its support infinite capacity.

Example 15. Let $d \in \mathbb{N}$, $d \geqslant 2$. Set Ω_d^+, the upper half space

$$\Omega_d^+ := \{x = (x', x_d)\colon x' \in \mathbb{R}^{d-1},\ x_d > 0\},$$

$\overline{\Omega}_d^+$ its closure. Set m_d the d-dimensional Lebesgue on $\mathbb{R}^d$, $L^2(\overline{\Omega}_d^+) := L^2(\overline{\Omega}_d^+, m_d)$ and $W^{1,2}(\Omega_d^+)$ the first order Sobolev space on Ω_d^+.

We consider the gradient energy form

$$D(\mathcal{E}) = W^{1,2}(\Omega_d^+), \qquad \mathcal{E}(f, f) = \int_{\Omega_d^+} |\nabla f|^2 \, dm_d \quad \forall f \in W^{1,2}(\Omega_d^+).$$

It is known that $\mathcal{E}$ is a regular Dirichlet form in $L^2(\overline{\Omega}_d^+)$.

Define the measure μ by

$$d\mu(x) = dm_{d-1}(x')\, d\delta_0(x_d). \tag{77}$$

The measure μ is just the restriction of the d-dimensional Lebesgue measure on $\mathbb{R}^{d-1}$ and has support as well as quasi-support equal to $\mathbb{R}^{d-1}$, which is the boundary of Ω_d^+ (see [7, p. 278]).

We claim that μ is $\mathcal{E}$-bounded. Indeed, let $\varphi \in C_c^\infty(\overline{\Omega}_d^+)$. Using the elementary identity

$$\varphi(x', 0) = -2 \int_0^\infty \varphi(x', t) \frac{\partial \varphi}{\partial t}(x', t)\, dt, \tag{78}$$

we achieve the inequality

$$\varphi^2(x', 0) \leqslant \int_0^\infty \left(\varphi^2(x', t) + \left(\frac{\partial \varphi}{\partial t}(x', t)\right)^2\right) dt, \tag{79}$$

which, integrated with respect to Lebesgue measure on $\mathbb{R}^{d-1}$, yields

$$\int_{\overline{\Omega}_d^+} \varphi^2 \, d\mu \leqslant \mathcal{E}_1(\varphi, \varphi) \quad \forall \varphi \in C_c^\infty(\overline{\Omega}_d^+), \tag{80}$$

and the claim follows from the denseness of $C_c^\infty(\overline{\Omega}_d^+)$ in $W^{1,2}(\Omega_d^+)$.

Finally we shall prove that $L(H, P_\mu) < \infty$. By Theorem 1, it is sufficient to show that

$$I_\mu K_1 f \in D(\check{H}) \quad \forall f \in L^2(\Omega_d^+, m_d).$$

Let $h \in D(\check{\mathcal{E}}_1)$ be arbitrary. Let $u \in D(\mathcal{E})$ be the (1-)harmonic function satisfying $I_\mu u = h$ μ-a.e. Then

$$\begin{aligned}\left|\check{\mathcal{E}}_1(I_\mu K_1 f, h)\right| &= \left|\mathcal{E}_1(K_1 f, u)\right| \\ &= \left|(f, u)\right| \\ &\leqslant \|f\|\|u\| \leqslant c\|f\|\|h\|_{L^2(\mathbb{R}^{d-1}, m_{d-1})}\end{aligned}$$

for some constant c which does not depend on h. Thus the mapping $h \mapsto \check{\mathcal{E}}_1(I_\mu K_1 f, h)$ extends to a bounded linear functional on $L^2(\mathbb{R}^{d-1}, m_{d-1})$ and there exists an $f^* \in L^2(\mathbb{R}^{d-1}, m_{d-1})$ such that $\check{\mathcal{E}}_1(I_\mu K_1 f, h) = (f^*, h)_{L^2(\mathbb{R}^{d-1}, m_{d-1})}$ for every $h \in D(\check{\mathcal{E}}_1)$. By Kato's representation theorem, this implies that $I_\mu K_1 f \in D(\check{H})$ and $\check{H} I_\mu K_1 f = f^*$.

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Appendix A

In Section 2 we have studied a special kind of perturbations P of the quadratic form $\mathcal{E}$. Actually, any nonnegative form perturbation of $\mathcal{E}$ leading to a selfadjoint operator is of this form:

Lemma. *Let $\mathcal{E}$ be a nonnegative closed quadratic form in the Hilbert space $\mathcal{H}$ and P a nonnegative quadratic form in $\mathcal{H}$ such that there exists a nonnegative selfadjoint operator H_1 in $\mathcal{H}$ satisfying*

$$\begin{aligned}&D(\sqrt{H_1}) = D(P) \cap D(\mathcal{E}), \\ &\|\sqrt{H_1} f\|^2 = \mathcal{E}(f, f) + P(f, f) \quad \forall f \in D(P) \cap D(\mathcal{E}).\end{aligned}$$

Then there exist a Hilbert space $\mathcal{H}_{\mathrm{aux}}$ and a closed operator J from $(D(\mathcal{E}), \mathcal{E}_1)$ to $\mathcal{H}_{\mathrm{aux}}$ such that

$$D(J) = D(P) \cap D(\mathcal{E}), \qquad P(f, f) = \|Jf\|^2_{\mathrm{aux}} \quad \forall f \in D(J),$$

and $\operatorname{ran}(J)$ *is dense in $\mathcal{H}_{\mathrm{aux}}$.*

Proof. Since $\|\sqrt{H_1} f\|^2 - \mathcal{E}(f, f)$ is a nonnegative quadratic form on $D(P) \cap D(\mathcal{E})$ it extends, by polarization, to a semi-scalar product $(\cdot,\cdot)_{\mathrm{s}}$ on $D(P) \cap D(\mathcal{E})$.

We define an equivalence relation $\sim$ on $D(P) \cap D(\mathcal{E})$ as follows: $f \sim g$ if and only if $(f - g, f - g)_{\mathrm{s}} = 0$. For every $f \in D(P) \cap D(\mathcal{E})$ let $[f]$ be the equivalence class of f with respect to this equivalence relation and denote by $\mathcal{H}_{\mathrm{aux}}$ the completion of the quotient space $(D(P) \cap D(\mathcal{E}), (\cdot,\cdot)_{\mathrm{s}})/\sim$. Then

$$\begin{aligned}&D(J) := D(P) \cap D(\mathcal{E}), \\ &Jf := [f] \quad \forall f \in D(J),\end{aligned}$$

defines a closed operator from $(D(\mathcal{E}), \mathcal{E}_1)$ to $\mathcal{H}_{\mathrm{aux}}$ with the required properties, as it is easily verified. $\square$

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Meromorphic continuation of dynamical zeta functions via transfer operators

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Abstract

We describe a method to prove meromorphic continuation of dynamical zeta functions to the entire complex plane under the condition that the corresponding partition functions are given via a dynamical trace formula from a family of transfer operators. Further we give general conditions for the partition functions associated with general spin chains to be of this type and provide various families of examples for which these conditions are satisfied.

Keywords: Dynamical zeta function; Transfer operator; Trace formulae; Thermodynamic formalism; Spin chain; Fock space; Regularized determinants; Weighted composition operator

Introduction

The *dynamical zeta functions* of interest in this paper are generating functions of the form

$$\zeta_{\mathrm{R}}(z) := \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} Z_n\right). \tag{1}$$

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associated with sequences $(Z_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$. If the Z_n arise as *partition functions* of a dynamical system, the notion of dynamical zeta functions or *Ruelle zeta functions* has been introduced by Ruelle in [21,22]. Naming and special form of these functions are motivated by concrete examples from statistical mechanics. The partition function encodes the statistical properties of a system in thermodynamic equilibrium. It depends on the temperature, the volume, and the microstates of a finite number of particles. We will consider partition functions of the form

$$Z_n = \sum_{\nu=0}^{n_1} (-1)^\nu \operatorname{trace} G_\nu^n \quad \text{or} \quad Z_n = \det\bigl(1 - \Lambda^n\bigr) \operatorname{trace} G^n \tag{2}$$

for large n. In the first case one has a (possibly) infinite family of compact operators G_ν, called *transfer operators*, such that the corresponding Schatten norms satisfy $\sum_{\nu=0}^{n_1} \|G_\nu^{n_o}\|_{\mathcal{S}_1(\mathcal{H}_\nu)} < \infty$. In the second case one has as a transfer operator G together with an auxiliary operator Λ. We will refer to formulae of the type (2) as *dynamical trace formulae*. Examples of such dynamical zeta functions derived from trace formulae of the type (2) have been treated repeatedly in the literature, see e.g. [4,5,8,10,11,13,15,18,19,26,27]. Apart from these specific references there is a vast literature on meromorphic continuation of dynamical zeta function via different types of trace formulae; for details we refer to Baladis memoir [2] and the references given therein.

Unlike other kinds of zeta functions such as Riemann's, Selberg's, or Artin's zeta function, our dynamical zeta function is an exponential of a power series, hence itself a power series. Considering $s \mapsto \zeta_{\mathrm{R}}(e^{-s})$ one obtains a function which is holomorphic in a right-half plane, provided ζ_{R} has a non-zero radius of convergence. Zeta functions typically occur as a kind of generating functions for collections of objects like prime numbers or prime geodesics and it is natural to ask for their analytic properties.

In such contexts it is important to know whether the zeta functions have meromorphic continuations to a larger set or even to the entire complex plane in order to prove asymptotic results for counting functions. In the case of spin chains counting functions are not a primary concern. But, via the location of the poles and zeros, meromorphic continuation of the zeta function in this case provides information on the spectrum of the associated transfer operator, which in turn encodes physical quantities of the system (e.g. the free energy or asymptotic properties of correlation functions). On the other hand, via symbolic dynamics, zeta functions of spin chains show up also for Axiom A flows, for which the counting functions are of considerable interest.

In fact, in the case of spin chains one observes that rapid decay of the interaction is linked to good continuation properties of the zeta function. The challenge is to prove continuation also for interactions with not so rapid decay. In this paper we present a class of interactions for meromorphic continuation of the zeta function to the entire plane can be proved. This class not only covers all spin chain examples treated in the literature so far, it also allows to treat new examples like the so-called hard-rod model.

We will show that this kind of information can indeed be provided *if* the partition function can be written via dynamical trace formulae. More precisely, we show that such dynamical zeta functions are quotients of regularized Fredholm determinants, one factor for each transfer operator (see Theorems 2.1 and 2.2). Thus they have representations as Euler products, and it is possible to give a spectral interpretation for its zeros and poles.

Finally, the approximation of the transfer operators by finite rank operators opens a path to a numerical analysis of the associated zeta functions. In special cases this was done in [18].

Results of the kind sketched above become interesting only if one has a sufficient supply of examples of partition functions with the desired properties. We provide a general principle

how to construct such examples from (classical) *spin chains*. To describe these, we consider a Hausdorff space F equipped with a finite measure ν, and let $\mathbb{M}: F \times F \to \{0,1\}$ be a $\nu \otimes \nu$-measurable function, which we call a *transition function*. Then $\Omega_{\mathbb{M}} := \{\underline{\xi} \in F^{\mathbb{N}} \mid \mathbb{M}(\xi_i, \xi_{i+1}) = 1 \ \forall i\}$ will be referred to as a *configuration space*. Further we fix a bounded continuous *interaction* $A \in \mathcal{C}_{\mathrm{b}}(\Omega_{\mathbb{M}})$. These data, together with the left shift $\tau: F^{\mathbb{N}} \to F^{\mathbb{N}}$, $(\tau\underline{\xi})_k := \xi_{k+1}$, are called a *spin chain* or, more technically, a *one-sided one-dimensional subshift*. With such a subshift we associate a *partition function*

$$Z_n(A) := \int_{F^n} \prod_{i=1}^{n} \mathbb{M}(\xi_i, \xi_{i+1}) \exp\left(\sum_{k=0}^{n-1} A\bigl(\tau^k(\overline{\xi_1 \dots \xi_n})\bigr) \right) d\nu^n(\xi_1, \dots, \xi_n), \tag{3}$$

where $\overline{\xi_1 \dots \xi_n} := (\xi_1, \dots, \xi_n, \xi_1, \dots, \xi_n, \dots) \in F^{\mathbb{N}}$ and $\xi_{n+1} := \xi_1$. Consider the special case of $A = 0$, and let $\rho_n : F^{\mathbb{N}} \to F^n$, $\underline{\xi} \mapsto (\xi_1, \dots, \xi_n)$ be the projection. Then

$$Z_n(0) = \nu^n\bigl(\rho_n\bigl(\{\underline{\xi} \in \Omega_{\mathbb{M}} \mid \tau^n \underline{\xi} = \underline{\xi}\}\bigr)\bigr),$$

which measures the number of closed τ-orbits in $\Omega_{\mathbb{M}}$ with period length n with respect to the a priori measure ν. In particular, if the system is a *full shift*, i.e., $\mathbb{M} \equiv 1$, then $Z_n(0) = \nu(F)^n$. For general non-interacting subshifts it is possible to show (see Proposition 5.1) that there is an operator $\mathcal{G}_{\mathbb{M}}$ such that $Z_n(0) = \operatorname{trace} \mathcal{G}_{\mathbb{M}}^n$ for $n \geqslant 2$. If the interaction A is non-zero we have to make more assumptions in order to guarantee such trace representations of the partition function. See Theorem 5.3 for a precise formulation. Its proof depends on a two special types of trace formulae. One (see Lemma 1.7) is elementary and deals with iterates of averages of Hilbert–Schmidt operators. The other (see Theorem 4.6) deals with special composition operators on Fock spaces and is based on a fixed point formula of Atiyah and Bott.

In order to satisfy the hypotheses of Theorem 5.3 one has to verify certain estimates which boil down to asking for a rather rapid decay of the interactions (see Theorem 6.2). Physicists would like to be able to treat models with polynomially decaying interactions,[2] which cannot be treated directly with the methods developed in this paper. One approach to polynomial interactions is to view them as limiting cases of interactions of the type polynomials times exponential decay. These models fall in the class covered by our theorem. Moreover, it is strong enough to also produce all the partial results scattered in the physics literature quoted above (cf. Example 6.5). Finally, it is worth noting that it is possible to reformulate the hard-rod model mentioned above as a spin system which can be treated by our methods (see [20]). Doing so one of the authors shows that the associated dynamical zeta function has a meromorphic continuation to the entire plane even in the interacting case. So far this has been way out of range, [14].

The paper is organized as follows. In Section 1 we recall some key definitions and provide a number of technical results concerning traces and determinants in infinite dimensions used later in the paper. In Section 2 we show how to continue dynamical zeta functions meromorphically if a dynamical trace formula holds. Section 3 contains a description of the composition operators that are instrumental in the construction of dynamical trace formulae for spin chains in Section 5. The underlying trace formulae for these operators are proven in Section 4. In Section 6 we apply our results to Ising type spin chains and give examples for interactions for which our results can be applied.

[2] This is not the same as polynomially decaying correlation functions.

1. Traces and determinants

Given a compact operator K on a Hilbert space $\mathcal{H}$ we will denote by $(\lambda_j(K))_{j\in\mathbb{N}}$ the sequence of its eigenvalues counted with multiplicities. Let $(s_j(K))_{j\in\mathbb{N}}$ be the sequence of the singular numbers of K, i.e., the eigenvalues of the positive compact operator $|K| = \sqrt{K^\star K}$. For $1 \leqslant p < \infty$ the *Schatten class* $\mathcal{S}_p(\mathcal{H})$ is defined as the space of all operators K such that

$$\|K\|_{\mathcal{S}_p(\mathcal{H})} := \big\|\big(s_n(K)\big)_{n\in\mathbb{N}}\big\|_{\ell^p(\mathbb{N})} < \infty.$$

The Schatten classes $\mathcal{S}_p(\mathcal{H}) \subset \mathrm{End}(\mathcal{H})$ for $1 \leqslant p < \infty$ are embedded subalgebras, i.e., they satisfy

$$\|A\|_{\mathrm{End}(\mathcal{H})} \leqslant \|A\|_{\mathcal{S}_p(\mathcal{H})}, \qquad \|AB\|_{\mathcal{S}_p(\mathcal{H})} \leqslant \|A\|_{\mathcal{S}_p(\mathcal{H})}\|B\|_{\mathcal{S}_p(\mathcal{H})},$$

and have the approximation property, i.e., the finite rank operators are dense with respect to $\|\cdot\|_{\mathcal{S}_p(\mathcal{H})}$ (cf. [3, Theorem XI.11.1]). For any $A_1, \dots, A_{\lceil p\rceil} \in \mathcal{S}_p(\mathcal{H})$ one has (cf. [3, Theorem IV.11.2])

$$\big|\mathrm{trace}(A_1 \cdots A_{\lceil p\rceil})\big| \leqslant \|A_1 \cdots A_{\lceil p\rceil}\|_{\mathcal{S}_1(\mathcal{H})} \leqslant \prod_{j=1}^{\lceil p\rceil} \|A_j\|_{\mathcal{S}_p(\mathcal{H})}. \tag{4}$$

By [3, Theorem XI.1.1] this estimate implies that for any $n_o \in \mathbb{N}_{\geqslant p}$ the *n_o-regularized determinant*

$$\det_{n_o}(1-F) := \det(1-F)\exp\left(\sum_{k=1}^{n_o-1} \frac{1}{k}\,\mathrm{trace}\,F^k\right) \tag{5}$$

defined on finite rank operators admits a continuous extension to $\mathcal{S}_p(\mathcal{H})$, also denoted by $A \mapsto \det_{n_o}(1-A)$. The following assertions are true on the level of finite rank operators and can be extended to $\mathcal{S}_p(\mathcal{H})$ by continuity (see [3, Theorem XI.2.1]).

Lemma 1.1. *Let $\mathcal{H}$ be a Hilbert space and $1 \leqslant p \leqslant n_o < \infty$. The function $z \mapsto \det_{n_o}(1 - zA)$ is entire for every fixed $A \in \mathcal{S}_p(\mathcal{H})$ and has the representation*

$$\det_{n_o}(1-zA) = 1 + \sum_{n=n_o}^{\infty} \frac{c_n(A)}{n!}(-1)^n z^n, \tag{6}$$

where the coefficients $c_n(A)$ are defined by

$$c_n(A) := \det\begin{pmatrix} b_1 & n-1 & 0 & \dots & 0 & 0 \\ b_2 & b_1 & n-2 & \dots & 0 & 0 \\ b_3 & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ b_{n-1} & b_{n-2} & b_{n-3} & \dots & b_1 & 1 \\ b_n & b_{n-1} & b_{n-2} & \dots & b_2 & b_1 \end{pmatrix}$$

and

$$b_n := \begin{cases} \operatorname{trace} A^n, & \text{if } n \geqslant n_o, \\ 0, & \text{otherwise.} \end{cases}$$

For $|z|$ sufficiently small one has

$$\det_{n_o}(1 - zA) = \exp\left(-\sum_{n=n_o}^{\infty} \frac{z^n}{n} \operatorname{trace} A^n\right). \tag{7}$$

Let $(\lambda_j)_j$ be the eigenvalues of $A \in \mathcal{S}_p(\mathcal{H})$, then one has the Euler product

$$\det_{n_o}(1 - zA) = \prod_j \left((1 - z\lambda_j) \exp\left(\sum_{k=1}^{n_o-1} \frac{\lambda_j^k}{k} z^k\right)\right) = \prod_j f_{n_o}(z\lambda_j), \tag{8}$$

where

$$f_{n_o}(z) = (1 - z) \exp\left(\sum_{k=1}^{n_o-1} \frac{z^k}{k}\right) \overset{(\star)}{=} \exp\left(-\sum_{k=n_o}^{\infty} \frac{z^k}{k}\right).$$

The identity $(\star)$ can be obtained as a consequence of the power series expansion of $\log(1 - z)$. The Euler product expansion (8) shows that the zeros of $z \mapsto \det_{n_o}(1 - zA)$ are in bijection with the eigenvalues of A.

Lemma 1.2. *Let $\mathcal{H}$ be a Hilbert space. Then for any $n_o \in \mathbb{N}$ there exists a constant $\Gamma_{n_o} > 0$ such that for all $A \in \mathcal{S}_{n_o}(\mathcal{H})$ the estimates below hold. One can choose $\Gamma_1 = 1$.*

$$\left|\det_{n_o}(1 + A)\right| \leqslant \exp\left(\Gamma_{n_o} \left\|A^{n_o}\right\|_{\mathcal{S}_1(\mathcal{H})}\right) \leqslant \exp\left(\Gamma_{n_o} \|A\|_{\mathcal{S}_{n_o}(\mathcal{H})}^{n_o}\right). \tag{9}$$

Proof. The inequality $|\det_{n_o}(1 + A)| \leqslant \exp(\Gamma_{n_o} \|A\|_{\mathcal{S}_{n_o}(\mathcal{H})}^{n_o})$ can be found in [3, Theorem XI.2.2]. The claim follows from this via (8). □

The following criterion for the convergence of infinite products of regularized determinants will turn out to be useful.

Lemma 1.3. *Let $(\mathcal{H}_\nu)_{\nu \in \mathbb{N}}$ be a family of Hilbert spaces. Fix $n_o \in \mathbb{N}$ and pick $G_\nu \in \mathcal{S}_{n_o}(\mathcal{H}_\nu)$ satisfying $\sum_{\nu=0}^{\infty} \|G_\nu^{n_o}\|_{\mathcal{S}_1(\mathcal{H}_\nu)} < \infty$. Then*

$$\prod_{\nu=0}^{\infty} \det_{n_o}(1 - zG_\nu)$$

converges absolutely and locally uniformly to an entire function of z.

Proof. Note that the function f_{n_o} from Lemma 1.1 is of the form $f_{n_o}(z) - 1 = z^{n_o} h_{n_o}(z)$ for some entire function h_{n_o}. Further, $c := \sup_{\nu \in \mathbb{N}_0} \|G_\nu\|$ is finite. Hence $|h_{n_o}(\lambda_j(zG_\nu))| \leqslant \sup_{|w| \leqslant |z|c} |h_{n_o}(w)| =: c_z$ for all eigenvalues $\lambda_j(zG_\nu)$ of zG_ν. Now the hypothesis implies that

$$\sum_{\nu=0}^{\infty} \sum_j \left| f_{n_o}\big(\lambda_j(zG_\nu)\big) - 1 \right| \leqslant c_z |z|^{n_o} \sum_{\nu=0}^{\infty} \left\| G_\nu^{n_o} \right\|_{\mathcal{S}_1(\mathcal{H}_\nu)}$$

is finite. Thus the infinite product $\prod_{\nu=0}^{\infty} \prod_j f_{n_o}(\lambda_j(zG_\nu))$ converges, and by Lemma 1.1 it is equal to $\prod_{\nu=0}^{\infty} \det_{n_o}(1 - zG_\nu)$. This proves the claim. $\square$

Proposition 1.4. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces and $n_o \in \mathbb{N}$. Pick $A \in \mathcal{S}_{n_o}(\mathcal{H}_1)$ and $B \in \mathcal{S}_{n_o}(\mathcal{H}_2)$. If we denote the eigenvalues of B by $\lambda_j(B)$, we have*

$$\det_{n_o}(1 - zA \otimes B) = \prod_j \det_{n_o}\big(1 - z\lambda_j(B)A\big).$$

Proof. For $|z| < \|A\|^{-1}_{\mathcal{S}_{n_o}(\mathcal{H}_1)} \|B\|^{-1}_{\mathcal{S}_{n_o}(\mathcal{H}_2)}$ the Lidskii trace theorem [3, Theorem IV.6.1] applied to the trace class operators B^n $(n \geqslant n_o)$ yields

$$\det_{n_o}(1 - zA \otimes B) \overset{(7)}{=} \prod_j \det_{n_o}\big(1 - z\lambda_j(B)A\big).$$

By Lemma 1.1 the left-hand side is an entire function in z. Therefore analytic continuation shows that the identity holds for all $z \in \mathbb{C}$ if we can show that also the right-hand side is an entire function in z. But that follows from Lemma 1.3 applied to the family $G_j := \lambda_j(B)A$. $\square$

Let $A : \mathcal{H} \to \mathcal{H}$ be a trace class operator on a Hilbert space $\mathcal{H}$ and $\wedge^r A : \bigwedge^r \mathcal{H} \to \bigwedge^r \mathcal{H}$ its r-fold exterior product. Then (cf. [23]) we have

$$\det(1 - A) = \sum_{r=0}^{\dim \mathcal{H}} (-1)^r \operatorname{trace} \wedge^r A \tag{10}$$

and the estimate

$$\left\| \wedge^r A \right\|_{\mathcal{S}_1(\bigwedge^r \mathcal{H})} \leqslant \frac{1}{r!} \|A\|^r_{\mathcal{S}_1(\mathcal{H})}. \tag{11}$$

For the special case of a finite rank operator B with spectrum $\lambda_1, \ldots, \lambda_d$ Proposition 1.4 implies that

$$\det_{n_o}\big(1 - zA \otimes \wedge^\nu B\big) = \prod_{\alpha \in \{0,1\}^d;\ |\alpha| = \nu} \det_{n_o}\big(1 - z\lambda^\alpha A\big), \tag{12}$$

where for $\alpha \in \{0,1\}^d$ we set $\lambda^\alpha := \prod_{j=1}^d \lambda_j^{\alpha_j}$. Approximating Schatten class operators by finite rank operators one derives the following proposition:

Proposition 1.5. *In the situation of Proposition* 1.4 *writing* $\lambda_j := \lambda_j(B)$ *we have*

(i) $$\det_{n_o}\bigl(1 - zA \otimes \wedge^\nu B\bigr) = \lim_{d\to\infty} \prod_{\alpha\in\{0,1\}^d;\, |\alpha|=\nu} \det_{n_o}\bigl(1 - z\lambda^\alpha A\bigr),$$

(ii) $$\prod_{\nu=0}^{\infty} \det_{n_o}\bigl(1 - zA \otimes \wedge^\nu B\bigr) = \lim_{d\to\infty} \prod_{\alpha\in\{0,1\}^d} \det_{n_o}\bigl(1 - z\lambda^\alpha A\bigr).$$

Proof. (i) This is a straightforward consequence of (12) and the approximation property.

(ii) Here the essential point is to verify the summability condition (17) from Lemma 1.3, i.e., the finiteness of

$$\sum_{\nu=0}^{\infty} \sum_{\alpha\in\{0,1\}^{\mathbb{N}};\, |\alpha|=\nu} \bigl\|z\lambda^\alpha A\bigr\|_{\mathcal{S}_{n_o}(\mathcal{H})} = \|zA\|_{\mathcal{S}_{n_o}(\mathcal{H})} \sum_{\nu=0}^{\infty} \sum_{\alpha\in\{0,1\}^{\mathbb{N}};\, |\alpha|=\nu} \bigl|\lambda^\alpha\bigr|.$$

To show this, we note that

$$\sum_{\nu=0}^{\infty} \sum_{\alpha\in\{0,1\}^{\mathbb{N}};\, |\alpha|=\nu} \bigl|\lambda^\alpha\bigr| \overset{(11)}{\leqslant} \sum_{\nu=0}^{\infty} \frac{1}{\nu!} \|B\|^{\nu}_{\mathcal{S}_1(\mathcal{H}_2)} < \infty. \qquad \square$$

The following lemma is proved by induction on n.

Lemma 1.6. *Let* $n \geqslant 2$ *and suppose that the functions* $a_k : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$ *satisfy* $\sum_{i,j=1}^{\infty} |a_k(i,j)|^2 < \infty$ *for* $k = 1, \ldots, n$. *Then*

$$\left| \sum_{i_1,\ldots,i_n=1}^{\infty} \prod_{k=1}^{n} a_k(i_k, i_{k+1}) \right| \leqslant \prod_{k=1}^{n} \left(\sum_{i,j=1}^{\infty} \bigl|a_k(i,j)\bigr|^2 \right)^{1/2}$$

using the convention that $i_{n+1} = i_1$.

Now we can prove the main result of this section.

Lemma 1.7. *Let* (F, ν) *be a measure space,* $g : F \times F \to \mathbb{C}$ *a measurable function, and* $(S_\xi)_{\xi\in F}$ *a measurable family of operators on a separable Hilbert space* $\mathcal{H}$. *The formula*

$$\bigl(T(f_1 \otimes f_2)\bigr)(\eta) := \int_F g(\xi,\eta) f_1(\xi) S_\xi f_2 \, d\nu(\xi) \tag{13}$$

defines a Hilbert–Schmidt operator $T : L^2(F, d\nu) \hat{\otimes} \mathcal{H} \to L^2(F, d\nu) \hat{\otimes} \mathcal{H}$ *iff*

$$\int_F \int_F \bigl|g(\xi,\eta)\bigr|^2 d\nu(\eta) \|S_\xi\|^2_{\mathcal{S}_2(\mathcal{H})} \, d\nu(\xi) < \infty. \tag{14}$$

In this case T satisfies

$$\|T\|^2_{\mathcal{S}_2(L^2(F,d\nu)\hat{\otimes}\mathcal{H})} = \int_F \int_F \left|g(\xi,\eta)\right|^2 d\nu(\eta) \|S_\xi\|^2_{\mathcal{S}_2(\mathcal{H})} d\nu(\xi) \tag{15}$$

and

$$\operatorname{trace} T^n = \int_{F^n} \left(\prod_{j=1}^{n-1} g(\xi_j,\xi_{j+1})\right) g(\xi_n,\xi_1) \operatorname{trace}(S_{\xi_n}\circ\cdots\circ S_{\xi_1})\, d\nu^n(\xi_1,\ldots,\xi_n)$$

for all $n \geqslant 2$. Moreover, for these n we have

$$\begin{aligned}\|T^n\|^2_{\mathcal{S}_2(L^2(F,d\nu)\hat{\otimes}\mathcal{H})} = \int_F \int_{F^n} &\left|\left(\prod_{j=1}^{n-1} g(\xi_j,\xi_{j+1})\right) g(\xi_n,\eta)\right|^2 \\ &\times \|S_{\xi_n}\circ\cdots\circ S_{\xi_1}\|^2_{\mathcal{S}_2(\mathcal{H})}\, d\nu^n(\xi_1,\ldots,\xi_n)\, d\nu(\eta).\end{aligned}$$

Proof. Suppose first that (13) defines a Hilbert–Schmidt operator. Then (15) follows from Parseval's identity. Conversely, if (14) holds, again Parseval's identity shows that not only the integral (13) converges for almost all η, but also that it defines a Hilbert–Schmidt operator on $L^2(F,d\nu)\,\hat{\otimes}\,\mathcal{H}$.

Now assume that T is Hilbert–Schmidt. Then for $n \geqslant 2$ the operator T^n is trace class and a simple induction argument shows that

$$\begin{aligned}&\big(T^n(e\otimes f)\big)(\eta) \\ &\quad= \int_{F^n} \left(\prod_{j=1}^{n-1} g(\xi_j,\xi_{j+1})\right) g(\xi_n,\eta) e(\xi_1) S_{\xi_n}\circ\cdots\circ S_{\xi_1} f\, d\nu^n(\xi_1,\ldots,\xi_n).\end{aligned}$$

By the first part of the proof the S_{ξ_j} are Hilbert–Schmidt (for almost all ξ_j), hence the compositions $S_{\xi_n}\circ\cdots\circ S_{\xi_1}$ are trace class. We claim that $\operatorname{trace} T^n$ can be rewritten as $\sum_{i=1}^{\infty}\langle\mathcal{G}_n e_i \mid e_i\rangle = \operatorname{trace}\mathcal{G}_n$ with

$$\begin{aligned}(\mathcal{G}_n f)(\eta) := \int_{F^n} &\left(\prod_{j=1}^{n-1} g(\xi_j,\xi_{j+1})\right) g(\xi_n,\eta) \\ &\times \operatorname{trace}(S_{\xi_n}\circ\cdots\circ S_{\xi_1}) f(\xi_1)\, d\nu^n(\xi_1,\ldots,\xi_n).\end{aligned}$$

Note that (by Fourier expansion and induction)

$$\operatorname{trace}(S_n\circ\cdots\circ S_1) = \sum_{i_1,\ldots,i_n=1}^{\infty} \left(\prod_{j=1}^{n-1} \langle S_j h_{i_j} \mid h_{i_{j+1}}\rangle\right) \langle S_n h_{i_n} \mid h_{i_1}\rangle$$

for any orthonormal basis $(h_i)_{i\in\mathbb{N}}$ for $\mathcal{H}$ and Hilbert–Schmidt operators S_i on $\mathcal{H}$. Setting

$$(\mathcal{G}_{i,j} f)(\eta) := \int_F g(\xi,\eta)\langle S_\xi h_i \mid h_j\rangle f(\xi)\,d\nu(\xi)$$

for $i, j \in \mathbb{N}$, we can rewrite $\mathcal{G}_n$ as

$$(\mathcal{G}_n f)(\eta) = \sum_{i_1,\dots,i_n=1}^{\infty} (\mathcal{G}_{i_n,i_1} \circ \mathcal{G}_{i_{n-1},i_n} \circ \cdots \circ \mathcal{G}_{i_1,i_2} f)(\eta).$$

The identity

$$\sum_{i,i=1}^{\infty} \|\mathcal{G}_{i,j}\|^2_{\mathcal{S}_2(L^2(F,d\nu))} = \int_{F^2} \bigl|g(\xi,\eta)\bigr|^2 \|S_\xi\|^2_{\mathcal{S}_2(\mathcal{H})}\, d\nu(\xi)\, d\nu(\eta) \tag{16}$$

implies that the $\mathcal{G}_{i,j}$ are Hilbert–Schmidt operators on $L^2(F,d\nu)$. Therefore, for each $(i_1,\dots,i_n) \in \mathbb{N}^n$ the integral operator $\mathcal{G}_{i_n,i_1} \circ \mathcal{G}_{i_{n-1},i_n} \circ \cdots \circ \mathcal{G}_{i_1,i_2}$ is trace class and by [7, Example X.1.18] its trace can be obtained by integrating the integral kernel along the diagonal. If $\mathcal{G}_n$ is trace class, we have

$$\begin{aligned}
\operatorname{trace} \mathcal{G}_n &= \sum_{i_1,\dots,i_n=1}^{\infty} \operatorname{trace}(\mathcal{G}_{i_n,i_1} \circ \mathcal{G}_{i_{n-1},i_n} \circ \cdots \circ \mathcal{G}_{i_1,i_2}) \\
&= \int_{F^n} \left(\prod_{j=1}^{n-1} g(\xi_j,\xi_{j+1})\right) g(\xi_n,\xi_1) \operatorname{trace}(S_{\xi_n} \circ \cdots \circ S_{\xi_1})\, d\nu^n(\xi_1,\dots,\xi_n) \\
&= \operatorname{trace} T^n.
\end{aligned}$$

Thus, to prove the claim it suffices to show that $\sum_{i_1,\dots,i_n=1}^{\infty} \mathcal{G}_{i_n,i_1} \circ \cdots \circ \mathcal{G}_{i_1,i_2}$ converges in $\mathcal{S}_1(L^2(F,d\nu))$. Using the estimate given in Lemma 1.6, we obtain the estimate

$$\begin{aligned}
\|\mathcal{G}_n\|_{\mathcal{S}_1(L^2(F,d\nu))} &\leqslant \sum_{i_1,\dots,i_n=1}^{\infty} \prod_{j=1}^{n} \|\mathcal{G}_{i_j,i_{j+1}}\|_{\mathcal{S}_2(L^2(F,d\nu))} \\
&\leqslant \left(\sum_{i,j=1}^{\infty} \|\mathcal{G}_{i_j,i_{j+1}}\|^2_{\mathcal{S}_2(L^2(F,d\nu))}\right)^{n/2} \\
&\overset{(16)}{=} \left(\int_F \int_F \bigl|g(\xi,\eta)\bigr|^2 \|S_\xi\|^2_{\mathcal{S}_2(\mathcal{H})}\, d\nu(\xi)\, d\nu(\eta)\right)^{n/2},
\end{aligned}$$

which proves the claim. To conclude the proof of the lemma one verifies the formula for $\|T^n\|^2_{\mathcal{S}_2(L^2(F,d\nu)\hat{\otimes}\mathcal{H})}$ for $n \geqslant 2$, which can be done similarly as in the case $n = 1$. $\square$

If ν is a finite measure on F, $g: F^2 \to \mathbb{C}$ is bounded, and $\int_F \|S_\xi\|^2_{\mathcal{S}_2(\mathcal{H})}\, d\nu(\xi)$ is finite, Lemma 1.7 shows that the associated operator T is Hilbert–Schmidt.

2. Meromorphic continuation

The following theorem is an analog of a result of D. Mayer (see [13, Theorem 7.17]) proven there in the context of generalized Perron–Frobenius operators associated with expanding maps.

Theorem 2.1. *Let $(\mathcal{H}_\nu)_{\nu\in\mathbb{N}_0}$ be a family of Hilbert spaces. Fix $n_o \in \mathbb{N}$ and pick $G_\nu \in \mathcal{S}_{n_o}(\mathcal{H}_\nu)$ such that*

$$\sum_{\nu=0}^{\infty} \left\| G_\nu^{n_o} \right\|_{\mathcal{S}_1(\mathcal{H}_\nu)} < \infty. \tag{17}$$

Let $(Z_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{C}$ such that $Z_n = \sum_{\nu=0}^{\infty}(-1)^\nu \operatorname{trace} G_\nu^n$ for all $n \geqslant n_o$. Then the dynamical zeta function ζ_{R} associated with $(Z_n)_{n\in\mathbb{N}}$ admits a meromorphic continuation to the entire plane. It is given by the formula

$$\zeta_{\mathrm{R}}(z) = \exp\left(\sum_{n=1}^{n_o-1} \frac{z^n}{n} Z_n\right) \prod_{\nu=0}^{\infty} \left(\det\nolimits_{n_o}(1 - zG_\nu)\right)^{(-1)^{\nu+1}}.$$

Proof. We treat the case of finitely many non-zero operators first, say $G_l = 0$ for all $l \geqslant k$. For $|z| < \min\{\|(G_\nu)^{n_o}\|^{-1}_{\mathcal{S}_1(\mathcal{H}_\nu)} \mid \nu = 0, \ldots, k\}$, using Lemma 1.1, one obtains a finite product of meromorphic functions

$$\zeta_{\mathrm{R}}(z) = \exp\left(\sum_{n=1}^{n_o-1} \frac{z^n}{n} Z_n\right) \prod_{\nu=0}^{k} \left(\det\nolimits_{n_o}(1 - zG_\nu)\right)^{(-1)^{\nu+1}}.$$

We turn to the general case. By (17) the sequence $\|G_\nu^{n_o}\|_{\mathcal{S}_1(\mathcal{H}_\nu)}$ tends to zero as $\nu \to \infty$, so the minimum $\min\{\|G_\nu^{n_o}\|^{-1}_{\mathcal{S}_1(\mathcal{H}_\nu)} \mid \nu \in \mathbb{N}_0\} > 0$ exists. Using the convergence criterion from Lemma 1.3 we see that the quotient of infinite products

$$\prod_{\nu=0}^{\infty} \left(\det\nolimits_{n_o}(1 - zG_\nu)\right)^{(-1)^{\nu+1}} = \frac{\prod_{\nu=0}^{\infty} \det_{n_o}(1 - zG_{2\nu+1})}{\prod_{\nu=0}^{\infty} \det_{n_o}(1 - zG_{2\nu})}$$

converges absolutely and locally uniformly for all $z \in \mathbb{C}$. $\square$

Theorem 2.2. *Given two Hilbert spaces $\mathcal{H}$ and $\mathcal{H}_o$, fix $n_o \in \mathbb{N}$, and consider $G \in \mathcal{S}_{n_o}(\mathcal{H})$ and $\Lambda \in \mathcal{S}_{n_o}(\mathcal{H}_o)$. Let $(Z_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{C}$ such that $Z_n = \det(1 - \Lambda^n) \operatorname{trace} G^n$ for $n \geqslant n_o$.*

(i) *The dynamical zeta function ζ_{R} associated with $(Z_n)_{n\in\mathbb{N}}$ admits a meromorphic continuation to the entire plane. It is given by the formula*

$$\zeta_{\mathrm{R}}(z) = \exp\left(\sum_{n=1}^{n_o-1} \frac{z^n}{n} Z_n\right) \prod_{\nu=0}^{\dim \mathcal{H}_o} \left(\det\nolimits_{n_o}\left(1 - zG \otimes \wedge^\nu \Lambda\right)\right)^{(-1)^{\nu+1}}.$$

(ii) *Denote the eigenvalues of Λ by $(\lambda_i)_{i\in\mathbb{N}}$, repeated according to multiplicity. For $\alpha \in \{0,1\}^d$ set $\lambda^\alpha := \prod_{\nu=1}^d \lambda_\nu^{\alpha_\nu}$. Then the meromorphic continuation of ζ_{R} is given by the formula*

$$\zeta_{\mathrm{R}}(z) = \exp\left(\sum_{n=1}^{n_o-1} \frac{z^n}{n} Z_n\right) \lim_{d\to\infty} \prod_{\alpha\in\{0,1\}^d} \left(\det_{n_o}\left(1 - z\lambda^\alpha G\right)\right)^{(-1)^{|\alpha|+1}}.$$

Proof. Since $\wedge^r(A^n) = (\wedge^r A)^n$ and $\operatorname{trace} A \operatorname{trace} B = \operatorname{trace}(A\otimes B)$ for all trace class operators A and B, the identity (10) implies that

$$Z_n = \sum_{\nu=0}^{\dim \mathcal{H}_o} (-1)^\nu \operatorname{trace} G_\nu^n$$

with $G_\nu := G \otimes \wedge^\nu \Lambda$ on $\mathcal{H}^{(\nu)} := \mathcal{H} \hat{\otimes} \bigwedge^\nu \mathcal{H}_o$ for $n \geqslant n_o$. Note here that the estimate (11) provides the summability condition (17). In fact,

$$\sum_{\nu=0}^{\dim \mathcal{H}_o} \left\|(G_\nu)^{n_o}\right\|_{\mathcal{S}_1(\mathcal{H}^{(\nu)})} \leqslant \left\|G^{n_o}\right\|_{\mathcal{S}_1(\mathcal{H})} \sum_{\nu=0}^{\infty} \frac{1}{\nu!} \left\|\Lambda^{n_o}\right\|_{\mathcal{S}_1(\mathcal{H}_o)}^\nu < \infty.$$

Now we can apply Theorem 2.1 to finish the proof of (i) which together with Proposition 1.5 also proves (ii). □

Suppose Λ belongs to $\mathcal{S}_{n_o}(\mathcal{H}_o)$ and $1 \notin \operatorname{spec}(\Lambda)$. Then the set $\bigcup_{d\in\mathbb{N}}\{\lambda^\alpha \in \mathbb{C} \mid \alpha \in \{0,1\}^d,\ \lambda_i \in \operatorname{spec}(\Lambda)\}$ contains a unique element of maximal modulus, namely λ^{α^*} where $\alpha_i^* = 1$ if $|\lambda_i| \geqslant 1$ and $\alpha_i^* = 0$ otherwise. Then Theorem 2.2(ii) implies that the physically important first pole of zeta, i.e. the pole of minimal modulus, is located at $(\lambda^{\alpha^*}\mu_1)^{-1}$ where μ_1 is the leading eigenvalue of G. Whereas the first pole cannot be cancelled by a zero, the others possibly can. So Theorem 2.2(ii) does in general not imply that the poles and zeros of ζ_{R} are precisely the z such that $z\lambda^\alpha G$ has 1 as an eigenvalue. On the other hand, we will see examples of physical systems with rapidly decaying interaction with a cancellation-free representation of the dynamical zeta function (cf. Proposition 5.1, Example 6.5(i) and (iii), where the product turns out to be simple). In these cases one has the desired spectral interpretation of zeta's poles given by a simple bijection between the poles and the eigenvalues of a Schatten class operator. If the interaction gets more complicated, cancellations in general cannot be avoided—see also Remark 6.9(iii).

3. Composition operators on Fock spaces

We start by briefly recalling some basic properties of reproducing kernel spaces of holomorphic functions on not necessarily finite-dimensional manifolds. Our basic reference for this material is [16], although we choose a different normalization.

Let $\mathcal{H} \subset \mathbb{C}^E$ be a Hilbert space consisting of complex valued functions on a set E. The space $\mathcal{H}$ is called a *reproducing kernel Hilbert space* (RKHS), if for each $x \in E$ the evaluation functional $\mathrm{ev}_x : \mathcal{H} \to \mathbb{C}$, $f \mapsto f(x)$ is continuous. A function $k : E \times E \to \mathbb{C}$ is called a *reproducing kernel* for $\mathcal{H}$, if for all $y \in E$ the function $k_y := k(\cdot, y) : E \to \mathbb{C}$ belongs to $\mathcal{H}$ and if for all $f \in \mathcal{H}$, $y \in E$ we have $f(y) = \langle f \mid k_y\rangle_{\mathcal{H}}$. Recall that a function $p : E \times E \to \mathbb{C}$ is *positive definite*, if

$\sum_{k,l=1}^{n} \overline{a_k} a_l p(x_k, x_l) \geqslant 0$ for all $n \in \mathbb{N}$, $a_j \in \mathbb{C}$, $x_j \in E$ $(j = 1, \dots, n)$. These concepts are connected by the following fact, cf. [16, I.1]: If $\mathcal{H} \subset \mathbb{C}^E$ is a RKHS, then the function $k : E \times E \to \mathbb{C}$ defined by $k(x, y) := \mathrm{ev}_x \circ \mathrm{ev}_y^*$ is a positive definite reproducing kernel for $\mathcal{H}$. Moreover, for all $f \in \mathcal{H}$, $x \in E$ one has

$$\left|f(x)\right| \leqslant \|f\| \sqrt{k(x, x)}. \tag{18}$$

Since the span of the kernel functions k_w $(w \in E)$ is dense in $\mathcal{H}$, a bounded operator T on $\mathcal{H}$ is uniquely determined by its "integral kernel"

$$k_T(z, w) := (T k_w)(z) = \langle T k_w \mid k_z \rangle. \tag{19}$$

The integral kernel of the adjoint T^* of T is obtained from

$$k_{T^*}(z, w) = \langle T^* k_w \mid k_z \rangle = \langle k_w \mid T k_z \rangle = \overline{\langle T k_z \mid k_w \rangle} = \overline{k_T(w, z)}.$$

Recall that the *Bargmann–Fock space* $\mathcal{F}(\mathbb{C}^m)$ is defined as the space of entire functions $F : \mathbb{C}^m \to \mathbb{C}$ with

$$\|f\|^2_{\mathcal{F}(\mathbb{C}^m)} := \int_{\mathbb{C}^m} \left|f(z)\right|^2 \exp\left(-\pi \|z\|^2\right) dz < \infty$$

where dz denotes Lebesgue measure on $\mathbb{C}^m$. It is a RKHS with reproducing kernel $k(z, w) = \exp(\pi \langle z \mid w \rangle)$.

Let $(\mathcal{H}, \langle \cdot \mid \cdot \rangle)$ be a separable Hilbert space, then the map $k : \mathcal{H} \times \mathcal{H} \to \mathbb{C}, (z, w) \mapsto \exp(\pi \langle z \mid w \rangle)$ is a positive definite kernel, see [16, I.2.2]. One defines the (*symmetric*) *Fock space* to be the unique reproducing kernel Hilbert space $\mathcal{F}(\mathcal{H}) \subset \mathbb{C}^{\mathcal{H}}$ associated with this kernel. Since the reproducing kernel is holomorphic in the first and anti-holomorphic in the second variable, [16, Proposition A.III.10] shows that $\mathcal{F}(\mathcal{H}) \subset \mathcal{O}(\mathcal{H})$. In particular, $\mathcal{F}(\mathcal{H})$ consists of continuous functions. Let $(e_i)_{i \in I}$ be an orthonormal basis for $\mathcal{H}$. Then the monomials $\zeta_\alpha(z) = (\frac{\pi^\alpha}{\alpha!})^{1/2} \prod_i \langle z \mid e_i \rangle^{\alpha_i}$ $(\alpha \in (\mathbb{N}_0)^I)$ form an orthonormal basis for the Fock space $\mathcal{F}(\mathcal{H})$, where we use the standard multi-index notations: $\alpha! := \prod_i \alpha_i!$ and $|\alpha| := \sum_i \alpha_i$. We will use composition operators to view the $\mathcal{F}(\mathbb{C}^m)$ as subspaces of $\mathcal{F}(\mathcal{H})$. In this context we note the following elementary lemma.

Lemma 3.1. *Let $\mathcal{H}_1$, $\mathcal{H}_2$ be separable Hilbert spaces and $\mathbb{B} : \mathcal{H}_1 \to \mathcal{H}_2$ a linear operator with $\|\mathbb{B}\| \leqslant 1$. Define $C_{\mathbb{B}} : \mathcal{F}(\mathcal{H}_2) \to \mathcal{F}(\mathcal{H}_1)$, $f \mapsto f \circ \mathbb{B}$. Then $C_{\mathbb{B}}$ is continuous with $\|C_{\mathbb{B}}\| \leqslant 1$ and $(C_{\mathbb{B}})^* = C_{\mathbb{B}^*}$. If $\mathbb{B}$ is surjective, then $C_{\mathbb{B}}$ is injective. In particular, if $\mathbb{B}\mathbb{B}^* = \mathrm{id}$, then $C_{\mathbb{B}}$ is an isometric embedding.*

Using Lemma 3.1 we obtain

(i) If $P : \mathcal{H} \to \mathcal{H}$ is a projection, then $C_P \in \mathrm{End}(\mathcal{F}(\mathcal{H}))$ is a projection.
(ii) If $P : \mathcal{H} \to \mathcal{H}$ is self-adjoint, then $C_P \in \mathrm{End}(\mathcal{F}(\mathcal{H}))$ is self-adjoint.
(iii) As a consequence of (i) and (ii) we see that if $P : \mathcal{H} \to \mathcal{H}$ is an orthogonal projection, then $C_P \in \mathrm{End}(\mathcal{F}(\mathcal{H}))$ is an orthogonal projection.

(iv) Let $P:\mathcal{H}\to\mathcal{H}$ be an orthogonal projection. For $p:\mathcal{H}\to P\mathcal{H}$, $z\mapsto Pz$ we have $P=p^*p$ and $pp^*=\mathrm{id}_{P\mathcal{H}}$. Hence $C_p:\mathcal{F}(P\mathcal{H})\to\mathcal{F}(\mathcal{H})$ is an isometric embedding.
(v) With the identification $\mathcal{F}(P\mathcal{H})\cong C_p(\mathcal{F}(P\mathcal{H}))\subset\mathcal{F}(\mathcal{H})$ we can view $\mathcal{F}(P\mathcal{H})$ as a subspace of $\mathcal{F}(\mathcal{H})$. Moreover, $\mathcal{F}(P\mathcal{H})$ has a reproducing kernel, namely the kernel of $\mathcal{F}(\mathcal{H})$ restricted to $P\mathcal{H}\times P\mathcal{H}$.
(vi) $C_{p*}:\mathcal{F}(\mathcal{H})\to\mathcal{F}(P\mathcal{H})$ is the adjoint of $C_p:\mathcal{F}(P\mathcal{H})\to\mathcal{F}(\mathcal{H})$, $C_P=C_p(C_p)^*$, and $1=\|C_p\|=\|C_{p*}\|$.

Lemma 3.2. *Let $P_n:\mathcal{H}\to\mathcal{H}$ be a sequence of orthogonal projections converging to the identity in the strong operator topology. Then the sequence $C_{P_n}\in\mathrm{End}(\mathcal{F}(\mathcal{H}))$ of orthogonal projections converges in the strong operator topology to the identity on $\mathcal{F}(\mathcal{H})$ as $n\to\infty$.*

Proof. For all $z\in\mathcal{H}$ one has $P_nz\to z$ as $n\to\infty$. Since $\mathcal{F}(\mathcal{H})\subset\mathcal{C}(\mathcal{H})$, we have for all $f\in\mathcal{F}(\mathcal{H})$

$$C_{P_n}f(z)=f(P_nz)\to f(z).$$

Using the reproducing kernel property, this can be rewritten as

$$\langle C_{P_n}f\mid k_z\rangle\to\langle f\mid k_z\rangle$$

for all $z\in\mathcal{H}$. Since the functions k_z ($z\in\mathcal{H}$) form a total subset of $\mathcal{F}(\mathcal{H})$, this implies weak operator convergence which on a Hilbert space coincides with the strong operator convergence. □

Using Lemma 3.2 it is not difficult to derive the following proposition part of which can also be found in [9, II].

Proposition 3.3. *Let $P_n:\mathcal{H}\to\mathcal{H}$ be an ascending sequence of orthogonal projections with n-dimensional range converging in the strong operator topology to the identity, i.e., $P_n\mathcal{H}\subset P_{n+1}\mathcal{H}$. Set $\mathrm{pr}_n:\mathcal{H}\to\mathcal{H}_n:=P_n\mathcal{H}$, $z\mapsto P_nz$. A function f belongs to the Fock space $\mathcal{F}(\mathcal{H})$, defined as the RKHS with reproducing kernel $k(z,w)=\exp(\pi\langle z\mid w\rangle)$, if and only if the following three conditions hold*:

(i) *$f:\mathcal{H}\to\mathbb{C}$ is continuous,*
(ii) *for all $m\in\mathbb{N}$ the map $f\circ\mathrm{pr}_m^*:\mathcal{H}_m\to\mathbb{C}$ is analytic, and*
(iii) *$\sup_{m\in\mathbb{N}}\int_{\mathcal{H}_m}|f\circ\mathrm{pr}_m^*(z)|^2\exp(-\pi\|z\|^2)\,dz<\infty$.*

In this case, this supremum equals

$$\|f\|^2_{\mathcal{F}(\mathcal{H})}=\lim_{m\in\mathbb{N}}\int\limits_{\mathcal{H}_m}\left|f\circ\mathrm{pr}_m^*(z)\right|^2\exp\left(-\pi\|z\|^2\right)dz.$$

Let E be a set and V a space of complex-valued functions on E. A (*weighted* or *generalized*) *composition operator* is an operator $T:V\to V$ of the form

$$(Tf)(z)=\phi(z)(f\circ\psi)(z),$$

where $\phi : E \to \mathbb{C}$, $\psi : E \to E$ are given functions. If the multiplication part is trivial, i.e., $\phi \equiv 1$, then T is called a (*classical*) *composition operator*.

Let E, F be non-empty sets. Let $\phi_x : E \to \mathbb{C}$, $\psi_x : E \to E$ for each $x \in F$, and $T_x : \mathbb{C}^E \to \mathbb{C}^E$, $(T_x f)(z) := \phi_x(z)(f \circ \psi_x)(z)$. Then a simple induction argument yields the composition law

$$(T_{x_n} \circ \cdots \circ T_{x_1} f)(z) = \prod_{k=1}^{n} (\phi_{x_k} \circ \psi_{x_{k+1}} \circ \cdots \circ \psi_{x_n})(z)(f \circ \psi_{x_1} \circ \cdots \circ \psi_{x_n})(z). \tag{20}$$

Remark 3.4. Let $0 < q < 1$ and $\psi : X \to X$ be a function on a normed space $(X, \|\cdot\|)$ with $\|\psi(z) - \psi(w)\| \leqslant q\|z - w\|$ for all $z, w \in X$. Then ψ is called a contraction. Set

$$r_\psi := \frac{\|\psi(0)\|}{1-q}. \tag{21}$$

Using standard estimates, one shows that the set $K_r := \{z \in X \mid \|z\| \leqslant r\}$ satisfies $\psi(K_r) \subset K_{qr+\|\psi(0)\|} \subset K_r$ for any $r > r_\psi$. Let $\psi^{(m)} := \psi \circ \cdots \circ \psi$ (m-times) be the mth iterate of ψ. Then for any $z \in X$ and $m \geqslant n_0 \geqslant \frac{\ln(\frac{r-r_\psi}{\|z\|})}{\ln q}$ we have

$$\left\|\psi^{(m)}(z)\right\| \leqslant \left\|\psi^{(m)}(z) - \psi^{(m)}(0)\right\| + \left\|\psi^{(m)}(0)\right\| \leqslant q^m\|z\| + r_\psi \leqslant r,$$

since $0 \in K_{r_\psi}$ implies $\psi^{(m)}(0) \in K_{r_\psi}$.

Remark 3.5. Let $(X, \|\cdot\|)$ be a normed space, $\psi : X \to X$ be a contraction in the sense of Proposition 3.4, and $\phi : X \to \mathbb{C}$ a continuous function. Let $r > r_\psi$ with r_ψ as in (21), and T be the weighted composition operator

$$T : \mathcal{C}(K_r) \to \mathcal{C}(K_r), \quad (Tf)(z) = \phi(z)(f \circ \psi)(z).$$

(i) Let $g \in \mathcal{C}(K_r)$, then Tg belongs to $\mathcal{C}(K_{\delta r})$ for some $\delta > 1$. In fact, if $z \in K_{\delta r}$ and $\delta < \frac{r-\|\psi(0)\|}{rq}$, then $|\psi(z)| \leqslant qr\delta + \|\psi(0)\| < r$. Since $\frac{r-\|\psi(0)\|}{rq} > 1$, we may choose $\delta > 1$.
(ii) Every eigenfunction of T for a non-zero eigenvalue belongs to $\mathcal{C}(X)$. To see this, let $f \in \mathcal{C}(K_r)$ be an eigenfunction of T for a non-zero eigenvalue ρ. Hence by iterating relation (i) n-times we get $f = \rho^{-n}T^n f \in \mathcal{C}(K_{\delta^n r})$ for some $\delta > 1$. Since $X = \bigcup_{r>0} K_r$, we conclude that $f \in \mathcal{C}(X)$.

4. A trace formula for composition operators

Let $U \subset \mathbb{C}^k$ be an open bounded complex domain. Let $A^\infty(U)$ denote the space of holomorphic functions on U which are continuous on the closure $\overline{U}$ of U. Clearly, $A^\infty(U)$ is a Banach space with respect to the supremum norm.

The following theorem, due to D. Ruelle ([21], see also [11, Appendix B], [12] for the infinite-dimensional case) is based on a fixed point formula of Atiyah and Bott (cf. [1]).

Theorem 4.1. *Let $U \subset \mathbb{C}^k$ be an open bounded complex domain. Let $\phi : U \to \mathbb{C}$ and $\psi : U \to U$ be holomorphic functions with continuous extensions to $\overline{U}$ and, moreover, $\psi(\overline{U}) \subset U$. Then ψ has a unique fixed point $z^* \in U$ and the weighted composition operator*

$$T : A^\infty(U) \to A^\infty(U), \quad (Tf)(z) = \phi(z)(f \circ \psi)(z)$$

is nuclear of order zero with trace given by the Atiyah–Bott type fixed point formula

$$\operatorname{trace}_{A^\infty(U)} T = \frac{\phi(z^*)}{\det(1 - \psi'(z^*))}.$$

Lemma 4.2. *Let $\psi : \mathbb{C}^m \to \mathbb{C}^m$ and $\phi : \mathbb{C}^m \to \mathbb{C}$ be entire functions, and ψ a contraction in the sense of Proposition* 3.4. *Let $r > r_\psi$ with r_ψ as in* (21) *and $T : A^\infty(B(0; r)) \to A^\infty(B(0; r))$ be the composition operator acting via*

$$(Tf)(z) = \phi(z)(f \circ \psi)(z).$$

Let f an eigenfunction of T for a non-zero eigenvalue ρ. Then f is entire and there exist $c_1, c_2 > 0$ such that for all $z \in \mathbb{C}^m$

$$\bigl|f(z)\bigr| \leqslant \|z\|^{-c_1 \ln \rho} \sup_{\|w\| \leqslant r} \bigl|f(w)\bigr| \max_{t \in [0,2\pi]} \bigl|\phi\bigl(e^{it} z\bigr)\bigr|^{c_2 \ln\|z\|}.$$

Moreover, if $A^2(U) := \mathcal{O}(U) \cap L^2(U, dz)$ denotes the Bergman space, then

$$\operatorname{trace}_{A^\infty(U)} T = \operatorname{trace}_{A^2(U)} T$$

for all ψ-invariant bounded domains $U \subset \mathbb{C}^m$.

Proof. Let f be an eigenfunction of T for a non-zero eigenvalue ρ. For $n \in \mathbb{N}$ we have $f = \rho^{-n} T^n f$ which by (20) is given as

$$f(z) = \rho^{-n} \prod_{k=0}^{n-1} \bigl(\phi \circ \psi^{(k)}\bigr)(z) \bigl(f \circ \psi^{(n)}\bigr)(z),$$

where $\psi^{(k)}$ is the kth iterate of ψ. As in Remark 3.5(ii) one shows that f is entire, thus belongs to $A^2(U)$ for all bounded domains $U \subset \mathbb{C}^m$. Hence every eigenvalue of $T|_{A^\infty(U)}$ belongs to the spectrum of $T|_{A^2(U)}$, thus by Lidskii's trace theorem the traces coincide.

Let $z \in \mathbb{C}^d$ with $\|z\| > r_\psi$ and

$$n(z) := \left\lceil \frac{\ln(\frac{r - r_\psi}{\|z\|})}{\ln q} \right\rceil.$$

One can find constants $c_1, c_2 > 0$ such that $c_1 \ln\|z\| \leqslant n(z) \leqslant c_2 \ln\|z\|$ for all $\|z\| > r_\psi$. Remark 3.4 implies that $\|\psi^{(n(z))}(z)\| \leqslant r$, and hence

$$\left|f(z)\right| \leqslant |\rho|^{-c_1 \ln\|z\|} \sup_{\|w\|\leqslant r} \left|f(w)\right| \sup_{\|w\|\leqslant \|z\|} \left|\phi(w)\right|^{c_2 \ln\|z\|}.$$

By the maximum principle we know that the supremum $\sup_{\|w\|\leqslant\|z\|}|\phi(w)|$ is attained for some w with $\|w\| = \|z\|$. □

Proposition 4.3. *Let $b \in \mathbb{C}^m$, $\mathbb{B} \in \mathrm{Gl}(m;\mathbb{C})$ with $\|\mathbb{B}\| < 1$, and $\phi : \mathbb{C}^m \to \mathbb{C}$ an entire function which can be estimated by $|\phi(z)| \leqslant c\exp(a\|z\|)$ for some constants $a, c > 0$. Let T be the composition operator given by*

$$(Tf)(z) = \phi(z) f(\mathbb{B}z + b).$$

Then $T : \mathcal{F}(\mathbb{C}^m) \to \mathcal{F}(\mathbb{C}^m)$ is a trace class operator with

$$\operatorname{trace}_{\mathcal{F}(\mathbb{C}^m)} T = \operatorname{trace}_{A^\infty(B(0;r))} T = \frac{\phi((1-\mathbb{B})^{-1}b)}{\det(1-\mathbb{B})}$$

for all $B(0;r) := \{z \in \mathbb{C}^m \mid \|z\| < r\}$ with $r > \frac{\|b\|}{1-\|\mathbb{B}\|}$.

Proof. A standard estimate shows that T is a trace class operator on $\mathcal{F}(\mathbb{C}^m)$. Let $f \in A^\infty(B(0;r))$ be an eigenfunction of T corresponding to a non-zero eigenvalue ρ. By Lemma 4.2 the eigenfunction f satisfies the estimate

$$\left|f(z)\right|^2 \exp\left(-\pi\|z\|^2\right) \leqslant \|z\|^{-c_1 \ln\rho} \exp\left(\left(a\|z\| + \ln c\right)c_2 \ln\|z\|\right) \exp\left(-\pi\|z\|^2\right).$$

This upper bound is Lebesgue-integrable on $\mathbb{C}^m$, and thus f belongs to $\mathcal{F}(\mathbb{C}^m)$. This shows that every non-zero eigenvalue of $T|_{A^\infty(B(0;r))}$ is an eigenvalue of $T|_{\mathcal{F}(\mathbb{C}^m)}$, hence the traces coincide, and by Theorem 4.1 they have the stated value. □

Let T be a trace class operator on a Hilbert space $\mathcal{H} \subset L^2(Z, dm)$ with reproducing kernel k. Then by general theory the trace is given by integrating the integral kernel (19) along the diagonal,

$$\operatorname{trace} T = \int_Z k_T(z,z)\,dm(z) = \int_Z (Tk_z)(z)\,dm(z).$$

Thus Proposition 4.3 yields the non-trivial integral identity

$$\frac{\phi((1-\mathbb{B})^{-1}b)}{\det(1-\mathbb{B})} = \int_{\mathbb{C}^m} \phi(z)\, e^{\pi\langle \mathbb{B}z+b|z\rangle} e^{-\pi\|z\|^2}\,dz. \tag{22}$$

Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{F}(\mathcal{H})$ the associated Fock space. Fix $a, b \in \mathcal{H}$, and $\mathbb{B} \in \mathrm{End}(\mathcal{H})$. Consider the (possibly unbounded) composition operator

$$\mathcal{K}_{a,b,\mathbb{B}} : \mathcal{F}(\mathcal{H}) \to \mathcal{F}(\mathcal{H}), \quad (\mathcal{K}_{a,b,\mathbb{B}} f)(z) = e^{\pi\langle z|a\rangle} f(\mathbb{B}z + b). \tag{23}$$

If $\mathcal{H}$ is finite-dimensional and $\|\mathbb{B}\| < 1$, then by Proposition 4.3 the operator $\mathcal{K}_{a,b,\mathbb{B}}$ is trace class, hence compact. Combining this with an argument from [12, III] yields the following proposition.

Proposition 4.4. *Let $(\mathcal{H}, \langle \cdot \mid \cdot \rangle)$ be a finite-dimensional Hilbert space.*

(i) *Let $a, b \in \mathcal{H}$, $\mathbb{B} \in \operatorname{End}(\mathcal{H})$ with $\|\mathbb{B}\| < 1$, and $\mathcal{K}_{a,b,\mathbb{B}} \in \operatorname{End}(\mathcal{F}(\mathcal{H}))$ be the corresponding composition operator* (23). *Then $(\mathcal{K}_{a,b,\mathbb{B}})^\star = \mathcal{K}_{b,a,\mathbb{B}^\star}$ and $\mathcal{K}_{a,b,\mathbb{B}}$ is selfadjoint if and only if $\mathbb{B}$ is selfadjoint and $a = b$.*

(ii) *If $\mathbb{B}$ is positive, then $\mathcal{K}_{b,b,\mathbb{B}}$ is positive and trace class with*

$$\operatorname{trace} \mathcal{K}_{\beta,\beta,\mathbb{B}} = \|\mathcal{K}_{\beta,\beta,\mathbb{B}}\|_{\mathcal{S}_1(\mathcal{F}(\mathcal{H}))} = \frac{\exp(\pi \|(1-\mathbb{B})^{-1/2}\beta\|^2)}{\det(1-\mathbb{B})}.$$

(iii) *Let $a_i, b_i \in \mathcal{H}$, $\mathbb{B}_i \in \operatorname{End}(\mathcal{H})$ with $\|\mathbb{B}_i\| < 1$ $(i = 1, 2)$, then*

$$\mathcal{K}_{a_1,b_1,\mathbb{B}_1}\mathcal{K}_{a_2,b_2,\mathbb{B}_2} = e^{\pi\langle b_1|a_2\rangle}\,\mathcal{K}_{a_1+\mathbb{B}_1^\star a_2,\mathbb{B}_2 b_1+b_2,\mathbb{B}_2\mathbb{B}_1}.$$

Lemma 4.5. *Let $(\mathcal{H}, \langle \cdot \mid \cdot \rangle)$ be a finite-dimensional Hilbert space, $\mathbb{B} \in \operatorname{End}(\mathcal{H})$ with $\|\mathbb{B}\| < 1$, and $a, b \in \mathcal{H}$. Set*

$$\Lambda = \sqrt{\mathbb{B}\mathbb{B}^\star}, \qquad \beta = \left(1 + \sqrt{\mathbb{B}\mathbb{B}^\star}\right)^{-1}(\mathbb{B}a + b), \qquad \gamma = \exp\left(\frac{\pi}{2}\left(\|a\|^2 - \|\beta\|^2\right)\right).$$

Let $\mathcal{K} := \mathcal{K}_{a,b,\mathbb{B}}$ and $K := \gamma\mathcal{K}_{\beta,\beta,\Lambda} \in \operatorname{End}(\mathcal{F}(\mathcal{H}))$ be the corresponding composition operators (23). *Then $K = |\mathcal{K}| = \sqrt{\mathcal{K}^\star\mathcal{K}}$, and*

$$\|\mathcal{K}\|_{\mathcal{S}_1(\mathcal{F}(\mathcal{H}))} = \frac{\gamma\exp(\pi \|(1-\Lambda)^{-1/2}\beta\|^2)}{\det(1-\Lambda)}.$$

Proof. Using Proposition 4.4 this is a straightforward verification. □

Similarly one shows that $\sqrt{\mathcal{K}\mathcal{K}^\star}$ is given by $K' := \gamma'\mathcal{K}_{\beta',\beta',\Lambda'} \in \operatorname{End}(\mathcal{F}(\mathcal{H}))$ with

$$\Lambda' := |\mathbb{B}| = \sqrt{\mathbb{B}^\star\mathbb{B}}, \qquad \beta' := \left(1 + |\mathbb{B}|\right)^{-1}\left(\mathbb{B}^\star b + a\right), \qquad \gamma' := \exp\left(\frac{\pi}{2}\left(\|b\|^2 - \|\beta_1\|^2\right)\right).$$

Theorem 4.6. *Let $\mathcal{H}$ be a separable Hilbert space and $a, b \in \mathcal{H}$. Fix $\mathbb{B} \in \mathcal{S}_1(\mathcal{H})$ with $\|\mathbb{B}\| < 1$, and consider the weighted composition operator*

$$\mathcal{K} : \mathcal{F}(\mathcal{H}) \to \mathcal{F}(\mathcal{H}), \quad (\mathcal{K}f)(z) = e^{\pi\langle z|a\rangle} f(\mathbb{B}z + b).$$

(i) *The operator $K := |\mathcal{K}| := \sqrt{\mathcal{K}^\star\mathcal{K}} : \mathcal{F}(\mathcal{H}) \to \mathcal{F}(\mathcal{H})$ is given by $(Kf)(z) := \gamma e^{\pi\langle z|\beta\rangle} f(\Lambda z + \beta)$, where*

$$\Lambda = \sqrt{\mathbb{B}\mathbb{B}^\star}, \qquad \beta = \left(1 + \sqrt{\mathbb{B}\mathbb{B}^\star}\right)^{-1}(\mathbb{B}a + b), \qquad \gamma = \exp\left(\frac{\pi}{2}\left(\|a\|^2 - \|\beta\|^2\right)\right).$$

(ii) *The operator $\mathcal{K}$ is trace class with*

$$\|\mathcal{K}\|_{\mathcal{S}_1(\mathcal{F}(\mathcal{H}))} = \operatorname{trace} K = \frac{\exp(\frac{\pi}{2}\|a\|^2 + \frac{\pi}{2}\|(1-\mathbb{B}\mathbb{B}^\star)^{-1/2}(\mathbb{B}a+b)\|^2)}{\det(1-|\mathbb{B}|)}$$

and

$$\operatorname{trace}\mathcal{K} = \frac{\exp(\pi\langle(1-\mathbb{B})^{-1}b \mid a\rangle)}{\det(1-\mathbb{B})}.$$

Proof. As in Lemma 4.5 one gets $K = |\mathcal{K}| = \sqrt{\mathcal{K}^\star\mathcal{K}}$. It remains to show that the trace norm of $\mathcal{K}$, i.e., the trace of K is finite. Let $P_n : \mathcal{H} \to \mathcal{H}$ be an ascending sequence of orthogonal projections with n-dimensional range converging to the identity in the strong operator topology. Set $\mathrm{pr}_n : \mathcal{H} \to \mathcal{H}_n := P_n\mathcal{H}$, $z \mapsto P_n z$. Fix $m \in \mathbb{N}$ and consider $K_m := C_{\mathrm{pr}_m^*} K C_{\mathrm{pr}_m} \in \operatorname{End}(\mathcal{F}(\mathcal{H}_m))$, which by the composition law acts via

$$(K_m f)(z) = \gamma e^{\pi\langle z|\,\mathrm{pr}_m\beta\rangle} f\big(\mathrm{pr}_m \Lambda\, \mathrm{pr}_m^* z + \mathrm{pr}_m\beta\big) = e^{\pi\langle z|\beta_m\rangle} f(\Lambda_m z + \beta_m)$$

with $\beta_m = \mathrm{pr}_m^*\beta$ and $\Lambda_m = \mathrm{pr}_m \Lambda\, \mathrm{pr}_m^*$. By the Atiyah–Bott formula from Proposition 4.4(ii) we have

$$\operatorname{trace} K_m = \gamma \frac{\exp(\pi\|(1-\Lambda_m)^{-1/2}\beta_m\|^2_{\mathcal{H}_m})}{\det_{\mathcal{H}_m}(1-\Lambda_m)} = \gamma \frac{\exp(\pi\|(1-\Lambda_m)^{-1/2}\beta_m\|^2_{\mathcal{H}})}{\det_{\mathcal{H}}(1-\Lambda_m)}$$

identifying $\Lambda_m \in \operatorname{End}(\mathcal{H}_m)$ with $\mathrm{pr}_m^* \Lambda_m \mathrm{pr}_m = P_m\Lambda P_m \in \operatorname{End}(\mathcal{H})$. Lemma 3.2, together with [3, Theorem IV5.5], shows that the pointwise convergence $P_m\Lambda P_m \to \Lambda$ is in trace norm. Therefore the following limit exists:

$$\lim_{m\to\infty} \operatorname{trace} K_m = \gamma \frac{\exp(\pi\|(1-\Lambda)^{-1/2}\beta\|^2_{\mathcal{H}})}{\det_{\mathcal{H}}(1-\Lambda)} < \infty.$$

Thus K and $\mathcal{K}$ are trace class. By Lemma 3.2 and [3, Theorem IV5.5] the sequence of trace class operators $C_{P_m}\mathcal{K}C_{P_m}$ converges to $\mathcal{K}$ in trace norm, hence

$$\begin{aligned}\operatorname{trace}\mathcal{K} &= \lim_{m\to\infty} \frac{\exp(\pi\langle(1-\mathbb{B}_m)^{-1}b_m \mid a_m\rangle_{\mathcal{H}_m})}{\det_{\mathcal{H}_m}(1-\mathbb{B}_m)}\\ &= \frac{\exp(\pi\langle(1-\mathbb{B})^{-1}b \mid a\rangle)}{\det_{\mathcal{H}}(1-\mathbb{B})},\end{aligned}$$

where we view $\mathbb{B}_m$, a_m, and b_m as operator on, respectively vectors in, $\mathcal{H}$. A similar approximation argument, combined with Lemma 4.5, also yields the formula for trace K. □

As a corollary we obtain an exact formula for the Hilbert–Schmidt norm of a weighted composition operator of the form $\mathcal{K}_{a,b,\mathbb{B}}$ which could also be obtained directly as a consequence of an identity on Gaussian integrals. We first consider the general situation: Let $\mathcal{H} \subset L^2(Z, dm)$ be a Hilbert space with reproducing kernel k. Consider the composition operator

$$(Tf)(z) = \phi(z)(f \circ \psi)(z),$$

where $\phi : Z \to \mathbb{C}$, $\psi : Z \to Z$ are fixed functions. Then the Hilbert–Schmidt norm of T is equal to

$$\int\limits_Z \left|\phi(z)\right|^2 k\big(\psi(z), \psi(z)\big)\, dm(z).$$

Corollary 4.7. *Let a, $b \in \mathcal{H}$, $\mathbb{B} \in \mathcal{S}_2(\mathcal{H})$ with $\|\mathbb{B}\| < 1$. Then the weighted composition operator*

$$\mathcal{K} : \mathcal{F}(\mathcal{H}) \to \mathcal{F}(\mathcal{H}), \quad (\mathcal{K} f)(z) = e^{\pi \langle z|a\rangle} f(\mathbb{B}z + b)$$

is Hilbert–Schmidt with

$$\|\mathcal{K}\|^2_{\mathcal{S}_2(\mathcal{F}(\mathcal{H}))} = \frac{\exp(\pi \|a\|^2 + \pi \|(1 - \mathbb{B}\mathbb{B}^*)^{-1/2}(\mathbb{B}a + b)\|^2)}{\det(1 - \mathbb{B}\mathbb{B}^*)}.$$

Proof. We use that $\|\mathcal{K}\|^2_{\mathcal{S}_2(\mathcal{F}(\mathcal{H}))} = \operatorname{trace} \mathcal{K}\mathcal{K}^*$ together with Theorem 4.6, where

$$\big(\mathcal{K}^\star \mathcal{K} f\big)(z) = e^{\pi \|a\|^2} e^{\pi \langle z|\mathbb{B}a+b\rangle} f\big(\mathbb{B}\mathbb{B}^\star z + \mathbb{B}a + b\big)$$

with $\|\mathbb{B}\mathbb{B}^\star\| = \|\mathbb{B}\|^2 < 1$ and $\mathbb{B}\mathbb{B}^\star \in \mathcal{S}_1(\mathcal{H})$. $\square$

5. Dynamical trace formulae for spin chains

The following proposition allows to describe the partition function of a non-interacting subshift as the trace of an operator, which is then called a *transfer operator*. In the special case of a finite alphabet F this result is well known.

Proposition 5.1. *Let $(F, \nu, \mathbb{M}, 0)$ be a subshift with vanishing interaction. Then for $n \geqslant 2$ the integral operator*

$$\mathcal{G}_{\mathbb{M}} : L^2(F, d\nu) \to L^2(F, d\nu), \quad (\mathcal{G}_{\mathbb{M}} f)(\xi) = \int\limits_F \mathbb{M}(\eta, \xi) f(\eta)\, d\nu(\eta)$$

associated with the transition function $\mathbb{M}$ satisfies the dynamical trace formula

$$Z_n(0) = \nu^n\big(\rho_n\big(\big\{\underline{\xi} \in \Omega_{\mathbb{M}} \bigm| \tau^n \underline{\xi} = \underline{\xi}\big\}\big)\big) = \operatorname{trace} \mathcal{G}^n_{\mathbb{M}}.$$

Proof. The operator $\mathcal{G}_{\mathbb{M}}$ can be seen as T in Lemma 1.7, where all the operators $S_\xi := \mathrm{id} : \mathbb{C} \to \mathbb{C}$ are trivial. Hence $\mathcal{G}_{\mathbb{M}}$ is Hilbert–Schmidt and the traces of its iterates are given by Lemma 1.7. Comparison of the formulae with (3) now proves the claim. $\square$

Let $(F, \nu, \mathbb{M}, A)$ be a subshift, where the interaction A is of the following form: Let $\mathbb{B} \in \mathcal{S}_p(\mathcal{H})$ for some $p < \infty$ with $\rho_{\mathrm{spec}}(\mathbb{B}) < 1$. For $\xi \in F$ assume that one has $a_\xi, b_\xi \in \mathcal{H}$, $q_\xi \in \mathbb{C}$ such that for $\underline{\xi} = (\xi_j)_{j \in \mathbb{N}} \in \Omega_{\mathbb{M}}$

$$A(\underline{\xi}) = q_{\xi_1} + \pi \sum_{j=2}^{\infty} \big\langle \mathbb{B}^{j-2} b_{\xi_j} \bigm| a_{\xi_1} \big\rangle. \tag{24}$$

Here we assume absolute convergence of the series. We interpret $A(\underline{\xi})$ as the sum of the potential term q_{ξ_1} and the sum of two-body interactions between the first particle and the particles at positions $i \in \mathbb{N}_{>1}$.

Let $\mathbb{B} \in \mathcal{S}_p(\mathcal{H})$ for some $p < \infty$ with $\rho_{\mathrm{spec}}(\mathbb{B}) < 1$, then one can choose $n \in \mathbb{N}$ large enough such that $\mathbb{B}^n$ has operator norm less than one and is trace class, since the spectral radius can be characterized via

$$\rho_{\mathrm{spec}}(\mathbb{B}) = \max\bigl\{|z| \in \mathbb{R} \bigm| z \in \mathrm{spec}(\mathbb{B})\bigr\} = \lim_{k\to\infty} \sqrt[k]{\|\mathbb{B}^k\|}$$

and $\mathbb{B}^n \in \mathcal{S}_{\max(1,p/n)}(\mathcal{H})$.

For all $n \in \mathbb{N}$ and $\xi^{(n)} = (\xi_n, \ldots, \xi_1) \in F^n$ we set

$$q\bigl(n;\xi^{(n)}\bigr) := \sum_{k=1}^{n} q_{\xi_k} + \pi \sum_{k=1}^{n}\sum_{j=1}^{n-k} \bigl\langle \mathbb{B}^{j-1} b_{\xi_{j+k}} \bigm| a_{\xi_k}\bigr\rangle, \tag{25}$$

$$a\bigl(n;\xi^{(n)}\bigr) := \sum_{k=1}^{n} \bigl(\mathbb{B}^{n-k}\bigr)^* a_{\xi_k} \quad \text{and} \quad b\bigl(n;\xi^{(n)}\bigr) := \sum_{j=0}^{n-1} \mathbb{B}^j b_{\xi_{j+1}}. \tag{26}$$

For $n \in \mathbb{N}$ such that $\mathbb{B}^n \in \mathcal{S}_2(\mathcal{H})$ with $\|\mathbb{B}^n\| < 1$ we define (depending on $a_\xi, b_\xi \in \mathcal{H}$, and $q_\xi \in \mathbb{C}$) a function $c(n;\cdot) : F^n \to \mathbb{R}$ via

$$c\bigl(n;\xi^{(n)}\bigr) = \frac{\exp(2\,\mathrm{Re}\,(q(n;\xi^{(n)})) + \pi\|a(n;\xi^{(n)})\|^2 + \pi\|(1-\mathbb{B}^n(\mathbb{B}^n)^*)^{-1/2}\mathbb{B}^n(a(n;\xi^{(n)}) + b(n;\xi^{(n)}))\|^2)}{\det(1-\mathbb{B}^n(\mathbb{B}^n)^*)}. \tag{27}$$

The following proposition describes the type of operators we will use to build our dynamical trace formula:

Proposition 5.2. *Let $F \neq \emptyset$ be an index set, $\mathcal{H}$ be a Hilbert space, and $\mathbb{B} \in \mathrm{End}(\mathcal{H})$. Given $a_\xi, b_\xi \in \mathcal{H}$, and $q_\xi \in \mathbb{C}$ for $\xi \in F$, consider the operator $\mathcal{M}_\xi : \mathcal{C}(\mathcal{H}) \to \mathcal{C}(\mathcal{H})$ defined by*

$$(\mathcal{M}_\xi f)(z) := \exp\bigl(q_\xi + \pi\langle z \mid a_\xi\rangle\bigr) f(b_\xi + \mathbb{B}z). \tag{28}$$

Fix $n \in \mathbb{N}$ and $\xi^{(n)} = (\xi_n, \ldots, \xi_1) \in F^n$. Then

$$(\mathcal{M}_{\xi_n} \circ \cdots \circ \mathcal{M}_{\xi_1} f)(z) = \exp\bigl(q\bigl(n;\xi^{(n)}\bigr) + \pi\bigl\langle z \bigm| a\bigl(n;\xi^{(n)}\bigr)\bigr\rangle\bigr) f\bigl(\mathbb{B}^n z + b\bigl(n;\xi^{(n)}\bigr)\bigr).$$

If $\mathbb{B}^n \in \mathcal{S}_2(\mathcal{H})$ with $\|\mathbb{B}^n\| < 1$, then $\mathcal{M}_{\xi_n} \circ \cdots \circ \mathcal{M}_{\xi_1}$ restricts to a Hilbert–Schmidt operator on $\mathcal{F}(\mathcal{H})$ which satisfies

$$\|\mathcal{M}_{\xi_n} \circ \cdots \circ \mathcal{M}_{\xi_1}\|^2_{\mathcal{S}_2(\mathcal{F}(\mathcal{H}))} = c\bigl(n;\xi^{(n)}\bigr).$$

Proof. For any family $\{\mathcal{M}_\xi \mid \xi \in F\}$ of weighted composition operators acting via $(\mathcal{M}_\xi f)(z) = \exp(A_\xi(z)) f(\psi_\xi(z))$, induction shows that

$$(\mathcal{M}_{\xi_n} \circ \cdots \circ \mathcal{M}_{\xi_1} f)(z) = \exp\left(\sum_{k=1}^{n} (A_{\xi_k} \circ \psi_{\xi_{k+1}} \circ \cdots \circ \psi_{\xi_n})(z)\right)(f \circ \psi_{\xi_1} \circ \cdots \circ \psi_{\xi_n})(z)$$

for all $\xi_1, \ldots, \xi_n \in F$. In particular, if $A_\xi(z) = q_\xi + \pi\langle z \mid a_\xi\rangle$ and $\psi_\xi(z) = b_\xi + \mathbb{B}z$ for some $a_\xi, b_\xi \in \mathcal{H}$, $q_\xi \in \mathbb{C}$, and $\mathbb{B} \in \mathrm{End}(\mathcal{H})$ as above, we have to consider mixed iterates of affine maps: Let V be a complex vector space, $\mathbb{B}: V \to V$ a linear operator, and $b_\xi \in V$ for $\xi \in F$. Then induction shows that $\psi_\xi : V \to V$, $z \mapsto b_\xi + \mathbb{B}z$ satisfies

$$(\psi_{\xi_1} \circ \cdots \circ \psi_{\xi_k})(z) = \mathbb{B}^k z + \sum_{j=0}^{k-1} \mathbb{B}^j b_{\xi_{j+1}} = \mathbb{B}^k z + b\big(k; (\xi_k, \ldots, \xi_1)\big)$$

for all $k \in \mathbb{N}$, $\xi_1, \ldots, \xi_k \in F$, and $z \in V$. This implies for $\xi^{(n)} = (\xi_n, \ldots, \xi_1) \in F^n$ that

$$\begin{aligned}
&(\mathcal{M}_{\xi_n} \circ \cdots \circ \mathcal{M}_{\xi_1} f)(z) \\
&\quad = \exp\left(\sum_{k=1}^{n} A_{\xi_k}\left(\mathbb{B}^{n-k} z + \sum_{j=1}^{n-k} \mathbb{B}^{j-1} b_{\xi_{j+k}}\right)\right) f\big(\mathbb{B}^n z + b\big(n; \xi^{(n)}\big)\big) \\
&\quad = \exp\big(q\big(n; \xi^{(n)}\big) + \pi\big\langle z \bigm| a\big(n; \xi^{(n)}\big)\big\rangle\big) f\big(\mathbb{B}^n z + b\big(n; \xi^{(n)}\big)\big),
\end{aligned}$$

which yields the first claim. Now Corollary 4.7 gives the stated Hilbert–Schmidt norm formula and the invariance of $\mathcal{F}(\mathcal{H})$. $\square$

Recall the subshift $(F, \nu, \mathbb{M}, A)$ with interaction function A from (24) and the operators $\mathcal{M}_\xi$ defined via (28). The following theorem provides a dynamical trace formula for the corresponding partition function $Z_n(A)$ defined via (3).

Theorem 5.3. *Suppose there exists $n_o \in \mathbb{N}$ such that $c(n_o; \cdot)$ is ν^{n_o}-integrable. Then there exists an index $n_1 \in \mathbb{N}$ such that for all $n \geqslant n_1$ the iterates $\mathcal{M}^n$ of the Ruelle–Mayer type transfer operator*

$$(\mathcal{M} f)(\sigma, z) = \int_F \mathbb{M}(\xi, \sigma) \exp\big(q_\xi + \pi\langle z \mid a_\xi\rangle\big) f(\xi, b_\xi + \mathbb{B}z)\, d\nu(\xi)$$

are trace class operators $\mathcal{M}^n \in \mathrm{End}(L^2(F, d\nu) \hat{\otimes} \mathcal{F}(\mathcal{H}))$. Moreover, the partition function $Z_n(A)$ can be expressed as

$$Z_n(A) = \det\big(1 - \mathbb{B}^n\big) \operatorname{trace} \mathcal{M}^n.$$

Proof. By Lemma 3.1 one has for $\tilde{\mathcal{H}} := L^2(F, d\nu) \hat{\otimes} \mathcal{F}(\mathcal{H})$ and $n \geqslant n_o$

$$\|\mathcal{M}^n\|^2_{\mathcal{S}_2(\tilde{\mathcal{H}})} = \int_F \int_{F^n} \mathbb{M}(\xi_2, \xi_1) \cdots \mathbb{M}(\xi_n, \xi_{n-1})\, \mathbb{M}(\sigma, \xi_n) \\ \times \|\mathcal{M}_{\xi_n} \circ \cdots \circ \mathcal{M}_{\xi_1}\|^2_{\mathcal{S}_2(\mathcal{F}(\mathcal{H}))}\, d\nu^n(\xi_1, \ldots, \xi_n)\, d\nu(\sigma) \\ \leqslant \nu(F) \int_{F^n} \|\mathcal{M}_{\xi_n} \circ \cdots \circ \mathcal{M}_{\xi_1}\|^2_{\mathcal{S}_2(\mathcal{F}(\mathcal{H}))}\, d\nu^n(\xi_1, \ldots, \xi_n).$$

There exists $n_1 \in \mathbb{N}$ such that for all $n \geqslant n_1$ we have $\mathbb{B}^n \in \mathcal{S}_1(\mathcal{H})$ and $\|\mathbb{B}^n\| < 1$. By Proposition 5.2 (and, possibly, by enlarging n_1) the operator $\mathcal{M}^n$ is trace class for all $n \geqslant n_1$. By Lemma 1.7 the trace of $\mathcal{M}^n$ is given by

$$\operatorname{trace} \mathcal{M}^n = \int_{F^n} \left(\prod_{j=1}^{n} \mathbb{M}(\xi_j, \xi_{j+1}) \right) \operatorname{trace}(\mathcal{M}_{\xi_n} \circ \cdots \circ \mathcal{M}_{\xi_1})\, d\nu^n(\xi_1, \ldots, \xi_n) \tag{29}$$

using the convention that $\xi_{n+1} = \xi_1$. Proposition 5.2 shows that for all choices of $\xi^{(n)} = (\xi_n, \ldots, \xi_1) \in F^n$ the trace formula from Theorem 4.6 can be applied to the operator $\mathcal{M}_{\xi_n} \circ \cdots \circ \mathcal{M}_{\xi_1}$. It yields

$$\operatorname{trace}(\mathcal{M}_{\xi_n} \circ \cdots \circ \mathcal{M}_{\xi_1}) = \frac{\exp(q(n; \xi^{(n)}) + \pi \langle (1 - \mathbb{B}^n)^{-1} b(n; \xi^{(n)}) \mid a(n; \xi^{(n)}) \rangle)}{\det(1 - \mathbb{B}^n)} \tag{30}$$

with $a(n; \xi^{(n)})$, $b(n; \xi^{(n)})$, and $q(n; \xi^{(n)})$ as in (25) and (26). The inner product occurring in (30) can be rewritten as

$$\bigl\langle \bigl(1 - \mathbb{B}^n\bigr)^{-1} b\bigl(n; \xi^{(n)}\bigr) \bigm| a\bigl(n; \xi^{(n)}\bigr) \bigr\rangle = \sum_{k=1}^{n} \sum_{j=0}^{n-1} \sum_{l=0}^{\infty} \bigl\langle \mathbb{B}^{j+n-k+ln} b_{\xi_{j+1}} \bigm| a_{\xi_k} \bigr\rangle.$$

Extending the finite sequence $b_{\xi_1}, \ldots, b_{\xi_n} \in \mathcal{H}$ to an n-periodic sequence, i.e., setting $b_{\xi_{k+ln}} := b_{\xi_k}$ for all $k = 1, \ldots, n$ and $l \in \mathbb{N}_0$, we obtain

$$q\bigl(n; \xi^{(n)}\bigr) + \pi \bigl\langle \bigl(1 - \mathbb{B}^n\bigr)^{-1} b\bigl(n; \xi^{(n)}\bigr) \bigm| a\bigl(n; \xi^{(n)}\bigr) \bigr\rangle = \sum_{k=0}^{n-1} A\bigl(\tau^k (\overline{\xi_1 \ldots \xi_n})\bigr).$$

This together with (29), (30), and (3) completes the proof. □

We note that unless $\mathbb{B} \in \mathcal{S}_2(\mathcal{H})$ with $\|\mathbb{B}\| < 1$, we cannot show $\mathcal{M}$ to be a bounded operator on $L^2(F, d\nu) \hat{\otimes} \mathcal{F}(\mathcal{H})$.

6. Ising type interactions

In this section we present some explicit models known from the physics literature for which Theorem 5.3 actually provides dynamical trace formulae. A detailed description of these models can be found in [19]. Another application of Theorem 5.3 are hard rod type models, which are studied in [19,20].

We consider a subshift $(F, \nu, \mathbb{M}, A)$, where the interaction is of the form $\beta A_{(q,r,d)}$ with

$$A_{(q,r,d)} : \Omega_{\mathbb{M}} \to \mathbb{C}, \quad \underline{\xi} \mapsto q(\xi_1) + \sum_{i=2}^{\infty} r(\xi_1, \xi_i)\, d(i-1). \tag{31}$$

Here $r : F \times F \to \mathbb{C}$ is called an *interaction function*, $d \in \ell^1\mathbb{N}$ a *distance function*, and $q \in \mathcal{C}_{\mathrm{b}}(F)$ a *potential*. The extra parameter $\beta \in \mathbb{C}$ is usually called the *inverse temperature*. In this context the partition function $Z_n(\beta A_{(q,r,d)})$ as defined in (3) coincides with the usual partition function for the two-body interaction with respect to (q, r, d) and *periodic boundary condition* (see [19, Corollary 1.11.3]). We will assume additional properties of the distance function and the interaction function. A list of examples will be given in Examples 6.1 and 6.5.

An interaction function r is said to be of *Ising type*, if there exists a finite number of functions $s_i, t_i : F \to \mathbb{C}$ such that

$$r(\xi, \eta) = \sum_{i=1}^{M} \overline{s_i(\xi)}\, t_i(\eta).$$

The minimal number M is called the *rank* of r.

Example 6.1.

(i) *Ising model*. Let $F \subset \mathbb{C}$ be a bounded set and $r(\xi, \eta) = \xi\eta$. In E. Ising's original model [6] he took $F = \{\pm 1\}$, the so called *spin-$\frac{1}{2}$ model*, in order to describe ferromagnetism of a solid, where the spins of the electrons can only take values in a set with two elements, "spin up" or "spin down."

(ii) Let $F \subset \mathcal{H}$ be a bounded subset of a finite-dimensional Hilbert space $(\mathcal{H}, \langle \cdot \mid \cdot \rangle)$. Then any bilinear form β on $\mathcal{H}$ defines an interaction function of Ising type via $r(\xi, \eta) := \beta(\xi, \eta)$. In fact, choose an orthonormal basis $(e_j)_{j=1,\dots,\dim\mathcal{H}}$ for $\mathcal{H}$, then

$$\beta(\xi, \eta) = \beta\left(\sum_{i=1}^{\dim\mathcal{H}} \langle \xi \mid e_i \rangle e_i, \eta\right) = \sum_{i=1}^{\dim\mathcal{H}} \langle \xi \mid e_i \rangle \beta(e_i, \eta).$$

Note that r has rank less or equal to $\dim\mathcal{H}$.

(iii) *The (generalized) Stanley M-vector model*, cf. [24]. It is a special case of (ii): Take $\mathcal{H} = \mathbb{R}^M$ with $r(\xi, \eta) = \langle \xi \mid \eta \rangle$ and $F := \{v \in \mathbb{R}^M \mid r(v, v) = s^2\}$ to be the $(M-1)$-sphere with radius $s > 0$ equipped with the (normalized) surface measure ν on F.
The following table gives a list of physical models which can be seen as applications of Stanley's M-vector model. Depending on the parameter M these models have special names.

Rank	Special name	System
1	Ising model	One-component fluid, binary alloy, mixture
2	Planar model	λ-Transition in a Bose fluid
3	Heisenberg model	(Anti-)ferromagnetism
$M > 3$	M-vector model	No physical system discovered yet

This table is taken from [25, p. 488] where one can also find a lot of references to the underlying physics. Note that the rank 1 case gives $F = \{\pm 1\}$ and hence the spin-$\frac{1}{2}$ Ising model.

(iv) If F is finite, then *every* interaction function is of Ising type, since

$$r(\xi, \eta) = \sum_{z \in F} r(\xi, z)\, \delta(\eta, z),$$

where $\delta : F \times F \to \mathbb{C}$ is Kronecker's delta on F. In particular, the finite-state *Potts model* is of Ising type: Let F be a finite set and $r(\xi, \eta) = \delta(\xi, \eta)$. This model is due to R. Potts [17] and describes the situation where only electrons with identical spin interact.

We remark that Ising, Potts, and Stanley have considered these models only for finite range interactions.

Consider the interactions $A_{(q,r,d)}$ with distance functions d belonging to subspaces $\mathcal{D}_p \subset \ell^1\mathbb{N}$ (for $p \in [1, \infty[$) which are defined as follows: $d = d_{\mathbb{B},v,w} \in \mathcal{D}_p$ if and only if there exist a Hilbert space $\mathcal{H}$ and a Schatten class operator $\mathbb{B} \in \mathcal{S}_p(\mathcal{H})$ with spectral radius $\rho_{\text{spec}}(\mathbb{B}) < 1$ and vectors $v, w \in \mathcal{H}$ such that

$$d : \mathbb{N} \to \mathbb{C}, \quad k \mapsto d(k) = \langle \mathbb{B}^{k-1} v \mid w \rangle_{\mathcal{H}}.$$

Now, let $d = d_{\mathbb{B},v,w} \in \mathcal{D}_p$ and $r(\xi, \eta) = \sum_{i=1}^{M} \overline{s_i(\xi)} t_i(\eta)$. We rewrite the Ising type observable $A_{(q,r,d)} : \Omega_{\mathbb{M}} \to \mathbb{C}$ in such a way that we can apply Theorem 5.3.

$$\begin{aligned} A_{(q,r,d)}(\underline{\xi}) &= q(\xi_1) + \sum_{i=2}^{\infty} \sum_{j=1}^{M} \overline{s_j(\xi_1)} t_j(\xi_i) \langle \mathbb{B}^{i-2} v \mid w \rangle_{\mathcal{H}} \\ &= q(\xi_1) + \sum_{i=2}^{\infty} \langle (\underline{\mathbb{B}_M})^{i-2} \underline{t_M}(\xi_i, v) \mid \underline{s_M}(\xi_1, w) \rangle_{\mathcal{H}^M}, \end{aligned}$$

where $\underline{\mathbb{B}_M} : \mathcal{H}^M \to \mathcal{H}^M$, $(z_1, \ldots, z_M) \mapsto (\mathbb{B}z_1, \ldots, \mathbb{B}z_M)$, $\underline{s_M} : F \times \mathcal{H} \to \mathcal{H}^M$, $\underline{s_M}(\xi, v) := (s_1(\xi)v, \ldots, s_M(\xi)v)$, and, similarly, $\underline{t_M} : F \times \mathcal{H} \to \mathcal{H}^M$, $\underline{t_M}(\xi, v) := (t_1(\xi)v, \ldots, t_M(\xi)v)$.

Theorem 6.2. *Let $(F, \nu, \mathbb{M}, A_{(q,r,d)})$ be a subshift where $q \in \mathcal{C}_b(F)$, $d = d_{\mathbb{B},v,w} \in \mathcal{D}_p$ and $r \in \mathcal{C}_b(F \times F)$ is an interaction function of Ising type, say $r(\xi, \eta) = \sum_{i=1}^{M} \overline{s_i(\xi)} t_i(\eta)$ with $s_i, t_j \in \mathcal{C}_b(F)$. Then there exists an index $n_o \in \mathbb{N}$ depending on $\mathbb{B}$ such that for all $n \geqslant n_o$ the iterates $\mathcal{M}_\beta^n \in \operatorname{End}(L^2(F, d\nu) \hat{\otimes} \mathcal{F}(\mathcal{H}^M))$ of the Ruelle–Mayer transfer operator*

$$(\mathcal{M}_\beta f)(\xi, z) = \int_F \mathbb{M}(\sigma, \xi) \exp\bigl(\beta q(\sigma) + \beta\langle z \mid \underline{s_M}(\sigma, w)\rangle\bigr) f\bigl(\sigma, \underline{t_M}(\sigma, v) + \underline{\mathbb{B}_M} z\bigr)\, d\nu(\sigma)$$

are of trace class and satisfy the dynamical trace formula

$$Z_n(\beta A_{(q,r,d)}) = \det\bigl(1 - \mathbb{B}^n\bigr)^M \operatorname{trace} \mathcal{M}_\beta^n.$$

Proof. By assumption the sets $\{a_\xi := \underline{s_M}(\xi, w) \in \mathcal{H}^M \mid \xi \in F\}$ and $\{b_\xi := \underline{t_M}(\xi, w) \in \mathcal{H}^M \mid \xi \in F\}$ are bounded. Choose $m \in \mathbb{N}$ large enough such that $\mathbb{B}^m$ has operator norm less than one and is Hilbert–Schmidt. Proposition 5.2, applied to $\underline{\mathbb{B}_M}$, shows that the associated function $c(m; \cdot)$ is bounded, hence integrable. Thus we can apply Theorem 5.3 to $\mathcal{M}_\beta$ which proves the claim. □

Corollary 6.3 *(Ising model). Let $F \subset \mathbb{C}$ be a bounded set equipped with a finite measure ν and $(F, \nu, \mathbb{M}, A_{(q,r,d)})$ be a subshift, where $q \in \mathcal{C}_b(F)$, $d = d_{\mathbb{B},v,w} \in \mathcal{D}_p$ and $r(\xi, \eta) = \xi\eta$. Then there exists an index $n_o \in \mathbb{N}$ depending on $\mathbb{B}$ such that for all $n \geqslant n_o$ the iterates $\mathcal{M}_\beta^n \in \operatorname{End}(L^2(F, d\nu) \hat{\otimes} \mathcal{F}(\mathcal{H}))$ of the Ruelle–Mayer transfer operator*

$$(\mathcal{M}_\beta f)(\xi, z) = \int_F \mathbb{M}(\sigma, \xi) \exp\bigl(\beta q(\sigma) + \beta\sigma \langle z|w\rangle\bigr) f(\sigma, \sigma v + \mathbb{B} z)\, d\nu(\sigma)$$

are trace class and satisfy $Z_n(\beta A_{(q,r,d)}) = \det(1 - \mathbb{B}^n) \operatorname{trace} \mathcal{M}_\beta^n$.

Corollary 6.4 *(Potts model). Let $F = \{1, \ldots, N\}$ be a finite alphabet, the measure ν on F be identified with its distribution vector, and $(F, \nu, \mathbb{M}, A_{(q,r,d)})$ be a subshift, where $q : F \to \mathbb{C}$, $d = d_{\mathbb{B},v,w} \in \mathcal{D}_p$ and $r(\xi, \eta) = \delta(\xi, \eta)$. Then there exists an index $n_0 \in \mathbb{N}$ depending on $\mathbb{B}$ such that for all $n \geqslant n_0$ the iterates $\mathcal{M}_\beta^n \in \operatorname{End}(L^2(F, d\nu) \hat{\otimes} \mathcal{F}(\mathcal{H}^N))$ of the Ruelle–Mayer transfer operator*

$$(\mathcal{M}_\beta f)(l; z_1, \ldots, z_N) = \sum_{k=1}^N \mathbb{M}(k, l)\nu_k \exp\bigl(\beta q(k) + \beta\langle z_k \mid w\rangle\bigr) f\bigl(k; (\delta_{k,m} v + \mathbb{B} z_m)_{m=1,\ldots,N}\bigr)$$

are trace class and satisfy $Z_n(\beta A_{(q,r,d)}) = \det(1 - \mathbb{B}^n)^N \operatorname{trace} \mathcal{M}_\beta^n$.

By the canonical identification $L^2(F, d\nu) \hat{\otimes} \mathcal{F}(\mathcal{H}^N) \cong \mathcal{F}(\mathcal{H}^N)^N$ the Ruelle–Mayer transfer operator can be rewritten as

$$\bigl(\mathcal{M}_\beta(f_1, \ldots, f_N)\bigr)_l(z_1, \ldots, z_N) = \sum_{k=1}^N \mathbb{M}(k, l)\nu_k \exp\bigl(\beta q(k) + \beta\langle z_k \mid w\rangle\bigr) f_k\bigl((\delta_{k,m} v + \mathbb{B} z_m)_{m=1,\ldots,N}\bigr).$$

Example 6.5.

(i) *Finite range*. There exists $\rho_0 \in \mathbb{N}$, the range of d, such that $d(k) = 0$ for all $k > \rho_0$. Remark 6.6 below shows that $d \in \mathcal{D}_1$.
(ii) *Polynomial-exponential*. $d : \mathbb{N} \to \mathbb{C}$, $k \mapsto \lambda^k p(k)$, where $p \in \mathbb{C}[z]$ is a polynomial and $\lambda \in \mathbb{C}$ with $0 < |\lambda| < 1$ is the decay rate. Remark 6.7 below shows that $d \in \mathcal{D}_1$.
(iii) *Superexponential*. Let $\gamma > 0$, $\delta > 1$ and $d : \mathbb{N} \to \mathbb{C}$, $k \mapsto a(k)\exp(-\gamma k^\delta)$, where $a : \mathbb{N} \to \mathbb{C}$ is of lower order such that $\lim_{k\to\infty} a(k)\exp(-\epsilon_1 k^{\epsilon_2}) = 0$ for all $\epsilon_1, \epsilon_2 > 0$. Proposition 6.8 below shows that $d \in \mathcal{D}_1$. (The decay estimate can be weakened, cf. Example 6.9.)
(iv) *Suitable infinite superpositions of exponentially decaying terms*:

$$d(k) = \sum_{i=1}^{\infty} c_i \lambda_i^k,$$

where $\lambda \in \ell^p\mathbb{N}$ ($1 \leqslant p < \infty$) and $c : \mathbb{N} \to \mathbb{C}$ such that $c\lambda : \mathbb{N} \to \mathbb{C}$, $n \mapsto c_n\lambda_n$ belongs to $\ell^1\mathbb{N}$. Obviously, $d \in \mathcal{D}_p$.
Example: Pick a holomorphic function f on the unit disk with $f(0) = 0$, $0 < |\lambda| < 1$, then $d(k) = f(\lambda^k)$ belongs to $\mathcal{D}_1$.

We conclude this section with the verification of the various claims made in Example 6.5. We start with some obvious facts on finite-range interactions.

Remark 6.6. For $\rho_0 \in \mathbb{N}_{>1}$ let

$$\mathbb{B}_{\rho_0} := \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} \in \mathrm{Mat}(\rho_0, \rho_0; \mathbb{Z}).$$

(i) Then $\mathbb{B}_{\rho_0}$ is a ρ_0-step nilpotent matrix with spectral radius $\rho_{\mathrm{spec}}(\mathbb{B}_{\rho_0}) = 0$.
(ii) Let $d : \mathbb{N} \to \mathbb{R}$ be a finite range distance function, say $d(k) = 0$ for all $k > \rho_0$, $\lambda \in \mathbb{C}^\times$, and $w^d \in \mathbb{C}^{\rho_0}$ with entries $w^d(k) = \lambda^{1-k} d(k)$. Then $d(k) = \langle (\lambda\mathbb{B}_{\rho_0})^{k-1} w^d \mid e_1 \rangle$ for all $k \in \mathbb{N}$, where $e_1 = (1, 0, \ldots, 0)$.

Fix an additional decay parameter λ with $|\lambda| < 1$. Then $\|(\lambda\mathbb{B}_{\rho_0})^k\| < 1$ for all $k \in \mathbb{N}$ and the dynamical trace formula for the Ruelle–Mayer transfer operator built from $\lambda\mathbb{B}_{\rho_0}$ holds for all $n \in \mathbb{N}$ (instead just for $n \geqslant \rho_0$ in the case $\lambda = 1$).

Remark 6.7. Let $p \in \mathbb{N}_0$ and $\mathbb{B}^{(p)} \in \mathrm{Mat}(p+1, p+1; \mathbb{R})$ be the unipotent (lower) triangular matrix with entries

$$\left(\mathbb{B}^{(p)}\right)_{i,j} = \begin{cases} \binom{i}{j}, & j \leqslant i, \\ 0, & \text{otherwise}. \end{cases}$$

Then $\det(1-(\lambda\mathbb{B}^{(p)})^n) = (1-\lambda^n)^{p+1}$ for any $\lambda \in \mathbb{C}$ and induction yields

$$\begin{pmatrix} 1 \\ k \\ \vdots \\ k^{p-1} \\ k^p \end{pmatrix} = \left(\mathbb{B}^{(p)}\right)^{k-1}\underline{1}$$

with $\underline{1} = (1,\dots,1) \in \mathbb{Z}^{p+1}$. Consequently, for $c := (c_0,\dots,c_p) \in \mathbb{C}^{p+1}$ we have

$$\lambda^k \sum_{i=0}^{p} c_i k^i = \left\langle \lambda^k \left(\mathbb{B}^{(p)}\right)^{k-1}\underline{1} \;\middle|\; \bar{c}\right\rangle.$$

Thus the distance functions $d(k) := \lambda^k \sum_{i=0}^{p} c_i k^i$ belongs to $\mathcal{D}_1$.

Proposition 6.8. *Let* $g : \mathbb{N} \to \mathbb{C} \setminus \{0\}$ *with* $\sum_{k=1}^{\infty} |\frac{g(k)}{g(k+1)}|^p < \infty$. *We define* $\mathbb{B}_g : \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$, $(\mathbb{B}_g z)_k := \frac{g(k)}{g(k+1)} z_{k+1}$. *Then*:

- (i) $\mathbb{B}_g$ *leaves the spaces* $\ell^q\mathbb{N}$ *invariant for* $1 \leqslant q < \infty$. *Moreover, it defines continuous operators with* $\|\mathbb{B}_g\|_{\ell^q\mathbb{N}\to\ell^q\mathbb{N}} \leqslant \sup_{k\in\mathbb{N}} |\frac{g(k)}{g(k+1)}|$ *on these spaces.*
- (ii) *For all* $z \in \ell^p\mathbb{N}$, $n \in \mathbb{N}_0$, $k \in \mathbb{N}$, *we have* $(\mathbb{B}_g^n z)_k = \frac{g(k)}{g(k+n)} z_{k+n}$.
- (iii) $\mathbb{B}_g : \ell^2\mathbb{N} \to \ell^2\mathbb{N}$ *belongs to the Schatten class* $\mathcal{S}_p(\ell^2\mathbb{N})$. *Moreover, it satisfies* $\rho_{\mathrm{spec}}(\mathbb{B}_g) < 1$ *as well as* $\det(1-\mathbb{B}_g^n) = 1$ *for all* $n \geqslant p$.
- (iv) *If, for* $d : \mathbb{N} \to \mathbb{C}$, *the function* $v^d : \mathbb{N} \to \mathbb{C}$, $v_m^d := g(m)\,d(m)$ *belongs to* $\ell^2\mathbb{N}$, *then*

$$d(k) = \frac{1}{g(1)} \left\langle \mathbb{B}_g^{k-1} v^d \;\middle|\; e_1\right\rangle_{\ell^2\mathbb{N}},$$

where $e_1 = (1,0,\dots,0)$. *In particular* $d = d_{\mathbb{B}_g, v^d, e_1} \in \mathcal{D}_p$ *for* $1 \leqslant p < \infty$.

Proof. Let $1 \leqslant q < \infty$ and $z \in \ell^q\mathbb{N}$, then

$$\|\mathbb{B}_g z\|_{\ell^q\mathbb{N}}^q = \sum_{k=1}^{\infty} \left|\frac{g(k)}{g(k+1)} z_{k+1}\right|^q \leqslant \sup_{k\in\mathbb{N}} \left|\frac{g(k)}{g(k+1)}\right|^q \|z\|_{\ell^q\mathbb{N}}^q.$$

This implies (i). Assertion (ii) is easily shown by induction. The $\ell^2\mathbb{N}$-adjoint $\mathbb{B}_g^*$ of $\mathbb{B}_g$ is given by

$$\mathbb{B}_g^* : \ell^2\mathbb{N} \to \ell^2\mathbb{N}, \quad \left(\mathbb{B}_g^* z\right)_i = \begin{cases} 0, & i = 1, \\ \frac{g(i-1)}{g(i)} z_{i-1}, & i \geqslant 2. \end{cases}$$

Therefore $((\mathbb{B}_g\mathbb{B}_g^*)(z))_k = |\frac{g(k)}{g(k+1)}|^2 z_k$, which shows that $\mathbb{B}_g\mathbb{B}_g^*$ is diagonal with respect to the standard basis. We can read off the singular numbers of $\mathbb{B}_g$ being the square roots of the diagonal

entries of $\mathbb{B}_g\mathbb{B}_g^*$. By assumption they belong to $\ell^p\mathbb{N}$. Using a telescope product argument one shows that for all $k\in\mathbb{N}$ the quotient $|\frac{g(k)}{g(k+n)}|$ tends to zero as $n\to\infty$. Hence we can find $n\in\mathbb{N}$ such that $\|\mathbb{B}_g^n\|<1$ so that $\rho_{\mathrm{spec}}(\mathbb{B}_g)<1$. With respect to the standard basis of $\ell^2\mathbb{N}$ the operator $\mathbb{B}_g$ is an upper triangular matrix with zeros along the diagonal, hence $\det(1-\mathbb{B}_g^n)=1$ for all $n\geqslant p$. As a consequence of (ii) we have

$$\left(\mathbb{B}_g^n v^d\right)_l=\frac{g(k)}{g(l+n)}v_{l+n}^d=\frac{g(l)}{g(l+n)}g(l+n)\,d(l+n)$$

for all $n\in\mathbb{N}_0$, $l\in\mathbb{N}$, which immediately implies that

$$\left\langle\mathbb{B}_g^{k-1}v^d\,\middle|\,e_1\right\rangle=\left(\mathbb{B}_g^{k-1}v^d\right)_1=g(1)\,d(k). \qquad \square$$

Remark 6.9.

(i) For any nowhere-vanishing sequence $s\in\ell^p\mathbb{N}$ setting $g:\mathbb{N}\to\mathbb{C}$, $g(k):=(\prod_{l=1}^{k-1}s(l))^{-1}$ one obtains a function g of the kind required in Proposition 6.8. In particular, $s(k)=\frac{g(k)}{g(k+1)}$ and $|s|:\mathbb{N}\to\mathbb{C}$, $n\mapsto|s(n)|$ is the sequence of singular numbers of the corresponding weighted shift operator $\mathbb{B}_g$. Functions $g:\mathbb{N}\to\mathbb{C}$ satisfying the summability condition $\sum_{k=1}^\infty|\frac{g(k)}{g(k+1)}|^p<\infty$ are, for instance, $g(k):=\exp(\gamma k^\delta)$ with $\gamma>0$, $\delta>1$.

(ii) More generally, consider the following distance function $d:\mathbb{N}\to\mathbb{C}$ which was studied by D. Mayer in [11, p. 100]: $d(k)=a(k)\exp(-\gamma\,k^\delta)$, where $\gamma>0$, $\delta>1$ and $a:\mathbb{N}\to\mathbb{C}$ is a lower order term, in the sense that $\lim_{k\to\infty}a(k)\exp(-\epsilon_1\,k^{\epsilon_2})=0$ for all $\epsilon_1,\epsilon_2>0$. We claim that

$$d(k)=\left\langle\mathbb{B}_g^{k-1}v^d\,\middle|\,e_1\right\rangle_{\ell^2\mathbb{N}},$$

where $v^d(m):=a(m)\exp(\gamma((m-1)^\delta-m^\delta))$ defines $v^d\in\ell^2\mathbb{N}$, and $g:\mathbb{N}\to\mathbb{C}$, $m\mapsto\exp(\gamma(m-1)^\delta)$ defines $\mathbb{B}_g\in\mathcal{S}_1(\ell^2\mathbb{N})$ with $\|\mathbb{B}_g\|<1$.

To prove the claim note first that $g(1)=1$ and g satisfies the summability condition from Proposition 6.8: For $\delta>1$ and $j\geqslant0$ we have

$$j^\delta-(1+j)^\delta=jj^{\delta-1}-(1+j)(1+j)^{\delta-1}\leqslant(j-1-j)j^{\delta-1}=-j^{\delta-1}.$$

Hence

$$\sum_{k=1}^\infty\left|\frac{\exp(\gamma(k-1)^\delta)}{\exp(\gamma\,k^\delta)}\right|^p\leqslant\sum_{k=1}^\infty\exp\left(-\gamma p(k-1)^{\delta-1}\right), \tag{32}$$

which is finite for all $p>0$. Hence the corresponding weighted shift operator $\mathbb{B}_g:\ell^2\mathbb{N}\to\ell^2\mathbb{N}$ is trace class. Moreover, $\mathbb{B}_g$ has operator norm bounded by $\exp(-\gamma)<1$. It remains to show that $v^d\in\ell^2\mathbb{N}$. We proceed similar to the previous estimate (32). For $0<\epsilon_1<\gamma$, $0<\epsilon_2\leqslant\delta-1$, by our assumptions on the lower order term a we can find a constant $C>0$ such that

$$\begin{aligned}\|v^d\|_{\ell^1\mathbb{N}} &= \sum_{k=1}^{\infty} \exp\bigl(-\gamma\bigl(k^\delta - (k-1)^\delta\bigr)\bigr)\bigl|a(k)\bigr| \\ &\leqslant C\sum_{k=1}^{\infty} \exp\bigl(-\gamma k^{\delta-1} + \epsilon_1 k^{\epsilon_2}\bigr) \\ &\leqslant C\sum_{k=1}^{\infty} \exp\bigl(-(\gamma-\epsilon_1)k^{\delta-1}\bigr) < \infty.\end{aligned}$$

This proves the claim and hence shows that Proposition 6.8(iv) applies to d.

(iii) For $d \in \mathcal{D}_q$ for some q it would be sufficient that $v^d \in \ell^2\mathbb{N}$ and (32) holds for some $p < \infty$. These observations allow to weaken the conditions on the lower order term. For instance, the sequence a might grow like $k \mapsto \exp(\gamma k^{\delta-1-\epsilon})$ for all $\epsilon > 0$.
This allows to extend Mayer's results. In particular, $\det(1-\mathbb{B}_g^n) = 1$ for almost all n together with Theorems 2.2 and 6.2 implies that

$$\zeta_{\mathrm{R}}(z) = \frac{\exp(\sum_{n=1}^{n_o-1} \frac{z^n}{n} Z_n)}{\det_{n_o}(1 - z\mathcal{M}_\beta)}.$$

This should be compared to interactions with finite range (Example 6.5(i)) or to vanishing interaction (Proposition 5.1) where a single transfer operator suffices, too. The simple form of the dynamical trace formula yields

$$\zeta_{\mathrm{R}}(z) = \frac{1}{\det(1 - z\mathcal{M}_\beta)}$$

and

$$\zeta_{\mathrm{R}}(z) = \frac{\exp(zZ_1)}{\det_{n_2}(1 - z\mathcal{G}_{\mathbb{M}})},$$

respectively, for the corresponding transfer operators. In all these cases we obtain a cancellation-free representation of zeta and a (complete) spectral interpretation of its poles. The zeta function for super-exponential interactions thus behaves like the zeta function associated with a finite range interaction up to a nowhere-vanishing entire function.
For more complicated interactions it seems plausible that the knowledge that Theorem 2.2 combined with the numerical analysis and analytical perturbation theory (with respect to β and a finite number of λ_j) still can be used to produce bounds for zeros. In fact, approximation by finite rank operators helps to locate the spectrum of the transfer operators, so for suitable (generic) $\mathbb{B}$ cancellations do not occur.

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Metaplectic representation on Wiener amalgam spaces and applications to the Schrödinger equation

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Abstract

We study the action of metaplectic operators on Wiener amalgam spaces, giving upper bounds for their norms. As an application, we obtain new fixed-time estimates in these spaces for Schrödinger equations with general quadratic Hamiltonians and Strichartz estimates for the Schrödinger equation with potentials $V(x) = \pm|x|^2$.

Keywords: Metaplectic representation; Wiener amalgam spaces; Modulation spaces; Schrödinger equation with quadratic Hamiltonians

1. Introduction

The Wiener amalgam spaces were introduced by Feichtinger [13] in 1980 and soon they revealed to be, together with the closely related modulation spaces, the natural framework for the time-frequency analysis; see, e.g., [14,15,17,18] and Gröchenig's book [21]. These spaces are modeled on the L^p spaces but they turn out to be much more flexible, since they control the local regularity of a function and its decay at infinity separately. For example, the Wiener amalgam space $W(B, L^q)$, in which typically $B = L^p$ or $B = \mathcal{F}L^p$, consists of functions which locally have the regularity of a function in B but globally display a L^q decay.

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In this paper we focus our attention on the action of the metaplectic representation on Wiener amalgam spaces. The metaplectic representation $\mu : Sp(d,\mathbb{R}) \to \mathcal{U}(L^2(\mathbb{R}^d))$ of the symplectic group $Sp(d,\mathbb{R})$ (see the subsequent Section 2 and [19] for details), was first constructed by Segal and Shale [30,31] in the framework of quantum mechanics (though on the algebra level the first construction is due to van Hove [42]) and by Weil [43] in number theory. Since then the metaplectic representation has attracted the attention of many people in different fields of mathematics and physics. In particular, we highlight the applications in the framework of reproducing formulae and wavelet theory [8], frame theory [16], quantum mechanics [12] and PDEs [24,25].

Fix a test function $g \in \mathcal{C}_0^\infty$ and $1 \leqslant p, q \leqslant \infty$. Then the *Wiener amalgam space* $W(\mathcal{F}L^p, L^q)$ with local component $\mathcal{F}L^p$ and global component L^q is defined as the space of all functions/tempered distributions f such that

$$\|f\|_{W(\mathcal{F}L^p,L^q)} := \big\| \|f T_x g\|_{\mathcal{F}L^p} \big\|_{L^q_x} < \infty,$$

where $T_x g(t) := g(t-x)$ and the $\mathcal{F}L^p$ norm is defined in (6). To give a flavor of the type of results:

If $1 \leqslant p \leqslant q \leqslant \infty$ *and* $\mathcal{A} = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp(d,\mathbb{R})$, *with* $\det B \neq 0$, *then the metaplectic operator* $\mu(\mathcal{A})$ *is a continuous mapping from* $W(\mathcal{F}L^q, L^p)$ *into* $W(\mathcal{F}L^p, L^q)$, *that is*

$$\big\|\mu(\mathcal{A}) f\big\|_{W(\mathcal{F}L^p,L^q)} \leqslant \alpha(\mathcal{A}, p, q) \|f\|_{W(\mathcal{F}L^q,L^p)}.$$

The norm upper bound $\alpha = \alpha(\mathcal{A}, p, q)$ is explicitly expressed in terms of the matrix $\mathcal{A}$ and the indices p, q (see Theorems 4.1 and 4.2).

This analysis generalizes the basic result [14]:

The Fourier transform $\mathcal{F}$ *is a continuous mapping between* $W(\mathcal{F}L^q, L^p)$ *and* $W(\mathcal{F}L^p, L^q)$ *if* (*and only if*) $1 \leqslant p \leqslant q \leqslant \infty$.

Indeed, the Fourier transform $\mathcal{F}$ is a special metaplectic operator. If we introduce the symplectic matrix

$$J = \begin{bmatrix} 0 & I_d \\ -I_d & 0 \end{bmatrix}, \tag{1}$$

then $\mathcal{F}$ is (up to a phase factor) the unitary metaplectic operator corresponding to J,

$$\mu(J) = (-i)^{d/2}\mathcal{F}.$$

A fundamental tool to achieve these estimates is represented by the analysis of the dilation operator $f(x) \mapsto f(Ax)$, for a real invertible $d \times d$ matrix $A \in GL(d,\mathbb{R})$, with bounds on its norm in terms of spectral invariants of A. In the framework of modulation spaces such an investigation was recently developed in the scalar case $A = \lambda I$ by Sugimoto and Tomita [35,36] and by Bényi and Okoudjou [4]. In Section 3 we study this problem for a general matrix $A \in GL(d,\mathbb{R})$ for both modulation and Wiener amalgam spaces. In particular, we extend the results in [35] to the case of a symmetric matrix A.

In the second part of the paper we present some natural applications to partial differential equations *with variable coefficients*. Precisely, we study the Cauchy problem for the Schrödinger equation with a quadratic Hamiltonian, namely

$$\begin{cases} i\dfrac{\partial u}{\partial t}+Hu=0,\\ u(0,x)=u_0(x), \end{cases} \tag{2}$$

where H is the Weyl quantization of a quadratic form on $\mathbb{R}^d\times\mathbb{R}^d$. The most interesting case is certainly the Schrödinger equation with a quadratic potential. Indeed, the solution $u(t,x)$ to (2) is given by

$$u(t,x)=e^{itH}u_0,$$

where the operator e^{itH} is a metaplectic operator, so that the estimates resulting from the previous sections provide at once fixed-time estimates for the solution $u(t,x)$, in terms of the initial datum u_0. An example is provided by the harmonic oscillator $H=-\frac{1}{4\pi}\Delta+\pi|x|^2$ (see, e.g., [19,23,29]), for which we deduce the dispersive estimate

$$\left\|e^{itH}u_0\right\|_{W(\mathcal{F}L^1,L^\infty)}\lesssim|\sin t|^{-d}\|u_0\|_{W(\mathcal{F}L^\infty,L^1)}. \tag{3}$$

Another Hamiltonian we take into account is $H=-\frac{1}{4\pi}\Delta-\pi|x|^2$ (see [6]). In this case, we show

$$\left\|e^{itH}u_0\right\|_{W(\mathcal{F}L^1,L^\infty)}\lesssim\left(\frac{1+|\sinh t|}{\sinh^2 t}\right)^{\frac{d}{2}}\|u_0\|_{W(\mathcal{F}L^\infty,L^1)}. \tag{4}$$

In Section 5 we shall combine these estimates with orthogonality arguments as in [9,27] to obtain space–time estimates: the so-called Strichartz estimates (for the classical theory in Lebesgue spaces, see [20,26,27,38,39,44]). For instance, the homogeneous Strichartz estimates achieved for the harmonic oscillator $H=-\frac{1}{4\pi}\Delta+\pi|x|^2$ read

$$\left\|e^{itH}u_0\right\|_{L^{q/2}([0,T])W(\mathcal{F}L^{r'},L^r)_x}\lesssim\|u_0\|_{L^2_x},$$

for every $T>0$, $4<q,\tilde{q}\leqslant\infty$, $2\leqslant r,\tilde{r}\leqslant\infty$, such that $2/q+d/r=d/2$, and, similarly, for $\tilde{q},\tilde{r}$. In the endpoint case $(q,r)=(4,2d/(d-1))$, $d>1$, we prove the same estimate with $\mathcal{F}L^{r'}$ replaced by the slightly larger $\mathcal{F}L^{r',2}$, where $L^{r',2}$ is a Lorentz space (Theorem 5.2).

The case of the Hamiltonian $H=-\frac{1}{4\pi}\Delta-\pi|x|^2$ will be detailed in Section 5.2. Finally, we shall compare all these estimates with the classical ones in the Lebesgue spaces (Section 5.3).

Our analysis combines techniques from time-frequency analysis (e.g., convolution relations, embeddings and duality properties of Wiener amalgam and modulation spaces) with methods from classical harmonic analysis and PDE's theory (interpolation results, Hölder-type inequalities, fractional integration theory).

This study carries on the one in [9], developed for the usual Schrödinger equation ($H=\Delta$).

We record that hybrid spaces like the Wiener amalgam ones had appeared before as a technical tool in PDEs (see, e.g., Tao [37]). Notice that fixed-time estimates between modulation spaces in the case $H=\Delta$ were first considered in [1] and, independently, in [3,4], and they were used to obtain well-posedness results on such spaces [2,5].

Finally we observe that, by combining the Strichartz estimates in the present paper with arguments from functional analysis as in [11], well-posedness in suitable Wiener amalgam spaces can also be proved for Schrödinger equations as above with an additional potential term in $L^\alpha_t W(\mathcal{F}L^{p'}, L^p)_x$ (see Remark 5.5).

Notation. We define $|x|^2 = x \cdot x$, for $x \in \mathbb{R}^d$, where $x \cdot y = xy$ is the scalar product on $\mathbb{R}^d$. The space of smooth functions with compact support is denoted by $\mathcal{C}_0^\infty(\mathbb{R}^d)$, the Schwartz class is $\mathcal{S}(\mathbb{R}^d)$, the space of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$. The Fourier transform is normalized to be $\hat{f}(\xi) = \mathcal{F}f(\xi) = \int f(t)e^{-2\pi i t\xi}\,dt$. Translation and modulation operators (*time and frequency shifts*) are defined, respectively, by

$$T_x f(t) = f(t-x) \quad \text{and} \quad M_\xi f(t) = e^{2\pi i \xi t} f(t).$$

We have the formulas $(T_x f)\hat{} = M_{-x}\hat{f}$, $(M_\xi f)\hat{} = T_\xi \hat{f}$, and $M_\xi T_x = e^{2\pi i x\xi} T_x M_\xi$. The notation $A \lesssim B$ means $A \leqslant cB$ for a suitable constant $c > 0$, whereas $A \asymp B$ means $c^{-1}A \leqslant B \leqslant cA$, for some $c \geqslant 1$. The symbol $B_1 \hookrightarrow B_2$ denotes the continuous embedding of the linear space B_1 into B_2.

2. Function spaces and preliminaries

2.1. Lorentz spaces [33,34]

We recall that the Lorentz space $L^{p,q}$ on $\mathbb{R}^d$ is defined as the space of temperate distributions f such that

$$\|f\|^*_{pq} = \left(\frac{q}{p}\int_0^\infty \left[t^{1/p} f^*(t)\right]^q \frac{dt}{t}\right)^{1/q} < \infty,$$

when $1 \leqslant p < \infty$, $1 \leqslant q < \infty$, and

$$\|f\|^*_{pq} = \sup_{t>0} t^{1/p} f^*(t) < \infty,$$

when $1 \leqslant p \leqslant \infty$, $q = \infty$. Here, as usual, $\lambda(s) = |\{|f| > s\}|$ denotes the distribution function of f and $f^*(t) = \inf\{s\colon \lambda(s) \leqslant t\}$.

One has $L^{p,q_1} \hookrightarrow L^{p,q_2}$ if $q_1 \leqslant q_2$, and $L^{p,p} = L^p$. Moreover, for $1 < p < \infty$ and $1 \leqslant q \leqslant \infty$, $L^{p,q}$ is a normed space and its norm $\|\cdot\|_{L^{p,q}}$ is equivalent to the above quasi-norm $\|\cdot\|^*_{pq}$.

We now recall the following generalized Hardy–Littlewood–Sobolev fractional integration theorem (see, e.g., [32, p. 119] and [41, Theorem 2, p. 139]), which will be used in the sequel (the original fractional integration theorem corresponds to the model case of convolution by $K(x) = |x|^{-\alpha} \in L^{d/\alpha,\infty}$, $0 < \alpha < d$).

Proposition 2.1. *Let $1 \leqslant p < q < \infty$, $0 < \alpha < d$, with $1/p = 1/q + 1 - \alpha/d$. Then,*

$$L^p(\mathbb{R}^d) * L^{d/\alpha,\infty}(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d). \tag{5}$$

2.2. Wiener amalgam spaces [13–15,17,18]

Let $g \in \mathcal{C}_0^\infty$ be a test function that satisfies $\|g\|_{L^2} = 1$. We will refer to g as a window function. For $1 \leqslant p \leqslant \infty$, recall the $\mathcal{F}L^p$ spaces, defined by

$$\mathcal{F}L^p(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d): \exists h \in L^p(\mathbb{R}^d),\ \hat{h} = f\};$$

they are Banach spaces equipped with the norm

$$\|f\|_{\mathcal{F}L^p} = \|h\|_{L^p}, \quad \text{with } \hat{h} = f. \tag{6}$$

Let B be one of the following Banach spaces: L^p, $\mathcal{F}L^p$, $1 \leqslant p \leqslant \infty$, $L^{p,q}$, $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, possibly valued in a Banach space, or also spaces obtained from these by real or complex interpolation. Let C be one of the following Banach spaces: L^p, $1 \leqslant p \leqslant \infty$, or $L^{p,q}$, $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, scalar-valued. For any given function f which is locally in B (i.e. $gf \in B$, $\forall g \in \mathcal{C}_0^\infty$), we set $f_B(x) = \|f T_x g\|_B$.

The *Wiener amalgam space* $W(B, C)$ with local component B and global component C is defined as the space of all functions f locally in B such that $f_B \in C$. Endowed with the norm $\|f\|_{W(B,C)} = \|f_B\|_C$, $W(B, C)$ is a Banach space. Moreover, different choices of $g \in \mathcal{C}_0^\infty$ generate the same space and yield equivalent norms.

If $B = \mathcal{F}L^1$ (the Fourier algebra), the space of admissible windows for the Wiener amalgam spaces $W(\mathcal{F}L^1, C)$ can be enlarged to the so-called Feichtinger algebra $W(\mathcal{F}L^1, L^1)$. Recall that the Schwartz class $\mathcal{S}$ is dense in $W(\mathcal{F}L^1, L^1)$.

We use the following definition of mixed Wiener amalgam norms. Given a measurable function F of the two variables (t, x) we set

$$\|F\|_{W(L^{q_1},L^{q_2})_t W(\mathcal{F}L^{r_1},L^{r_2})_x} = \big\| \|F(t,\cdot)\|_{W(\mathcal{F}L^{r_1},L^{r_2})_x} \big\|_{W(L^{q_1},L^{q_2})_t}.$$

Observe that [9]

$$\|F\|_{W(L^{q_1},L^{q_2})_t W(\mathcal{F}L^{r_1},L^{r_2})_x} = \|F\|_{W(L_t^{q_1}(W(\mathcal{F}L_x^{r_1},L_x^{r_2})),L_t^{q_2})}.$$

The following properties of Wiener amalgam spaces will be frequently used in the sequel.

Lemma 2.1. *Let B_i, C_i, $i = 1, 2, 3$, be Banach spaces such that $W(B_i, C_i)$ are well defined. Then*:

(i) (Convolution) *If $B_1 * B_2 \hookrightarrow B_3$ and $C_1 * C_2 \hookrightarrow C_3$, we have*

$$W(B_1, C_1) * W(B_2, C_2) \hookrightarrow W(B_3, C_3). \tag{7}$$

In particular, for every $1 \leqslant p, q \leqslant \infty$, we have

$$\|f * u\|_{W(\mathcal{F}L^p,L^q)} \leqslant \|f\|_{W(\mathcal{F}L^\infty,L^1)} \|u\|_{W(\mathcal{F}L^p,L^q)}. \tag{8}$$

(ii) (Inclusions) *If $B_1 \hookrightarrow B_2$ and $C_1 \hookrightarrow C_2$,*

$$W(B_1, C_1) \hookrightarrow W(B_2, C_2).$$

Moreover, the inclusion of B_1 into B_2 need only hold "locally" and the inclusion of C_1 into C_2 "globally." In particular, for $1 \leqslant p_i, q_i \leqslant \infty$, $i = 1, 2$, we have

$$p_1 \geqslant p_2 \quad \textit{and} \quad q_1 \leqslant q_2 \quad \Rightarrow \quad W\big(L^{p_1}, L^{q_1}\big) \hookrightarrow W\big(L^{p_2}, L^{q_2}\big). \tag{9}$$

(iii) (Complex interpolation) *For $0 < \theta < 1$, we have*

$$\big[W(B_1, C_1), W(B_2, C_2)\big]_{[\theta]} = W\big([B_1, B_2]_{[\theta]}, [C_1, C_2]_{[\theta]}\big),$$

if C_1 or C_2 has absolutely continuous norm.

(iv) (Duality) *If B', C' are the topological dual spaces of the Banach spaces B, C, respectively, and the space of test functions $\mathcal{C}_0^\infty$ is dense in both B and C, then*

$$W(B, C)' = W(B', C'). \tag{10}$$

Proposition 2.2. *For every $1 \leqslant p \leqslant q \leqslant \infty$, the Fourier transform $\mathcal{F}$ maps $W(\mathcal{F}L^q, L^p)$ in $W(\mathcal{F}L^p, L^q)$ continuously.*

The proof of all these results can be found in [13–15,22].

The subsequent result of real interpolation is proved in [9].

Proposition 2.3. *Given two local components B_0, B_1 as above, for every $1 \leqslant p_0, p_1 < \infty$, $0 < \theta < 1$, $1/p = (1-\theta)/p_0 + \theta/p_1$, and $p \leqslant q$ we have*

$$W\big((B_0, B_1)_{\theta,q}, L^p\big) \hookrightarrow \big(W\big(B_0, L^{p_0}\big), W\big(B_1, L^{p_1}\big)\big)_{\theta,q}.$$

2.3. Modulation spaces [21]

Let $g \in \mathcal{S}$ be a non-zero window function. The short-time Fourier transform (STFT) $V_g f$ of a function/tempered distribution f with respect to the window g is defined by

$$V_g f(z, \xi) = \int e^{-2\pi i \xi y} f(y) g(y - z)\, dy,$$

i.e., the Fourier transform $\mathcal{F}$ applied to $f T_z g$.

For $1 \leqslant p, q \leqslant \infty$, the modulation space $M^{p,q}(\mathbb{R}^n)$ is defined as the space of temperate distributions f on $\mathbb{R}^n$ such that the norm

$$\|f\|_{M^{p,q}} = \big\| \big\| V_g f(\cdot, \xi) \big\|_{L^p} \big\|_{L^q_\xi}$$

is finite. Among the properties of modulation spaces, we record that $M^{2,2} = L^2$, $M^{p_1,q_1} \hookrightarrow M^{p_2,q_2}$, if $p_1 \leqslant p_2$ and $q_1 \leqslant q_2$. If $p, q < \infty$, then $(M^{p,q})' = M^{p',q'}$.

For comparison, notice that the norm in the Wiener amalgam spaces $W(\mathcal{F}L^p, L^q)$ reads

$$\|f\|_{W(\mathcal{F}L^p, L^q)} = \big\| \big\| V_g f(z, \cdot) \big\|_{L^p} \big\|_{L^q_z}.$$

The relationship between modulation and Wiener amalgam spaces is expressed by the following result.

Proposition 2.4. *The Fourier transform establishes an isomorphism* $\mathcal{F}: M^{p,q} \to W(\mathcal{F}L^p, L^q)$.

Consequently, convolution properties of modulation spaces can be translated into point-wise multiplication properties of Wiener amalgam spaces, as shown below.

Proposition 2.5. *For every* $1 \leqslant p, q \leqslant \infty$ *we have*

$$\|fu\|_{W(\mathcal{F}L^p,L^q)} \leqslant \|f\|_{W(\mathcal{F}L^1,L^\infty)}\|u\|_{W(\mathcal{F}L^p,L^q)}.$$

Proof. From Proposition 2.4, the estimate to prove is equivalent to

$$\|\hat{f} * \hat{u}\|_{M^{p,q}} \leqslant \|\hat{f}\|_{M^{1,\infty}}\|\hat{u}\|_{M^{p,q}},$$

but this is a special case of [7, Proposition 2.4]. □

The characterization of the $M^{2,\infty}$-norm in [35, Lemma 3.4], see also [28], can be rephrased in our context as follows.

Lemma 2.2. *Suppose that* $\varphi \in \mathcal{S}(\mathbb{R}^d)$ *is a real-valued function satisfying* $\varphi \geqslant C$ *on* $[-1/2, 1/2]^d$*, for some constant* $C > 0$, $\operatorname{supp}\varphi \subset [-1, 1]^d$, $\varphi(t) = \phi(-t)$ *and* $\sum_{k\in\mathbb{Z}^d} \varphi(t-k) = 1$ *for all* $t \in \mathbb{R}^d$*. Then*

$$\|f\|_{M^{2,\infty}} \asymp \sup_{k\in\mathbb{Z}^d} \big\|(M_k\Phi) * f\big\|_{L^2}, \tag{11}$$

for all $f \in M^{2,\infty}$*, where* $\Phi = \mathcal{F}^{-1}\varphi$.

To compute the $M^{p,q}$-norm we shall often use the *duality* technique, justified by the result below (see [21, Proposition 11.3.4 and Theorem 11.3.6] and [35, relation (2.1)]).

Lemma 2.3. *Let* $\varphi \in \mathcal{S}(\mathbb{R}^d)$*, with* $\|\varphi\|_2 = 1$, $1 \leqslant p, q < \infty$*. Then* $(M^{p,q})^* = M^{p',q'}$*, under the duality*

$$\langle f, g\rangle = \langle V_\varphi f, V_\varphi g\rangle = \int_{\mathbb{R}^{2d}} V_\varphi f(x,\omega)\overline{V_\varphi g(x,\omega)}\,dx\,d\xi, \tag{12}$$

for $f \in M^{p,q}$, $g \in M^{p',q'}$.

Lemma 2.4. *Assume* $1 < p, q \leqslant \infty$ *and* $f \in M^{p,q}$*. Then*

$$\|f\|_{M^{p,q}} = \sup_{\|g\|_{M^{p',q'}} \leqslant 1} \big|\langle f, g\rangle\big|. \tag{13}$$

Notice that (13) still holds true whenever $p = 1$ or $q = 1$ and $f \in \mathcal{S}(\mathbb{R}^d)$, simply by extending [21, Theorem 3.2.1] to the duality ${}_{\mathcal{S}'}\langle\cdot,\cdot\rangle_{\mathcal{S}}$.

Finally we recall the behaviour of modulation spaces with respect to complex interpolation (see [14, Corollary 2.3]).

Proposition 2.6. *Let $1 \leqslant p_1, p_2, q_1, q_2 \leqslant \infty$, with $q_2 < \infty$. If T is a linear operator such that, for $i = 1, 2$,*

$$\|Tf\|_{M^{p_i,q_i}} \leqslant A_i \|f\|_{M^{p_i,q_i}} \quad \forall f \in M^{p_i,q_i},$$

then

$$\|Tf\|_{M^{p,q}} \leqslant C A_1^{1-\theta} A_2^{\theta} \|f\|_{M^{p,q}} \quad \forall f \in M^{p,q},$$

where $1/p = (1-\theta)/p_1 + \theta/p_2$, $1/q = (1-\theta)/q_1 + \theta/q_2$, $0 < \theta < 1$, and C is independent of T.

2.4. *The metaplectic representation [19]*

The symplectic group is defined by

$$Sp(d,\mathbb{R}) = \left\{ g \in GL(2d,\mathbb{R}) \colon {}^t g J g = J \right\},$$

where the symplectic matrix J is defined in (1). The metaplectic or Shale–Weil representation μ is a unitary representation of the (double cover of the) symplectic group $Sp(d,\mathbb{R})$ on $L^2(\mathbb{R}^d)$. For elements of $Sp(d,\mathbb{R})$ in special form, the metaplectic representation can be computed explicitly. For $f \in L^2(\mathbb{R}^d)$, we have

$$\mu\left(\begin{bmatrix} A & 0 \\ 0 & {}^t A^{-1} \end{bmatrix}\right) f(x) = (\det A)^{-1/2} f\left(A^{-1}x\right),$$
$$\mu\left(\begin{bmatrix} I & 0 \\ C & I \end{bmatrix}\right) f(x) = \pm e^{i\pi\langle Cx,x\rangle} f(x). \tag{14}$$

The symplectic algebra $\mathfrak{sp}(d,\mathbb{R})$ is the set of all $2d \times 2d$ real matrices $\mathcal{A}$ such that $e^{t\mathcal{A}} \in Sp(d,\mathbb{R})$ for all $t \in \mathbb{R}$.

The following formulae for the metaplectic representation can be found in [19, Theorems 4.51 and 4.53].

Proposition 2.7. *Let $\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(d,\mathbb{R})$.*

(i) *If $\det B \neq 0$ then*

$$\mu(\mathcal{A}) f(x) = i^{d/2} (\det B)^{-1/2} \int e^{-\pi i x\cdot DB^{-1}x + 2\pi i y\cdot B^{-1}x - \pi i y\cdot B^{-1}Ay} f(y)\, dy. \tag{15}$$

(ii) *If $\det A \neq 0$,*

$$\mu(\mathcal{A}) f(x) = (\det A)^{-1/2} \int e^{-\pi i x\cdot CA^{-1}x + 2\pi i \xi\cdot A^{-1}x + \pi i \xi\cdot A^{-1}B\xi} \hat{f}(\xi)\, d\xi. \tag{16}$$

The following hybrid formula will be also used in the sequel.

Proposition 2.8. *If* $\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(d, \mathbb{R})$, $\det B \neq 0$ *and* $\det A \neq 0$, *then*

$$\mu(\mathcal{A})f(x) = (-i \det B)^{-1/2} e^{-\pi i x \cdot CA^{-1}x} \left(e^{-\pi i y \cdot B^{-1}Ay} * f\right)\left(A^{-1}x\right). \tag{17}$$

Proof. By (16) we can write

$$\begin{aligned}\mu(\mathcal{A})f(x) &= (\det A)^{-1/2} e^{-\pi i x \cdot CA^{-1}x} \int e^{2\pi i \xi \cdot A^{-1}x} \mathcal{F}\left(\mathcal{F}^{-1} e^{\pi i \xi \cdot A^{-1}B\xi}\right) \hat{f}(\xi)\, d\xi \\ &= (-i \det B)^{-1/2} e^{-\pi i x \cdot CA^{-1}x} \int e^{2\pi i \xi \cdot A^{-1}x} \mathcal{F}\left(e^{-\pi i y \cdot B^{-1}Ay} * f\right)(\xi)\, d\xi,\end{aligned}$$

where we used the formula (see [19, Theorem 2, p. 257])

$$\mathcal{F}^{-1}\left(e^{i\pi \xi \cdot A^{-1}B\xi}\right)(y) = \left(-i \det A^{-1}B\right)^{-1/2} e^{-\pi i y \cdot B^{-1}Ay}.$$

Hence, from the Fourier inversion formula we obtain (17). □

3. Dilation of modulation and Wiener amalgam spaces

Given a function f on $\mathbb{R}^d$ and $A \in GL(d, \mathbb{R})$, we set $f_A(t) = f(At)$. We also consider the unitary operator $\mathcal{U}_A$ on $L^2(\mathbb{R}^d)$ defined by

$$\mathcal{U}_A f(t) = |\det A|^{1/2} f(At) = |\det A|^{1/2} f_A(t). \tag{18}$$

In this section we study the boundedness of this operator on modulation and Wiener amalgam spaces. We need the following three lemmata.

Lemma 3.1. *Let* $A \in GL(d, \mathbb{R})$, $\varphi(t) = e^{-\pi |t|^2}$, *then*

$$V_\varphi \varphi_A(x, \xi) = \left(\det(A^*A + I)\right)^{-1/2} e^{-\pi(I - (A^*A+I)^{-1})x \cdot x} M_{-((A^*A+I)^{-1})x} e^{-\pi(A^*A+I)^{-1}\xi \cdot \xi}.$$

Proof. By definition of the STFT,

$$\begin{aligned}V_\varphi \varphi_A(x, \xi) &= \int_{\mathbb{R}^d} e^{-\pi Ay \cdot Ay} e^{-2\pi i \xi \cdot y} e^{-\pi (y-x)^2}\, dy \\ &= e^{-\pi |x|^2} \int_{\mathbb{R}^d} e^{-\pi(A^*A+I)y \cdot y + 2\pi x \cdot y} e^{-2\pi i \xi \cdot y}\, dy.\end{aligned}$$

Now, we rewrite the generalized Gaussian above using the translation and dilation operators, that is

$$e^{-\pi(A^*A+I)y \cdot y + 2\pi x \cdot y} = e^{\pi(A^*A+I)^{-1}x \cdot x} \left(\det(A^*A + I)\right)^{-1/4} (T_{(A^*A+I)^{-1}x} \mathcal{U}_{(A^*A+I)^{1/2}})\varphi(y)$$

and use the properties $\mathcal{F}\mathcal{U}_B = \mathcal{U}_{(B^*)^{-1}}\mathcal{F}$, for every $B \in GL(d, \mathbb{R})$ and $\mathcal{F}T_x = M_{-x}\mathcal{F}$. Thereby,

$$V_\varphi\varphi_A(x,\xi) = e^{-\pi(I-(A^*A+I)^{-1})x\cdot x}\big(\det(A^*A+I)\big)^{-1/4}\mathcal{F}(T_{(A^*A+I)^{-1}x}\mathcal{U}_{(A^*A+I)^{1/2}}\varphi(\xi)$$
$$= e^{-\pi(I-(A^*A+I)^{-1})x\cdot x}\big(\det(A^*A+I)\big)^{-1/2}M_{-(A^*A+I)^{-1}x}e^{-\pi(A^*A+I)^{-1}\xi\cdot\xi},$$

as desired. □

The result below generalizes [40, Lemma 1.8], recaptured in the special case $A=\lambda I$, $\lambda>0$.

Lemma 3.2. *Let* $1\leqslant p,q\leqslant\infty$, $A\in GL(d,\mathbb{R})$ *and* $\varphi(t)=e^{-\pi|t|^2}$. *Then,*

$$\|\varphi_A\|_{M^{p,q}} = p^{-d/(2p)}q^{-d/(2q)}|\det A|^{-1/p}\big(\det(A^*A+I)\big)^{-(1-1/q-1/p)/2}. \tag{19}$$

Proof. Since the modulation space norm is independent of the choice of the window function, we choose the Gaussian φ, so that $\|\varphi_A\|_{M^{p,q}}\asymp\|V_\varphi\varphi_A\|_{L^{p,q}}$. Since

$$\int_{\mathbb{R}^d} e^{-\pi p(I-(A^*A+I)^{-1})x\cdot x}\,dx = \det\big(I-(A^*A+I)^{-1}\big)^{-1/2}p^{-d/2}$$
$$= p^{-d/2}|\det A|^{-1}\big(\det(A^*A+I)\big)^{1/2}$$

and, analogously, $\int_{\mathbb{R}^d} e^{-\pi q(A^*A+I)^{-1}\xi\cdot\xi}\,d\xi = (\det(A^*A+I))^{1/2}q^{-d/2}$, the result immediately follows from Lemma 3.1. □

We record [21, Lemma 11.3.3]

Lemma 3.3. *Let* $f\in\mathcal{S}'(\mathbb{R}^d)$ *and* $\varphi,\psi,\gamma\in\mathcal{S}(\mathbb{R}^d)$. *Then,*

$$\big|V_\varphi f(x,\xi)\big| \leqslant \frac{1}{\langle\gamma,\psi\rangle}\big(|V_\psi f|*|V_\varphi\gamma|\big)(x,\xi)\quad\forall(x,\xi)\in\mathbb{R}^{2d}.$$

The results above are the ingredients for the first dilation property of modulation spaces we are going to present.

Proposition 3.1. *Let* $1\leqslant p,q\leqslant\infty$ *and* $A\in GL(d,\mathbb{R})$. *Then, for every* $f\in M^{p,q}(\mathbb{R}^d)$,

$$\|f_A\|_{M^{p,q}} \lesssim |\det A|^{-(1/p-1/q+1)}\big(\det(I+A^*A)\big)^{1/2}\|f\|_{M^{p,q}}. \tag{20}$$

Proof. The proof follows the guidelines of [35, Lemma 3.2]. First, by a change of variable, the dilation is transferred from the function f to the window φ:

$$V_\varphi f_A(x,\xi) = |\det A|^{-1}V_{\varphi_{A^{-1}}}f\big(Ax,(A^*)^{-1}\xi\big).$$

Whence, performing the change of variables $Ax=u$, $(A^*)^{-1}\xi=v$,

$$\|f_A\|_{M^{p,q}} = |\det A|^{-1}\bigg(\int_{\mathbb{R}^d}\bigg(\int_{\mathbb{R}^d}\big|V_{\varphi_{A^{-1}}} f\big(Ax,(A^*)^{-1}\xi\big)\big|^p\,dx\bigg)^{q/p} d\xi\bigg)^{1/q}$$

$$= |\det A|^{-(1/p-1/q+1)}\|V_{\varphi_{A^{-1}}} f\|_{L^{p,q}}.$$

Now, Lemma 3.3, written for $\psi(t)=\gamma(t)=\varphi(t)=e^{-\pi t^2}$, yields the following majorization

$$\big|V_{\varphi_{A^{-1}}} f(x,\xi)\big| \leqslant \|\varphi\|_{L^2}^{-2}\big(|V_\varphi f| * |V_{\varphi_{A^{-1}}}\varphi|\big)(x,\xi).$$

Finally, Young's Inequality and Lemma 3.2 provide the desired result:

$$\begin{aligned}\|f_A\|_{M^{p,q}} &\lesssim |\det A|^{-(1/p-1/q+1)}\big\||V_\varphi f| * |V_{\varphi_{A^{-1}}}\varphi|\big\|_{L^{p,q}}\\ &\lesssim |\det A|^{-(1/p-1/q+1)}\|V_\varphi f\|_{L^{p,q}}\|V_{\varphi_{A^{-1}}}\varphi\|_{L^1}\\ &\asymp |\det A|^{-(1/p-1/q+1)}\big(\det(I+A^*A)\big)^{1/2}\|f\|_{M^{p,q}}. \qquad \Box\end{aligned}$$

Proposition 3.1 generalizes [35, Lemma 3.2], that can be recaptured by choosing the matrix $A=\lambda I$, $\lambda>0$.

Corollary 3.2. *Let $1 \leqslant p,q \leqslant \infty$ and $A \in GL(d,\mathbb{R})$. Then, for every $f \in W(\mathcal{F}L^p, L^q)(\mathbb{R}^d)$,*

$$\|f_A\|_{W(\mathcal{F}L^p,L^q)} \lesssim |\det A|^{(1/p-1/q-1)}\big(\det(I+A^*A)\big)^{1/2}\|f\|_{W(\mathcal{F}L^p,L^q)}. \tag{21}$$

Proof. It follows immediately from the relation between Wiener amalgam spaces and modulation spaces given by $W(\mathcal{F}L^p,L^q)=\mathcal{F}M^{p,q}$ and by the relation $\widehat{(f_A)} = |\det A|^{-1}(\hat{f})_{(A^*)^{-1}}$. $\Box$

In what follows we give a more precise result about the behaviour of the operator norm $\|D_A\|_{M^{p,q}\to M^{p,q}}$ in terms of A, when A is a symmetric matrix, extending the diagonal case $A=\lambda I$, $\lambda>0$, treated in [35]. We shall use the set and index terminology of the paper above. Namely, for $1\leqslant p\leqslant\infty$, let p' be the conjugate exponent of p ($1/p+1/p'=1$). For $(1/p,1/q)\in[0,1]\times[0,1]$, we define the subsets

$$\begin{aligned} I_1 &= \max(1/p,1/p') \leqslant 1/q, & I_1^* &= \min(1/p,1/p') \geqslant 1/q,\\ I_2 &= \max(1/q,1/2) \leqslant 1/p', & I_2^* &= \min(1/q,1/2) \geqslant 1/p',\\ I_3 &= \max(1/q,1/2) \leqslant 1/p, & I_3^* &= \min(1/q,1/2) \geqslant 1/p, \end{aligned}$$

as shown in Fig. 1.

We introduce the indices:

$$\mu_1(p,q) = \begin{cases} -1/p & \text{if } (1/p,1/q)\in I_1^*,\\ 1/q-1 & \text{if } (1/p,1/q)\in I_2^*,\\ -2/p+1/q & \text{if } (1/p,1/q)\in I_3^*, \end{cases}$$

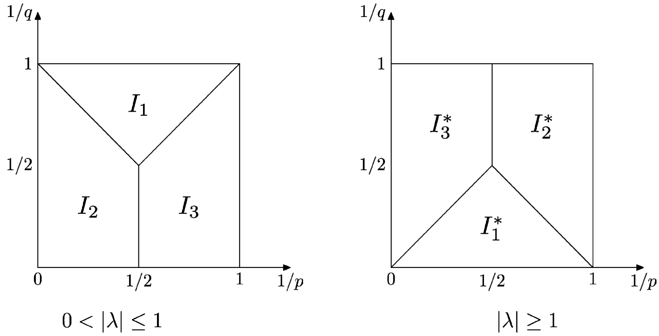

Fig. 1. The index sets.

and

$$\mu_2(p,q) = \begin{cases} -1/p & \text{if } (1/p, 1/q) \in I_1, \\ 1/q - 1 & \text{if } (1/p, 1/q) \in I_2, \\ -2/p + 1/q & \text{if } (1/p, 1/q) \in I_3. \end{cases}$$

The above mentioned result by [35, Theorem 3.1] reads as follows:

Theorem 3.3. *Let* $1 \leqslant p, q \leqslant \infty$, *and* $A = \lambda I$, $\lambda \neq 0$.

(i) *We have*

$$\|f_A\|_{M^{p,q}} \lesssim |\lambda|^{d\mu_1(p,q)} \|f\|_{M^{p,q}}, \quad \forall |\lambda| \geqslant 1, \ \forall f \in M^{p,q}(\mathbb{R}^d).$$

Conversely, if there exists $\alpha \in \mathbb{R}$ *such that*

$$\|f_A\|_{M^{p,q}} \lesssim |\lambda|^{\alpha} \|f\|_{M^{p,q}}, \quad \forall |\lambda| \geqslant 1, \ \forall f \in M^{p,q}(\mathbb{R}^d),$$

then $\alpha \geqslant d\mu_1(p,q)$.

(ii) *We have*

$$\|f_A\|_{M^{p,q}} \lesssim |\lambda|^{d\mu_2(p,q)} \|f\|_{M^{p,q}}, \quad \forall 0 < |\lambda| \leqslant 1, \ \forall f \in M^{p,q}(\mathbb{R}^d).$$

Conversely, if there exists $\beta \in \mathbb{R}$ *such that*

$$\|f_A\|_{M^{p,q}} \lesssim |\lambda|^{\beta} \|f\|_{M^{p,q}}, \quad \forall 0 < |\lambda| \leqslant 1, \ \forall f \in M^{p,q}(\mathbb{R}^d),$$

then $\beta \leqslant d\mu_2(p,q)$.

Here is our extension.

Theorem 3.4. *Let* $1 \leqslant p, q \leqslant \infty$. *There exists a constant* $C > 0$ *such that, for every symmetric matrix* $A \in GL(d, \mathbb{R})$, *with eigenvalues* $\lambda_1, \dots, \lambda_d$, *we have*

$$\|f_A\|_{M^{p,q}} \leqslant C \prod_{j=1}^{d} \left(\max\{1, |\lambda_j|\}\right)^{\mu_1(p,q)} \left(\min\{1, |\lambda_j|\}\right)^{\mu_2(p,q)} \|f\|_{M^{p,q}}, \tag{22}$$

for every $f \in M^{p,q}(\mathbb{R}^d)$.

Conversely, if there exist $\alpha_j \in \mathbb{R}$, $\beta_j \in \mathbb{R}$ *such that, for every* $\lambda_j \neq 0$,

$$\|f_A\|_{M^{p,q}} \leqslant C \prod_{j=1}^{d} \left(\max\{1, |\lambda_j|\}\right)^{\alpha_j} \left(\min\{1, |\lambda_j|\}\right)^{\beta_j} \|f\|_{M^{p,q}}, \quad \forall f \in M^{p,q}(\mathbb{R}^d),$$

with $A = \operatorname{diag}[\lambda_1, \dots, \lambda_d]$, *then* $\alpha_j \geqslant \mu_1(p,q)$ *and* $\beta_j \leqslant \mu_2(p,q)$.

Proof. The necessary conditions are an immediate consequence of the one-dimensional case, already contained in Theorem 3.3. Indeed, it can be seen by taking f as tensor product of functions of one variable and by leaving free to vary just one eigenvalue, the remaining eigenvalues being all equal to one.

Let us come to the first part of the theorem. It suffices to prove it in the diagonal case $A = D = \operatorname{diag}[\lambda_1, \dots, \lambda_d]$. Indeed, since A is symmetric, there exists an orthogonal matrix T such that $A = T^{-1}DT$, and D is a diagonal matrix. On the other hand, by Proposition 3.1, we have $\|f_A\|_{M^{p,q}} \lesssim \|f_{T^{-1}D}\|_{M^{p,q}} = \|(f_{T^{-1}})_D\|_{M^{p,q}}$ and $\|f_{T^{-1}}\|_{M^{p,q}} \lesssim \|f\|_{M^{p,q}}$; hence the general case in (22) follows from the diagonal case $A = D$, with f replaced by $f_{T^{-1}}$.

From now onward, $A = D = \operatorname{diag}[\lambda_1, \dots, \lambda_d]$.

If the theorem holds true for a pair (p,q), with $(1/p, 1/q) \in [0,1] \times [0,1]$, then it is also true for their dual pair (p', q') (with $f \in \mathcal{S}$ if $p' = 1$ or $q' = 1$, see (13)). Indeed,

$$\begin{aligned}
\|f_D\|_{M^{p',q'}} &= \sup_{\|g\|_{M^{p,q}} \leqslant 1} \left|\langle f_D, g\rangle\right| = |\det D|^{-1} \sup_{\|g\|_{M^{p,q}} \leqslant 1} \left|\langle f, g_{D^{-1}}\rangle\right| \\
&\leqslant |\det D|^{-1} \|f\|_{M^{p',q'}} \sup_{\|g\|_{M^{p,q}} \leqslant 1} \|g_{D^{-1}}\|_{M^{p,q}} \\
&\lesssim \prod_{j=1}^{d} |\lambda_j|^{-1} \prod_{j=1}^{d} \left(\max\{1, |\lambda_j|^{-1}\}\right)^{\mu_1(p,q)} \left(\min\{1, |\lambda_j|^{-1}\}\right)^{\mu_2(p,q)} \|f\|_{M^{p',q'}} \\
&= \prod_{j=1}^{d} \left(\max\{1, |\lambda_j|\}\right)^{\mu_1(p',q')} \left(\min\{1, |\lambda_j|\}\right)^{\mu_2(p',q')} \|f\|_{M^{p',q'}},
\end{aligned}$$

for the index functions μ_1 and μ_2 fulfill

$$\mu_1(p', q') = -1 - \mu_2(p,q), \qquad \mu_2(p', q') = -1 - \mu_1(p,q). \tag{23}$$

Hence it suffices to prove the estimate (22) for the case $p \geqslant q$. Notice that the estimate in $M^{1,q'}$, $q' > 1$, are proved for Schwartz functions only, but they extend to all functions in $M^{1,q'}$, $q' < \infty$, for $\mathcal{S}(\mathbb{R}^d)$ is dense in $M^{1,q'}$. The uncovered case $(1, \infty)$ will be verified directly at the end of the proof.

From Fig. 1 it is clear that the estimate (22) for the points in the upper triangles follows by complex interpolation (Proposition 2.6) from the diagonal case $p = q$, and the two cases $(p,q) = (\infty, 1)$ and $(p,q) = (2,1)$, see Fig. 2.

Case $p = q$. If $d = 1$ the claim is true by Theorem 3.3 in dimension $d = 1$. We then use the induction method. Namely, we assume that (22) is fulfilled in dimension $d - 1$ and prove that still holds in dimension d.

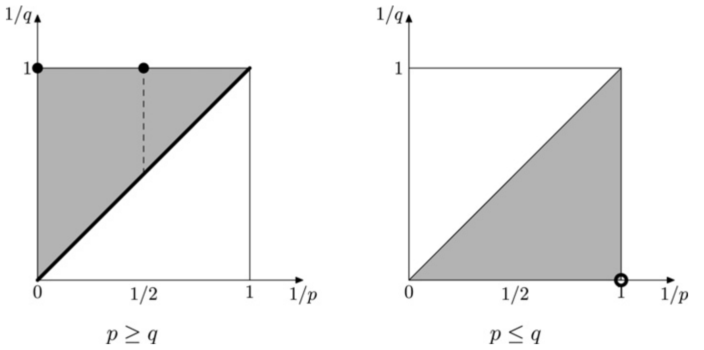

Fig. 2. The complex interpolation and the duality method.

For $x, \xi \in \mathbb{R}^d$, we write $x = (x', x_d)$, $\xi = (\xi', \xi_d)$, with $x', \xi' \in \mathbb{R}^{d-1}$, $x_d, \xi_d \in \mathbb{R}$, $D' = \mathrm{diag}[\lambda_1, \ldots, \lambda_{d-1}]$, and choose the Gaussian $\varphi(x) = e^{-\pi|x|^2} = e^{-\pi|x'|^2} e^{-\pi|x_d|^2} = \varphi'(x')\varphi_d(x_d)$ as window function. Observe that $V_\varphi f_D$ admits the two representations

$$\begin{aligned} V_\varphi f_D(x', x_d, \xi', \xi_d) &= \int_{\mathbb{R}^d} f(\lambda_1 t_1, \ldots, \lambda_d t_d) \overline{M_{\xi'} T_{x'} \varphi'(t')}\, \overline{M_{\xi_d} T_{x_d} \varphi_d(t_d)}\, dt'\, dt_d \\ &= V_{\varphi'}\big((F_{x_d,\xi_d,\lambda_d})_{D'}\big) \\ &= V_{\varphi_d}\big((G_{x',\xi',D'})_{\lambda_d}\big), \end{aligned}$$

where

$$F_{x_d,\xi_d,\lambda_d}(t') = V_{\varphi_d}\big(f(t', \lambda_d \cdot)\big)(x_d, \xi_d), \qquad G_{x',\xi',D'}(t_d) = V_{\varphi'}\big(f(D' \cdot, t_d)\big)(x', \xi').$$

By the inductive hypothesis we have

$$\begin{aligned} \|f_D\|_{M^{p,p}(\mathbb{R}^d)} &= \|V_\varphi f_D\|_{L^p(\mathbb{R}^{2d})} \\ &= \bigg(\int_{\mathbb{R}^2} \bigg(\int_{\mathbb{R}^{2(d-1)}} \big| V_{\varphi'}\big((F_{x_d,\xi_d,\lambda_d})_{D'}\big)(x', \xi') \big|^p\, dx'\, d\xi' \bigg) dx_d\, d\xi_d \bigg)^{1/p} \\ &\lesssim \prod_{j=1}^{d-1} \big(\max\{1, |\lambda_j|\}\big)^{\mu_1(p,p)} \big(\min\{1, |\lambda_j|\}\big)^{\mu_2(p,p)} \\ &\quad \cdot \bigg(\int_{\mathbb{R}^{2d}} \big| V_{\varphi'}(F_{x_d,\xi_d,\lambda_d})(x', \xi') \big|^p\, dx\, d\xi \bigg)^{1/p} \\ &= \prod_{j=1}^{d-1} \big(\max\{1, |\lambda_j|\}\big)^{\mu_1(p,p)} \big(\min\{1, |\lambda_j|\}\big)^{\mu_2(p,p)} \end{aligned}$$

$$\cdot\biggl(\int\limits_{\mathbb{R}^{2(d-1)}}\biggl(\int\limits_{\mathbb{R}^2}\bigl|V_{\varphi_d}\bigl((G_{x',\xi',I})_{\lambda_d}\bigr)(x_d,\xi_d)\bigr|^p\,dx_d\,d\xi_d\biggr)\,dx'\,d\xi'\biggr)^{1/p}$$

$$\lesssim\prod_{j=1}^d\bigl(\max\{1,|\lambda_j|\}\bigr)^{\mu_1(p,p)}\bigl(\min\{1,|\lambda_j|\}\bigr)^{\mu_2(p,p)}\|f\|_{M^{p,p}(\mathbb{R}^d)},$$

where in the last row we used Theorem 3.3 for $d=1$.

Case $(p,q)=(2,1)$. First, we prove the case $(p,q)=(2,\infty)$ and then obtain the claim by duality as above, since $\mathcal{S}$ is dense in $M^{2,1}$. Namely, we want to show that

$$\|f_D\|_{M^{2,\infty}}\lesssim\prod_{j=1}^d\bigl(\max\{1,|\lambda_j|\}\bigr)^{-1/2}\bigl(\min\{1,|\lambda_j|\}\bigr)^{-1}\|f\|_{M^{2,\infty}},\quad \forall f\in M^{2,\infty}.$$

The arguments are similar to [35, Lemma 3.5]. We use the characterization of the $M^{2,\infty}$-norm in (11)

$$\begin{aligned}\|f_D\|_{M^{2,\infty}}&\lesssim|\det D|^{-1/2}\sup_{k\in\mathbb{Z}^d}\bigl\|\varphi(D\cdot-k)\hat f\bigr\|_{L^2}\\&=|\det D|^{-1/2}\sup_{k\in\mathbb{Z}^d}\biggl\|\varphi(D\cdot-k)\biggl(\sum_{l\in\mathbb{Z}^d}\varphi(\cdot-l)\biggr)\hat f\biggr\|_{L^2}.\end{aligned}\tag{24}$$

Observe that

$$\begin{aligned}\biggl|\varphi(Dt-k)\biggl(\sum_{l\in\mathbb{Z}^d}\varphi(t-l)\biggr)\hat f(t)\biggr|^2&\leqslant 4^d\sum_{l\in\mathbb{Z}^d}\bigl|\varphi(Dt-k)\varphi(t-l)\hat f(t)\bigr|^2\\&=4^d\sum_{l\in\Lambda_k}\bigl|\varphi(Dt-k)\varphi(t-l)\hat f(t)\bigr|^2,\end{aligned}$$

where

$$\Lambda_k=\biggl\{l\in\mathbb{Z}^d\colon\ \biggl|l_j-\frac{k_j}{\lambda_j}\biggr|\leqslant 1+\frac{1}{|\lambda_j|}\biggr\}$$

and

$$\#\Lambda_k\leqslant C\prod_{j=1}^d\min\{1,|\lambda_j|\}^{-1},\quad \forall k\in\mathbb{Z}^d$$

(C being a constant depending on d only). Since $|\lambda_j|=\max\{1,|\lambda_j|\}\min\{1,|\lambda_j|\}$, the expression on the right-hand side of (24) is dominated by

$$C'\prod_{j=1}^d\bigl(\max\{1,|\lambda_j|\}\bigr)^{-1/2}\bigl(\min\{1,|\lambda_j|\}\bigr)^{-1}\sup_{m\in\mathbb{Z}^d}\bigl\|(M_m\Phi)*f\bigr\|_{L^2}.$$

Thereby the norm equivalence (11) gives the desired estimate.

Case $(p,q)=(\infty,1)$. We have to prove that

$$\|f_D\|_{M^{\infty,1}} \lesssim \prod_{j=1}^{d} \max\{1,|\lambda_j|\}\|f\|_{M^{\infty,1}}, \quad \forall f\in M^{\infty,1}.$$

This estimate immediately follows from (20), written for $A=D=\operatorname{diag}[\lambda_1,\dots,\lambda_d]$:

$$\|f_D\|_{M^{\infty,1}} \lesssim \prod_{j=1}^{d}\left(1+\lambda_j^2\right)^{1/2} \lesssim \prod_{j=1}^{d} \max\{1,|\lambda_j|\}\|f\|_{M^{\infty,1}}.$$

Case $(p,q)=(1,\infty)$. We are left to prove that

$$\|f_D\|_{M^{1,\infty}} \lesssim \prod_{j=1}^{d}\left(\max\{1,|\lambda_j|\}\right)^{-1}\left(\min\{1,|\lambda_j|\}\right)^{-2}\|f\|_{M^{1,\infty}}, \quad \forall f\in M^{1,\infty}.$$

This is again the estimate (20), written for $A=D=\operatorname{diag}[\lambda_1,\dots,\lambda_d]$:

$$\|f_D\|_{M^{1,\infty}} \lesssim \prod_{j=1}^{d}|\lambda_j|^{-2}\prod_{j=1}^{d} \max\{1,|\lambda_j|\}\|f\|_{M^{1,\infty}}. \qquad \square$$

Corollary 3.5. *Let $1\leqslant p,q\leqslant\infty$. There exists a constant $C>0$ such that, for every symmetric matrix $A\in GL(d,\mathbb{R})$, with eigenvalues $\lambda_1,\dots,\lambda_d$, we have*

$$\|f_A\|_{W(\mathcal{F}L^p,L^q)} \leqslant C\prod_{j=1}^{d}\left(\max\{1,|\lambda_j|\}\right)^{\mu_1(p',q')}\left(\min\{1,|\lambda_j|\}\right)^{\mu_2(p',q')}\|f\|_{W(\mathcal{F}L^p,L^q)}, \tag{25}$$

for every $f\in W(\mathcal{F}L^p,L^q)(\mathbb{R}^d)$.

Conversely, if there exist $\alpha_j\in\mathbb{R}$, $\beta_j\in\mathbb{R}$ such that, for every $\lambda_j\neq 0$,

$$\|f_A\|_{W(\mathcal{F}L^p,L^q)} \leqslant C\prod_{j=1}^{d}\left(\max\{1,|\lambda_j|\}\right)^{\alpha_j}\left(\min\{1,|\lambda_j|\}\right)^{\beta_j}\|f\|_{W(\mathcal{F}L^p,L^q)},$$

for every $f\in W(\mathcal{F}L^p,L^q)(\mathbb{R}^d)$, with $A=\operatorname{diag}[\lambda_1,\dots,\lambda_d]$, then $\alpha_j\geqslant\mu_1(p',q')$ and $\beta_j\leqslant\mu_2(p',q')$.

Proof. It is a mere consequence of Theorem 3.4 and the index relation (23). Namely,

$$\begin{aligned}\|f_A\|_{W(\mathcal{F}L^p,L^q)} &= \|\widehat{f_A}\|_{M^{p,q}} = |\det A|^{-1}\|\hat{f}_{A^{-1}}\|_{M^{p,q}}\\ &\leqslant C\prod_{j=1}^{d}|\lambda_j|^{-1}\prod_{j=1}^{d}\left(\max\{1,|\lambda_j|^{-1}\}\right)^{\mu_1(p,q)}\left(\min\{1,|\lambda_j|^{-1}\}\right)^{\mu_2(p,q)}\|\hat{f}\|_{M^{p,q}}\end{aligned}$$

$$= C \prod_{j=1}^{d} \big(\max\{1, |\lambda_j|\}\big)^{\mu_1(p',q')} \big(\min\{1, |\lambda_j|\}\big)^{\mu_2(p',q')} \|f\|_{W(\mathcal{F}L^p, L^q)}.$$

The necessary conditions use the same argument. □

4. Action of metaplectic operators on Wiener amalgam spaces

In this section we study the continuity property of metaplectic operators on Wiener amalgam spaces, giving bounds on their norms. Here is our first result.

Theorem 4.1. *Let* $\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(d, \mathbb{R})$, *and* $1 \leqslant p \leqslant q \leqslant \infty$.

(i) *If* $\det B \neq 0$, *then*

$$\big\|\mu(\mathcal{A}) f\big\|_{W(\mathcal{F}L^p, L^q)} \lesssim \alpha(\mathcal{A}, p, q) \|f\|_{W(\mathcal{F}L^q, L^p)}, \tag{26}$$

where

$$\alpha(\mathcal{A}, p, q) = |\det B|^{1/q - 1/p - 3/2} \big|\det(I + B^* B)(B + iA)(B + iD)\big|^{1/2}. \tag{27}$$

(ii) *If* $\det A, \det B \neq 0$, *then*

$$\big\|\mu(\mathcal{A}) f\big\|_{W(\mathcal{F}L^1, L^\infty)} \lesssim \beta(\mathcal{A}) \|f\|_{W(\mathcal{F}L^\infty, L^1)}, \tag{28}$$

with

$$\beta(\mathcal{A}) = |\det A|^{-3/2} |\det B|^{-1} \big|\det(I + A^* A)(B + iA)(A + iC)\big|^{1/2}. \tag{29}$$

If the matrices A or B are symmetric, Theorem 4.1 can be sharpened as follows.

Theorem 4.2. *Let* $\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(d, \mathbb{R})$, *and* $1 \leqslant p \leqslant q \leqslant \infty$.

(i) *If* $\det B \neq 0$, $B^* = B$, *with eigenvalues* $\lambda_1, \dots, \lambda_d$, *then*

$$\big\|\mu(\mathcal{A}) f\big\|_{W(\mathcal{F}L^p, L^q)} \lesssim \alpha'(\mathcal{A}, p, q) \|f\|_{W(\mathcal{F}L^q, L^p)}, \tag{30}$$

where

$$\alpha'(\mathcal{A}, p, q) = \big|\det(B + iA)(B + iD)\big|^{1/2} \cdot \prod_{j=1}^{d} \big(\max\{1, |\lambda_j|\}\big)^{\mu_1(p,q) - 1/2} \big(\min\{1, |\lambda_j|\}\big)^{\mu_2(p,q) - 1/2}. \tag{31}$$

(ii) *If* $\det A, \det B \neq 0$, *and* $A^* = A$ *with eigenvalues* $\nu_1, \dots, \nu_d$, *then*

$$\big\|\mu(\mathcal{A}) f\big\|_{W(\mathcal{F}L^1, L^\infty)} \lesssim \beta'(\mathcal{A}) \|f\|_{W(\mathcal{F}L^\infty, L^1)}, \tag{32}$$

with

$$\beta'(\mathcal{A}) = |\det B|^{-1}\bigl|\det(B+iA)(A+iC)\bigr|^{1/2} \cdot \prod_{j=1}^{d}\bigl(\max\{1,|\nu_j|\}\bigr)^{-1/2}\bigl(\min\{1,|\nu_j|\}\bigr)^{-3/2}. \tag{33}$$

We now prove Theorems 4.1 and 4.2. We need the following preliminary result.

Lemma 4.1. *Let R be a $d\times d$ real symmetric matrix, and $f(y)=e^{-\pi i Ry\cdot y}$. Then,*

$$\|f\|_{W(\mathcal{F}L^1,L^\infty)} = \bigl|\det(I+iR)\bigr|^{1/2}. \tag{34}$$

Proof. We first compute the short-time Fourier transform of f, with respect to the window $g(y)=e^{-\pi|y|^2}$. We have

$$\begin{aligned} V_g f(x,\xi) &= \int e^{-2\pi i y\xi} e^{-i\pi Ry\cdot y} e^{-\pi|y-x|^2}\,dy \\ &= e^{-\pi|x|^2}\int e^{-2\pi i y\cdot(\xi+ix)-\pi(I+iR)y\cdot y}\,dy \\ &= e^{-\pi|x|^2}\bigl(\det(I+iR)\bigr)^{-1/2} e^{-\pi(I+iR)^{-1}(\xi+ix)\cdot(\xi+ix)}, \end{aligned}$$

where we used [19, Theorem 1, p. 256]. Hence

$$\bigl|V_g f(x,\xi)\bigr| = \bigl|\det(I+iR)\bigr|^{-1/2} e^{-\pi(I+R^2)^{-1}(\xi+Rx)\cdot(\xi+Rx)},$$

and, performing the change of variables $(I+R^2)^{-1/2}(\xi+Rx)=y$, with $d\xi = |\det(I+R^2)|^{1/2}\,dy$, we obtain

$$\int_{\mathbb{R}^d} V_g f(x,\xi)\,d\xi = \bigl|\det(I+iR)\bigr|^{-1/2}\bigl(\det\bigl(I+R^2\bigr)\bigr)^{1/2}\int_{\mathbb{R}^d} e^{-\pi|y|^2}\,dy = \bigl|\det(I+iR)\bigr|^{1/2}. \tag{35}$$

The last equality follows from $(I+iR)=(I+R^2)(I-iR)^{-1}$, so that $\det(I+iR)^{-1}=\det(I+R^2)^{-1}\det(I-iR)$. Now, relation (34) is proved by taking the supremum with the respect to $x\in\mathbb{R}^d$ in (35). $\square$

Proof of Theorem 4.1. (i) We use the expression of $\mu(\mathcal{A})f$ in formula (15). The estimates below are obtained by using (in order): Proposition 2.5 with Lemma 4.1, the estimate (21), Proposition 2.2, and, finally, Proposition 2.5 combined with Lemma 4.1 again:

$$\begin{aligned} \bigl\|\mu(\mathcal{A})f\bigr\|_{W(\mathcal{F}L^p,L^q)} &= |\det B|^{-1/2}\bigl\|e^{-\pi i x\cdot DB^{-1}x}\mathcal{F}^{-1}\bigl(e^{-\pi i y\cdot B^{-1}Ay}f\bigr)\bigl(B^{-1}x\bigr)\bigr\|_{W(\mathcal{F}L^p,L^q)} \\ &\leqslant |\det B|^{-1/2}\bigl\|e^{-\pi i x\cdot DB^{-1}x}\bigr\|_{W(\mathcal{F}L^1,L^\infty)} \\ &\quad\cdot \bigl\|\bigl(\mathcal{F}^{-1}\bigl(e^{-\pi i y\cdot B^{-1}Ay}f\bigr)\bigr)_{B^{-1}}\bigr\|_{W(\mathcal{F}L^p,L^q)} \\ &\lesssim |\det B|^{1/q-1/p-1/2}\bigl(\det(B^*B+I)\bigr)^{1/2}\bigl|\det\bigl(I+iDB^{-1}\bigr)\bigr|^{1/2} \end{aligned}$$

$$
\begin{aligned}
&\cdot \big\| \mathcal{F}^{-1}\big(e^{-\pi i y\cdot B^{-1}Ay} f\big)\big\|_{W(\mathcal{F}L^p,L^q)}\\
&\lesssim |\det B|^{1/q-1/p-1/2}\big(\det(B^*B+I)\big)^{1/2}\big|\det\big(I+iDB^{-1}\big)\big|^{1/2}\\
&\quad \cdot \big\|e^{-\pi i y\cdot B^{-1}Ay} f\big\|_{W(\mathcal{F}L^q,L^p)}\\
&\lesssim \alpha(\mathcal{A},p,q)\|f\|_{W(\mathcal{F}L^q,L^p)}
\end{aligned}
$$

with $\alpha(\mathcal{A},p,q)$ given by (27).

(ii) In this case, we use formula (17). Then, proceeding likewise the case (i), we majorize as follows:

$$
\begin{aligned}
\big\|\mu(\mathcal{A})f\big\|_{W(\mathcal{F}L^1,L^\infty)} &= |\det B|^{-1/2}\big\|e^{-\pi i x\cdot CA^{-1}x}\big(e^{-\pi i y\cdot B^{-1}Ay} * f\big)\big(A^{-1}x\big)\big\|_{W(\mathcal{F}L^1,L^\infty)}\\
&\leqslant |\det B|^{-1/2}\big\|e^{-\pi i x\cdot CA^{-1}x}\big\|_{W(\mathcal{F}L^1,L^\infty)}\\
&\quad \cdot \big\|\big(e^{-\pi i y\cdot B^{-1}Ay} * f\big)_{A^{-1}}\big\|_{W(\mathcal{F}L^1,L^\infty)}\\
&\lesssim |\det B|^{-1/2}|\det A|^{-1}\big(\det(A^*A+I)\big)^{1/2}\big|\det\big(I+iCA^{-1}\big)\big|^{1/2}\\
&\quad \cdot \big\|e^{-\pi i y\cdot B^{-1}Ay} * f\big\|_{W(\mathcal{F}L^1,L^\infty)}\\
&\lesssim \beta(\mathcal{A})\|f\|_{W(\mathcal{F}L^\infty,L^1)},
\end{aligned}
$$

where the last row is due to (8), with $\beta(\mathcal{A})$ defined in (29). $\square$

Proof of Theorem 4.2. The proof uses the same arguments as in Theorem 4.1. Here, the estimate (21) is replaced by (25). Besides, the index relation (23) is applied in the final step. In details,

$$
\begin{aligned}
\big\|\mu(\mathcal{A})f\big\|_{W(\mathcal{F}L^p,L^q)} &\leqslant |\det B|^{-1/2}\big\|e^{-\pi i x\cdot DB^{-1}x}\big\|_{W(\mathcal{F}L^1,L^\infty)}\\
&\quad \cdot \big\|\big(\mathcal{F}^{-1}\big(e^{-\pi i y\cdot B^{-1}Ay} f\big)\big)_{B^{-1}}\big\|_{W(\mathcal{F}L^p,L^q)}\\
&\lesssim \prod_{j=1}^d |\lambda_j|^{-1/2}\big|\det\big(I+iDB^{-1}\big)\big(I+iB^{-1}A\big)\big|^{1/2}\\
&\quad \cdot \prod_{j=1}^d \big(\max\big\{1,|\lambda_j|^{-1}\big\}\big)^{\mu_1(p',q')}\big(\min\big\{1,|\lambda_j|^{-1}\big\}\big)^{\mu_2(p',q')}\|f\|_{W(\mathcal{F}L^q,L^p)}\\
&= \big|\det(B+iD)(B+iA)\big|^{1/2}\\
&\quad \cdot \prod_{j=1}^d \big(\max\big\{1,|\lambda_j|\big\}\big)^{\mu_1(p,q)-1/2}\big(\min\big\{1,|\lambda_j|\big\}\big)^{\mu_2(p,q)-1/2}\|f\|_{W(\mathcal{F}L^q,L^p)},
\end{aligned}
$$

that is case (i). Case (ii) indeed is not an improvement of (28) but is just (28) rephrased in terms of the eigenvalues of A. $\square$

Remark 4.3. The above theorems require the condition $\det B \neq 0$. However, in some special cases with $\det B = 0$, the previous results can still be used to obtain estimates between Wiener amalgam spaces. For example, if $\mathcal{A} = \left(\begin{smallmatrix} I & 0 \\ C & I \end{smallmatrix}\right)$, with $C = C^*$, then $\mu(\mathcal{A}) f(x) = \pm e^{-\pi i C x \cdot x} f(x)$ (see (14)), so that, for every $1 \leqslant p, q \leqslant \infty$, Proposition 2.5 and the estimate (34) give

$$\left\|\mu(\mathcal{A}) f\right\|_{W(\mathcal{F}L^p, L^q)} \lesssim \prod_{j=1}^{d} \left(1 + \lambda_j^2\right)^{1/4} \|f\|_{W(\mathcal{F}L^p, L^q)},$$

where the λ_j's are the eigenvalues of C (incidentally, this estimate was already shown in [1,3,9]).

5. Applications to the Schrödinger equation

In this section we apply the previous results to the analysis of the Cauchy problem of Schrödinger equations with quadratic Hamiltonians, i.e.

$$\begin{cases} i \dfrac{\partial u}{\partial t} + H_{\mathcal{A}} u = 0, \\ u(0, x) = u_0(x), \end{cases} \tag{36}$$

where $H_{\mathcal{A}}$ is the Weyl quantization of a quadratic form on the phase space $\mathbb{R}^{2d}$, defined from a matrix $\mathcal{A}$ in the Lie algebra $\mathfrak{sp}(d, \mathbb{R})$ of the symplectic group as follows (see [12,19]).

Any given matrix $\mathcal{A} \in \mathfrak{sp}(d, \mathbb{R})$ defines a quadratic form $\mathcal{P}_{\mathcal{A}}(x, \xi)$ in $\mathbb{R}^{2d}$ via the formula

$$\mathcal{P}_{\mathcal{A}}(x, \xi) = -\frac{1}{2} {}^t(x, \xi) \mathcal{A} \mathcal{J} (x, \xi),$$

where, as usual, $\mathcal{J} = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$ (notice that $\mathcal{A}\mathcal{J}$ is symmetric). Explicitly, if $\mathcal{A} = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathfrak{sp}(d, \mathbb{R})$ then

$$P_{\mathcal{A}}(x, \xi) = \frac{1}{2} \xi \cdot B \xi - \xi \cdot A x - \frac{1}{2} x \cdot C x. \tag{37}$$

From the Weyl quantization, the quadratic polynomial $P_{\mathcal{A}}$ in (37) corresponds to the Weyl operator $\mathcal{P}^w_{\mathcal{A}}(D, X)$ defined by

$$2\pi \mathcal{P}^w_{\mathcal{A}}(D, X) = -\frac{1}{4\pi} \sum_{j,k=1}^{d} B_{j,k} \frac{\partial^2}{\partial x_j \partial x_k} + i \sum_{j,k=1}^{d} A_{j,k} x_k \frac{\partial}{\partial x_j} + \frac{i}{2} \operatorname{Tr}(A) - \pi \sum_{j,k=1}^{d} C_{j,k} x_j x_k.$$

The operator $H_{\mathcal{A}} := 2\pi \mathcal{P}^w_{\mathcal{A}}(D, X)$ is called the Hamiltonian operator.

The evolution operator for (36) is related to the metaplectic representation via the following key formula

$$e^{itH_{\mathcal{A}}} = \mu\left(e^{t\mathcal{A}}\right).$$

Consequently, Theorems 4.1 and 4.2 can be used in the study of fixed-time estimates for the solution $u(t) = e^{itH_{\mathcal{A}}} u_0$ to (36).

As an example, consider the matrix $\mathcal{A}=\left(\begin{smallmatrix}0 & B\\ 0 & 0\end{smallmatrix}\right)\in\mathfrak{sp}(d,\mathbb{R})$, with $B=B^*$. Then the Hamiltonian operator is $H_{\mathcal{A}}=-\frac{1}{4\pi}B\nabla\cdot\nabla$ and $e^{it\mathcal{A}}=\left(\begin{smallmatrix}I & tB\\ 0 & I\end{smallmatrix}\right)\in Sp(d,\mathbb{R})$.

Fix $t\neq 0$. If $\det B\neq 0$, and B has eigenvalues $\lambda_1,\dots,\lambda_d$, then the expression of $\beta'(e^{it\mathcal{A}})$ in (33) is given by

$$\beta'\big(e^{it\mathcal{A}}\big)=2^{d/4}|\det tB|^{-1}\big|\det(tB+iI)\big|^{1/2}=2^{d/4}\prod_{j=1}^{d}\left(\frac{1+t^2\lambda_j^2}{t^4\lambda_j^4}\right)^{1/4}.$$

Consequently, the fixed-time estimate (32) is

$$\big\|e^{itH_{\mathcal{A}}}f\big\|_{W(\mathcal{F}L^1,L^\infty)}\lesssim\prod_{j=1}^{d}\left(\frac{1+t^2\lambda_j^2}{t^4\lambda_j^4}\right)^{1/4}\|f\|_{W(\mathcal{F}L^\infty,L^1)},$$

which generalizes the dispersive estimate in [9], corresponding to $B=I$.

In the next two sections we present new fixed-time estimates, and also Strichartz estimates, in the cases of the Hamiltonian $H_{\mathcal{A}}=-\frac{1}{4\pi}\Delta+\pi|x|^2$ and $H_{\mathcal{A}}=-\frac{1}{4\pi}\Delta-\pi|x|^2$.

5.1. Schrödinger equation with Hamiltonian $H_{\mathcal{A}}=-\frac{1}{4\pi}\Delta+\pi|x|^2$

Here we consider the Cauchy problem (36) with the Hamiltonian $H_{\mathcal{A}}$ corresponding to the matrix $\mathcal{A}=\left(\begin{smallmatrix}0 & I\\ -I & 0\end{smallmatrix}\right)\in\mathfrak{sp}(d,\mathbb{R})$, namely $H_{\mathcal{A}}=-\frac{1}{4\pi}\Delta+\pi|x|^2$. As a consequence of the estimates proved in the previous section we obtain the following fixed-time estimates.

Proposition 5.1. *For $2\leqslant r\leqslant\infty$, we have the fixed-time estimates*

$$\big\|e^{itH_{\mathcal{A}}}u_0\big\|_{W(\mathcal{F}L^{r'},L^r)}\lesssim|\sin t|^{-2d(\frac{1}{2}-\frac{1}{r})}\|u_0\|_{W(\mathcal{F}L^r,L^{r'})}. \tag{38}$$

Proof. The symplectic matrix $e^{t\mathcal{A}}$ reveals to be $e^{t\mathcal{A}}=\left(\begin{smallmatrix}(\cos t)I & (\sin t)I\\ (-\sin t)I & (\cos t)I\end{smallmatrix}\right)$.

First, using the estimate (32) we get

$$\big\|e^{itH_{\mathcal{A}}}u_0\big\|_{W(\mathcal{F}L^1,L^\infty)}\lesssim|\sin t|^{-d}|\cos t|^{-\frac{3}{2}d}\|u_0\|_{W(\mathcal{F}L^\infty,L^1)}. \tag{39}$$

On the other hand, the estimate (30), for $p=1$, $q=\infty$, reads

$$\big\|e^{itH_{\mathcal{A}}}u_0\big\|_{W(\mathcal{F}L^1,L^\infty)}\lesssim|\sin t|^{-5d/2}\|u_0\|_{W(\mathcal{F}L^\infty,L^1)}. \tag{40}$$

Since $\min\{|\sin t|^{-d}|\cos t|^{-3d/2},|\sin t|^{-5d/2}\}\asymp|\sin t|^{-d}$, we obtain (38) for $r=\infty$, which is the dispersive estimate.

The estimates (38) for $2\leqslant r\leqslant\infty$ follow by complex interpolation from the dispersive estimate and the L^2–L^2 estimate

$$\big\|e^{itH_{\mathcal{A}}}f\big\|_{L^2}=\|f\|_{L^2}.\qquad\Box \tag{41}$$

The Strichartz estimates for the solutions to (36) are detailed as follows.

Theorem 5.2. *Let $T > 0$ and $4 < q, \tilde{q} \leqslant \infty$, $2 \leqslant r, \tilde{r} \leqslant \infty$, such that*

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}, \tag{42}$$

and similarly for $\tilde{q}, \tilde{r}$. Then we have the homogeneous Strichartz estimates

$$\left\| e^{itH_{\mathcal{A}}} u_0 \right\|_{L^{q/2}([0,T]) W(\mathcal{F}L^{r'}, L^r)_x} \lesssim \|u_0\|_{L^2_x}, \tag{43}$$

the dual homogeneous Strichartz estimates

$$\left\| \int_0^T e^{-isH_{\mathcal{A}}} F(s)\, ds \right\|_{L^2} \lesssim \|F\|_{L^{(\tilde{q}/2)'}([0,T]) W(\mathcal{F}L^{\tilde{r}}, L^{\tilde{r}'})_x}, \tag{44}$$

and the retarded Strichartz estimates

$$\left\| \int_{0 \leqslant s < t} e^{i(t-s)H_{\mathcal{A}}} F(s)\, ds \right\|_{L^{q/2}([0,T]) W(\mathcal{F}L^{r'}, L^r)_x} \lesssim \|F\|_{L^{(\tilde{q}/2)'}([0,T]) W(\mathcal{F}L^{\tilde{r}}, L^{\tilde{r}'})_x}. \tag{45}$$

Consider then the endpoint $P := (4, 2d/(d-1))$. For $(q, r) = P$, $d > 1$, we have

$$\left\| e^{itH_{\mathcal{A}}} u_0 \right\|_{L^2([0,T]) W(\mathcal{F}L^{r',2}, L^r)_x} \lesssim \|u_0\|_{L^2_x}, \tag{46}$$

$$\left\| \int_0^T e^{-isH_{\mathcal{A}}} F(s)\, ds \right\|_{L^2} \lesssim \|F\|_{L^2([0,T]) W(\mathcal{F}L^{r,2}, L^{r'})_x}. \tag{47}$$

The retarded estimates (45) *still hold with (q, r) satisfying* (42)*, $q > 4$, $r \geqslant 2$, $(\tilde{q}, \tilde{r}) = P$, if one replaces $\mathcal{F}L^{\tilde{r}'}$ by $\mathcal{F}L^{\tilde{r}',2}$. Similarly it holds for $(q, r) = P$ and $(\tilde{q}, \tilde{r}) \neq P$ as above if one replaces $\mathcal{F}L^{r'}$ by $\mathcal{F}L^{r',2}$. It holds for both $(p, r) = (\tilde{p}, \tilde{r}) = P$ if one replaces $\mathcal{F}L^{r'}$ by $\mathcal{F}L^{r',2}$ and $\mathcal{F}L^{\tilde{r}'}$ by $\mathcal{F}L^{\tilde{r}',2}$.*

In the previous theorem the bounds may depend on T.

Proof. The arguments are essentially the ones in [9,27]. For the convenience of the reader, we present the guidelines of the proof.

Due to the property group of the evolution operator $e^{itH_{\mathcal{A}}}$, we can limit ourselves to the case $T = 1$. Indeed, observe that, if (43) holds for a given $T > 0$, it holds for any $0 < T' \leqslant T$ as well, so that it suffices to prove (43) for $T = N$ integer. Since

$$\left\| e^{itH_{\mathcal{A}}} u_0 \right\|^{\frac{q}{2}}_{L^{q/2}([0,N]) W(\mathcal{F}L^{r'}, L^r)_x} = \sum_{k=0}^{N-1} \left\| e^{itH_{\mathcal{A}}} e^{ikH_{\mathcal{A}}} u_0 \right\|^{\frac{q}{2}}_{L^{q/2}([0,1]) W(\mathcal{F}L^{r'}, L^r)_x},$$

the $T = N$ case is reduced to the $T = 1$ case by using (43) for $T = 1$ and the conservation law (41). The other estimates can be treated analogously. Whence from now on $T = 1$.

Consider first the non-endpoint case. Set $U(t) = \chi_{[0,1]}(t) e^{itH_{\mathcal{A}}}$. For $2 \leqslant r \leqslant \infty$, using relation (38), we get

$$\big\| U(t)\big(U(s)\big)^* f \big\|_{W(\mathcal{F}L^{r'}, L^r)} \lesssim |t-s|^{-2d(\frac{1}{2}-\frac{1}{r})} \|f\|_{W(\mathcal{F}L^r, L^{r'})}. \tag{48}$$

By the TT^* method[1] (see, e.g., [20, Lemma 2.1] or [32, p. 353]) the estimate (43) is equivalent to

$$\left\| \int U(t)\big(U(s)\big)^* F(s)\, ds \right\|_{L_t^{q/2} W(\mathcal{F}L^{r'}, L^r)_x} \lesssim \|F\|_{L_t^{(q/2)'} W(\mathcal{F}L^{\tilde{r}}, L^{r'})_x}. \tag{49}$$

The estimate above is attained by applying Minkowski's Inequality and the Hardy–Littlewood–Sobolev inequality (5) to the estimate (48). The dual homogeneous estimates (44) follow by duality. Finally, the retarded estimates (45), with $(1/q, 1/r)$, $(1/\tilde{q}, 1/\tilde{r})$ and $(1/\infty, 1/2)$ collinear, follow by complex interpolation from the three cases $(\tilde{q}, \tilde{r}) = (q, r)$, $(q, r) = (\infty, 2)$ and $(\tilde{q}, \tilde{r}) = (\infty, 2)$, which in turns are a consequence of (49) (with $\chi_{s<t} F$ in place of F), (44) (with $\chi_{s<t} F$ in place of F) and the duality argument, respectively.

We are left to the endpoint case: $(q, r) = (2, 2d/(d-1))$. The estimate (46) is equivalent to the bilinear estimate

$$\left| \iint \big\langle \big(U(s)\big)^* F(s), \big(U(t)\big)^* G(t) \big\rangle\, ds\, dt \right| \lesssim \|F\|_{L_t^2 W(\mathcal{F}L^{r,2}, L^{r'})_x} \|G\|_{L_t^2 W(\mathcal{F}L^{r,2}, L^{r'})_x}.$$

By symmetry, it is enough to prove

$$\big|T(F, G)\big| \lesssim \|F\|_{L_t^2 W(\mathcal{F}L^{r,2}, L^{r'})_x} \|G\|_{L_t^2 W(\mathcal{F}L^{r,2}, L^{r'})_x}, \tag{50}$$

where

$$T(F, G) = \iint_{s<t} \big\langle \big(U(s)\big)^* F(s), \big(U(t)\big)^* G(t) \big\rangle\, ds\, dt.$$

To this aim, $T(F, G)$ is decomposed dyadically as $T = \sum_{j \in \mathbb{Z}} T_j$, with

$$T_j(F, G) = \iint_{t-2^{j+1} < s \leqslant t-2^j} \big(U(s)\big)^* F(s), \big(U(t)\big)^* G(t) \rangle\, ds\, dt. \tag{51}$$

By resorting on (44) one can prove exactly as in [27, Lemma 4.1] the following estimates:

$$\big|T_j(F, G)\big| \lesssim 2^{-j\beta(a,b)} \|F\|_{L_t^2 W(\mathcal{F}L^a, L^{a'})} \|G\|_{L_t^2 W(\mathcal{F}L^b, L^{b'})}, \tag{52}$$

for $(1/a, 1/b)$ in a neighborhood of $(1/r, 1/r)$, with $\beta(a, b) = d - 1 - \frac{d}{a} - \frac{d}{b}$.

[1] This duality argument is generally established for L^p spaces. Its use for Wiener amalgam spaces is similarly justified thanks to the duality defined by the Hölder-type inequality [9]

$$\big|\langle F, G\rangle_{L_t^2 L_x^2}\big| \leqslant \|F\|_{W(L^s, L^q)_t W(\mathcal{F}L^{r'}, L^r)_x} \|G\|_{W(L^{s'}, L^{q'})_t W(\mathcal{F}L^r, L^{r'})_x}.$$

The estimate (50) is achieved by means of a real interpolation result, detailed in [27, Lemma 6.1], and applied to the vector-valued bilinear operator $T = (T_j)_{j\in\mathbb{Z}}$. Here, however, we must observe that, if $A_k = L^2_t W(\mathcal{F}L^{a_k}, L^{a_k'})_x$, $k = 0, 1$, and θ_0 fulfills $1/r = (1-\theta_0)/a_0 + \theta_0/a_1$, then

$$L^2_t W\big(\mathcal{F}L^{r,2}, L^{r'}\big)_x \subset (A_0, A_1)_{\theta_0,2}.$$

The above inclusion follows by [41, Theorem 1.18.4, p. 129] (with $p = p_0 = p_1 = 2$) and Proposition 2.3. This gives (46) and (47).

Consider now the endpoint retarded estimates. The case $(\tilde{q}, \tilde{r}) = (q, r) = P$ is exactly (50). The case $(\tilde{q}, \tilde{r}) = P$, $(q, r) \neq P$, can be obtained by a repeated use of Hölder's inequality to interpolate from the case $(\tilde{q}, \tilde{r}) = (q, r) = P$ and the case $(\tilde{q}, \tilde{r}) = P$, $(q, r) = (\infty, 2)$ (that is clear from (47)). Finally, the retarded estimate in the case $(q, r) = P$, $(\tilde{q}, \tilde{r}) \neq P$, follows by applying the arguments above to the adjoint operator $G \mapsto \int_{t>s} (U(t))^* U(s) G(t)\, dt$, which gives the dual estimate. □

5.2. *Schrödinger equation with Hamiltonian $H_{\mathcal{A}} = -\frac{1}{4\pi}\Delta - \pi|x|^2$*

The Hamiltonian operator $H_{\mathcal{A}} = -\frac{1}{4\pi}\Delta - \pi|x|^2$ corresponds to the matrix $\mathcal{A} = \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right) \in \mathfrak{sp}(d, \mathbb{R})$. In this case,

$$e^{t\mathcal{A}} = \begin{pmatrix} (\cosh t)I & (\sinh t)I \\ (\sinh t)I & (\cosh t)I \end{pmatrix} \in Sp(d, \mathbb{R}).$$

Fixed-time estimates for $H_{\mathcal{A}}$ are as follows.

Proposition 5.3. *For $2 \leqslant r \leqslant \infty$,*

$$\big\|e^{itH_{\mathcal{A}}} u_0\big\|_{W(\mathcal{F}L^{r'}, L^r)} \lesssim \left(\frac{1 + |\sinh t|}{\sinh^2 t}\right)^{d(\frac{1}{2} - \frac{1}{r})} \|u_0\|_{W(\mathcal{F}L^r, L^{r'})}. \tag{53}$$

Proof. The estimate (32) yields the dispersive estimate

$$\big\|e^{itH_{\mathcal{A}}} u_0\big\|_{W(\mathcal{F}L^1, L^\infty)} \lesssim \left(\frac{1 + |\sinh t|}{\sinh^2 t}\right)^{\frac{d}{2}} \|u_0\|_{W(\mathcal{F}L^\infty, L^1)}. \tag{54}$$

(Observe that (30), with $p = 1$, $q = \infty$, gives a bound worse than (54).)

The estimates (53) follow by complex interpolation between the dispersive estimate (54) and the conservation law (41). □

We can now establish the corresponding Strichartz estimates.

Theorem 5.4. *Let $4 < q, \tilde{q} \leqslant \infty$, $2 \leqslant r, \tilde{r} \leqslant \infty$, such that*

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}, \tag{55}$$

and similarly for $\tilde{q}, \tilde{r}$. Then we have the homogeneous Strichartz estimates

$$\left\|e^{itH_{\mathcal{A}}}u_0\right\|_{W(L^{q/2},L^2)_t W(\mathcal{F}L^{r'},L^r)_x} \lesssim \|u_0\|_{L^2_x}, \tag{56}$$

the dual homogeneous Strichartz estimates

$$\left\|\int e^{-isH_{\mathcal{A}}}F(s)\,ds\right\|_{L^2} \lesssim \|F\|_{W(L^{(\tilde{q}/2)'},L^2)_t W(\mathcal{F}L^{\tilde{r}},L^{\tilde{r}'})_x}, \tag{57}$$

and the retarded Strichartz estimates

$$\left\|\int_{s<t} e^{i(t-s)H_{\mathcal{A}}}F(s)\,ds\right\|_{W(L^{q/2},L^2)_t W(\mathcal{F}L^{r'},L^r)_x} \lesssim \|F\|_{W(L^{(\tilde{q}/2)'},L^2)_t W(\mathcal{F}L^{\tilde{r}},L^{\tilde{r}'})_x}. \tag{58}$$

Consider then the endpoint $P := (4, 2d/(d-1))$. For $(q,r) = P$, $d > 1$, we have

$$\left\|e^{itH_{\mathcal{A}}}u_0\right\|_{L^2_t W(\mathcal{F}L^{r',2},L^r)_x} \lesssim \|u_0\|_{L^2_x}, \tag{59}$$

$$\left\|\int e^{-isH_{\mathcal{A}}}F(s)\,ds\right\|_{L^2} \lesssim \|F\|_{L^2_t W(\mathcal{F}L^{r,2},L^{r'})_x}. \tag{60}$$

The retarded estimates (58) *still hold with (q,r) satisfying* (55)*, $q > 4$, $r \geqslant 2$, $(\tilde{q},\tilde{r}) = P$, if one replaces $\mathcal{F}L^{\tilde{r}'}$ by $\mathcal{F}L^{\tilde{r}',2}$. Similarly it holds for $(q,r) = P$ and $(\tilde{q},\tilde{r}) \neq P$ as above if one replaces $\mathcal{F}L^{r'}$ by $\mathcal{F}L^{r',2}$. It holds for both $(p,r) = (\tilde{p},\tilde{r}) = P$ if one replaces $\mathcal{F}L^{r'}$ by $\mathcal{F}L^{r',2}$ and $\mathcal{F}L^{\tilde{r}'}$ by $\mathcal{F}L^{\tilde{r}',2}$.*

Proof. Let us first prove (56). By the TT^* method it suffices to prove

$$\left\|\int e^{i(t-s)H_{\mathcal{A}}}F(s)\,ds\right\|_{W(L^{q/2},L^2)_t W(\mathcal{F}L^{r'},L^r)_x} \lesssim \|F\|_{W(L^{(q/2)'},L^2)_t W(\mathcal{F}L^{r},L^{r'})_x}. \tag{61}$$

For $0 < \alpha < 1/2$, let $\phi_\alpha(t) = |\sinh t|^{-\alpha} + |\sinh t|^{-2\alpha}$, $t \in \mathbb{R}$, $t \neq 0$. A direct computation shows that $\phi_\alpha \in W(L^{1/(2\alpha),\infty}, L^1)$. Since $L^1 * L^2 \hookrightarrow L^2$ (Young's Inequality) and $L^{(\frac{1}{\alpha})'} * L^{\frac{1}{2\alpha},\infty} \hookrightarrow L^{\frac{1}{\alpha}}$ (Proposition 2.1), Lemma 2.1(i) gives the convolution relation

$$\|F * \phi_\alpha\|_{W(L^{1/\alpha},L^{2/\alpha})} \lesssim \|F\|_{W(L^{(1/\alpha)'},L^{(2/\alpha)'})}. \tag{62}$$

Fix now $\alpha = d(1/2 - 1/r) = 2/q$; then, by (53), (62) and Minkowski's Inequality,

$$\begin{aligned}
&\left\|\int e^{i(t-s)H_{\mathcal{A}}}F(s)\,ds\right\|_{W(L^{q/2},L^2)_t W(\mathcal{F}L^{r'},L^r)_x} \\
&\quad\leqslant \left\|\int \left\|e^{i(t-s)H_{\mathcal{A}}}F(s)\right\|_{W(\mathcal{F}L^{r'},L^r)_x}\,ds\right\|_{W(L^{q/2},L^2)_t} \\
&\quad\lesssim \left\|\,\|F(t)\|_{W(\mathcal{F}L^{r'},L^r)_x} * \phi_\alpha(t)\right\|_{W(L^{q/2},L^2)_t} \\
&\quad\lesssim \|F\|_{W(L^{(q/2)'},L^2)_t W(\mathcal{F}L^{r},L^{r'})_x}.
\end{aligned}$$

This proves (61) and whence (56). The estimate (57) follows from (56) by duality. The proof of (58) is analogous to (45) in Theorem 5.2.

For the endpoint case one can repeat essentially verbatim the arguments in the proof of Theorem 5.2, upon setting $U(t) = e^{itH_{\mathcal{A}}}$. To avoid repetitions, we omit the details (see also the proof of [9, Theorem 1.2]). $\square$

Remark 5.5. As an application of the previous Strichartz estimates for the operators $H = -\frac{1}{4\pi}\Delta \pm \pi|x|^2$ we can study the well-posedness in L^2 for the following Cauchy problem:

$$\begin{cases} i\partial_t u + Hu = V(t,x)u, & t \in [0,T] = I_T,\ x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), \end{cases} \tag{63}$$

for the class of potentials

$$V \in L^\alpha\big(I_T; W\big(\mathcal{F}L^{p'}, L^p\big)_x\big), \quad \frac{1}{\alpha} + \frac{d}{p} \leqslant 1,\ 1 \leqslant \alpha < \infty,\ d < p \leqslant \infty. \tag{64}$$

Namely, we have the following result.

Theorem 5.6. *Let V satisfy* (64). *Then, for all (q,r) such that $2/q + d/r = d/2$, $q > 4$, $r \geqslant 2$, the Cauchy problem* (63) *has a unique solution*

(i) $u \in \mathcal{C}(I_T; L^2(\mathbb{R})) \cap L^{q/2}(I_T; W(\mathcal{F}L^{r'}, L^r))$, *if* $d = 1$;
(ii) $u \in \mathcal{C}(I_T; L^2(\mathbb{R}^d)) \cap L^{q/2}(I_T; W(\mathcal{F}L^{r'}, L^r)) \cap L^2(I_T; W(\mathcal{F}L^{2d/(d+1),2}, L^{2d/(d-1)}))$, *if* $d > 1$.

The proof is omitted, since it goes through exactly in the same manner as that detailed in [10, Theorem 6.1], for the case $H = \Delta$. Indeed, it relies entirely on the Strichartz estimates proved above.

5.3. Comparison with the classical estimates in Lebesgue spaces

Here we compare the above estimates with the classical ones between Lebesgue spaces. For the convenience of the reader we recall the following very general result by Keel and Tao [27, Theorem 1.2].

Given $\sigma > 0$, we say that an exponent pair (q,r) is *sharp σ-admissible* if $1/q + \sigma/r = \sigma/2$, $q \geqslant 2, r \geqslant 2$, $(q,r,\sigma) \neq (2,\infty,1)$.

Theorem 5.7. *Let $(X, \mathcal{S}, \mu)$ be a σ-finite measured space, and $U : \mathbb{R} \to B(L^2(X,\mathcal{S},\mu))$ be a weakly measurable map satisfying, for some $\sigma > 0$,*

$$\big\|U(t)f\big\|_{L^2} \lesssim \|f\|_{L^2}, \quad t \in \mathbb{R},$$

and

$$\big\|U(s)U(t)^* f\big\|_{L^\infty} \lesssim |t-s|^{-\sigma}\|f\|_{L^1}, \quad t,s \in \mathbb{R}.$$

Then for every sharp σ-admissible pairs (q,r), $(\tilde q,\tilde r)$, one has

$$\|U(t)f\|_{L^q_t L^r_x} \lesssim \|f\|_{L^2},$$

$$\left\|\int U(s)^* F(s)\,ds\right\|_{L^2} \lesssim \|F\|_{L^{q'}_t L^{r'}_x},$$

$$\left\|\int_{s<t} U(t)U(s)^* F(s)\,ds\right\|_{L^q_t L^r_x} \lesssim \|F\|_{L^{\tilde q'} L^{\tilde r'}}.$$

First we fix the attention to the case of the Hamiltonian $H_{\mathcal A} = -\frac{1}{4\pi}\Delta + \pi|x|^2$. One has the following explicit formula for $e^{itH_{\mathcal A}}u_0 = \mu(e^{it\mathcal A})u_0$ in (15):

$$e^{itH_{\mathcal A}}u_0 = i^{d/2}(\sin t)^{-d/2}\int e^{-\pi i(\cot t)(|x|^2+|y|^2)+2\pi i(\operatorname{cosec} t)y\cdot x}u_0(y)\,dy.$$

Hereby it follows immediately the dispersive estimate

$$\|e^{itH_{\mathcal A}}u_0\|_{L^\infty} \leqslant |\sin t|^{-d/2}\|u\|_{L^1}. \tag{65}$$

Notice that (3) (i.e. (38) with $r=\infty$) represents an improvement of (65) for every fixed $t\neq 0$, since $L^1 \hookrightarrow W(\mathcal F L^\infty, L^1)$ and $W(\mathcal F L^1, L^\infty) \hookrightarrow L^\infty$. However, as might be expected, the bound on the norm in (3) becomes worse than that in (65) as $t\to k\pi$, $k\in\mathbb Z$.

As a consequence of (65), Theorem 5.7 with $U(t) = e^{itH_{\mathcal A}}\chi_{[0,1]}(t)$ and $\sigma = d/2$, and the group property of the operator $e^{itH_{\mathcal A}}$ (as in the proof of Theorem 5.2 above) one deduces, for example, the homogeneous Strichartz estimate

$$\|e^{itH_{\mathcal A}}u_0\|_{L^q([0,T])L^r_x} \lesssim \|u_0\|_{L^2_x}, \tag{66}$$

for every pair (q,r) satisfying $2/q + d/r = d/2$, $q\geqslant 2$, $r\geqslant 2$, $(q,r,d)\neq(2,\infty,2)$. These estimates were also obtained recently in [29] by different methods.

Hence, one sees that (43) predicts, for the solution to (36), a better local spatial regularity than (66), but just after averaging on $[0,T]$ by the $L^{q/2}$ norm, which is smaller than the L^q norm.

We now consider the case of the Hamiltonian $H_{\mathcal A} = -\frac{1}{4\pi}\Delta - \pi|x|^2$.

The dispersive estimate here reads

$$\|e^{itH_{\mathcal A}}u_0\|_{L^\infty(\mathbb R^d)} \leqslant |\sinh t|^{-d/2}\|u_0\|_{L^1(\mathbb R^d)}. \tag{67}$$

This estimate follows immediately from the explicit expression of $e^{itH_{\mathcal A}}u_0 = \mu(e^{it\mathcal A})u_0$ in (15):

$$e^{itH_{\mathcal A}}u_0 = i^{d/2}(\sinh t)^{-d/2}\int e^{-\pi i(\coth t)(|x|^2+|y|^2)+2\pi i(\operatorname{cosech} t)y\cdot x}u_0(y)\,dy.$$

The corresponding Strichartz estimates between the Lebesgue spaces read

$$\|e^{itH_{\mathcal A}}u_0\|_{L^q_t L^r_x} \lesssim \|u_0\|_{L^2_x}, \tag{68}$$

for $q \geqslant 2$, $r \geqslant 2$, with $2/q + d/r = d/2$, $(q, r, d) \neq (2, \infty, 2)$. These estimates are the issues of Theorem 5.7 with $U(t) = e^{itH_{\mathcal{A}}}$, and the dispersive estimate (67) (indeed, $|\sinh t|^{-d/2} \leqslant |t|^{-d/2}$). These estimates are to be compared with (53) (with $r = \infty$) and (56), respectively.

One can do the same remarks as in the previous case. In addition here one should observe that (56) displays a better time decay at infinity than the classical one (L^2 instead of L^r), for a norm, $\|u(t,\cdot)\|_{W(\mathcal{F}L^{r'},L^r)}$, which is even bigger than L^r. Notice however that our range of exponents is restricted to $q \geqslant 4$.

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Symmetry of minimizers for some nonlocal variational problems

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Abstract

We present a new approach to study the symmetry of minimizers for a large class of nonlocal variational problems. This approach which generalizes the Reflection method is based on the existence of some integral identities. We study the identities that lead to symmetry results, the functionals that can be considered and the function spaces that can be used. Then we use our method to prove the symmetry of minimizers for a class of variational problems involving the fractional powers of Laplacian, for the generalized Choquard functional and for the standing waves of the Davey–Stewartson equation.

Keywords: Symmetry of minimizers; Nonlocal functional; Minimization under constraints; Fractional powers of Laplacian; Choquard functional; Davey–Stewartson equation

1. Introduction

Many important partial differential equations arising in physics are Euler–Lagrange equations of variational problems. Among the solutions of these equations those that correspond to a minimum of the associated functional (e.g. the “energy”) subject to some constraint are of particular interest. For example in many situations the set of such solutions is orbitally stable (see [9]).

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In this paper we address the general question of whether, or not, the fact that the underlying problem has some symmetries is reflected on the minimizers. Namely if a problem is invariant under the action of a group of transformations, is it true that the corresponding minimizers are also invariant under the action of this group (or, perhaps, a subgroup of it)? As it is shown in [14], this may not be the case.

A classical approach to radial symmetry of minimizers is Schwarz symmetrization (or spherical decreasing rearrangement, see [16]). For a nonnegative function $u \in H^1(\mathbf{R}^N)$ its symmetrization u^* is a radially-decreasing function from $\mathbf{R}^N$ into $\mathbf{R}$ which has the property that $\operatorname{meas}(\{x \in \mathbf{R}^N \mid u(x) > \lambda\} = \operatorname{meas}(\{x \in \mathbf{R}^N \mid u^*(x) > \lambda\}$ for any $\lambda > 0$. It is well known that u^* satisfies (among others) the following properties:

$$\int_{\mathbf{R}^N} \left|\nabla u^*(x)\right|^2 dx \leqslant \int_{\mathbf{R}^N} \left|\nabla u(x)\right|^2 dx \quad \text{and}$$
$$\int_{\mathbf{R}^N} F\big(u^*(x)\big)\, dx = \int_{\mathbf{R}^N} F\big(u(x)\big)\, dx, \tag{1.1}$$

where F is, say, a smooth function from $\mathbf{R}$ into itself such that $F(u) \in L^1(\mathbf{R}^N)$ (see [16]). As a simple application of symmetrization, consider the problem of minimizing

$$E(u) = \frac{1}{2} \int_{\mathbf{R}^N} \left|\nabla u(x)\right|^2 dx + \int_{\mathbf{R}^N} F\big(u(x)\big)\, dx$$

subject to the constraint

$$\int_{\mathbf{R}^N} G\big(u(x)\big)\, dx = \lambda,$$

where $F, G \in C^1(\mathbf{R}, \mathbf{R})$ have the property that $F(u), G(u) \in L^1(\mathbf{R}^N)$ whenever $u \in H^1(\mathbf{R}^N)$. If $u \in H^1(\mathbf{R}^N)$ is a nonnegative minimizer, then it follows from (1.1) that u^* also satisfies the constraint and $E(u^*) \leqslant E(u)$; therefore, u^* is also a minimizer. To show that $u \equiv u^*$ except for translation is a more delicate question and this follows from a result in [6] and the Unique Continuation Principle.

In the case of vector-valued minimizers $u : \mathbf{R}^N \to \mathbf{R}^k$, symmetrization can also be used provided that each component of the minimizer is nonnegative, the function $F : \mathbf{R}^k \to \mathbf{R}$ satisfies a cooperative condition $F_{x_i x_j} \leqslant 0$ for $i \neq j$ and the constraint is of the form $\int_{\mathbf{R}^N} G_1(u_1) + G_2(u_2) + \cdots + G_k(u_k)\, dx = \mathit{constant}$. Notice that the function defining the constraint must have a special form because we want the value of the constraint to be preserved by symmetrization.

Another tool to prove radial symmetry of minimizers is the result by Gidas, Ni and Nirenberg [11] about the radial symmetry of positive solutions of the semilinear elliptic equation

$$-\Delta u + f(u) = 0.$$

In the case of systems, an extension of that result has been proved in [7,25] assuming a cooperative condition for the nonlinearity. In [11] as well as in its generalizations the nonlinearities are also allowed to depend on the space variable in a radial and monotonic way.

As we can see, in the vector case, besides the need to know in advance that the components of the minimizer are positive, both methods described above require the nonlinearity to satisfy a cooperative condition and the function defining the constraint to have a special form. To avoid these two restrictions, the Reflection method has been developed in [18,19]. We now briefly describe this method.

Consider the problem of minimizing

$$E(u,v)=\frac{1}{2}\int_{\mathbf{R}^N}\left(\left|\nabla u(x)\right|^2+\left|\nabla v(x)\right|^2\right)dx+\int_{\mathbf{R}^N}F\big(u(x),v(x)\big)\,dx$$

subject to

$$\int_{\mathbf{R}^N}G\big(u(x),v(x)\big)\,dx=\lambda\neq 0.$$

To show that any minimizer (u,v) is symmetric with respect to x_1 (except possibly for a translation), we first make a translation in the x_1 variable in such a way that

$$\int_{\{x_1<0\}}G\big(u(x),v(x)\big)\,dx=\int_{\{x_1>0\}}G\big(u(x),v(x)\big)\,dx=\frac{\lambda}{2}. \tag{1.2}$$

Next, setting $x=(x_1,x')$, where $x'\in\mathbf{R}^{N-1}$, we define the functions u_1 and u_2 by

$$u_1(x)=u_1(x_1,x')=\begin{cases}u(x_1,x') & \text{if } x_1<0,\\ u(-x_1,x') & \text{if } x_1\geqslant 0\end{cases}\quad\text{and}$$

$$u_2(x)=\begin{cases}u(-x_1,x') & \text{if } x_1<0,\\ u(x_1,x') & \text{if } x_1\geqslant 0.\end{cases}$$

In a similar way we define v_1 and v_2. According to (1.2), the pairs (u_1,v_1) and (u_2,v_2) also satisfy the constraint (i.e. they are admissible). Moreover, a change of variables shows that

$$E(u_1,v_1)+E(u_2,v_2)=2E(u,v). \tag{1.3}$$

Thus (u_1,v_1) and (u_2,v_2) are also minimizers. This shows that there exist minimizers which are symmetric with respect to x_1. In fact, by using the Euler–Lagrange equations and the Unique Continuation Principle we can show that necessarily $(u_1,v_1)=(u,v)=(u_2,v_2)$. Clearly, this implies that any minimizer (u,v) is symmetric with respect to the first variable. Replacing the x_1-direction by any other direction in $\mathbf{R}^N$ and repeating the same argument, we can show that (u,v) is radially symmetric except for translation (details will be given later). Notice that to use this argument there is no need to know the sign of components of the minimizers.

The main point of this paper is to extend the Reflection method to a class of nonlocal functionals. To be more specific, consider the problem of minimizing

$$E(u)=\int_{\mathbf{R}^N}m(\xi)\left|\widehat{u}(\xi)\right|^2d\xi+\int_{\mathbf{R}^N}F(u)\,dx \tag{1.4}$$

subject to the constraint

$$Q(u) = \int_{\mathbf{R}^N} G(u)\,dx = \lambda \neq 0. \tag{1.5}$$

Defining $W(u) = \int_{\mathbf{R}^N} m(\xi)|\widehat{u}(\xi)|^2\,d\xi$ and u_1, u_2 as above, instead of (1.3) we have

$$E(u_1) + E(u_2) - 2E(u) = W(u_1) + W(u_2) - 2W(u).$$

Therefore, to show that u_1 and u_2 are also minimizers we need to know that

$$W(u_1) + W(u_2) - 2W(u) \leqslant 0. \tag{1.6}$$

The key to the method developed here is to show that inequality (1.6) holds true (see Theorem 2.7). In this article we will use this extended Reflection method to prove the symmetry of all minimizers of the following functionals:

- the Hamiltonian of a coupled system between a multidimensional Korteweg–de Vries equation and a Benjamin–Ono equation. Here minimizers correspond to solitary waves;
- the generalized Choquard functional. In this case the minimizers give rise to standing waves for the generalized Hartree equation;
- the Hamiltonian of the generalized Davey–Stewartson equation. Here again, minimizers correspond to standing waves.

The existence of minimizers for these problems can be proved by using the concentration–compactness method [17] or the alternative method presented in [20] and will not be discussed here.

Notice that the symmetrization approach, in general, does not apply to the problems above. Indeed, in the first two examples, symmetrization cannot be used to prove the existence of a radially symmetric minimizer under the general assumptions on the nonlinearities made in this paper. Furthermore, with the tools available at the present time, it is not clear how to prove the radial symmetry of *all minimizers*, even in the cases where symmetrization can be used to prove the existence of a radially symmetric minimizer. Finally, in the last example, symmetrization cannot be used because one term of the Hamiltonian of the Davey–Stewartson equation is a singular integral operator whose kernel changes sign.

This paper is organized as follows: in the next section we present some integral identities for functionals of the form $W(u) = \int_{\mathbf{R}^N} m(\xi)|\widehat{u}(\xi)|^2\,d\xi$. These identities are first proved for functions $u \in C_c^\infty$ and are crucial for our approach to symmetry. It will also appear clearly what kind of symbols $m(\xi)$ we may consider. In Section 3 we search for appropriate function spaces on which our method can be applied. It will be proved that we may work on $H^s(\mathbf{R}^N)$ or on $\dot{H}^s(\mathbf{R}^N)$ if $-\frac{1}{2} < s < \frac{3}{2}$. We will extend the integral identities obtained in Section 2 to these function spaces. In Section 4 we apply our results to the concrete problems presented above. We end this article with some open problems.

2. Some identities

In what follows, $x = (x_1, x_2, \dots, x_N) = (x_1, x')$ denotes a point of $\mathbf{R}^N$, $x' = (x_2, \dots, x_N) \in \mathbf{R}^{N-1}$, $\xi = (\xi_1, \xi_2, \dots, \xi_N) = (\xi_1, \xi') \in \mathbf{R}^N$ with $\xi' = (\xi_2, \dots, \xi_N) \in \mathbf{R}^{N-1}$. We denote the Fourier transform either by $\widehat{\ }$ or by $\mathcal{F}$.

The aim of this section is to prove an identity for some functionals of the type

$$W(u) = \int_{\mathbf{R}^N} m(\xi)\left|\widehat{u}(\xi)\right|^2 d\xi \tag{2.1}$$

which play a very important role in proving symmetries.

Consider a function $u \in C_c^\infty(\mathbf{R}^N)$. We define the reflected functions u_1 and u_2 as follows:

$$u_1(x) = u_1(x_1, x') = \begin{cases} u(x_1, x') & \text{if } x_1 < 0, \\ u(-x_1, x') & \text{if } x_1 \geqslant 0 \end{cases} \quad \text{and}$$
$$u_2(x) = \begin{cases} u(-x_1, x') & \text{if } x_1 < 0, \\ u(x_1, x') & \text{if } x_1 \geqslant 0. \end{cases} \tag{2.2}$$

We also define

$$g(x) = \frac{1}{2}\big(u(x_1, x') + u(-x_1, x')\big) \quad \text{and} \quad f(x) = \frac{1}{2}\big(u(x_1, x') - u(-x_1, x')\big). \tag{2.3}$$

Clearly, $f, g \in C_c^\infty(\mathbf{R}^N)$, g is even and f is odd with respect to x_1 and $u = f + g$. Let

$$f_*(x) = \begin{cases} f(-x_1, x') = -f(x) & \text{if } x_1 < 0, \\ f(x_1, x') & \text{if } x_1 \geqslant 0. \end{cases} \tag{2.4}$$

Then f_* is even with respect to x_1, $u_1 = g - f_*$ and $u_2 = g + f_*$.

We want to study the quantity $W(u_1) + W(u_2) - 2W(u)$, where W is given by (2.1). Later in Theorem 2.7 we specify the class of multipliers under consideration but, at this early stage, besides integrability conditions, we assume that

$$m(\xi) \text{ is real} \quad \text{and} \quad m(-\xi_1, \xi') = m(\xi_1, \xi'). \tag{2.5}$$

It is easy to see that

$$\widehat{g}(-\xi_1, \xi') = \widehat{g}(\xi_1, \xi') \quad \text{and} \quad \widehat{f}(-\xi_1, \xi') = -\widehat{f}(\xi_1, \xi'). \tag{2.6}$$

Therefore

$$\begin{aligned} &W(u_1) + W(u_2) - 2W(u) \\ &\quad = \int_{\mathbf{R}^N} m(\xi_1, \xi')\big(\left|\widehat{g}(\xi) - \widehat{f_*}(\xi)\right|^2 + \left|\widehat{g}(\xi) + \widehat{f_*}(\xi)\right|^2 - 2\left|\widehat{g}(\xi) + \widehat{f}(\xi)\right|^2\big)\, d\xi \end{aligned}$$

$$= \int_{\mathbf{R}^N} m(\xi_1, \xi')\big(2\big|\widehat{f_*}(\xi)\big|^2 - 2\big|\widehat{f}(\xi)\big|^2 - 4\,\mathrm{Re}\big(\widehat{g}(\xi)\,\overline{\widehat{f}(\xi)}\big)\big)\,d\xi$$

$$= 2\int_{\mathbf{R}^N} m(\xi_1, \xi')\big(\big|\widehat{f_*}(\xi)\big|^2 - \big|\widehat{f}(\xi)\big|^2\big)\,d\xi = 2W(f_*) - 2W(f) \tag{2.7}$$

because $\int_{\mathbf{R}^N} m(\xi_1, \xi')\,\mathrm{Re}(\widehat{g}(\xi)\,\overline{\widehat{f}(\xi)})\,d\xi = 0$ in view of (2.5) and (2.6).

It is obvious that

$$\widehat{f}(\xi_1, \xi') = \int_{\mathbf{R}} \int_{\mathbf{R}^{N-1}} e^{-ix_1\xi_1 - ix'\cdot\xi'} f(x_1, x')\,dx'\,dx_1$$

$$= \int_0^\infty \int_{\mathbf{R}^{N-1}} \big(e^{-ix_1\xi_1} - e^{ix_1\xi_1}\big)e^{-ix'\cdot\xi'} f(x_1, x')\,dx'\,dx_1$$

$$= -2i\int_0^\infty \int_{\mathbf{R}^{N-1}} \sin(x_1\xi_1)e^{-ix'\cdot\xi'} f(x_1, x')\,dx'\,dx_1$$

and similarly

$$\widehat{f_*}(\xi_1, \xi') = 2\int_0^\infty \int_{\mathbf{R}^{N-1}} \cos(x_1\xi_1)e^{-ix'\cdot\xi'} f(x_1, x')\,dx'\,dx_1.$$

We denote by $\mathcal{F}_{N-1}$ the partial Fourier transform in the last $N-1$ variables, that is

$$\mathcal{F}_{N-1} f(x_1, \xi') = \int_{\mathbf{R}^{N-1}} e^{-ix'\cdot\xi'} f(x_1, x')\,dx'.$$

Since $f \in C_c^\infty(\mathbf{R}^N)$ we may use Fubini's theorem to get

$$\big|\widehat{f}(\xi_1, \xi')\big|^2 = \widehat{f}(\xi_1, \xi')\,\overline{\widehat{f}(\xi_1, \xi')}$$

$$= 4\int_0^\infty \int_0^\infty \sin(x_1\xi_1)\sin(y_1\xi_1)(\mathcal{F}_{N-1} f)(x_1, \xi')\overline{(\mathcal{F}_{N-1} f)(y_1, \xi')}\,dx_1\,dy_1.$$

In the same way,

$$\big|\widehat{f_*}(\xi_1, \xi')\big|^2 = 4\int_0^\infty \int_0^\infty \cos(x_1\xi_1)\cos(y_1\xi_1)(\mathcal{F}_{N-1} f)(x_1, \xi')\overline{(\mathcal{F}_{N-1} f)(y_1, \xi')}\,dx_1\,dy_1.$$

Consequently,

$$
\begin{aligned}
&W(f_*) - W(f) \\
&\quad = 4 \int_{\mathbf{R}^N} m(\xi_1, \xi') \int_0^\infty \int_0^\infty \big[\cos(x_1\xi_1)\cos(y_1\xi_1) - \sin(x_1\xi_1)\sin(y_1\xi_1)\big] \\
&\qquad \times (\mathcal{F}_{N-1} f)(x_1, \xi') \overline{(\mathcal{F}_{N-1} f)(y_1, \xi')}\, dx_1\, dy_1\, d\xi \\
&\quad = 4 \int_{\mathbf{R}^N} m(\xi_1, \xi') \int_0^\infty \int_0^\infty \cos\big((x_1 + y_1)\xi_1\big)(\mathcal{F}_{N-1} f)(x_1, \xi') \overline{(\mathcal{F}_{N-1} f)(y_1, \xi')}\, dx_1\, dy_1\, d\xi. \qquad (2.8)
\end{aligned}
$$

For an arbitrary (but fixed) $\xi' \in \mathbf{R}^{N-1}$, we define $\varphi_{\xi'}(t) = (\mathcal{F}_{N-1} f)(t, \xi')$. Since $f \in C_c^\infty(\mathbf{R}^N)$, it is clear that $\varphi_{\xi'} \in C_c^\infty(\mathbf{R})$. If $\operatorname{supp}(f) \subset B_{\mathbf{R}^N}(0, R)$, then $\operatorname{supp}(\varphi_{\xi'}) \subset [-R, R]$. For $z \in \mathbf{C}$, we define

$$
h_{\xi'}(z) = \int_0^\infty \int_0^\infty e^{i(x_1+y_1)z} \varphi_{\xi'}(x_1) \overline{\varphi_{\xi'}(y_1)}\, dx_1\, dy_1. \tag{2.9}
$$

Since $\varphi_{\xi'}$ is bounded and has compact support, $h_{\xi'}$ is well defined and is an holomorphic function on $\mathbf{C}$. For any $z \in \mathbf{R}$ we have

$$
\overline{h_{\xi'}(z)} = \int_0^\infty \int_0^\infty e^{-i(x_1+y_1)z} \overline{\varphi_{\xi'}(x_1)} \varphi_{\xi'}(y_1)\, dx_1\, dy_1 = h_{\xi'}(-z) \quad \text{and}
$$

$$
\operatorname{Re}\big(h_{\xi'}(z)\big) = \frac{1}{2}\big(h_{\xi'}(z) + \overline{h_{\xi'}(z)}\big) = \int_0^\infty \int_0^\infty \cos\big((x_1 + y_1)z\big) \varphi_{\xi'}(x_1) \overline{\varphi_{\xi'}(y_1)}\, dx_1\, dy_1.
$$

From (2.7) and (2.8) we get

$$
W(u_1) + W(u_2) - 2W(u) = 2W(f_*) - 2W(f) = 8 \int_{\mathbf{R}^{N-1}} \int_{-\infty}^\infty m(\xi_1, \xi') h_{\xi'}(\xi_1)\, d\xi_1\, d\xi'. \tag{2.10}
$$

Some properties of the function $h_{\xi'}$ are given in the next lemma. To simplify the notation, we shall write h instead of $h_{\xi'}$.

Lemma 2.1. *For any fixed ξ', the function $h = h_{\xi'}$ given by* (2.9) *has the following properties*:

(i) *h is bounded in the upper half-plane* $\{z \in \mathbf{C} \mid \operatorname{Im}(z) \geqslant 0\}$.
(ii) *There exists a constant $C > 0$* (*depending on f and ξ'*) *such that for any $z \neq 0$ with* $\operatorname{Im}(z) \geqslant 0$ *we have*:

$$
\big|h(z)\big| \leqslant \frac{C}{|z|^4} \quad \textit{and} \quad \big|h'(z)\big| \leqslant \frac{C}{|z|^5}. \tag{2.11}
$$

Proof. (i) If $b \geqslant 0$ and $x \geqslant 0$ then $|e^{iax-bx}| \leqslant 1$ and we have

$$\left|h(a+ib)\right| = \left| \int_0^\infty \int_0^\infty e^{i(x_1+y_1)a-(x_1+y_1)b} \varphi_{\xi'}(x_1)\overline{\varphi_{\xi'}(y_1)}\, dx_1\, dy_1 \right|$$
$$\leqslant \left(\int_0^\infty \left|e^{iat-bt}\right| \cdot \left|\varphi_{\xi'}(t)\right| dt \right)^2 \leqslant \left(\int_0^\infty \left|\varphi_{\xi'}(t)\right| dt \right)^2.$$

(ii) It is clear that

$$h(z) = \int_0^\infty e^{ix_1 z} \varphi_{\xi'}(x_1)\, dx_1 \cdot \int_0^\infty e^{iy_1 z} \overline{\varphi_{\xi'}(y_1)}\, dy_1 = \Psi_1(z)\Psi_2(z), \tag{2.12}$$

where $\Psi_1(z)$ and $\Psi_2(z)$ are defined in an obvious way. Notice that $\varphi_{\xi'}(0) = (\mathcal{F}_{N-1} f)(0,\xi') = 0$ because $f(0,x') = 0$ (recall that f is odd with respect to x_1). Moreover, for any $k \in \mathbf{N}$,

$$\frac{d^k}{dt^k}\varphi_{\xi'}(t) = \int_{\mathbf{R}^{N-1}} e^{-ix'\xi'} \frac{\partial^k f}{\partial x_1^k}(t,x')\, dx' = \left(\mathcal{F}_{N-1} \frac{\partial^k f}{\partial x_1^k} \right)(t,\xi')$$

is a C_c^∞ function of t, uniformly bounded for $(t,\xi') \in \mathbf{R} \times \mathbf{R}^{N-1}$. Integrating by parts, we get:

$$\Psi_1(z) = \int_0^\infty e^{itz}\varphi_{\xi'}(t)\, dt = \frac{1}{iz} e^{itz}\varphi_{\xi'}(t) \bigg|_{t=0}^{\infty} - \frac{1}{iz}\int_0^\infty e^{itz}\varphi'_{\xi'}(t)\, dt$$
$$= -\frac{e^{itz}}{(iz)^2}\varphi'_{\xi'}(t)\bigg|_{t=0}^{\infty} + \frac{1}{(iz)^2}\int_0^\infty e^{itz}\varphi''_{\xi'}(t)\, dt = -\frac{1}{z^2}\left[\varphi'_{\xi'}(0) + \int_0^\infty e^{itz}\varphi''_{\xi'}(t)\, dt \right].$$

It is clear that an analogous estimate is true for $\Psi_2(z)$ and the first inequality in (2.11) holds.

Similarly one can prove that $|\psi'_j(z)| \leqslant \frac{C_j}{|z|^3}$ for $j = 1, 2$ and $\mathrm{Re}(z) \geqslant 0$. Since $h'(z) = \Psi'_1(z)\Psi_2(z) + \Psi_1(z)\Psi'_2(z)$, the second estimate in (2.11) follows. □

Remark 2.2. In general, $\frac{\partial f}{\partial x_1}(0,x')$ does not vanish identically; hence $\mathcal{F}_{N-1} f(0,\xi') \neq 0$ for some ξ', i.e. there exists ξ' such that $\varphi'_{\xi'}(0) \neq 0$. For such ξ', the functions Ψ_1 and Ψ_2 do not decay faster than $\frac{1}{|z|^2}$ and the estimate (2.11) is optimal.

Remark 2.3. Note that for any $t \in \mathbf{R}$ we have $h(it) = |\int_0^\infty e^{-x_1 t}\varphi_{\xi'}(x_1)\, dx_1|^2 \in [0,\infty)$. Suppose that for any fixed $\xi' \in \mathbf{R}^{N-1}$, $m(\xi_1,\xi')$ admits an holomorphic extension $z \mapsto m(z,\xi')$ to the upper half-plane $\{z \in \mathbf{C} \mid \mathrm{Im}(z) > 0\}$, with possibly some singularities on the imaginary axis $\{it \mid t \in [0,\infty)\}$. If $|m(z,\xi')|$ increases more slowly than $|z|^3$ as $|z| \to \infty$, then $\int_{-\infty}^{\infty} m(\xi_1,\xi')h(\xi_1)\, d\xi_1$ should depend only on the values of h on the singular set of $m(\cdot,\xi')$. This simple idea will enable us to prove the identities that will be crucial in symmetry problems.

In order to clarify what kind of symbols may be considered, we start with some auxiliary technical results about holomorphic functions in a half-plane and their boundary values.

Given a function $\alpha \in L^p(\mathbf{R})$, $1 \leqslant p < \infty$, we recall that its Hilbert transform is defined by

$$(H\alpha)(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{\{|y|>\varepsilon\}} \frac{\alpha(x-y)}{y}\,dy \quad \text{or equivalently} \quad \widehat{H\alpha}(\xi) = -i\,\mathrm{sgn}(\xi)\widehat{\alpha}(\xi).$$

It is well known that H is a bounded linear mapping from $L^p(\mathbf{R})$ into $L^p(\mathbf{R})$ (see, e.g., [23, Chapter II], or [24, inequality (2.11), p. 188]).

In the next two lemmas we collect some classical facts that will be very useful in the sequel. Proofs can be found in [24, Chapters I, II, VI] or in [23].

Lemma 2.4. *Consider $\alpha \in L^p(\mathbf{R})$, $1 < p < \infty$, and let $\beta = H\alpha$. For $x > 0$ and $y \in \mathbf{R}$ define*

$$a(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x}{x^2 + (y-t)^2} \alpha(t)\,dt = \int_{-\infty}^{\infty} P(y-t,x)\alpha(t)\,dt \quad \text{and}$$

$$b(x,y) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y-t}{x^2 + (y-t)^2} \alpha(t)\,dt = -\int_{-\infty}^{\infty} Q(y-t,x)\alpha(t)\,dt,$$

where $P(s,k) = \frac{1}{\pi}\frac{k}{s^2+k^2}$ and $Q(s,k) = \frac{1}{\pi}\frac{s}{s^2+k^2}$ are the Poisson kernel, respectively the conjugate Poisson kernel.

Then we have:

(i) $b(x,y) = -\int_{-\infty}^{\infty} P(y-t,x)\beta(t)\,dt$ *for any $x > 0$ and $t \in \mathbf{R}$.*

(ii) $\|a(x,\cdot)\|_{L^p(\mathbf{R})} \leqslant \|\alpha\|_{L^p(\mathbf{R})}$, $\|b(x,\cdot)\|_{L^p(\mathbf{R})} \leqslant \|\beta\|_{L^p(\mathbf{R})}$ *and* $\|a(x,\cdot) - \alpha\|_{L^p(\mathbf{R})} \to 0$, $\|b(x,\cdot) + \beta\|_{L^p(\mathbf{R})} \to 0$ *as $x \to 0$. Moreover, $a(x,y) \to \alpha(y)$ for any y in the Lebesgue set of α* (*hence almost everywhere*) *and $b(x,y) \to -\beta(y)$ for any y in the Lebesgue set of β.*

(iii) *The functions a and b are harmonic in $\{(x,y) \in \mathbf{R}^2 \mid x > 0\}$ and $r(z) = r(x+iy) := a(x,y) + ib(x,y)$ is holomorphic in $\{z \in \mathbf{C} \mid \mathrm{Re}(z) > 0\}$.*

(iv) *There exists a constant $A > 0$ such that*

$$\left|a(x,y)\right| \leqslant \frac{A\|\alpha\|_{L^p}}{x^{\frac{1}{p}}} \quad \text{and} \quad \left|b(x,y)\right| \leqslant \frac{A\|\alpha\|_{L^p}}{x^{\frac{1}{p}}} \quad \text{for any } x > 0 \text{ and } y \in \mathbf{R}, \tag{2.13}$$

and for any $\delta > 0$ we have

$$\lim_{|(x,y)| \to \infty,\, x \geqslant \delta} a(x,y) = 0 \quad \text{and} \quad \lim_{|(x,y)| \to \infty,\, x \geqslant \delta} b(x,y) = 0.$$

Lemma 2.5. *Let μ be a finite Borel measure on $\mathbf{R}$. For $x > 0$ and $y \in \mathbf{R}$ define*

$$a(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x}{x^2 + (y-t)^2}\,d\mu(t) = \int_{-\infty}^{\infty} P(y-t,x)\,d\mu(t) \quad \text{and}$$

$$b(x, y) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y-t}{x^2+(y-t)^2}\, d\mu(t) = -\int_{-\infty}^{\infty} Q(y-t, x)\, d\mu(t),$$

where $P(s,k)$ and $Q(s,k)$ are the Poisson kernel, respectively the conjugate Poisson kernel. Then:

(i) *The functions a and b are harmonic in $\{(x, y) \in \mathbf{R}^2 \mid x > 0\}$ and $r(z) = r(x+iy) := a(x, y) + ib(x, y)$ is holomorphic in the right half-plane $\{z \in \mathbf{C} \mid \mathrm{Re}(z) > 0\}$.*

(ii) *For any $x > 0$ and any p, $1 \leqslant p \leqslant \infty$, we have*

$$\left\|a(x,\cdot)\right\|_{L^p(\mathbf{R})} \leqslant \frac{1}{\pi^{\frac{1}{q}} x^{\frac{1}{q}}} \|\mu\|, \tag{2.14}$$

where q is the conjugate exponent of p and $\|\mu\|$ is the total variation of μ. Furthermore,

$$\lim_{x\to 0} \int_{\mathbf{R}} a(x, y)\phi(y)\, dy = \int_{\mathbf{R}} \phi(y)\, d\mu(y) \tag{2.15}$$

for any function ϕ which is continuous on $\mathbf{R}$ and tends to zero at $\pm\infty$.

(iii) *For any $x > 0$ we have $b(x,\cdot) = -Ha(x,\cdot)$ and $|b(x, y)| \leqslant \frac{1}{2\pi x}\|\mu\|$. Moreover, for any $p \in (1,\infty)$ there exists $A_p > 0$ such that*

$$\left\|b(x,\cdot)\right\|_{L^p(\mathbf{R})} \leqslant A_p x^{-\frac{p-1}{p}} \|\mu\|.$$

(iv) *For any $\delta > 0$ we have $\lim_{|(x,y)|\to\infty,\, x\geqslant\delta} a(x, y) = 0$ and $\lim_{|(x,y)|\to\infty,\, x\geqslant\delta} b(x, y) = 0$.*

(v) *Suppose in addition that $\mu(S) = \mu(-S)$ and $\mu(S \cap [-\varepsilon, \varepsilon]) = 0$ for any Borel measurable set S. Then a and b are well defined, bounded and holomorphic in the strip $\{(x, y) \in \mathbf{R}^2 \mid -\frac{\varepsilon}{2} < y < \frac{\varepsilon}{2}\}$, the function $r(x+iy) = a(x, y) + ib(x, y)$ is holomorphic in that strip and $r(0) = 0$.*

After this preparation, we come back to the study of the integral $\int_{\mathbf{R}} m(\xi_1, \xi')h_{\xi'}(\xi_1)\, d\xi_1$ which appears in the right-hand side of (2.10).

Lemma 2.6. *Suppose that for a given $\xi' \in \mathbf{R}^{N-1}$ the symbol $m(\xi_1, \xi')$ can be written as*

$$\begin{aligned} m(\xi_1, \xi') = {} & A_0(\xi') + A_1(\xi')|\xi_1| + A_2(\xi')\xi_1^2 + \frac{1}{\pi}\xi_1^4 \int_{\mathbf{R}} \frac{1}{\xi_1^2+t^2} \alpha_{\xi'}(t)\, dt \\ & + \frac{1}{\pi}\left[\int_{\mathbf{R}} \frac{1}{\xi_1^2+t^2}\, d\mu_{\xi',0}(t) + \xi_1^2 \int_{\mathbf{R}} \frac{1}{\xi_1^2+t^2}\, d\mu_{\xi',1}(t) + \xi_1^4 \int_{\mathbf{R}} \frac{1}{\xi_1^2+t^2}\, d\mu_{\xi',2}(t)\right], \end{aligned} \tag{2.16}$$

where:

(a) $A_0(\xi')$, $A_1(\xi')$, $A_2(\xi') \in \mathbf{R}$,

(b) *$\alpha_{\xi'} \in L^p(\mathbf{R})$ for some $p \in (1, \infty)$ and $\alpha_{\xi'}$ is an even function,*
(c) *$\mu_{\xi',i}$ are finite Borel measures on $\mathbf{R}$ such that $\mu_{\xi',i}(S) = \mu_{\xi',i}(-S)$ for any Borel measurable set $S \subset \mathbf{R}$, $i = 0, 1, 2$. Moreover, there exists $\eta > 0$ such that $\mu_{\xi',0}(S) = 0$ for any Borel measurable set $S \subset [-\eta, \eta]$.*

Let $h = h_{\xi'}$ be given by (2.9). *Then we have the identity*:

$$\frac{1}{2}\int_{-\infty}^{\infty} m(\xi_1, \xi')h(\xi_1)\,d\xi_1 = -A_1(\xi')\int_0^{\infty} th(it)\,dt + \int_0^{\infty} t^3\alpha_{\xi'}(t)h(it)\,dt + \int_0^{\infty} \frac{h(it)}{t}\,d\mu_{\xi',0}(t) - \int_0^{\infty} th(it)\,d\mu_{\xi',1}(t) + \int_0^{\infty} t^3 h(it)\,d\mu_{\xi',2}(t). \tag{2.17}$$

Proof. For $z = x + iy \in \mathbf{C}$ with $\operatorname{Re}(z) > 0$ we define

$$r(z) = \frac{1}{\pi}\int_{\mathbf{R}} \frac{x}{x^2 + (y-t)^2}\alpha_{\xi'}(t)\,dt - \frac{i}{\pi}\int_{\mathbf{R}} \frac{y-t}{x^2 + (y-t)^2}\alpha_{\xi'}(t)\,dt \quad \text{and}$$

$$p_i(z) = \frac{1}{\pi}\int_{\mathbf{R}} \frac{x}{x^2 + (y-t)^2}\,d\mu_{\xi',i}(t) - \frac{i}{\pi}\int_{\mathbf{R}} \frac{y-t}{x^2 + (y-t)^2}\,d\mu_{\xi',i}(t) \quad \text{for } i = 0, 1, 2.$$

It follows from Lemmas 2.4 and 2.5 that r and p_i are well defined and holomorphic in the right half-plane $\{z \in \mathbf{C} \mid \operatorname{Re}(z) > 0\}$. Moreover, assumption (c) and Lemma 2.5(v) imply that p_0 admits an holomorphic extension to the domain $\{z \in \mathbf{C} \mid \operatorname{Re}(z) > 0 \text{ or } |\operatorname{Im}(z)| < \frac{\eta}{2}\}$, and $p_0(0) = 0$. Consequently, $\frac{p_0(z)}{z}$ is holomorphic in this domain and is bounded in a neighbourhood of zero.

Finally, we define

$$m_{\xi'}(z) = A_0(\xi') + A_1(\xi')z + A_2(\xi')z^2 + z^3 r(z) + \frac{p_0(z)}{z} + zp_1(z) + z^3 p_2(z). \tag{2.18}$$

It is obvious that $m_{\xi'}$ is well defined and holomorphic in the right half-plane. Since $\alpha_{\xi'}$ and $\mu_{\xi',i}$ are “even” and $t \mapsto \frac{t}{\xi_1^2 + t^2}$ is odd, for any $\xi_1 > 0$ we have $\operatorname{Im}(m_{\xi'}(\xi_1)) = 0$ and

$$m_{\xi'}(\xi_1) = \operatorname{Re}\big(m_{\xi'}(\xi_1)\big) = m(\xi_1, \xi').$$

For $\varepsilon, R > 0$, consider the closed continuous path $\gamma_{\varepsilon,R}$ composed by the following pieces:

$$\begin{aligned}
&\gamma_{1,\varepsilon,R}(t) = t, && t \in [\varepsilon, \varepsilon + R],\\
&\gamma_{2,\varepsilon,R}(\theta) = \varepsilon + Re^{i\theta}, && \theta \in [0, \tfrac{\pi}{2}],\\
&\gamma_{3,\varepsilon,R}(t) = \varepsilon + i(R - t), && t \in [0, R].
\end{aligned}$$

The function $z \mapsto m_{\xi'}(z)h(z)$ being holomorphic in the right half-plane we have $\int_{\gamma_{\varepsilon,R}} m_{\xi'}(z)h(z)\,dz = 0$, that is

$$\int_{\varepsilon}^{R} m(\xi_1,\xi')h(\xi_1)\,d\xi_1 + \int_{\gamma_{2,\varepsilon,R}} m_{\xi'}(z)h(z)\,dz + \int_{\gamma_{3,\varepsilon,R}} m_{\xi'}(z)h(z)\,dz = 0. \tag{2.19}$$

It follows from (2.18), Lemmas 2.4(iv) and 2.5(iv) that $\lim_{|z|\to\infty,\,\mathrm{Re}(z)\geqslant\varepsilon} \frac{m_{\xi'}(z)}{z^3} = 0$; hence, $\lim_{R\to\infty} \frac{m_{\xi'}(\varepsilon+Re^{i\theta})}{(\varepsilon+Re^{i\theta})^3} = 0$ uniformly with respect to $\theta \in [0,\frac{\pi}{2}]$. On the other hand, from Lemma 2.1(ii), we have $|h(\varepsilon+Re^{i\theta})| \leqslant \frac{C}{|\varepsilon+Re^{i\theta}|^4}$ and then $|(\varepsilon+Re^{i\theta})^3 h(\varepsilon+Re^{i\theta})\cdot iRe^{i\theta}| \leqslant \frac{CR}{|\varepsilon+Re^{i\theta}|} \leqslant \frac{CR}{R-\varepsilon} \leqslant 2C$ for any $R \geqslant 2\varepsilon$. We infer that $\lim_{R\to\infty}\int_{\gamma_{2,\varepsilon,R}} m_{\xi'}(z)h(z)\,dz = 0$.

From (2.13), (2.14) and the boundedness of $\frac{p_0(z)}{z}$ near 0 it follows that $|m(\xi_1,\xi')| \leqslant C$ for $0<\xi_1<1$ and $|m(\xi_1,\xi')| \leqslant C|\xi_1|^{3-\delta}$ for large ξ_1 and some $C,\delta>0$. Since h is continuous and $|h(\xi_1)| \leqslant \frac{C}{|\xi_1|^4}$, the integral $\int_0^\infty m(\xi_1,\xi')h(\xi_1)\,d\xi_1$ converges absolutely.

Clearly we have $\int_{\gamma_{3,\varepsilon,R}} m_{\xi'}(z)h(z)\,dz = -i\int_0^R m_{\xi'}(\varepsilon+iy)h(\varepsilon+iy)\,dy$. Passing to the limit as $R\to\infty$ in (2.19) we infer that $\int_0^\infty m_{\xi'}(\varepsilon+iy)h(\varepsilon+iy)\,dy$ converges and

$$\int_{\varepsilon}^{\infty} m(\xi_1,\xi')h(\xi_1)\,d\xi_1 = i\int_0^\infty m_{\xi'}(\varepsilon+iy)h(\varepsilon+iy)\,dy. \tag{2.20}$$

Since $m(\xi_1,\xi')$ is real and symmetric with respect to ξ_1 and $h(-\xi_1) = \overline{h(\xi_1)}$, we have

$$\int_{-\infty}^{-\varepsilon} m(\xi_1,\xi')h(\xi_1)\,d\xi_1 = \int_{\varepsilon}^{\infty} m(\xi_1,\xi')\overline{h(\xi_1)}\,d\xi_1,$$

and then, taking (2.20) into account, we get

$$\int_{-\infty}^{-\varepsilon} m(\xi_1,\xi')h(\xi_1)\,d\xi_1 + \int_{\varepsilon}^{\infty} m(\xi_1,\xi')h(\xi_1)\,d\xi_1 = -2\int_0^\infty \mathrm{Im}\big(m_{\xi'}(\varepsilon+iy)h(\varepsilon+iy)\big)\,dy; \tag{2.21}$$

hence

$$\int_{-\infty}^{\infty} m(\xi_1,\xi')h(\xi_1)\,d\xi_1 = -2\lim_{\varepsilon\to 0}\int_0^\infty \mathrm{Im}\big(m_{\xi'}(\varepsilon+iy)h(\varepsilon+iy)\big)\,dy. \tag{2.22}$$

Since $h(iy) \in \mathbf{R}$ for $y \in [0,\infty)$, using Lemma 2.1 and the Dominated Convergence Theorem we find

$$\lim_{\varepsilon\to 0}\int_0^\infty \mathrm{Im}\big[\big(A_0(\xi')+A_1(\xi')(\varepsilon+iy)+A_2(\xi')(\varepsilon+iy)^2\big)h(\varepsilon+iy)\big]\,dy$$
$$= A_1(\xi')\int_0^\infty yh(iy)\,dy. \tag{2.23}$$

It is easy to see that $|(\varepsilon+iy)^\ell h(\varepsilon+iy)-(iy)^\ell h(iy)|\leqslant C_1\varepsilon\min(1,\frac{1}{y^2})$ for $y\in(0,\infty)$, $\ell\in\{0,1,2,3\}$ and $\varepsilon\in[0,1]$. Hence there exists $C_2>0$ such that

$$\big\|(\varepsilon+iy)^\ell h(\varepsilon+iy)-(iy)^\ell h(iy)\big\|_{L^q(0,\infty)}\leqslant C_2\varepsilon \tag{2.24}$$

for any $\varepsilon\in[0,1]$, $\ell\in\{0,1,2,3\}$ and $q\in[1,\infty]$. This implies that

$$\begin{aligned}
&\left|\int_0^\infty \mathrm{Im}\big((\varepsilon+iy)^3h(\varepsilon+iy)r(\varepsilon+iy)\big)\,dy-\int_0^\infty \mathrm{Im}\big((iy)^3h(iy)r(\varepsilon+iy)\big)\,dy\right|\\
&\quad\leqslant\big(\big\|\mathrm{Re}\big(r(\varepsilon+i\cdot)\big)\big\|_{L^p}+\big\|\mathrm{Im}\big(r(\varepsilon+i\cdot)\big)\big\|_{L^p}\big)\big\|(\varepsilon+iy)^3h(\varepsilon+iy)-(iy)^3h(iy)\big\|_{L^{p'}(0,\infty)}\\
&\quad\leqslant\big(\|\alpha_{\xi'}\|_{L^p}+\|H\alpha_{\xi'}\|_{L^p}\big)C_2\varepsilon\to 0\quad\text{as }\varepsilon\to 0.
\end{aligned}$$

On the other hand, by Lemma 2.4(ii) we obtain

$$\begin{aligned}
\lim_{\varepsilon\to 0}\int_0^\infty \mathrm{Im}\big[(iy)^3h(iy)r(\varepsilon+iy)\big]\,dy &= -\lim_{\varepsilon\to 0}\int_0^\infty y^3h(iy)\,\mathrm{Re}\big[r(\varepsilon+iy)\big]\,dy\\
&= -\int_0^\infty y^3h(iy)\alpha_{\xi'}(y)\,dy.
\end{aligned}$$

Therefore we have

$$\lim_{\varepsilon\to 0}\int_0^\infty \mathrm{Im}\big[(\varepsilon+iy)^3h(\varepsilon+iy)r(\varepsilon+iy)\big]\,dy=-\int_0^\infty y^3h(iy)\alpha_{\xi'}(y)\,dy. \tag{2.25}$$

Let $\chi\in C_c^\infty(\mathbf{R},\mathbf{R}_+)$ be such that $\mathrm{supp}(\chi)\subset[-\frac{\eta}{4},\frac{\eta}{4}]$ and $\chi\equiv 1$ on $[-\frac{\eta}{8},\frac{\eta}{8}]$. Since the function $z\mapsto\frac{p_0(z)}{z}h(z)$ is uniformly continuous on $[-1,1]\times[-\frac{\eta}{4},\frac{\eta}{4}]$ we have

$$\begin{aligned}
\lim_{\varepsilon\to 0}\int_0^\infty \mathrm{Im}\left[\frac{p_0(\varepsilon+iy)}{\varepsilon+iy}h(\varepsilon+iy)\chi(y)\right]dy &= \int_0^\infty \mathrm{Im}\left(\frac{p_0(iy)}{iy}h(iy)\chi(y)\right)dy\\
&= -\int_0^\infty \frac{\mathrm{Re}(p_0(iy))}{y}h(iy)\chi(y)\,dy=0.
\end{aligned}\tag{2.26}$$

By Lemma 2.1 we infer that there exists $C_3 > 0$ such that $|h(\varepsilon+iy)-h(iy)| \leqslant \varepsilon C_3 \min(1, \frac{1}{|y|^5})$ for any $y \in (0,\infty)$ and $\varepsilon \in [0,1]$. It is easy to see that

$$\left|\left(\frac{h(\varepsilon+iy)}{\varepsilon+iy} - \frac{h(iy)}{iy}\right)\big(1-\chi(y)\big)\right| \leqslant C_4\varepsilon \min\left(\frac{1}{y^6}, 1\right)$$

for any $y \in (0,\infty)$ and some $C_4 > 0$. Consequently there exists $C_5 > 0$ such that

$$\left\|\left(\frac{h(\varepsilon+iy)}{\varepsilon+iy} - \frac{h(iy)}{iy}\right)\big(1-\chi(y)\big)\right\|_{L^p(0,\infty)} \leqslant C_5\varepsilon$$

for any $p \in [1,\infty]$. Using the Cauchy–Schwarz inequality and Lemma 2.5(ii) and (iii), we get

$$\begin{aligned}
&\left|\int_0^\infty p_0(\varepsilon+iy)\left(\frac{h(\varepsilon+iy)}{\varepsilon+iy} - \frac{h(iy)}{iy}\right)\big(1-\chi(y)\big)\,dy\right| \\
&\quad \leqslant \big\|p_0(\varepsilon+i\cdot)\big\|_{L^2(\mathbf{R})}\left|\left(\frac{h(\varepsilon+iy)}{\varepsilon+iy} - \frac{h(iy)}{iy}\right)\big(1-\chi(y)\big)\right|_{L^2(0,\infty)} \\
&\quad \leqslant C_6\varepsilon^{\frac{1}{2}} \to 0 \quad \text{as } \varepsilon \to 0.
\end{aligned} \tag{2.27}$$

We also have by (2.15) and assumption (c),

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_0^\infty \operatorname{Im}\left[p_0(\varepsilon+iy)\frac{h(iy)}{iy}\big(1-\chi(y)\big)\right]dy \\
&\quad = -\lim_{\varepsilon\to 0}\int_0^\infty \operatorname{Re}\big(p_0(\varepsilon+iy)\big)\frac{h(iy)}{y}\big(1-\chi(y)\big)\,dy \\
&\quad = -\int_0^\infty \frac{h(iy)}{y}\big(1-\chi(y)\big)\,d\mu_{\xi',0}(y) = -\int_0^\infty \frac{h(iy)}{y}\,d\mu_{\xi',0}(y).
\end{aligned} \tag{2.28}$$

From (2.26)–(2.28) we get

$$\lim_{\varepsilon\to 0}\int_0^\infty \operatorname{Im}\left[\frac{p_0(\varepsilon+iy)}{\varepsilon+iy}h(\varepsilon+iy)\right]dy = -\int_0^\infty \frac{h(iy)}{y}\,d\mu_{\xi',0}(y). \tag{2.29}$$

Similarly we find

$$\lim_{\varepsilon\to 0}\int_0^\infty \operatorname{Im}\big((\varepsilon+iy)p_1(\varepsilon+iy)h(\varepsilon+iy)\big)\,dy = \int_0^\infty yh(iy)\,d\mu_{\xi',1}(y) \quad \text{and} \tag{2.30}$$

$$\lim_{\varepsilon\to 0}\int_0^\infty \operatorname{Im}\big((\varepsilon+iy)^3 p_2(\varepsilon+iy)h(\varepsilon+iy)\big)\,dy = -\int_0^\infty y^3 h(iy)\,d\mu_{\xi',2}(y). \tag{2.31}$$

Since $m_{\xi'}(z)$ is given by (2.18), replacing (2.23), (2.25), (2.29)–(2.31) into (2.22) we obtain the conclusion of Lemma 2.6. □

Now we are ready to state and prove the main result of this section.

Theorem 2.7. *Suppose that for any $\xi' \in \mathbf{R}^{N-1}$, $m(\xi_1,\xi')$ satisfies the assumptions of Lemma* 2.6. *For $u \in C_c^\infty(\mathbf{R}^N)$ define u_1, u_2, f, g and W as in* (2.1)–(2.4). *Then we have the identity*:

$$\begin{aligned}
&\frac{\pi^2}{16}\big(W(u_1)+W(u_2)-2W(u)\big)\\
&\quad= -\int_{\mathbf{R}^{N-1}} A_1(\xi')\int_0^\infty t\left|\int_0^\infty \widehat{f}(\xi_1,\xi')\frac{\xi_1}{t^2+\xi_1^2}\,d\xi_1\right|^2 dt\,d\xi'\\
&\qquad+\int_{\mathbf{R}^{N-1}}\int_0^\infty t^3\alpha_{\xi'}(t)\left|\int_0^\infty \widehat{f}(\xi_1,\xi')\frac{\xi_1}{t^2+\xi_1^2}\,d\xi_1\right|^2 dt\,d\xi'\\
&\qquad+\int_{\mathbf{R}^{N-1}}\int_0^\infty \frac{1}{t}\left|\int_0^\infty \widehat{f}(\xi_1,\xi')\frac{\xi_1}{t^2+\xi_1^2}\,d\xi_1\right|^2 d\mu_{\xi',0}(t)\,d\xi'\\
&\qquad-\int_{\mathbf{R}^{N-1}}\int_0^\infty t\left|\int_0^\infty \widehat{f}(\xi_1,\xi')\frac{\xi_1}{t^2+\xi_1^2}\,d\xi_1\right|^2 d\mu_{\xi',1}(t)\,d\xi'\\
&\qquad+\int_{\mathbf{R}^{N-1}}\int_0^\infty t^3\left|\int_0^\infty \widehat{f}(\xi_1,\xi')\frac{\xi_1}{t^2+\xi_1^2}\,d\xi_1\right|^2 d\mu_{\xi',2}(t)\,d\xi'.
\end{aligned} \tag{2.32}$$

Proof. Since $\mathcal{F}_{N-1}f \in \mathcal{S}(\mathbf{R}^N)$, the integral $\int_0^\infty e^{-x_1 t}(\mathcal{F}_{N-1}f)(x_1,\xi')\,dx_1$ is well defined for all $t>0$ and $\xi' \in \mathbf{R}^{N-1}$. Using Plancherel's theorem we get

$$\begin{aligned}
\int_0^\infty e^{-x_1 t}(\mathcal{F}_{N-1}f)(x_1,\xi')\,dx_1 &= \big\langle \mathcal{F}_{N-1}f(\cdot,\xi'), e^{-(\cdot)t}\chi_{[0,\infty)}(\cdot)\big\rangle_{L^2(\mathbf{R})}\\
&= (2\pi)^{-1}\big\langle \mathcal{F}_1\big(\mathcal{F}_{N-1}f(\cdot,\xi')\big), \mathcal{F}_1\big(e^{-(\cdot)t}\chi_{[0,\infty)}(\cdot)\big)\big\rangle_{L^2(\mathbf{R})}.
\end{aligned} \tag{2.33}$$

Moreover, we have

$$\mathcal{F}_1\big(e^{-(\cdot)t}\chi_{[0,\infty)}(\cdot)\big)(\xi_1) = \int_0^\infty e^{-ix_1\xi_1}e^{-x_1 t}\,dx_1 = -\frac{1}{t+i\xi_1}e^{-(t+i\xi_1)x_1}\bigg|_{x_1=0}^{\infty} = \frac{1}{t+i\xi_1}$$

and then, using (2.33) and the oddness of $\widehat{f}$ with respect to ξ_1 we get:

$$h_{\xi'}(it) = \left| \int_0^\infty e^{-x_1 t} (\mathcal{F}_{N-1} f)(x_1, \xi')\, dx_1 \right|^2 = (2\pi)^{-2} \left| \int_{-\infty}^\infty \widehat{f}(\xi_1, \xi') \cdot \frac{1}{t - i\xi_1}\, d\xi_1 \right|^2$$
$$= \frac{1}{(2\pi)^2} \left| \int_0^\infty \widehat{f}(\xi_1, \xi') \left(\frac{1}{t - i\xi_1} - \frac{1}{t + i\xi_1} \right) d\xi_1 \right|^2$$
$$= \frac{1}{\pi^2} \left| \int_0^\infty \widehat{f}(\xi_1, \xi') \frac{\xi_1}{t^2 + \xi_1^2}\, d\xi_1 \right|^2. \tag{2.34}$$

Identity (2.32) is a simple consequence of (2.10), (2.17) and (2.34) and Theorem 2.7 is proved. □

Remark 2.8. It is worth to note that we can prove an identity analogous to (2.32) whenever we work with a symbol $m(\xi) = m(\xi_1, \xi')$ symmetric with respect to ξ_1 and such that for any $\xi' \in \mathbf{R}^{N-1}$, $m(\cdot, \xi')$ admits an holomorphic extension $m_{\xi'}(z)$ to the domain $\{z \in \mathbf{C} \mid \mathrm{Re}(z) > 0,\ \mathrm{Im}(z) > 0\}$ having the following properties:

P1. $\lim_{z \to \xi_1, \mathrm{Im}(z) > 0} m_{\xi'}(z) = m(\xi_1, \xi')$.
P2. For any $\varepsilon > 0$, $\lim_{|z| \to \infty, \mathrm{Re}(z) \geqslant \varepsilon} \frac{m_{\xi'}(z)}{z^3} = 0$.
P3. $\lim_{\varepsilon \to 0} \int_0^\infty m_{\xi'}(\varepsilon + it) h_{\xi'}(\varepsilon + it)\, dt$ exists (and depends on ξ' and the values taken by $h_{\xi'}$ on the imaginary axis).

Note that assumption P1 implies that $m(\cdot, \xi')$ admits an holomorphic extension to the whole right half-plane. Indeed, it follows from Schwarz' reflection principle [8, p. 75] that the function

$$\widetilde{m}_{\xi'} = \begin{cases} m_{\xi'}(z) & \text{if } \mathrm{Im}(z) \geqslant 0, \\ \overline{m_{\xi'}(\bar{z})} & \text{if } \mathrm{Im}(z) < 0 \end{cases}$$

is holomorphic in $\{z \in \mathbf{C} \mid \mathrm{Re}(z) > 0\}$.

Assumption P2 is needed in the proof of Lemma 2.6 to show that

$$\lim_{R \to \infty} \int_{\gamma_{2,\varepsilon,R}} m_{\xi'}(z) h_{\xi'}(z)\, dz = 0$$

(where $\gamma_{2,\varepsilon,R}(\theta) = \varepsilon + Re^{i\theta}$, $\theta \in [0, \frac{\pi}{2}]$). We recall that $|h_{\xi'}(z)|$ behaves like $\frac{1}{|z|^4}$ as $|z| \to \infty$ (see Lemma 2.1 and Remark 2.2). This assumption could be replaced by a weaker one that guarantees at least that

$$\lim_{n \to \infty} \int_{\gamma_{2,\varepsilon,R_n}} m_{\xi'}(z) h_{\xi'}(z)\, dz = 0 \quad \text{for some sequence } R_n \to \infty.$$

In Theorem 2.7 assumption P3 is satisfied because of the special form of $m(\cdot,\xi')$ (see (2.16)).

Conversely, suppose that a function $m(z)$ has the properties P1–P3 above. Let $\widetilde{m}$ be the holomorphic extension of m to the right half-plane and define $q(z)=\frac{\widetilde{m}(z)}{z^3}$. Clearly, q is an holomorphic function in the right half-plane and $\lim_{|z|\to\infty,\,\mathrm{Re}(z)\geqslant\varepsilon} q(z)=0$ for any $\varepsilon>0$. Thus for any $x>\varepsilon$ we have the Poisson representation formulae

$$q(x+iy)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x-\varepsilon}{(x-\varepsilon)^2+(t-y)^2}\,\mathrm{Re}\big(q(\varepsilon+it)\big)\,dt + \frac{i}{\pi}\int_{-\infty}^{\infty}\frac{t-y}{(x-\varepsilon)^2+(t-y)^2}\,\mathrm{Re}\big(q(\varepsilon+it)\big)\,dt \tag{2.35}$$

and

$$q(x+iy)=\frac{-1}{\pi}\int_{-\infty}^{\infty}\frac{t-y}{(x-\varepsilon)^2+(t-y)^2}\,\mathrm{Im}\big(q(\varepsilon+it)\big)\,dt + \frac{i}{\pi}\int_{-\infty}^{\infty}\frac{x-\varepsilon}{(x-\varepsilon)^2+(t-y)^2}\,\mathrm{Im}\big(q(\varepsilon+it)\big)\,dt. \tag{2.36}$$

Multiplying (2.35) (respectively (2.36)) by $(x+iy)^3$, we find the expression of $m(x+iy)$ in terms of $\mathrm{Re}(q(\varepsilon+it))$ (respectively in terms of $\mathrm{Im}(q(\varepsilon+it))$). If $\mathrm{Re}(q(\varepsilon+it))\to\alpha(t)$ as $\varepsilon\to 0$ and if it is possible to pass to the limit as $\varepsilon\to 0$ in (2.35) we obtain, at least formally,

$$m(\xi_1)=\xi_1^3 q(\xi_1)=\frac{\xi_1^4}{\pi}\int_{-\infty}^{\infty}\frac{\alpha(t)}{\xi_1^2+t^2}\,dt.$$

However, as it will be seen later in applications, the function q may be singular at the origin. In this case it is not possible to pass to the limit as $\varepsilon\to 0$ in (2.35) or in (2.36) in order to express the function q (hence the function m) in terms of its "boundary values" on the imaginary axis. For this reason we have introduced "lower order terms" in the expression of $m_{\xi'}(z)$ in (2.16).

It is now clear that Theorem 2.7 can be generalized. For example, if the expression (2.16) of $m(\xi_1,\xi')$ contains other terms

$$\frac{1}{\pi}\sum_{k=0}^{3}|\xi_1|^k\int_{\mathbf{R}}\frac{1}{\xi_1^2+t^2}\alpha_{\xi',k}(t)\,dt,$$

where $\alpha_{\xi',k}\in L^{p_k}(\mathbf{R})$ for some $p_k\in(1,\infty)$, $\alpha_{\xi',k}$ are even functions and $\alpha_{\xi',0}$ vanishes in a neighborhood of zero, then we have to add terms

$$\int_{\mathbf{R}^{N-1}}\int_0^{\infty}\bigg[\frac{\alpha_{\xi',0}(t)}{t}+\beta_{\xi',1}(t)-t\alpha_{\xi',2}(t)-t^2\beta_{\xi',3}(t)\bigg]\bigg|\int_0^{\infty}\widehat{f}(\xi_1,\xi')\frac{\xi_1}{t^2+\xi_1^2}\,d\xi_1\bigg|^2\,dt\,d\xi'$$

in the right-hand side of (2.32), where $\beta_{\xi',1}$ and $\beta_{\xi',3}$ are Hilbert transforms of $\alpha_{\xi',1}$ and $\alpha_{\xi',3}$, respectively.

We give now some examples illustrating several situations that may be encountered in applications. Throughout $u \in C_c^\infty(\mathbf{R}^N)$ and we keep the notation introduced in (2.1)–(2.4).

Example 2.9. If the symbol m is of the form $m(\xi_1, \xi') = A_1(\xi')|\xi_1|$, then Theorem 2.7 gives

$$W(u_1) + W(u_2) - 2W(u) = -\frac{16}{\pi^2} \int_{\mathbf{R}^{N-1}} A_1(\xi') \int_0^\infty t \left| \int_0^\infty \widehat{f}(\xi_1, \xi') \frac{\xi_1}{t^2 + \xi_1^2} \, d\xi_1 \right|^2 dt \, d\xi'. \quad (2.37)$$

This kind of symbol appears in problems involving operators of the type $H_1 \frac{\partial}{\partial x_1} P(\frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_N})$, where H_1 is the Hilbert transform with respect to the x_1 variable and P is a pseudo-differential operator in the last $N-1$ variables.

Example 2.10. (i) Consider the symbol $m(\xi) = \frac{1}{|\xi|^2}$ appearing in Choquard's problem. It can be written as

$$m(\xi_1, \xi') = \frac{1}{\xi_1^2 + |\xi'|^2} = \frac{1}{\pi} \int_{\mathbf{R}} \frac{1}{\xi_1^2 + t^2} \, d\mu_{\xi',0}(t),$$

where $\mu_{\xi',0} = \frac{\pi}{2}(\delta_{-|\xi'|} + \delta_{|\xi'|})$ and δ_a is the Dirac measure with support $\{a\}$. From Theorem 2.7 we get the identity

$$W(u_1) + W(u_2) - 2W(u) = \frac{8}{\pi} \int_{\mathbf{R}^{N-1}} \frac{1}{|\xi'|} \left| \int_0^\infty \widehat{f}(\xi_1, \xi') \frac{\xi_1}{|\xi'|^2 + \xi_1^2} \, d\xi_1 \right|^2 d\xi'. \quad (2.38)$$

The same identity could be obtained by observing that the function $m_{\xi'}(z) = \frac{1}{z^2 + |\xi'|^2}$ is meromorphic in $\mathbf{C}$ and has exactly one pole in the upper half-plane, namely $i|\xi'|$. Using Residue's Theorem it is not hard to see that

$$\int_{-\infty}^{\infty} m_{\xi'}(z) h_{\xi'}(z) \, dz = 2\pi i \operatorname{Res}\bigl(m_{\xi'} h_{\xi'}, i|\xi'|\bigr),$$

and integrating this identity over $\mathbf{R}^{N-1}$ we get (2.38).

(ii) Consider the symbol $m(\xi) = \frac{1}{|\xi|^2 + a^2} = \frac{1}{\xi_1^2 + |\xi'|^2 + a^2}$ corresponding to the operator $(-\Delta + a^2)^{-1}$. It is obvious that

$$m(\xi_1, \xi') = \frac{1}{\pi} \int_{\mathbf{R}} \frac{1}{\xi_1^2 + t^2} \, d\mu_{\xi',0}(t),$$

where $\mu_{\xi',0} = \frac{\pi}{2}(\delta_{-\sqrt{|\xi'|^2 + a^2}} + \delta_{\sqrt{|\xi'|^2 + a^2}})$. From Theorem 2.7 we get the identity

$$W(u_1)+W(u_2)-2W(u)=\frac{8}{\pi}\int\limits_{\mathbf{R}^{N-1}}\frac{1}{\sqrt{|\xi'|^2+a^2}}\left|\int\limits_0^\infty \widehat{f}(\xi_1,\xi')\frac{\xi_1}{a^2+|\xi'|^2+\xi_1^2}\,d\xi_1\right|^2 d\xi'. \tag{2.39}$$

The same identity could be obtained by applying Residue's Theorem to the meromorphic function $z\mapsto \frac{1}{z^2+|\xi'|^2+a^2}h_{\xi'}(z)$.

(iii) More generally, consider a symbol of the form $m(\xi_1,\xi')=\frac{c(\xi')}{\xi_1^2+r^2(\xi')}$. It can be written as

$$m(\xi_1,\xi')=\frac{1}{\pi}\int\limits_{\mathbf{R}}\frac{1}{\xi_1^2+t^2}\,d\mu_{\xi',0}(t),$$

where $\mu_{\xi',0}=\frac{\pi}{2}c(\xi')(\delta_{-r(\xi')}+\delta_{r(\xi')})$. Using Theorem 2.7 we obtain the identity

$$W(u_1)+W(u_2)-2W(u)=\frac{8}{\pi}\int\limits_{\mathbf{R}^{N-1}}\frac{c(\xi')}{r(\xi')}\cdot\left|\int\limits_0^\infty \widehat{f}(\xi_1,\xi')\frac{\xi_1}{r^2(\xi')+\xi_1^2}\,d\xi_1\right|^2 d\xi'. \tag{2.40}$$

In particular, for the symbol $m(\xi_1,\xi')=\frac{\xi_j^{2k}}{\xi_1^2+|\xi'|^2+a^2}$, $j=2,\ldots,N$ (corresponding to the operator $(-1)^k\frac{\partial^{2k}}{\partial x_j^{2k}}(-\Delta+a^2)^{-1})$), we get

$$W(u_1)+W(u_2)-2W(u)=\frac{8}{\pi}\int\limits_{\mathbf{R}^{N-1}}\frac{\xi_j^{2k}}{\sqrt{|\xi'|^2+a^2}}\left|\int\limits_0^\infty \widehat{f}(\xi_1,\xi')\frac{\xi_1}{a^2+|\xi'|^2+\xi_1^2}\,d\xi_1\right|^2 d\xi'. \tag{2.41}$$

(iv) The symbol $m(\xi_1,\xi')=\frac{\xi_1^2}{\xi_1^2+|\xi'|^2+a^2}$ can be expressed as

$$m(\xi_1,\xi')=\frac{\xi_1^2}{\pi}\int\limits_{\mathbf{R}}\frac{1}{\xi_1^2+t^2}\,d\mu_{\xi',1}(t),$$

where $\mu_{\xi',1}=\frac{\pi}{2}(\delta_{-\sqrt{|\xi'|^2+a^2}}+\delta_{\sqrt{|\xi'|^2+a^2}})$. From Theorem 2.7 we find the identity

$$W(u_1)+W(u_2)-2W(u)=-\frac{8}{\pi}\int\limits_{\mathbf{R}^{N-1}}\sqrt{|\xi'|^2+a^2}\left|\int\limits_0^\infty \widehat{f}(\xi_1,\xi')\frac{\xi_1}{a^2+|\xi'|^2+\xi_1^2}\,d\xi_1\right|^2 d\xi'. \tag{2.42}$$

Notice that the right-hand side in (2.42) is nonpositive, while in (2.41) it is nonnegative.

(v) The symbol $m(\xi_1, \xi') = \frac{\xi_1^4}{\xi_1^2 + |\xi'|^2 + a^2}$ (corresponding to the operator $\frac{\partial^4}{\partial x_1^4}(-\Delta + a^2)^{-1}$) can be written as

$$m(\xi_1, \xi') = \frac{\xi_1^4}{\pi} \int_{\mathbf{R}} \frac{1}{\xi_1^2 + t^2} \, d\mu_{\xi',2}(t),$$

where $\mu_{\xi',2} = \frac{\pi}{2}(\delta_{-\sqrt{|\xi'|^2+a^2}} + \delta_{\sqrt{|\xi'|^2+a^2}})$. By Theorem 2.7 we have the identity

$$W(u_1) + W(u_2) - 2W(u) = \frac{8}{\pi} \int_{\mathbf{R}^{N-1}} \left(|\xi'|^2 + a^2\right)^{\frac{3}{2}} \left| \int_0^\infty \widehat{f}(\xi_1, \xi') \frac{\xi_1}{a^2 + |\xi'|^2 + \xi_1^2} \, d\xi_1 \right|^2 d\xi'. \tag{2.43}$$

Obviously all the identities in (2.40)–(2.43) could be obtained by using the Residue Theorem.

Example 2.11. Consider the symbol $m(\xi) = |\xi|^{2s}$, corresponding to the operator $(-\Delta)^s$.

The complex logarithm $\log(z) = \ln|z| + i \arg(z)$ is well defined and holomorphic on $\mathbf{C} \setminus (-\infty, 0]$. For $z \in \Omega_{\xi'} := \mathbf{C} \setminus \{it \mid t \in (-\infty, -|\xi'|] \cup [|\xi'|, \infty)\}$, we have $z^2 + |\xi'|^2 \notin (-\infty, 0]$; hence we may define

$$m_{\xi'}(z) = e^{s \log(z^2 + |\xi'|^2)} = \left|z^2 + |\xi'|^2\right|^s e^{is \arg(z^2 + |\xi'|^2)}.$$

The function $m_{\xi'}$ is holomorphic in $\Omega_{\xi'}$ and $|m_{\xi'}(z)| = |z^2 + |\xi'|^2|^s$ for any $z \in \Omega_{\xi'}$. It is easy to see that, for $\xi' \neq 0$,

$$m_{\xi'}(z) = |\xi'|^{2s} \left(1 + s\frac{z^2}{|\xi'|^2} + \sum_{k=2}^{\infty} C_s^k \frac{z^{2k}}{|\xi'|^{2k}}\right), \tag{2.44}$$

where $C_s^k = \frac{s(s-1)\cdots(s-k+1)}{k!}$ and the series converges in the open ball $B_{\mathbf{C}}(0, |\xi'|)$.

For $s < \frac{3}{2}$ and $\xi' \neq 0$, the function $z \mapsto \frac{m_{\xi'}(z)}{z^3}$ is holomorphic in $\Omega_{\xi'} \setminus \{0\}$, tends to zero as $|z| \to \infty$ and has a third order pole at the origin. Consider the function $r_{\xi'}(z) = \frac{1}{z^3}(m_{\xi'}(z) - |\xi'|^{2s} - s|\xi'|^{2s-2}z^2)$. According to (2.44), $r_{\xi'}$ is a holomorphic function in $\Omega_{\xi'}$. If $s < \frac{3}{2}$, we have $r_{\xi'}(z) \to 0$ as $|z| \to \infty$. Consequently, the Poisson representation formula (2.35) holds for $r_{\xi'}$. Since $r_{\xi'}(\bar{z}) = \overline{r_{\xi'}(z)}$, the function $t \mapsto \mathrm{Re}(r_{\xi'}(\varepsilon + it))$ is even and we have, in particular,

$$\begin{aligned} m_{\xi'}(\xi_1) &= |\xi'|^{2s} + s|\xi'|^{2s-2}\xi_1^2 + \xi_1^3 r_{\xi'}(\xi_1) \\ &= |\xi'|^{2s} + s|\xi'|^{2s-2}\xi_1^2 + \frac{\xi_1^3}{\pi} \int_{-\infty}^{\infty} \frac{\xi_1 - \varepsilon}{(\xi_1 - \varepsilon)^2 + (t - y)^2} \mathrm{Re}\big(r_{\xi'}(\varepsilon + it)\big) \, dt. \end{aligned} \tag{2.45}$$

It is clear that for any $t \in (-|\xi'|, |\xi'|)$ we have $\lim_{\varepsilon \to 0} \mathrm{Re}(r_{\xi'}(\varepsilon + it)) = \mathrm{Re}(r_{\xi'}(it)) = 0$. For any $t > |\xi'|$ we have $\lim_{\varepsilon \downarrow 0} m_{\xi'}(\varepsilon + it) = (t^2 - |\xi'|^2)^s e^{is\pi}$ and $\lim_{\varepsilon \downarrow 0} \mathrm{Re}(r_{\xi'}(\varepsilon + it)) = -\sin(s\pi)\frac{(t^2 - |\xi'|^2)^s}{t^3}$.

On the other hand, it is straightforward to check that for $-1 < s < \frac{3}{2}$, there exists $p_s \in (1, \infty)$ and $C_{s,\xi'} > 0$ such that

$$\left\| r_{\xi'}(\varepsilon + i\cdot) \right\|_{L^{p_s}(\mathbf{R})} \leqslant C_{s,\xi'} \quad \text{for any } \varepsilon \in \left(0, \frac{|\xi'|}{2}\right). \tag{2.46}$$

It follows from (2.46) and [24, Theorem 2.5, p. 50] that there exists $k_{\xi'} \in L^{p_s}(\mathbf{R})$ such that $\mathrm{Re}(r_{\xi'}(x+iy)) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x}{x^2+(y-t)^2} k_{\xi'}(t)\,dt$. Moreover, from [24, Theorem 2.1, p. 47] we have $\lim_{\varepsilon\downarrow 0} \mathrm{Re}(r_{\xi'}(\varepsilon + it)) = k_{\xi'}(t)$ for almost every $t \in \mathbf{R}$ and $\|\mathrm{Re}(r_{\xi'}(\varepsilon + i\cdot)) - k_{\xi'}\|_{L^{p_s}} \to 0$ as $\varepsilon \to 0$. In view of the pointwise convergence, we infer that $k_{\xi'}(-t) = k_{\xi'}(t)$ a.e. and

$$k_{\xi'}(t) = \begin{cases} 0 & \text{if } t \in (-|\xi'|, |\xi'|), \\ -\sin(s\pi)\frac{(t^2-|\xi'|^2)^s}{|t|^3} & \text{if } |t| > |\xi'|, \end{cases} \qquad \text{a.e. on } \mathbf{R}.$$

Now it is clear that the symbol $m(\xi_1, \xi') = (\xi_1^2 + |\xi'|^2)^s$ can be written as

$$\begin{aligned} m(\xi_1, \xi') &= |\xi'|^{2s} + s|\xi'|^{2s-2}\xi_1^2 + \xi_1^3 r_{\xi'}(\xi_1) \\ &= |\xi'|^{2s} + s|\xi'|^{2s-2}\xi_1^2 + \frac{\xi_1^4}{\pi}\int_{-\infty}^{\infty} \frac{1}{\xi_1^2 + t^2} k_{\xi'}(t)\,dt. \end{aligned} \tag{2.47}$$

Thus we may apply Theorem 2.7 to get, for any $u \in C_c^\infty(\mathbf{R}^N)$ and $s \in (-1, \frac{3}{2})$,

$$\begin{aligned} &W(u_1) + W(u_2) - 2W(u) \\ &\quad = \frac{16}{\pi^2}\int_{\mathbf{R}^{N-1}}\int_0^\infty t^3 k_{\xi'}(t)\left|\int_0^\infty \widehat{f}(\xi_1, \xi')\frac{\xi_1}{t^2+\xi_1^2}\,d\xi_1\right|^2 dt\,d\xi' \\ &\quad = -\frac{16\sin(s\pi)}{\pi^2}\int_{\mathbf{R}^{N-1}}\int_{|\xi'|}^\infty \left(t^2 - |\xi'|^2\right)^s \left|\int_0^\infty \widehat{f}(\xi_1, \xi')\frac{\xi_1}{t^2+\xi_1^2}\,d\xi_1\right|^2 dt\,d\xi'. \end{aligned} \tag{2.48}$$

Similarly, if we consider the symbol $m(\xi) = (|\xi|^2 + a^2)^s$ we get the identity

$$\begin{aligned} &W(u_1) + W(u_2) - 2W(u) \\ &\quad = -\frac{16\sin(s\pi)}{\pi^2}\int_{\mathbf{R}^{N-1}}\int_{\sqrt{|\xi'|^2+a^2}}^\infty \left(t^2 - |\xi'|^2 - a^2\right)^s \left|\int_0^\infty \widehat{f}(\xi_1, \xi')\frac{\xi_1}{t^2+\xi_1^2}\,d\xi_1\right|^2 dt\,d\xi'. \end{aligned} \tag{2.49}$$

3. Symmetry and function spaces

For any $u \in C_c^\infty(\mathbf{R}^N)$ we define u_1 and u_2 as in (2.2) and we put $T_1 u = u_1$, $T_2 u = u_2$. Clearly, T_1 and T_2 are linear continuous mappings from $C_c^\infty(\mathbf{R}^N)$ to $C_c^0(\mathbf{R}^N)$. In this section we consider the following intimately related problems.

Problem 1. Determine significant subspaces $\mathcal{X} \subset \mathcal{D}'(\mathbf{R}^N)$ such that T_1 and T_2 can be extended to linear continuous mappings from $\mathcal{X}$ to $\mathcal{X}$. (Or, equivalently, find subspaces $\mathcal{X}$ such that $u \in \mathcal{X}$ implies $T_1 u, T_2 u \in \mathcal{X}$ and $u \mapsto T_1 u$, $u \mapsto T_2 u$ are continuous for the $\mathcal{X}$ topology.)

Problem 2. If $\mathcal{X}$ is a subspace as above, how the identities proved in the previous section can be extended to $\mathcal{X}$?

The answer to these questions is of great importance in symmetry problems. For instance, suppose that a function space $\mathcal{X}$ has the two properties described above and that the solutions of the variational problem

$$\begin{aligned} &\text{minimize} \quad E(u) := \int_{\mathbf{R}^N} m(\xi)\big|\widehat{u}(\xi)\big|^2 \, d\xi + \int_{\mathbf{R}^N} F(u)\, dx \\ &\text{under the constraint} \quad \int_{\mathbf{R}^N} G(u)\, dx = \lambda \neq 0 \end{aligned} \tag{3.1}$$

belong to $\mathcal{X}$. As before, the symbol $m(\xi) = m(\xi_1, \xi')$ is assumed to be symmetric with respect to ξ_1. Defining $W(u) := \int_{\mathbf{R}^N} m(\xi)|\widehat{u}(\xi)|^2 \, d\xi$, we suppose also that an identity of type (2.32) holds for $W(u)$ and it can be extended to $\mathcal{X}$ in such a way that

$$W(T_1 u) + W(T_2 u) - 2W(u) < 0 \quad \text{whenever } T_1 u \neq u,\ T_2 u \neq u.$$

(We will see later that most of the symbols in Examples 2.9–2.11 have this property.) Then, we claim that after a translation in the x_1 direction, any solution of (3.1) is symmetric with respect to x_1. Indeed, let u be a minimizer. After a translation in the x_1 direction, we may assume that

$$\int_{\{x_1<0\}} G\big(u(x)\big)\, dx = \int_{\{x_1>0\}} G\big(u(x)\big)\, dx = \frac{\lambda}{2}.$$

Denoting $u_1 = T_1 u$, $u_2 = T_2 u$, this implies

$$\int_{\mathbf{R}^N} G\big(u_1(x)\big)\, dx = 2 \int_{\{x_1<0\}} G\big(u(x)\big)\, dx = \lambda \quad \text{and} \quad \int_{\mathbf{R}^N} G\big(u_2(x)\big)\, dx = 2 \int_{\{x_1>0\}} G\big(u(x)\big)\, dx = \lambda;$$

consequently u_1 and u_2 (which belong to $\mathcal{X}$) also satisfy the constraint. It is obvious that

$$\int_{\mathbf{R}^N} F\big(u_1(x)\big)\, dx + \int_{\mathbf{R}^N} F\big(u_2(x)\big)\, dx = 2\int_{\mathbf{R}^N} F\big(u(x)\big)\, dx.$$

Suppose by contradiction that u is not symmetric with respect to x_1. Then we get

$$E(u_1) + E(u_2) - 2E(u) = W(u_1) + W(u_2) - 2W(u) < 0,$$

and this implies that either $E(u_1) < E(u)$ or $E(u_2) < E(u)$. Therefore u cannot be a minimizer and this proves the claim.

Given the motivation above, we will study the behavior of T_1 and T_2 from $H^s(\mathbf{R}^N)$ to $H^s(\mathbf{R}^N)$, respectively from $\dot{H}^s(\mathbf{R}^N)$ to $\dot{H}^s(\mathbf{R}^N)$, where

$$H^s(\mathbf{R}^N) = \left\{ u \in \mathcal{S}'(\mathbf{R}^N) \mid \widehat{u} \in L^1_{\mathrm{loc}}(\mathbf{R}^N) \text{ and } \int_{\mathbf{R}^N} (1 + |\xi|^2)^s \left|\widehat{u}(\xi)\right|^2 d\xi < \infty \right\},$$

$$\dot{H}^s(\mathbf{R}^N) = \left\{ u \in \mathcal{S}'(\mathbf{R}^N) \mid \widehat{u} \in L^1_{\mathrm{loc}}(\mathbf{R}^N) \text{ and } \int_{\mathbf{R}^N} |\xi|^{2s} \left|\widehat{u}(\xi)\right|^2 d\xi < \infty \right\}.$$

It happens that T_1 and T_2 are not well defined from $H^s(\mathbf{R}^N)$ to $H^s(\mathbf{R}^N)$ (respectively from $\dot{H}^s(\mathbf{R}^N)$ to $\dot{H}^s(\mathbf{R}^N)$) if $s \geqslant \frac{3}{2}$ or if $s \leqslant -\frac{1}{2}$, as it can be seen in the following example.

Example 3.1. (i) Define $\varphi : \mathbf{R} \to \mathbf{R}$, $\varphi(x) = xe^{-|x|}$. An easy computation shows that $\widehat{\varphi}(\xi) = \frac{-4i\xi}{(1+\xi^2)^2}$, hence $\varphi \in H^s(\mathbf{R})$ for any $s < \frac{5}{2}$ and $\varphi \in \dot{H}^s(\mathbf{R})$ for any $s \in (-\frac{3}{2}, \frac{5}{2})$. It is clear that $(T_1\varphi)(x) = -|x|e^{-|x|}$ and $\widehat{T_1\varphi}(\xi) = \frac{2(\xi^2-1)}{(1+\xi^2)^2}$. Consequently, $T_1\varphi \in H^s(\mathbf{R})$ for $s < \frac{3}{2}$ (respectively $T_1\varphi \in \dot{H}^s(\mathbf{R})$ for $-\frac{1}{2} < s < \frac{3}{2}$), but $T_1\varphi \notin H^s(\mathbf{R})$ and $T_1\varphi \notin \dot{H}^s(\mathbf{R})$ for $s \geqslant \frac{3}{2}$.

In dimension $N \geqslant 2$ it suffices to take $\psi(x) = \varphi(x_1)\varphi_1(x_2, \ldots, x_N)$, where $\varphi_1 \in C_c^\infty(\mathbf{R}^{N-1})$, to see that T_1 and T_2 are not well defined from $H^s(\mathbf{R}^N)$ to $H^s(\mathbf{R}^N)$ (respectively from $\dot{H}^s(\mathbf{R}^N)$ to $\dot{H}^s(\mathbf{R}^N)$) if $\frac{3}{2} \leqslant s < \frac{5}{2}$.

(ii) If $s < 0$, the elements of $H^s(\mathbf{R}^N)$ or $\dot{H}^s(\mathbf{R}^N)$ are not necessarily measurable functions. In this case we extend T_1 and T_2 to $H^s(\mathbf{R}^N)$ or $\dot{H}^s(\mathbf{R}^N)$ by duality. For $u, \varphi \in C_c^\infty(\mathbf{R}^N)$ we have

$$\begin{aligned}
\langle T_1 u, \varphi\rangle_{\mathcal{S}',\mathcal{S}} &= \int_{\mathbf{R}^N} (T_1 u)(x)\varphi(x)\,dx = \int_{\{x_1<0\}} u(x)\varphi(x)\,dx + \int_{\{x_1>0\}} u(-x_1, x')\varphi(x)\,dx \\
&= \int_{\{x_1<0\}} u(x)\varphi(x)\,dx + \int_{\{x_1<0\}} u(x_1, x')\varphi(-x_1, x')\,dx = \langle u, T_1^*\varphi\rangle_{L^2,L^2},
\end{aligned}$$

where $(T_1^*\varphi)(x) = \chi_{\{x_1<0\}}(\varphi(x_1, x') + \varphi(-x_1, x'))$. Hence, for $u \in H^s(\mathbf{R}^N)$ with $s < 0$ we should define $T_1 u$ by

$$\langle T_1 u, \varphi\rangle_{H^s,H^{-s}} = \langle u, T_1^*\varphi\rangle_{H^s,H^{-s}}$$

for any test function $\varphi \in C_c^\infty(\mathbf{R}^N)$. However, the operator T_1^* does not map $H^k(\mathbf{R}^N)$ into $H^k(\mathbf{R}^N)$ if $k \geqslant \frac{1}{2}$ (as it can be easily seen by taking the function $\eta(x) = e^{-|x|}$ in one dimension, respectively $\eta(x_1)\eta_1(x_2, \ldots, x_N)$, where $\eta_1 \in C_c^\infty(\mathbf{R}^{N-1})$ in dimension $N \geqslant 2$). This shows that we cannot define T_1 and T_2 on $H^s(\mathbf{R}^N)$ and on $\dot{H}^s(\mathbf{R}^N)$ if $s \leqslant -\frac{1}{2}$.

Our next goal is to prove that the operators T_1 and T_2 are well defined and continuous from $H^s(\mathbf{R}^N)$ to $H^s(\mathbf{R}^N)$ (respectively from $\dot{H}^s(\mathbf{R}^N)$ to $\dot{H}^s(\mathbf{R}^N)$) if $-\frac{1}{2} < s < \frac{3}{2}$. It is obvious that T_1 and T_2 are well defined and continuous from $L^2(\mathbf{R}^N)$ to $L^2(\mathbf{R}^N)$. It is well known that $H^1(\mathbf{R}^N) = W^{1,2}(\mathbf{R}^N) = \{\varphi \in L^2(\mathbf{R}^N) \mid \frac{\partial\varphi}{\partial x_i} \in L^2(\mathbf{R}^N),\ i = 1, \ldots, N\}$ and that $T_1, T_2 : W^{1,2}(\mathbf{R}^N) \to W^{1,2}(\mathbf{R}^N)$ are well defined and continuous. Using interpolation theory we

conclude that T_1 and T_2 are well defined and continuous from $H^s(\mathbf{R}^N)$ to $H^s(\mathbf{R}^N)$ if $0 \leqslant s \leqslant 1$. However, interpolation gives no information if either $s < 0$ or $s > 1$. Our next result deals with any value of s in $(-\frac{1}{2}, \frac{3}{2})$.

Theorem 3.2. *The operators T_1 and T_2 are well defined and continuous from $H^s(\mathbf{R}^N)$ to $H^s(\mathbf{R}^N)$ and from $\dot{H}^s(\mathbf{R}^N)$ to $\dot{H}^s(\mathbf{R}^N)$ for any $s \in (-\frac{1}{2}, \frac{3}{2})$.*

Proof. We will prove that there exists $C_s > 0$ such that for any $u \in C_c^\infty(\mathbf{R}^N)$ we have

$$\|T_i u\|_{H^s} \leqslant C_s \|u\|_{H^s}, \quad \text{respectively } \|T_i u\|_{\dot{H}^s} \leqslant C_s \|u\|_{\dot{H}^s}, \quad i = 1, 2, \tag{3.2}$$

and then the theorem will follow by density.

Therefore, suppose $u \in C_c^\infty(\mathbf{R}^N)$. By (2.48) and (2.49) we have

$$\begin{aligned} &\|T_1 u\|_{\dot{H}^s}^2 + \|T_2 u\|_{\dot{H}^s}^2 - 2\|u\|_{\dot{H}^s}^2 \\ &\quad = -\frac{16\sin(s\pi)}{\pi^2} \int\limits_{\mathbf{R}^{N-1}} \int\limits_{|\xi'|}^{\infty} \left(t^2 - |\xi'|^2\right)^s \left| \int\limits_0^\infty \widehat{f}(\xi_1, \xi') \frac{\xi_1}{t^2 + \xi_1^2}\, d\xi_1 \right|^2 dt\, d\xi', \end{aligned} \tag{3.3}$$

respectively

$$\begin{aligned} &\|T_1 u\|_{H^s}^2 + \|T_2 u\|_{H^s}^2 - 2\|u\|_{H^s}^2 \\ &\quad = -\frac{16\sin(s\pi)}{\pi^2} \int\limits_{\mathbf{R}^{N-1}} \int\limits_{\sqrt{|\xi'|^2+1}}^{\infty} \left(t^2 - |\xi'|^2 - 1\right)^s \left| \int\limits_0^\infty \widehat{f}(\xi_1, \xi') \frac{\xi_1}{t^2 + \xi_1^2}\, d\xi_1 \right|^2 dt\, d\xi'. \end{aligned} \tag{3.4}$$

If $N = 1$ we use the convention $\mathbf{R}^0 = \{0\}$ and the measure of $\{0\}$ is 1.

We begin by proving that T_1 and T_2 are bounded from $\dot{H}^s(\mathbf{R})$ to $\dot{H}^s(\mathbf{R})$, $-\frac{1}{2} < s < \frac{3}{2}$. For $N = 1$, the integral in the right-hand side of (3.3) can be formally written as

$$\int\limits_0^\infty \int\limits_0^\infty \int\limits_0^\infty t^{2s} \frac{\xi}{t^2 + \xi^2} \cdot \frac{\eta}{t^2 + \eta^2} \widehat{f}(\xi) \overline{\widehat{f}(\eta)}\, d\xi\, d\eta\, dt. \tag{3.5}$$

Our strategy is as follows: first we compute explicitly the integral

$$I_s(\xi, \eta) = \int\limits_0^\infty t^{2s} \frac{\xi}{t^2 + \xi^2} \cdot \frac{\eta}{t^2 + \eta^2}\, dt = \xi\eta \int\limits_0^\infty t^{2s} \frac{1}{t^2 + \xi^2} \cdot \frac{1}{t^2 + \eta^2}\, dt. \tag{3.6}$$

Observe that $I_s(\xi, \eta) > 0$ if $\xi > 0$, $\eta > 0$. Then we will prove that for any $s \in (-\frac{1}{2}, \frac{3}{2})$ and any $\varphi, \psi \in L^2(0, \infty)$ we have

$$\left| \int\limits_0^\infty \int\limits_0^\infty \xi^{-s} \eta^{-s} I_s(\xi, \eta) \varphi(\xi) \psi(\eta)\, d\xi\, d\eta \right| \leqslant C(s) \|\varphi\|_{L^2(0,\infty)} \cdot \|\psi\|_{L^2(0,\infty)}.$$

This will be done in Lemma 3.3. Thereafter it will be clear that for any $f \in \dot{H}^s(\mathbf{R})$ we have

$$\begin{aligned}
&\int_0^\infty\int_0^\infty I_s(\xi,\eta)\big|\widehat{f}(\xi)\big|\cdot\big|\overline{\widehat{f}(\eta)}\big|\,d\xi\,d\eta \\
&\quad=\int_0^\infty\int_0^\infty \xi^{-s}\eta^{-s}I_s(\xi,\eta)\big|\xi^s\widehat{f}(\xi)\big|\cdot\big|\eta^s\overline{\widehat{f}(\eta)}\big|\,d\xi\,d\eta \\
&\quad\leqslant C(s)\big\||\cdot|^s\widehat{f}\big\|^2_{L^2(0,\infty)}\leqslant C(s)\|f\|^2_{\dot{H}^s(\mathbf{R})}.
\end{aligned}\tag{3.7}$$

This justifies the use of Fubini's Theorem in evaluating (3.5) and proves that the right-hand side of (3.3) is less than $C_1(s)\|f\|^2_{\dot{H}^s(\mathbf{R})}$, where $C_1(s)$ is a constant depending only on s. Thus we infer that there exists $C_s > 0$ such that $\|T_1u\|_{\dot{H}^s(\mathbf{R})} \leqslant C_s\|u\|_{\dot{H}^s(\mathbf{R})}$ and $\|T_2u\|_{\dot{H}^s(\mathbf{R})} \leqslant C_s\|u\|_{\dot{H}^s(\mathbf{R})}$ for any $u \in C_c^\infty(\mathbf{R})$. Consequently, T_1 and T_2 can be extended as continuous linear mappings form $\dot{H}^s(\mathbf{R})$ to $\dot{H}^s(\mathbf{R})$, $-\frac{1}{2} < s < \frac{3}{2}$, as claimed.

To carry out the first step of this strategy, we come back to $I_s(\xi,\eta)$ given by (3.6). Since the complex logarithm can be defined analytically on $\mathbf{C}\setminus\{it \mid t\in(-\infty,0]\}$, we may define the holomorphic function $z \mapsto z^{2s} := e^{2s\log(z)} = |z|^{2s}e^{2is\arg(z)}$ on $\mathbf{C}\setminus\{it \mid t\in(-\infty,0]\}$. With this definition the function $k(z) = \frac{z^{2s}}{(z^2+\xi^2)(z^2+\eta^2)}$ is meromorphic on $\mathbf{C}\setminus\{it \mid t\in(-\infty,0]\}$. If $\xi\neq\eta$, k has two simple poles, namely $i\xi$ and $i\eta$; if $\xi=\eta$ it has a double pole at $i\xi$. For $0<\varepsilon<\min(\xi,\eta)$, and $R>\max(\xi,\eta)$, consider the closed path $\beta_{\varepsilon,R}$ composed by the following pieces:

$$\begin{aligned}
&\beta_{1,\varepsilon,R}(t)=t, && t\in[-R,-\varepsilon],\\
&\beta_{2,\varepsilon}(\theta)=\varepsilon e^{i(\pi-\theta)}, && \theta\in[0,\pi],\\
&\beta_{3,\varepsilon,R}(t)=t, && t\in[\varepsilon,R],\\
&\beta_{4,R}(\theta)=Re^{i\theta}, && \theta\in[0,\pi].
\end{aligned}$$

If $\xi\neq\eta$, using the Residue Theorem we get

$$\int_{\beta\varepsilon,R} k(z)\,dz = 2\pi i\big[\operatorname{Res}(k,i\xi)+\operatorname{Res}(k,i\eta)\big]=\pi e^{is\pi}\left[\frac{\xi^{2s}}{\xi(\eta^2-\xi^2)}+\frac{\eta^{2s}}{\eta(\xi^2-\eta^2)}\right].\tag{3.8}$$

Since $s>-\frac{1}{2}$ we have $\lim_{\varepsilon\to 0}\int_{\beta_{2,\varepsilon}}k(z)\,dz=0$. We have also $\lim_{R\to\infty}\int_{\beta_{4,R}}k(z)\,dz=0$ because $s<\frac{3}{2}$. Passing to the limit as $\varepsilon\to 0$ in (3.8) and then passing to the limit as $R\to\infty$ in the resulting equation, we get

$$\int_{-\infty}^0 k(z)\,dz+\int_0^\infty k(z)\,dz=\pi e^{is\pi}\frac{\xi^{2s-1}-\eta^{2s-1}}{\eta^2-\xi^2},$$

that is

$$\left(e^{2is\pi}+1\right)\int_0^\infty \frac{t^{2s}}{(t^2+\xi^2)(t^2+\eta^2)}\,dt = \pi e^{is\pi}\frac{\xi^{2s-1}-\eta^{2s-1}}{\eta^2-\xi^2}.$$

For $s \neq \frac{1}{2}$ we obtain

$$\int_0^\infty \frac{t^{2s}}{(t^2+\xi^2)(t^2+\eta^2)}\,dt = \frac{\pi}{2\cos(s\pi)}\frac{\xi^{2s-1}-\eta^{2s-1}}{\eta^2-\xi^2}. \tag{3.9}$$

For $s = \frac{1}{2}$ we compute directly

$$\int_0^\infty \frac{t}{(t^2+\xi^2)(t^2+\eta^2)}\,dt = \frac{1}{\eta^2-\xi^2}\int_0^\infty \frac{t}{t^2+\xi^2}-\frac{t}{t^2+\eta^2}\,dt = \frac{\ln\eta-\ln\xi}{\eta^2-\xi^2}. \tag{3.10}$$

Hence

$$I_s(\xi,\eta) = \frac{\pi}{2\cos(s\pi)}\frac{\xi\eta(\xi^{2s-1}-\eta^{2s-1})}{\eta^2-\xi^2} \quad \text{if } s\neq\frac{1}{2}, \quad \text{and}$$
$$I_{\frac{1}{2}}(\xi,\eta) = \frac{\xi\eta(\ln\eta-\ln\xi)}{\eta^2-\xi^2}. \tag{3.11}$$

This gives

$$\xi^{-s}\eta^{-s}I_s(\xi,\eta) = \frac{\pi}{2\cos(s\pi)}\frac{\xi^s\eta^{1-s}-\xi^{1-s}\eta^s}{\eta^2-\xi^2} \quad \text{if } s\neq\frac{1}{2},$$

and

$$\xi^{-\frac{1}{2}}\eta^{-\frac{1}{2}}I_{\frac{1}{2}}(\xi,\eta) = \xi^{\frac{1}{2}}\eta^{\frac{1}{2}}\frac{\ln\eta-\ln\xi}{\eta^2-\xi^2}.$$

An interesting property of these functions is given by the next lemma.

Lemma 3.3. *Let $K_s(\xi,\eta) = \frac{\xi^s\eta^{1-s}-\xi^{1-s}\eta^s}{\eta^2-\xi^2}$ if $s\neq\frac{1}{2}$, respectively $K_{\frac{1}{2}}(\xi,\eta) = \xi^{\frac{1}{2}}\eta^{\frac{1}{2}}\frac{\ln\eta-\ln\xi}{\eta^2-\xi^2}$. For any $s\in(-\frac{1}{2},\frac{3}{2})$ there exists a constant $C(s)$ (depending only on s) such that for any $\varphi,\psi\in L^2(0,\infty)$ we have*

$$\left|\int_0^\infty\int_0^\infty \varphi(\xi)K_s(\xi,\eta)\psi(\eta)\,d\xi\,d\eta\right| \leqslant C(s)\|\varphi\|_{L^2(0,\infty)}\|\psi\|_{L^2(0,\infty)}.$$

Proof. Using polar coordinates we write $\xi = r\cos(\theta)$, $\eta = r\sin(\theta)$, where $r = \sqrt{\xi^2+\eta^2}$ and $\theta = \arctan\frac{\eta}{\xi}$. It is easy to see that $K_s(\xi,\eta) = \frac{1}{r}L_s(\theta)$, where

$$L_s(\theta) = \frac{(\sin\theta)^s(\cos\theta)^{1-s} - (\cos\theta)^s(\sin\theta)^{1-s}}{\cos^2\theta - \sin^2\theta} \quad \text{if } s \neq \frac{1}{2},$$

and

$$L_{\frac{1}{2}}(\theta) = \frac{-\ln\tan\theta}{(1-\tan^2\theta)\cos^2\theta}(\sin\theta)^{\frac{1}{2}}(\cos\theta)^{\frac{1}{2}}.$$

By a change of variables we get

$$\int_0^\infty\int_0^\infty \left|\varphi(\xi)K_s(\xi,\eta)\psi(\eta)\right| d\xi\, d\eta = \int_0^{\frac{\pi}{2}}\int_0^\infty \left|\varphi(r\cos\theta)\psi(r\sin\theta)\right| dr \left|L_s(\theta)\right| d\theta.$$

Using the Cauchy–Schwarz inequality we have

$$\int_0^\infty \left|\varphi(r\cos\theta)\psi(r\sin\theta)\right| dr \leqslant \left\|\varphi(\cdot\cos\theta)\right\|_{L^2(0,\infty)}\left\|\psi(\cdot\sin\theta)\right\|_{L^2(0.\infty)} = \frac{\|\varphi\|_{L^2(0,\infty)}\|\psi\|_{L^2(0,\infty)}}{\sqrt{\cos\theta\cdot\sin\theta}}.$$

Consequently,

$$\int_0^\infty\int_0^\infty \left|\varphi(\xi)K_s(\xi,\eta)\psi(\eta)\right| d\xi\, d\eta \leqslant \|\varphi\|_{L^2(0,\infty)}\|\psi\|_{L^2(0,\infty)} \int_0^{\frac{\pi}{2}} \frac{|L_s(\theta)|}{\sqrt{\cos\theta\cdot\sin\theta}}\, d\theta. \tag{3.12}$$

The lemma will be proved if we show that the last integral in (3.12) is finite. If $s \neq \frac{1}{2}$ we have

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \frac{|L_s(\theta)|}{\sqrt{\cos\theta\cdot\sin\theta}}\, d\theta &= \int_0^{\frac{\pi}{2}} \left|\frac{(\sin\theta)^{s-\frac{1}{2}}(\cos\theta)^{\frac{1}{2}-s} - (\cos\theta)^{s-\frac{1}{2}}(\sin\theta)^{\frac{1}{2}-s}}{\cos^2\theta-\sin^2\theta}\right| d\theta \\
&= \int_0^{\frac{\pi}{2}} \left|\frac{(\tan\theta)^{s-\frac{1}{2}} - (\tan\theta)^{\frac{1}{2}-s}}{1-\tan^2\theta}\right| \cdot \frac{1}{\cos^2\theta}\, d\theta \\
&= \int_0^\infty \left|\frac{t^{s-\frac{1}{2}} - t^{\frac{1}{2}-s}}{1-t^2}\right| dt.
\end{aligned} \tag{3.13}$$

Using l'Hôspital's rule it is easy to see that $\lim_{t\to 1}\frac{t^{s-1/2}-t^{1/2-s}}{1-t^2} = \frac{1}{2} - s$; hence the function $t \mapsto \frac{t^{s-1/2}-t^{1/2-s}}{1-t^2}$ is bounded near 1. Since $s - \frac{1}{2} \in (-1,1)$, the last integral in (3.13) converges.

If $s = \frac{1}{2}$ we have

$$\int_0^{\frac{\pi}{2}} \frac{|L_{\frac{1}{2}}(\theta)|}{\sqrt{\cos\theta \cdot \sin\theta}}\, d\theta = \int_0^{\frac{\pi}{2}} \left| \frac{-\ln \tan\theta}{1 - \tan^2\theta} \right| \cdot \frac{1}{\cos^2\theta}\, d\theta = \int_0^{\infty} \left| \frac{\ln y}{y^2 - 1} \right| dy. \tag{3.14}$$

Note that $\lim_{y \to 1} \frac{\ln y}{y^2-1} = \frac{1}{2}$ and this implies that the last integral in (3.14) converges. This completes the proof of Lemma 3.3. $\square$

In view of (3.3), (3.5), (3.7), (3.11) and Lemma 3.3, it follows that T_1 and T_2 are well defined and continuous from $\dot{H}^s(\mathbf{R})$ to $\dot{H}^s(\mathbf{R})$ for $-\frac{1}{2} < s < \frac{3}{2}$.

Next we prove that T_1 and T_2 are continuous from $H^s(\mathbf{R})$ to $H^s(\mathbf{R})$. We estimate the integral in the right-hand side of (3.4) for $N = 1$. If $s \in [0, \frac{3}{2})$ we have by (3.5)–(3.7)

$$\begin{aligned} \int_1^{\infty} \left(t^2 - 1\right)^s \left| \int_0^{\infty} \widehat{f}(\xi) \frac{\xi}{t^2 + \xi^2}\, d\xi \right|^2 dt &\leqslant \int_0^{\infty} t^{2s} \left| \int_0^{\infty} \widehat{f}(\xi) \frac{\xi}{t^2 + \xi^2}\, d\xi \right|^2 dt \\ &\leqslant C(s) \|f\|^2_{\dot{H}^s} \leqslant C(s) \|f\|^2_{H^s}. \end{aligned} \tag{3.15}$$

If $s \in (-\frac{1}{2}, 0)$, using the change of variable $\tau = \sqrt{t^2 - 1}$ and (3.9) we get

$$\begin{aligned} \int_1^{\infty} \frac{(t^2 - 1)^s}{(t^2 + \xi^2)(t^2 + \eta^2)}\, dt &= \int_0^{\infty} \frac{\tau^{2s}}{(\tau^2 + 1 + \xi^2)(\tau^2 + 1 + \eta^2)} \cdot \frac{\tau}{\sqrt{\tau^2 + 1}}\, d\tau \\ &\leqslant \int_0^{\infty} \frac{\tau^{2s}}{(\tau^2 + 1 + \xi^2)(\tau^2 + 1 + \eta^2)}\, d\tau \\ &= \frac{\pi}{2\cos(s\pi)} \cdot \frac{(1 + \xi^2)^{\frac{2s-1}{2}} - (1 + \eta^2)^{\frac{2s-1}{2}}}{\eta^2 - \xi^2}. \end{aligned} \tag{3.16}$$

Consequently,

$$\begin{aligned} &\int_1^{\infty} \left(t^2 - 1\right)^s \left| \int_0^{\infty} \widehat{f}(\xi) \frac{\xi}{t^2 + \xi^2}\, d\xi \right|^2 dt \\ &\quad \leqslant \int_0^{\infty} \int_0^{\infty} \left|\widehat{f}(\xi)\right| \cdot \left|\overline{\widehat{f}(\eta)}\right| \int_1^{\infty} \left(t^2 - 1\right)^s \frac{\xi\eta}{(t^2 + \xi^2)(t^2 + \eta^2)}\, dt\, d\xi\, d\eta \\ &\quad \leqslant \frac{\pi}{2\cos(s\pi)} \int_0^{\infty} \int_0^{\infty} \left|\widehat{f}(\xi)\right| \cdot \left|\overline{\widehat{f}(\eta)}\right| \cdot \xi\eta \frac{(1 + \xi^2)^{\frac{2s-1}{2}} - (1 + \eta^2)^{\frac{2s-1}{2}}}{\eta^2 - \xi^2}\, d\xi\, d\eta \end{aligned}$$

$$= \frac{\pi}{2\cos(s\pi)} \int_0^\infty \int_0^\infty \left(1+\xi^2\right)^{\frac{s}{2}} \left|\widehat{f}(\xi)\right| \cdot \left(1+\eta^2\right)^{\frac{s}{2}} \left|\overline{\widehat{f}(\eta)}\right|$$

$$\times \frac{\xi\eta}{\eta^2-\xi^2} \cdot \frac{(1+\xi^2)^{\frac{2s-1}{2}} - (1+\eta^2)^{\frac{2s-1}{2}}}{(1+\xi^2)^{\frac{s}{2}}(1+\eta^2)^{\frac{s}{2}}}\, d\xi\, d\eta. \tag{3.17}$$

It is elementary to prove that for any $\xi, \eta > 0$, $\xi \neq \eta$ we have

$$\frac{\xi\eta}{\eta^2-\xi^2} \cdot \frac{(1+\xi^2)^{\frac{2s-1}{2}} - (1+\eta^2)^{\frac{2s-1}{2}}}{(1+\xi^2)^{\frac{s}{2}}(1+\eta^2)^{\frac{s}{2}}} \leqslant \frac{\xi^s\eta^{1-s} - \xi^{1-s}\eta^s}{\eta^2-\xi^2} = K_s(\xi,\eta). \tag{3.18}$$

Coming back to (3.17) and using Lemma 3.3 we obtain

$$\int_1^\infty \left(t^2-1\right)^s \left|\int_0^\infty \widehat{f}(\xi)\frac{\xi}{t^2+\xi^2}\, d\xi\right|^2 dt \leqslant \frac{\pi C(s)}{2\cos(s\pi)} \left\|\left(1+|\cdot|^2\right)^{\frac{s}{2}}\widehat{f}\right\|^2_{L^2(0,\infty)}$$

$$\leqslant C'(s)\|f\|^2_{H^s}. \tag{3.19}$$

From (3.4) and (3.15) if $s \in [0, \frac{3}{2})$, respectively from (3.4) and (3.19) if $s \in (-\frac{1}{2}, 0)$, we infer that T_1 and T_2 can be extended as linear continuous operators from $H^s(\mathbf{R})$ to $H^s(\mathbf{R})$.

Now we prove Theorem 3.2 in the case $N \geqslant 2$.

If $s \in [0, \frac{3}{2})$, arguing as in (3.5)–(3.7) and using Lemma 3.3 we have

$$\int_{|\xi'|}^\infty \left(t^2-|\xi'|^2\right)^s \left|\int_0^\infty \widehat{f}(\xi_1,\xi')\frac{\xi_1}{t^2+\xi_1^2}\, d\xi_1\right|^2 dt$$

$$\leqslant \int_0^\infty t^{2s} \left|\int_0^\infty \widehat{f}(\xi_1,\xi')\frac{\xi_1}{t^2+\xi_1^2}\, d\xi_1\right|^2 dt$$

$$\leqslant \int_0^\infty \int_0^\infty \left|\widehat{f}(\xi_1,\xi')\right|\xi_1^s \cdot \left|\overline{\widehat{f}(\eta_1,\xi')}\right|\eta_1^s \cdot \left(\xi_1^{-s}\eta_1^{-s} I_s(\xi_1,\eta_1)\right) d\xi_1\, d\eta_1$$

$$\leqslant C(s)\left\||\cdot|^s \widehat{f}(\cdot,\xi')\right\|^2_{L^2(0,\infty)} \leqslant C(s) \int_{-\infty}^\infty \left(\xi_1^2+|\xi'|^2\right)^s \left|\widehat{f}(\xi_1,\xi')\right|^2 d\xi_1. \tag{3.20}$$

If $s \in (-\frac{1}{2}, 0)$, using the change of variable $\tau = \sqrt{t^2-|\xi'|^2}$, arguing as in the proof of (3.16), then taking (3.9) into account we obtain

$$\int_{|\xi'|}^\infty \frac{(t^2-|\xi'|^2)^s}{(t^2+\xi^2)(t^2+\eta^2)}\, dt = \int_0^\infty \frac{\tau^{2s}}{(\tau^2+|\xi'|^2+\xi_1^2)(\tau^2+|\xi'|^2+\eta_1^2)} \cdot \frac{\tau}{\sqrt{\tau^2+|\xi'|^2}}\, d\tau$$

$$\leqslant \int_0^\infty \frac{\tau^{2s}}{(\tau^2+|\xi'|^2+\xi_1^2)(\tau^2+|\xi'|^2+\eta_1^2)}\,d\tau$$

$$= \frac{\pi}{2\cos(s\pi)} \cdot \frac{(|\xi'|^2+\xi_1^2)^{\frac{2s-1}{2}} - (|\xi'|^2+\eta_1^2)^{\frac{2s-1}{2}}}{\eta_1^2-\xi_1^2}.$$

We also have

$$\frac{\xi_1\eta_1}{\eta_1^2-\xi_1^2} \cdot \frac{(\xi_1^2+|\xi'|^2)^{\frac{2s-1}{2}} - (\eta_1^2+|\xi'|^2)^{\frac{2s-1}{2}}}{(\xi_1^2+|\xi'|^2)^{\frac{s}{2}}(\eta_1^2+|\xi'|^2)^{\frac{s}{2}}} \leqslant K_s(\xi_1,\eta_1)$$

(this inequality is analogous to (3.18)). Arguing as in (3.17), using the two previous inequalities and Lemma 3.3 we get

$$\begin{aligned}
&\int_{|\xi'|}^\infty \big(t^2-|\xi'|^2\big)^s \left| \int_0^\infty \widehat{f}(\xi_1,\xi')\frac{\xi_1}{t^2+\xi_1^2}\,d\xi_1\right|^2 dt \\
&\quad\leqslant \frac{\pi C(s)}{2\cos(s\pi)} \big\| \big(|\xi'|^2+|\cdot|^2\big)^{\frac{s}{2}}\widehat{f}(\cdot,\xi')\big\|^2_{L^2(0,\infty)} \\
&\quad\leqslant C'(s)\int_{-\infty}^\infty \big(\xi_1^2+|\xi'|^2\big)^s\big|\widehat{f}(\xi_1,\xi')\big|^2\,d\xi_1.
\end{aligned} \tag{3.21}$$

Integrating (3.20), respectively (3.21), over $\mathbf{R}^{N-1}$ we infer that the integral in the right-hand side of (3.3) is less than $C''(s)\|f\|^2_{\dot{H}^s}$. This proves that T_1 and T_2 can be extended by continuity from $\dot{H}^s(\mathbf{R}^N)$ to $\dot{H}^s(\mathbf{R}^N)$ for $s \in (-\frac{1}{2},\frac{3}{2})$.

In a similar way we show that T_1 and T_2 can be extended by continuity from $H^s(\mathbf{R}^N)$ to $H^s(\mathbf{R}^N)$ for $s \in (-\frac{1}{2},\frac{3}{2})$. Theorem 3.2 is now proved. □

For a measurable function u defined on $\mathbf{R}^N$, we define its antisymmetric part in the x_1 direction by $Au(x_1,x') = \frac{1}{2}(u(x_1,x') - u(-x_1,x'))$. If u is a tempered distribution, we define Au by $\langle Au,\phi\rangle_{\mathcal{S}',\mathcal{S}} = \langle u, A\phi\rangle_{\mathcal{S}',\mathcal{S}}$ for any $\phi \in \mathcal{S}$. Obviously, Au is odd with respect to x_1 (for distributions, this means that $\langle Au,\phi(-x_1,x')\rangle_{\mathcal{S}',\mathcal{S}} = -\langle Au,\phi\rangle_{\mathcal{S}',\mathcal{S}}$). It is clear from the definition that A defines a linear continuous map from $H^s(\mathbf{R}^N)$ to $H^s(\mathbf{R}^N)$ (respectively from $\dot{H}^s(\mathbf{R}^N)$ to $\dot{H}^s(\mathbf{R}^N)$) for any s. Moreover, for any tempered distribution u, the distribution $\mathcal{F}(Au)$ is odd with respect to x_1.

It follows from the proof of Theorem 3.2 that for any $s \in (-\frac{1}{2},\frac{3}{2})$, the following complex bilinear forms are continuous:

$$B_{N,s} : \dot{H}^s\big(\mathbf{R}^N\big) \times \dot{H}^s\big(\mathbf{R}^N\big) \to \mathbf{C},$$

$$B_{N,s}(u,v)=\int\limits_{\mathbf{R}^{N-1}}\int\limits_{|\xi'|}^{\infty}\left(t^2-|\xi'|^2\right)^s\int\limits_0^{\infty}\widehat{Au}(\xi_1,\xi')\frac{\xi_1}{t^2+\xi_1^2}\,d\xi_1$$
$$\times\int\limits_0^{\infty}\overline{\widehat{Av}(\eta_1,\xi')}\frac{\eta_1}{t^2+\eta_1^2}\,d\eta_1\,dt\,d\xi',$$

$$\widetilde{B}_{N,s}:H^s\left(\mathbf{R}^N\right)\times H^s\left(\mathbf{R}^N\right)\to\mathbf{C},$$

$$\widetilde{B}_{N,s}(u,v)=\int\limits_{\mathbf{R}^{N-1}}\int\limits_{\sqrt{|\xi'|^2+1}}^{\infty}\left(t^2-|\xi'|^2-1\right)^s\int\limits_0^{\infty}\widehat{Au}(\xi_1,\xi')\frac{\xi_1}{t^2+\xi_1^2}\,d\xi_1$$
$$\times\int\limits_0^{\infty}\overline{\widehat{Av}(\eta_1,\xi')}\frac{\eta_1}{t^2+\eta_1^2}\,d\eta_1\,dt\,d\xi'.$$

Moreover, from (3.3) and (3.4) we have the identities

$$\|T_1u\|^2_{\dot{H}^s(\mathbf{R}^N)}+\|T_2u\|^2_{\dot{H}^s(\mathbf{R}^N)}-2\|u\|^2_{\dot{H}^s(\mathbf{R}^N)}=-\frac{16\sin(s\pi)}{\pi^2}B_{N,s}(Au,Au),\tag{3.22}$$

$$\|T_1u\|^2_{H^s(\mathbf{R}^N)}+\|T_2u\|^2_{H^s(\mathbf{R}^N)}-2\|u\|^2_{H^s(\mathbf{R}^N)}=-\frac{16\sin(s\pi)}{\pi^2}\widetilde{B}_{N,s}(Au,Au)\tag{3.23}$$

for any $u\in C_c^\infty(\mathbf{R}^N)$. From Theorem 3.2, the continuity of $B_{N,s}$ and of $\widetilde{B}_{N,s}$ and the density of $C_c^\infty(\mathbf{R}^N)$ in $\dot{H}^s(\mathbf{R}^N)$ and in $H^s(\mathbf{R}^N)$ we infer that we have the following.

Corollary 3.4. *Let $s\in(-\frac{1}{2},\frac{3}{2})$. The identity* (3.22) *holds for any $u\in\dot{H}^s(\mathbf{R}^N)$ and* (3.23) *holds for any $u\in H^s(\mathbf{R}^N)$.*

Our next aim is to show that the quadratic forms $B_{N,s}$ and $\widetilde{B}_{N,s}$ define norms in some spaces of odd functions. We start with the following proposition.

Lemma 3.5. *Assume that $g:\mathbf{R}\to\mathbf{C}$ is a measurable function, $g(-t)=-g(t)$ a.e. and*

- *either $g\in L^p(\mathbf{R})$ for some $p\in(1,\infty)$,*
- *or $(k^2+\xi^2)^{\frac{s}{2}}g(\xi)\in L^2(\mathbf{R})$ for some $k\in\mathbf{R}$ and $s\in(-\frac{1}{2},\frac{3}{2})$.*

Suppose that the set

$$A=\left\{x>0\;\middle|\;\int\limits_0^{\infty}\frac{\xi}{x^2+\xi^2}g(\xi)\,d\xi=0\right\}$$

has a limit point $x_0>0$.

Then $g=0$ almost everywhere on $\mathbf{R}$.

Proof. We may suppose without loss of generality that g is real (otherwise we carry out the proof for its real and imaginary parts).

First we deal with the simpler case $g \in L^p(\mathbf{R})$ for some p, $1 < p < \infty$. We define the Poisson integrals for g,

$$a(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x}{x^2 + (y-t)^2} g(t)\,dt \quad \text{and}$$

$$b(x,y) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y-t}{x^2 + (y-t)^2} g(t)\,dt. \tag{3.24}$$

It follows from Lemma 2.4(iii) that the functions a and b are well defined and harmonic in the right half-plane and $r(x+iy) := a(x,y) + ib(x,y)$ is holomorphic in $\{z \in \mathbf{C} \mid \mathrm{Re}(z) > 0\}$. Since g is odd, we have $a(x,0) = 0$ for any $x > 0$. If $x \in A$, we have also $b(x,0) = 0$. Consequently, $r(x) = 0$ for any $x \in A$. But r is holomorphic and A has a limit point $x_0 > 0$, thus necessarily $r \equiv 0$. By Lemma 2.4(ii) we know that $a(x,y) \to g(y)$ as $x \to 0$ for almost every y, hence $g = 0$ a.e. on $\mathbf{R}$.

Suppose that $(k^2 + |\cdot|^2)^{\frac{s}{2}} g \in L^2(\mathbf{R})$ for some $k \neq 0$ and $s \in (-\frac{1}{2}, \frac{3}{2})$. We may assume that $k = 1$. If $s \in [0, \frac{3}{2})$, then obviously $g \in L^2(\mathbf{R})$ and the conclusion of the lemma follows from the above considerations. If $s \in (-\frac{1}{2}, 0)$, then for any $x > 0$ and $y \in \mathbf{R}$ the functions $\varphi_{x,y}(t) = (1+t^2)^{-\frac{s}{2}} \frac{x}{x^2+(y-t)^2}$ and $\psi_{x,y}(t) = (1+t^2)^{-\frac{s}{2}} \frac{y-t}{x^2+(y-t)^2}$ belong to $L^2(\mathbf{R})$. We may write

$$\int_{-\infty}^{\infty} \frac{x}{x^2 + (y-t)^2} g(t)\,dt = \int_{-\infty}^{\infty} \varphi_{x,y}(t)\big(1+t^2\big)^{\frac{s}{2}} g(t)\,dt$$

and

$$\int_{-\infty}^{\infty} \frac{y-t}{x^2 + (y-t)^2} g(t)\,dt = \int_{-\infty}^{\infty} \psi_{x,y}(t)\big(1+t^2\big)^{\frac{s}{2}} g(t)\,dt.$$

Using the Cauchy–Schwarz inequality, we see that the functions a and b in (3.24) are well defined in the right half-plane (in particular, $\int_0^\infty \frac{\xi}{x^2+\xi^2} g(\xi)\,d\xi$ exists for any $x > 0$). Clearly the function $r(x+iy) := a(x,y) + ib(x,y)$ is holomorphic and, as above we have $r(x) = 0$ for $x \in A$. Since A has a limit point $x_0 > 0$, we infer that $r \equiv 0$. Next, we have $\lim_{x \downarrow 0} a(x,y) = g(y)$ whenever y is a Lebesgue point of g (the proof of this fact follows from standard arguments and it is quite similar to the proof of [24, Theorem 1.25, p. 15]; for brevity, we omit it). This obviously implies $g = 0$ a.e., as desired.

Now let us consider the case $k = 0$. If $|\cdot|^s g \in L^2(\mathbf{R})$ and $s \in (-\frac{1}{2}, \frac{1}{2})$, we may repeat almost word by word the proof above (we have only to replace the functions $\varphi_{x,y}$ and $\psi_{x,y}$ by $t \mapsto t^{-s} \frac{x}{x^2+(y-t)^2}$, respectively by $t \mapsto t^{-s} \frac{y-t}{x^2+(y-t)^2}$).

If $|\cdot|^s g \in L^2(\mathbf{R})$ and $s \in [\frac{1}{2}, \frac{3}{2})$, the integrals defining a and b in (3.24) do not necessarily converge. In this case we define

$$a_1(x,y)=\frac{1}{\pi}\int_0^\infty \frac{4xyt}{[x^2+(y-t)^2][x^2+(y+t)^2]}g(t)\,dt \quad \text{and}$$

$$b_1(x,y)=\frac{1}{\pi}\int_0^\infty \frac{2t(t^2+x^2-y^2)}{[x^2+(y-t)^2][x^2+(y+t)^2]}g(t)\,dt. \tag{3.25}$$

Notice that if $g\in L^1_{\mathrm{loc}}(\mathbf{R})$ is odd and $\frac{g(t)}{t}\in L^1([1,\infty))$, then $a=a_1$ and $b=b_1$. It is obvious that for fixed $x>0$, $y\in\mathbf{R}$ and $s\in(-\frac12,\frac32)$, the functions $\varphi_1(t)=t^{-s}\frac{4xyt}{[x^2+(y-t)^2][x^2+(y+t)^2]}$ and $\psi_1(t)=t^{-s}\frac{2t(t^2+x^2-y^2)}{[x^2+(y-t)^2][x^2+(y+t)^2]}$ belong to $L^2((0,\infty))$ and this implies that a_1 and b_1 are well defined. It is straightforward that $r_1(x+iy):=a_1(x,y)+ib_1(x,y)$ is holomorphic in the right half-plane. Obviously $a_1(x,0)=0$ for any $x>0$ and $b_1(x,0)=\frac{2}{\pi}\int_0^\infty \frac{t}{x^2+t^2}g(t)\,dt=0$ for $x\in A$. Consequently $r=0$ on A. Since A has a limit point $x_0>0$, we infer that $r\equiv 0$ in the right half-plane. Let $y>0$ be a Lebesgue point of g. Since

$$\int_0^\infty \frac{4xyt}{[x^2+(y-t)^2][x^2+(y+t)^2]}\,dt=2\arctan\frac{y}{x},$$

proceeding as in the previous cases, one can show that $|a_1(x,y)-\frac{2}{\pi}(\arctan\frac{y}{x})g(y)|\to 0$ as $x\to 0$, hence $\lim_{x\downarrow 0}a_1(x,y)=g(y)$. Consequently we have $\lim_{x\downarrow 0}a_1(x,y)=g(y)$ for almost every y and the lemma is proved. □

We set

$$H^s_{1,\mathrm{odd}}(\mathbf{R}^N)=\{f\in H^s(\mathbf{R}^N)\mid f \text{ is odd with respect to } x_1\}=\{f\in H^s(\mathbf{R}^N)\mid f=Af\},$$
$$\dot H^s_{1,\mathrm{odd}}(\mathbf{R}^N)=\{f\in \dot H^s(\mathbf{R}^N)\mid f \text{ is odd with respect to } x_1\}=\{f\in \dot H^s(\mathbf{R}^N)\mid f=Af\},$$

where, as before, Af is the antisymmetric part of f in the x_1 direction. For $f\in\dot H^s_{1,\mathrm{odd}}(\mathbf{R}^N)$ we define $N_s(f)=(B_{N,s}(f,f))^{\frac12}$ and for $f\in H^s_{1,\mathrm{odd}}(\mathbf{R}^N)$ we define $\widetilde N_s(f)=(\widetilde B_{N,s}(f,f))^{\frac12}$.

Theorem 3.6. *$\widetilde N_s$ is a norm on $H^s_{1,\mathrm{odd}}(\mathbf{R}^N)$, continuous with respect to the usual H^s norm, and N_s is a norm on $\dot H^s_{1,\mathrm{odd}}(\mathbf{R}^N)$, continuous with respect to the $\dot H^s$ norm.*

Endowed with these norms, $H^s_{1,\mathrm{odd}}(\mathbf{R}^N)$ and $\dot H^s_{1,\mathrm{odd}}(\mathbf{R}^N)$ are pre-Hilbert spaces.

Proof. It is clear that $\widetilde B_{N,s}$ and $B_{N,s}$ are complex-symmetric bilinear forms on $H^s(\mathbf{R}^N)$ (respectively on $\dot H^s(\mathbf{R}^N)$) and that $\widetilde B_{N,s}(f,f)\geqslant 0$ and $B_{N,s}(f,f)\geqslant 0$ for any f (thus, in particular, $\widetilde N_s$ and N_s are well defined). Suppose, for instance, that $f\in H^s_{1,\mathrm{odd}}(\mathbf{R}^N)$ and $\widetilde B_{N,s}(f,f)=0$. This implies that for almost every $\xi'\in\mathbf{R}^{N-1}$ we have $\widehat f(-\cdot,\xi')=-\widehat f(\cdot,\xi')$ a.e., $(|\cdot|^2+|\xi'|^2)^{\frac{s}{2}}\widehat f(\cdot,\xi')\in L^2(\mathbf{R})$ and

$$\int_{\sqrt{|\xi'|^2+1}}^{\infty} \left(t^2 - |\xi'|^2 - 1\right)^s \left| \int_0^{\infty} \widehat{f}(\xi_1, \xi') \frac{\xi_1}{t^2 + \xi_1^2} \, d\xi_1 \right| dt = 0.$$

For such ξ' we must have

$$\int_0^{\infty} \widehat{f}(\xi_1, \xi') \frac{\xi_1}{t^2 + \xi_1^2} \, d\xi_1 = 0 \quad \text{for almost every } t \in \left(\sqrt{|\xi'|^2 + 1}, \infty\right)$$

and using Lemma 3.5 we infer that $\widehat{f}(\cdot, \xi') = 0$ a.e. on $\mathbf{R}$, so $\int_{\mathbf{R}} (\xi_1^2 + |\xi'|^2)^s |\widehat{f}(\xi_1, \xi')|^2 \, d\xi_1 = 0$. Consequently

$$\|f\|_{H^s}^2 = \int_{\mathbf{R}^{N-1}} \int_{\mathbf{R}} \left(\xi_1^2 + |\xi'|^2\right)^s \left|\widehat{f}(\xi_1, \xi')\right|^2 d\xi_1 \, d\xi' = 0,$$

i.e. $f = 0$ a.e. The proof is the same for $f \in \dot{H}^s(\mathbf{R}^N)$. Finally, the continuity of $\widetilde{N}_s$ and N_s with respect to the usual norms follows from Theorem 3.2 and Corollary 3.4. □

4. Applications

In this section we illustrate how the results in Sections 2 and 3 can be used to prove the symmetry of minimizers in some concrete examples.

4.1. Problems involving fractional powers of the Laplace operator

Theorem 4.1. *Let $s \in (0, 1)$ and assume that $F, G : \mathbf{R} \to \mathbf{R}$ are such that $u \mapsto F(u)$ and $u \mapsto G(u)$ map $\dot{H}^s(\mathbf{R}^N)$ (or $H^s(\mathbf{R}^N)$) into $L^1(\mathbf{R}^N)$. Suppose that either*:

Case A. *$u \in \dot{H}^s(\mathbf{R}^N)$ and u is a solution of the minimization problem*

$$\textit{minimize} \quad E(u) := \int_{\mathbf{R}^N} |\xi|^{2s} \left|\widehat{u}(\xi)\right|^2 d\xi + \int_{\mathbf{R}^N} F\big(u(x)\big) \, dx$$

$$\textit{under the constraint} \quad I(u) = \int_{\mathbf{R}^N} G\big(u(x)\big) \, dx = \lambda \neq 0, \quad \textit{or}$$

Case B. *$u \in H^s(\mathbf{R}^N)$ and u is a solution of the minimization problem*

$$\textit{minimize} \quad E(u) := \int_{\mathbf{R}^N} \left(1 + |\xi|^2\right)^s \left|\widehat{u}(\xi)\right|^2 d\xi + \int_{\mathbf{R}^N} F\big(u(x)\big) \, dx$$

$$\textit{under the constraint} \quad I(u) = \int_{\mathbf{R}^N} G\big(u(x)\big) \, dx = \lambda \neq 0.$$

Then, after a translation in $\mathbf{R}^N$, u is radially symmetric.

Proof. Let us prove first that u is symmetric with respect to x_1. Making a translation in the x_1 direction if necessary, we may assume that $\int_{\{x_1<0\}} G(u(x))\,dx = \int_{\{x_1>0\}} G(u(x))\,dx = \frac{\lambda}{2}$. Let $u_1 = T_1 u$ and $u_2 = T_2 u$. It follows from Theorem 3.2 that $u_1, u_2 \in \dot{H}^s(\mathbf{R}^N)$ in case A, respectively $u_1, u_2 \in H^s(\mathbf{R}^N)$ in case B. It is obvious that we have $\int_{\mathbf{R}^N} G(u_1(x))\,dx = 2\int_{\{x_1<0\}} G(u(x))\,dx = \lambda$ and $\int_{\mathbf{R}^N} G(u_2(x))\,dx = 2\int_{\{x_1>0\}} G(u(x))\,dx = \lambda$; hence u_1 and u_2 also satisfy the constraint. From (3.22) and (3.23) we have

$$E(u_1) + E(u_2) - 2E(u) = -\frac{16\sin(s\pi)}{\pi^2} N_s^2(Au) \quad \text{in case A,} \quad \text{respectively}$$

$$E(u_1) + E(u_2) - 2E(u) = -\frac{16\sin(s\pi)}{\pi^2} \widetilde{N}_s^2(Au) \quad \text{in case B,}$$

where, as before, $Au(x_1, x') = \frac{1}{2}(u(x_1, x') - u(-x_1, x'))$ is the antisymmetric part of u in the x_1 direction. If $Au \not\equiv 0$, then Theorem 3.6 implies $N_s^2(Au) > 0$ (respectively $\widetilde{N}_s^2(Au) > 0$) and we infer that $E(u_1) + E(u_2) - 2E(u) < 0$, contradicting the fact that u is a minimizer. Thus necessarily $Au \equiv 0$ and this means that u is symmetric with respect to x_1.

Arguing similarly with the remaining variables $x_2, \ldots, x_N$, we find a new origin O' such that u is symmetric with respect to any of the variables $x_1, \ldots, x_N$; in particular, $u(-x) = u(x)$ a.e. on $\mathbf{R}^N$. Now let Π be any hyperplane containing the new origin O' and let Π_+ and Π_- be the halfspaces determined by Π. Since the transformation $x \mapsto -x$ maps Π_- into Π_+, we see that $\int_{\Pi_-} G(u(x))\,dx = \int_{\Pi_+} G(u(x))\,dx = \frac{\lambda}{2}$. Arguing as above we conclude that u must be symmetric with respect to Π. This implies that u is radially symmetric with respect to the new origin O'. $\square$

An application of Theorem 4.1 concerns the solitary waves to the generalized Benjamin–Ono equation

$$A_t + \alpha A A_x - \beta(-\Delta)^{\frac{1}{2}} A_x = 0, \quad (x, y) \in \mathbf{R}^2,\ t \in \mathbf{R},$$

where $\alpha, \beta > 0$. Solitary waves are solutions of the form $A(t, x, y) = u(x - ct, y)$. After a scale change, a solitary wave $u(x, y)$ satisfies the equation

$$u + (-\Delta)^{\frac{1}{2}} u = u^2 \quad \text{in } \mathbf{R}^2.$$

The existence of solitary waves was proved in [21] by minimizing the functional

$$V(u) = \frac{1}{2}\int_{\mathbf{R}^2} \left|(-\Delta)^{\frac{1}{4}} u\right|^2 dx + \int_{\mathbf{R}^2} u^2\,dx = \frac{1}{2(2\pi)^2}\int_{\mathbf{R}^2} |\xi|\left|\widehat{u}(\xi)\right|^2 d\xi + \int_{\mathbf{R}^2} u^2\,dx$$

under the constraint $I(u) = \frac{1}{3}\int_{\mathbf{R}^2} u^3\,dx = \mathit{constant}$. It has been shown in [21] that any solution u_* of the above problem also minimizes

$$E(v) := \frac{1}{2}\int_{\mathbf{R}^2} \left|(-\Delta)^{\frac{1}{4}} v\right|^2 dx - \frac{1}{3}\int_{\mathbf{R}^2} v^3\,dx$$

under the constraint $Q(v) = Q(u_*)$, where $Q(v) = \frac{1}{2}\int_{\mathbf{R}^2} |u|^2\,dx$.

It follows directly from Theorem 4.1 that, except for translation, any minimizer of these problems is radially symmetric.

Next we apply our method to a variational problem involving two unknown functions (the vector case). Consider the functionals

$$E(u,v)=\frac{1}{2}\int\limits_{\mathbf{R}^N}\left(\left|(-\Delta)^{\frac{s}{2}}u\right|^2+|\nabla v|^2\right)dx+\int\limits_{\mathbf{R}^N}F(u,v)\,dx$$

where $0<s<1$, and

$$Q(u,v)=\int\limits_{\mathbf{R}^N}G(u,v)\,dx.$$

We make the following assumptions:

A1. $F,G:\mathbf{R}^2\to\mathbf{R}$ are C^2 functions satisfying $F(0,0)=\partial_1F(0,0)=\partial_2F(0,0)=0$, $G(0,0)=\partial_1G(0,0)=\partial_2G(0,0)=0$ and the growth conditions

$$\left|\partial_iF(u,v)\right|\leqslant C\left(|u|^{p-1}+|v|^{q-1}\right)\quad\text{and}\quad\left|\partial_iG(u,v)\right|\leqslant C\left(|u|^{p-1}+|v|^{q-1}\right)\quad\text{if }\left|(u,v)\right|\geqslant 1,$$

where $i\in\{1,2\}$, C is a positive constant, $2<p<\frac{2N}{N-2s}$ and $2<q<\frac{2N}{N-2}$.

A2. If $(u,v)\in H^s(\mathbf{R}^N)\times H^1(\mathbf{R}^N)$ and $(u,v)\not\equiv(0,0)$, then either $\partial_1G(u,v)\not\equiv 0$ or $\partial_2G(u,v)\not\equiv 0$ (a manifold condition).

Theorem 4.2. *Under assumptions* A1 *and* A2*, any minimizer* $(u,v)\in H^s(\mathbf{R}^N)\times H^1(\mathbf{R}^N)$ *of* $E(u,v)$ *subject to the constraint* $Q(u,v)=\lambda\neq 0$ *is radially symmetric* (*except for translation*).

Proof. First we prove that after a translation, (u,v) is symmetric with respect to x_1. In fact, after possibly a translation in the x_1 direction we may assume that

$$\int\limits_{\{x_1<0\}}G(u,v)\,dx=\int\limits_{\{x_1>0\}}G(u,v)\,dx=\frac{\lambda}{2}.\tag{4.1}$$

We put $u_1=T_1u$, $u_2=T_2u$, $v_1=T_1v$ and $v_2=T_2v$. By Theorem 3.2, the pairs (u_1,v_1) and (u_2,v_2) belong to $H^s(\mathbf{R}^N)\times H^1(\mathbf{R}^N)$ and in view of (4.1) they also satisfy the constraint $Q(u_1,v_1)=Q(u_2,v_2)=\lambda$. Moreover, defining $W(\varphi)=\int_{\mathbf{R}^N}|\xi|^{2s}|\widehat{\varphi}(\xi)|^2\,d\xi$ and using (3.22) we see that

$$\begin{aligned}E(u_1,v_1)+E(u_2,v_2)-2E(u,v)&=\frac{1}{2}\frac{1}{(2\pi)^N}\left(W(u_1)+W(u_2)-2W(u)\right)\\&=-\frac{1}{(2\pi)^N}\frac{8\sin(s\pi)}{\pi^2}B_{N,s}(Au,Au)\leqslant 0.\end{aligned}$$

We conclude that (u_1,v_1) and (u_2,v_2) are also minimizers and we must have $B_{N,s}(Au,Au)=0$. Theorem 3.6 implies that $Au=0$, that is u is symmetric with respect to x_1, i.e. $u=u_1=u_2$.

Since (u, v) and $(u_1, v_1) = (u, v_1)$ are minimizers, they satisfy the Euler–Lagrange equations

$$\begin{cases} (-\Delta)^s u + \partial_1 F(u, v) + \alpha \partial_1 G(u, v) = 0, \\ -\Delta v + \partial_2 F(u, v) + \alpha \partial_2 G(u, v) = 0, \end{cases} \tag{4.2}$$

respectively

$$\begin{cases} (-\Delta)^s u + \partial_1 F(u, v_1) + \beta \partial_1 G(u, v_1) = 0, \\ -\Delta v_1 + \partial_2 F(u, v_1) + \beta \partial_2 G(u, v_1) = 0. \end{cases} \tag{4.3}$$

From (4.2), A1, the elliptic regularity for the Laplacian and its fractional powers and the usual boot-strap argument we get $u \in H^{2s}(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$ and $v \in H^2(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$. Of course that the same conclusion holds for (u, v_1). Notice that the L^p elliptic regularity for fractional powers of the Laplacian and for $1 < p < \infty$ follows from the fact that the multiplier $m(\xi) = \frac{(1+|\xi|^2)^s}{1+|\xi|^{2s}}$ satisfies the estimate $|D^\alpha m(\xi)| \leqslant \frac{B(\alpha)}{|\xi|^\alpha}$ and from the theorem of Mihlin–Hörmander.

We recall the following well-known result.

Unique Continuation Principle. *Assume that $\Phi \in H^2(\mathbf{R}^N, \mathbf{R}^m)$ solves the linear system*

$$-\Delta \Phi + A(x)\Phi(x) = 0 \quad \textit{in } \mathbf{R}^N, \tag{4.4}$$

where $A(x)$ is an $m \times m$ matrix whose elements belong to $L^\infty(\mathbf{R}^N)$. If $\Phi \equiv 0$ in some open set $\omega \subset \mathbf{R}^N$, then $\Phi \equiv 0$ in $\mathbf{R}^N$.

A proof for the Unique Continuation Principle is given in [13, Chapter VIII] in the scalar case and in the appendix of [18] in the vector case. Notice that the Unique Continuation Principle is essentially a local result. Although it is stated for functions $\Phi \in H^2(\mathbf{R}^N)$, it is also valid for functions $\Phi \in W^{2,p}(\mathbf{R}^N)$ with $p > 2$ because $W^{2,p}_{\mathrm{loc}}(\mathbf{R}^N) \subset H^2_{\mathrm{loc}}(\mathbf{R}^N)$. This observation will be useful later.

Now let us come back to the proof of Theorem 4.2.

If $(u_1, v_1) = (0, 0)$, since $G(0, 0) = 0$ we have $\lambda = Q(u_1, v_1) = 0$, a contradiction. Thus $(u_1, v_1) \neq (0, 0)$ and it follows from A2 that there exists $(x_1, x') \in (-\infty, 0) \times \mathbf{R}^{N-1}$ such that $\partial_1 G(u_1, v_1)(x_1, x') \neq 0$ or $\partial_2 G(u_1, v_1)(x_1, x') \neq 0$. Since $v = v_1$ for $x_1 < 0$, we infer from (4.2) and (4.3) that $\alpha = \beta$. Moreover, using the regularity of u, v, v_1 we get $\partial_2 F(u, v) - \partial_2 F(u, v_1) = b(x)(v(x) - v_1(x))$ and $\partial_2 G(u, v) - \partial_2 G(u, v_1) = c(x)(v(x) - v_1(x))$ where $b, c \in L^\infty(\mathbf{R}^N)$. Let $w(x) = v(x) - v_1(x)$. Using the second components of (4.2) and (4.3) and the fact that $\alpha = \beta$, we see that w satisfies the linear equation $-\Delta w(x) + a(x)w(x) = 0$ in $\mathbf{R}^N$, where $a = b + \alpha c \in L^\infty(\mathbf{R}^N)$. Since w vanishes on a half-space, by the Unique Continuation Principle we conclude that w vanishes everywhere, and this implies $v = v_1$ in $\mathbf{R}^N$. Thus we have shown that (u, v) is symmetric with respect to x_1.

Repeating this argument with the variables $x_2, \ldots, x_N$, we find a new origin O' such that (u, v) is symmetric with respect to $x_1, \ldots, x_N$. Then as in the proof of Theorem 4.1 we show that (u, v) is symmetric with respect to any hyperplane Π containing O', consequently (u, v) is radially symmetric with respect to the new origin O'. $\square$

Remark 4.3. Symmetrization inequalities for functions in the space $H^{1/2}(\mathbf{R}^N)$ have been proved in [3]. Therefore if $s = \frac{1}{2}$, the function F in Theorem 4.2 satisfies the cooperative condition

$\partial_{1,2}^2 F(u,v) \leqslant 0$ (see [5]), G has a special form and it is known in advance that the components u, v of the minimizer are nonnegative, then using symmetrization one can conclude that *there exists* a radially symmetric minimizer.

Remark 4.4. In the case $F(u,v) = u^2 + v^2$, $G(u,v) = u^2 v$, by using symmetrization and Riesz' inequality it has been proved in [3] that *there exists* a radially symmetric minimizer. The fact that F and G are homogeneous plays a crucial role in their proof.

As an example of application for Theorem 4.2, we consider the Hamiltonian system

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x_1}\big((-\Delta)^{1/2} u + \partial_1 F(u,v)\big), \\ \dfrac{\partial v}{\partial t} = \dfrac{\partial}{\partial x_1}\big(-\Delta v + \partial_2 F(u,v)\big). \end{cases} \tag{4.5}$$

The generalized multidimensional Benjamin–Ono equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x_1}\big((-\Delta)^{1/2} u + g(u)\big) \tag{4.6}$$

with $g(u) = u^2$ and the generalized multidimensional Korteweg–de Vries equation

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial x_1}\big(-\Delta v + f(v)\big) \tag{4.7}$$

have been considered in [21] and in [4], respectively; in these papers, references giving the physical motivation for the above equations can also be found. System (4.5) can be considered a Hamiltonian coupling between (4.6) and (4.7).

Formally, the system (4.5) has the following conserved quantities:

$$E(u,v) = \frac{1}{2} \int_{\mathbf{R}^N} \big|(-\Delta)^{1/4} u\big|^2 + |\nabla v|^2 \, dx + \int_{\mathbf{R}^N} F(u,v)\, dx \quad \text{and}$$

$$Q(u,v) = \frac{1}{2} \int_{\mathbf{R}^N} \big(u^2 + v^2\big)\, dx.$$

If we minimize $E(u,v)$ subject to the constraint $Q(u,v) = \lambda$, where $\lambda > 0$, then according to [9] the set S_λ containing the elements of $H^{\frac{1}{2}}(\mathbf{R}^N) \times H^1(\mathbf{R}^N)$ where the minimum is achieved is invariant and orbitally stable with respect to (4.5). Since any element $(\phi, \psi) \in S_\lambda$ satisfies the Euler–Lagrange system

$$\begin{cases} (-\Delta)^{1/2}\phi + \partial_1 F(\phi,\psi) + c\phi = 0, \\ -\Delta\psi + \partial_2 F(\phi,\psi) + c\psi = 0, \end{cases}$$

we see that (ϕ, ψ) gives rise to a travelling wave solution of (4.5) of the form $(u(t,x), v(t,x)) = (\phi(x_1 - ct, x'), \psi(x_1 - ct, x'))$, $x' \in \mathbf{R}^{N-1}$. As a consequence of Theorem 4.2, the elements (ϕ, ψ) obtained in this way are radially symmetric (after a translation).

4.2. Minimizers of the generalized Choquard functional

In this paragraph we consider the problem of minimizing the generalized Choquard functional

$$E(u)=\frac{1}{2}\int_{\mathbf{R}^N}|\nabla u|^2\,dx-\int_{\mathbf{R}^N}\int_{\mathbf{R}^N}F\bigl(u(x)\bigr)\frac{1}{|x-y|^{N-2}}F\bigl(u(y)\bigr)\,dx\,dy+\int_{\mathbf{R}^N}H\bigl(u(x)\bigr)\,dx \tag{4.8}$$

subject to the constraint $Q(u)=\int_{\mathbf{R}^N}G(u(x))\,dx=constant\neq 0$.

It is worth to note that the complex version of E,

$$\widetilde{E}(u)=\frac{1}{2}\int_{\mathbf{R}^N}|\nabla u|^2\,dx-\int_{\mathbf{R}^N}\int_{\mathbf{R}^N}F_1\bigl(\bigl|u(x)\bigr|^2\bigr)\frac{1}{|x-y|^{N-2}}F_1\bigl(\bigl|u(y)\bigr|^2\bigr)\,dx\,dy$$
$$+\int_{\mathbf{R}^N}H_1\bigl(\bigl|u(x)\bigr|^2\bigr)\,dx$$

is the Hamiltonian for the generalized Hartree equation

$$iu_t+\Delta u+4\biggl(\int_{\mathbf{R}^N}\frac{F_1(|u(y)|^2)}{|x-y|^{N-2}}\,dy\biggr)F_1'\bigl(|u|^2\bigr)(x)u(x)-2H_1'\bigl(\bigl|u(x)\bigr|^2\bigr)u(x)=0, \tag{4.9}$$

and $\widetilde{Q}(u)=\int_{\mathbf{R}^N}|u^2(x)|\,dx$ is a conserved quantity for this evolution equation. The critical points of $\widetilde{E}+\omega\widetilde{Q}$ give rise to standing waves for (4.9). As far as minimization is concerned, using an argument of T. Cazenave and P.-L. Lions (see the proof of Theorem II.1 in [9, p. 555]), we may restrict ourselves to the real functionals $E(u)$ and $Q(u)$.

In the case $N=3$, $F(u)=G(u)=u^2$ and $H(u)=0$, the problem of minimizing $E(u)$ subject to $Q(u)=\lambda$ has been studied in [15], where the existence, the radial symmetry and the uniqueness of the minimizer have been proved. The symmetry was proved by using a sharp inequality for spherical rearrangements. This can still be used in our case if we know that the minimizer is nonnegative and if we assume that F is increasing on $[0,\infty)$ (because the equality $F(u^*)=(F(u))^*$ is needed). Using the results in Sections 2 and 3, we will show the radial symmetry of minimizers in dimension $N\geqslant 3$ under more general assumptions on F, G and H.

We begin by studying some properties of the nonlocal term appearing in (4.8).

Lemma 4.5. *Let $N\geqslant 3$ and let $F:\mathbf{R}\to\mathbf{R}$ be a function of class C^2 satisfying $F(0)=F'(0)=0$ and*

$$\bigl|F'(x)\bigr|\leqslant C|x|^{\sigma}\quad \text{for } |x|\geqslant 1,$$

where $C>0$ is a constant and $\sigma<\frac{4}{N-2}$. Then the singular integral operator

$$I(\varphi)(x)=\int_{\mathbf{R}^N}\frac{1}{|x-y|^{N-2}}\varphi(y)\,dy$$

and the functional

$$M(\varphi)=\int_{\mathbf{R}^N}\int_{\mathbf{R}^N} F\big(\varphi(x)\big)\frac{1}{|x-y|^{N-2}}F\big(\varphi(y)\big)\,dx\,dy$$

have the following properties:

(i) *I is continuous from $L^p(\mathbf{R}^N)$ to $L^q(\mathbf{R}^N)$ if $1<p<q<\infty$ and $\frac{1}{q}=\frac{1}{p}-\frac{2}{N}$.*

(ii) *If $1\leqslant p_1<\frac{N}{2}<p_2\leqslant\infty$, then I is continuous from $L^{p_1}(\mathbf{R}^N)\cap L^{p_2}(\mathbf{R}^N)$ to $L^\infty(\mathbf{R}^N)\cap C^0(\mathbf{R}^N)$.*

(iii) *If $1\leqslant r_1<\frac{2N}{N+2}<r_2\leqslant 2$ and $\varphi\in L^{r_1}(\mathbf{R}^N)\cap L^{r_2}(\mathbf{R}^N)$, then*

$$\widehat{I(\varphi)}(\xi)=\frac{4\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2}-1)}\cdot\frac{1}{|\xi|^2}\widehat{\varphi}(\xi)\quad \text{in } \mathcal{S}'\big(\mathbf{R}^N\big).$$

(iv) *M is well defined and differentiable on $H^1(\mathbf{R}^N)$ and*

$$M'(u).\varphi=2\int_{\mathbf{R}^N}\bigg(\int_{\mathbf{R}^N}\frac{F(u(y))}{|x-y|^{N-2}}\,dy\bigg)F'\big(u(x)\big)\varphi(x)\,dx.$$

(v) *For any $u\in H^1(\mathbf{R}^N)$ we have*

$$M(u)=c_N\int_{\mathbf{R}^N}\frac{1}{|\xi|^2}\big|\widehat{F(u)}(\xi)\big|^2\,d\xi,\quad \text{where } c_N=\frac{1}{2^{N-2}\pi^{\frac{N}{2}}\Gamma(\frac{N}{2}-1)}.$$

Proof. (i) follows directly from Theorem 1 in [23, pp. 119–120].

(ii) We write $\frac{1}{|x|^{N-2}}$ as $a_1(x)+a_2(x)$, where $a_1(x)=\frac{1}{|x|^{N-2}}\chi_{\{|x|>1\}}$ and $a_2(x)=\frac{1}{|x|^{N-2}}\chi_{\{|x|\leqslant 1\}}$. Then we have $I(\varphi)=a_1*\varphi+a_2*\varphi$. It is obvious that $a_1\in L^q(\mathbf{R}^N)$ for $q\in(\frac{N}{N-2},\infty]$ and $a_2\in L^q(\mathbf{R}^N)$ for $q\in[1,\frac{N}{N-2})$. Let p_1' and p_2' be the conjugate exponents of p_1 and p_2. Then $p_1'>\frac{N}{N-2}$ and $p_2'<\frac{N}{N-2}$, so that $a_1\in L^{p_1'}(\mathbf{R}^N)$ and $a_2\in L^{p_2'}(\mathbf{R}^N)$. We infer that $I(\varphi)$ is continuous and by Young's inequality we get

$$\big\|I(\varphi)\big\|_{L^\infty}\leqslant\|a_1\|_{L^{p_1'}}\cdot\|\varphi\|_{L^{p_1}}+\|a_2\|_{L^{p_2'}}\cdot\|\varphi\|_{L^{p_2}}.$$

(iv) First we consider the bilinear form

$$P(\varphi,\psi)=\int_{\mathbf{R}^N}\int_{\mathbf{R}^N}\varphi(x)\frac{1}{|x-y|^{N-2}}\overline{\psi(y)}\,dx\,dy.$$

Notice that P is well defined and continuous on $L^{\frac{2N}{N+2}}(\mathbf{R}^N) \times L^{\frac{2N}{N+2}}(\mathbf{R}^N)$. Indeed, it follows from (i) that I is well defined and continuous from $L^{\frac{2N}{N+2}}(\mathbf{R}^N)$ to $L^{\frac{2N}{N-2}}(\mathbf{R}^N)$ and we have

$$\left|P(\varphi,\psi)\right| = \left|\int_{\mathbf{R}^N} I(\varphi)(x)\overline{\psi(x)}\,dx\right| \leqslant \left\|I(\varphi)\right\|_{L^{\frac{2N}{N-2}}} \cdot \|\psi\|_{L^{\frac{2N}{N+2}}} \leqslant A_N\|\varphi\|_{L^{\frac{2N}{N+2}}}\|\psi\|_{L^{\frac{2N}{N+2}}}.$$

Without loss of generality we may assume that $\sigma > \frac{2}{N}$. From the assumptions on F we have $|F(u)| \leqslant C|u|^2$ if $|u| \leqslant 1$ and $|F(u)| \leqslant C|u|^{1+\sigma}$ if $|u| > 1$. It is well known that $H^1(\mathbf{R}^N)$ is continuously embedded in $L^p(\mathbf{R}^N)$ for $p \in [2, \frac{2N}{N-2}]$ and then it is standard (see, e.g. [26, Appendix A]) that $u \mapsto F(u)$ is continuously differentiable from $H^1(\mathbf{R}^N)$ to $L^q(\mathbf{R}^N)$ for $q \in [\max(1, \frac{2}{1+\sigma}), \frac{2N}{(N-2)(1+\sigma)}]$. In particular, $u \mapsto F(u)$ is continuously differentiable from $H^1(\mathbf{R}^N)$ to $L^{\frac{2N}{N+2}}(\mathbf{R}^N)$ (because $\frac{2}{1+\sigma} < \frac{2N}{N+2} < \frac{2N}{(N-2)(1+\sigma)}$). Since $M(u) = P(F(u), F(u))$, (iv) follows.

(iii) and (v). Let $K(x) = \frac{1}{|x|^{N-2}}$. Then $K \in \mathcal{S}'(\mathbf{R}^N)$ and it follows from Theorem 4.1 in [24, p. 160] or from Lemma 1 in [23, p. 117] that $\widehat{K}(\xi) = \frac{4\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2}-1)} \cdot \frac{1}{|\xi|^2}$. From Lemma 1 in [23, p. 117] we have

$$P(\varphi,\psi) = \frac{1}{(2\pi)^N}\int_{\mathbf{R}^N} \widehat{I(\varphi)}(\xi)\overline{\widehat{\psi}(\xi)}\,d\xi = c_N\int_{\mathbf{R}^N} \frac{1}{|\xi|^2}\widehat{\varphi}(\xi)\overline{\widehat{\psi}(\xi)}\,d\xi \tag{4.10}$$

whenever $\varphi, \psi \in \mathcal{S}(\mathbf{R}^N)$. We claim that (4.10) holds for any $\varphi, \psi \in L^{r_1}(\mathbf{R}^N) \cap L^{r_2}(\mathbf{R}^N)$ with $1 \leqslant r_1 < \frac{2N}{N+2} < r_2 \leqslant 2$. This assertion implies both (iii) and (v).

Now let us prove the claim. Since (4.10) holds on $\mathcal{S} \times \mathcal{S}$, the bilinear form P is continuous on $L^{\frac{2N}{N+2}}(\mathbf{R}^N) \times L^{\frac{2N}{N+2}}(\mathbf{R}^N)$ and $L^{r_1}(\mathbf{R}^N) \cap L^{r_2}(\mathbf{R}^N)$ is continuously embedded into $L^{\frac{2N}{N+2}}(\mathbf{R}^N)$, all we have to do is to show that the bilinear form

$$P_1(\varphi,\psi) = \int_{\mathbf{R}^N} \frac{1}{|\xi|^2}\widehat{\varphi}(\xi)\overline{\widehat{\psi}(\xi)}\,d\xi$$

is continuous on $(L^{r_1}(\mathbf{R}^N) \cap L^{r_2}(\mathbf{R}^N)) \times (L^{r_1}(\mathbf{R}^N) \cap L^{r_2}(\mathbf{R}^N))$; then the claim follows by density of $\mathcal{S}$ in $L^{r_1}(\mathbf{R}^N) \cap L^{r_2}(\mathbf{R}^N)$.

Let r_1', r_2' be the conjugate exponents of r_1, r_2 and let q_1, q_2 be such that $\frac{1}{r_1'} + \frac{1}{q_1} = \frac{1}{2}$, respectively $\frac{1}{r_2'} + \frac{1}{q_2} = \frac{1}{2}$. Let $b_1(\xi) = \frac{1}{|\xi|}\chi_{\{|\xi| \leqslant 1\}}$ and $b_2(\xi) = \frac{1}{|\xi|}\chi_{\{|\xi| > 1\}}$. We have $2 \leqslant q_1 < N$ and $q_2 > N$, so that $b_1 \in L^{q_1}(\mathbf{R}^N)$ and $b_2 \in L^{q_2}(\mathbf{R}^N)$. Since the Fourier transform maps continuously $L^{r_1}(\mathbf{R}^N)$ into $L^{r_1'}(\mathbf{R}^N)$ and $L^{r_2}(\mathbf{R}^N)$ into $L^{r_2'}(\mathbf{R}^N)$, we have:

$$\begin{aligned}
\left|P_1(\varphi,\psi)\right| &\leqslant \left|\int_{\{|\xi| \leqslant 1\}} \frac{1}{|\xi|^2}\widehat{\varphi}(\xi)\overline{\widehat{\psi}(\xi)}\,d\xi\right| + \left|\int_{\{|\xi|>1\}} \frac{1}{|\xi|^2}\widehat{\varphi}(\xi)\overline{\widehat{\psi}(\xi)}\,d\xi\right| \\
&\leqslant \|b_1\widehat{\varphi}\|_{L^2} \cdot \|b_1\widehat{\psi}\|_{L^2} + \|b_2\widehat{\varphi}\|_{L^2} \cdot \|b_2\widehat{\psi}\|_{L^2} \\
&\leqslant \|b_1\|_{L^{q_1}}^2\|\widehat{\varphi}\|_{L^{r_1'}}\|\widehat{\psi}\|_{L^{r_1'}} + \|b_2\|_{L^{q_2}}^2\|\widehat{\varphi}\|_{L^{r_2'}}\|\widehat{\psi}\|_{L^{r_2'}}
\end{aligned}$$

$$\leqslant C(N, r_1, r_2)\big(\|\varphi\|_{L^{r_1}}\|\psi\|_{L^{r_1}} + \|\varphi\|_{L^{r_2}}\|\psi\|_{L^{r_2}}\big).$$

This proves the continuity of P_1 and our claim. Thus the proof of Lemma 4.5 is complete. □

Theorem 4.6. *Let $N \geqslant 3$ and let $F, G, H : \mathbf{R} \to \mathbf{R}$ be C^2 functions satisfying the following assumptions*:

(a) $F(0) = F'(0) = 0$ *and there exist $\sigma < \frac{4}{N-2}$ and $C > 0$ such that*

$$\big|F'(u)\big| \leqslant C|u|^{\sigma} \quad \text{if } |u| \geqslant 1.$$

(b) *There exist $\sigma_1 \in [1, \frac{N+2}{N-2})$ and $C_1 > 0$ such that*

$$\big|G'(u)\big| \leqslant C_1|u|^{\sigma_1} \quad \text{and} \quad \big|H'(u)\big| \leqslant C_1|u|^{\sigma_1} \quad \text{for any } u \in \mathbf{R}.$$

Moreover, if $\sigma_1 < 2$ then we assume that $\sigma_1 \geqslant \max(\frac{(N-2)(1+2\sigma)-4}{N}, 1)$.

(c) *For any $\varepsilon > 0$, $G' \not\equiv 0$ on $(-\varepsilon, 0)$ and on $(0, \varepsilon)$.*

Then any minimizer $u \in H^1(\mathbf{R}^N)$ of the functional E given by (4.8) *subject to the constraint $Q(u) = \lambda \neq 0$ is radially symmetric* (*after a translation in $\mathbf{R}^N$*).

Proof. First of all, notice that the functionals E and Q are well defined and of class C^1 on $H^1(\mathbf{R}^N)$. Let $u \in H^1(\mathbf{R}^N)$ be a minimizer. We will show that, except for translation, u is symmetric with respect to x_1. The same proof is valid for any other direction in $\mathbf{R}^N$ and the radial symmetry of u follows as in the proof of Theorem 4.1.

After a translation in the x_1 direction we may suppose that

$$\int_{\{x_1<0\}} G\big(u(x)\big)\,dx = \int_{\{x_1>0\}} G\big(u(x)\big)\,dx = \frac{\lambda}{2}.$$

As before, we define $u_1 = T_1 u$ and $u_2 = T_2 u$. We know that $u_1, u_2 \in H^1(\mathbf{R}^N)$. In view of assumption (a), it is obvious that $F(u) \in L^1(\mathbf{R}^N)$ and we have $T_1(F(u)) = F(u_1)$, $T_2(F(u)) = F(u_2)$, $Q(u_1) = Q(u_2) = \lambda$. Defining $W(\varphi) = \int_{\mathbf{R}^N} \frac{1}{|\xi|^2}|\widehat{\varphi}(\xi)|^2\,d\xi$, from Lemma 4.5(v) we get

$$\begin{aligned} E(u_1) + E(u_2) - 2E(u) &= -\big[M(u_1) + M(u_2) - 2M(u)\big] \\ &= -c_N\big[W\big(T_1\big(F(u)\big)\big) + W\big(T_2\big(F(u)\big)\big) - 2W\big(F(u)\big)\big]. \end{aligned} \tag{4.11}$$

Recall that by (2.38) we have for any $\varphi \in C_c^\infty(\mathbf{R}^N)$,

$$W(T_1\varphi) + W(T_2\varphi) - 2W(\varphi) = \frac{8}{\pi}\int_{\mathbf{R}^{N-1}} \frac{1}{|\xi'|}\left|\int_0^\infty \widehat{A\varphi}(\xi_1, \xi')\frac{\xi_1}{|\xi'|^2 + \xi_1^2}\,d\xi_1\right|^2 d\xi'. \tag{4.12}$$

To show that this identity also holds for $F(u)$ we need the following lemma.

Lemma 4.7. *Let $N \geqslant 3$ and let r_1, r_2 be such that $1 < r_1 < \frac{2N}{N+2} < r_2 < 2$. The bilinear form*

$$R(\varphi,\psi)=\int\limits_{\mathbf{R}^{N-1}}\frac{1}{|\xi'|}\int\limits_0^\infty\widehat{\varphi}(\xi_1,\xi')\frac{\xi_1}{|\xi'|^2+\xi_1^2}\,d\xi_1\cdot\int\limits_0^\infty\overline{\widehat{\psi}}(\eta_1,\xi')\frac{\eta_1}{|\xi'|^2+\eta_1^2}\,d\eta_1\,d\xi'$$

is continuous on $(L^{r_1}(\mathbf{R}^N)\cap L^{r_2}(\mathbf{R}^N))\times(L^{r_1}(\mathbf{R}^N)\cap L^{r_2}(\mathbf{R}^N))$.

Proof. Consider $\varphi,\psi\in L^{r_1}(\mathbf{R}^N)\cap L^{r_2}(\mathbf{R}^N)$. Then $\widehat{\varphi},\widehat{\psi}\in L^{r_1'}(\mathbf{R}^N)\cap L^{r_2'}(\mathbf{R}^N)$, where r_1' and r_2' are the conjugate exponents of r_1 and r_2. Using Hölder's inequality and the change of variable $\xi_1=t|\xi'|$, we get for $\xi'\neq 0$ and $i=1,2$,

$$\begin{aligned}\left|\int\limits_0^\infty\widehat{\varphi}(\xi_1,\xi')\frac{\xi_1}{|\xi'|^2+\xi_1^2}\,d\xi_1\right|&\leqslant\left(\int\limits_0^\infty\left|\widehat{\varphi}(\xi_1,\xi')\right|^{r_i'}d\xi_1\right)^{\frac{1}{r_i'}}\left(\int\limits_0^\infty\frac{\xi_1^{r_i}}{(|\xi'|^2+\xi_1^2)^{r_i}}\,d\xi_1\right)^{\frac{1}{r_i}}\\&=|\xi'|^{\frac{1-r_i}{r_i}}\left(\int\limits_0^\infty\frac{t^{r_i}}{(1+t^2)^{r_i}}\,dt\right)^{\frac{1}{r_i}}\left(\int\limits_0^\infty\left|\widehat{\varphi}(\xi_1,\xi')\right|^{r_i'}d\xi_1\right)^{\frac{1}{r_i'}}\\&=C_i|\xi'|^{\frac{1-r_i}{r_i}}\left(\int\limits_0^\infty\left|\widehat{\varphi}(\xi_1,\xi')\right|^{r_i'}d\xi_1\right)^{\frac{1}{r_i'}}.\end{aligned}\tag{4.13}$$

A similar estimate holds for ψ. Let q_i be the conjugate exponent of $\frac{r_i'}{2}$, i.e. $q_i=\frac{r_i}{2-r_i}$. Using (4.13), Hölder's inequality and the estimate $\|\widehat{\varphi}\|_{L^{r_i'}}\leqslant A_i\|\varphi\|_{L^{r_i}}$ we have

$$\begin{aligned}&\left|\int\limits_{B_{\mathbf{R}^{N-1}}(0,1)}\frac{1}{|\xi'|}\int\limits_0^\infty\widehat{\varphi}(\xi_1,\xi')\frac{\xi_1}{|\xi'|^2+\xi_1^2}\,d\xi_1\cdot\int\limits_0^\infty\overline{\widehat{\psi}}(\eta_1,\xi')\frac{\eta_1}{|\xi'|^2+\eta_1^2}\,d\eta_1\,d\xi'\right|\\&\leqslant C_1^2\int\limits_{B_{\mathbf{R}^{N-1}}(0,1)}|\xi'|^{\frac{2-2r_1}{r_1}-1}\left(\int\limits_0^\infty\left|\widehat{\varphi}(\xi_1,\xi')\right|^{r_1'}d\xi_1\right)^{\frac{1}{r_1'}}\left(\int\limits_0^\infty\left|\widehat{\psi}(\eta_1,\xi')\right|^{r_1'}d\eta_1\right)^{\frac{1}{r_1'}}d\xi'\\&\leqslant C_1^2\left(\int\limits_{B_{\mathbf{R}^{N-1}}(0,1)}|\xi'|^{\frac{q_1(2-3r_1)}{r_1}}\,d\xi'\right)^{\frac{1}{q_1}}\left(\int\limits_{B_{\mathbf{R}^{N-1}}(0,1)}\int\limits_0^\infty\left|\widehat{\varphi}(\xi_1,\xi')\right|^{r_1'}d\xi_1\,d\xi'\right)^{\frac{1}{r_1'}}\\&\quad\times\left(\int\limits_{B_{\mathbf{R}^{N-1}}(0,1)}\int\limits_0^\infty\left|\widehat{\psi}(\eta_1,\xi')\right|^{r_1'}d\eta_1\,d\xi'\right)^{\frac{1}{r_1'}}\\&\leqslant C_1^2A_1^2\left(\int\limits_{B_{\mathbf{R}^{N-1}}(0,1)}|\xi'|^{\frac{q_1(2-3r_1)}{r_1}}\,d\xi'\right)^{\frac{1}{q_1}}\|\varphi\|_{L^{r_1}}\|\psi\|_{L^{r_1}}\end{aligned}\tag{4.14}$$

and

$$\begin{aligned}
&\left| \int_{\{|\xi'|>1\}} \frac{1}{|\xi'|} \int_0^\infty \widehat{\varphi}(\xi_1, \xi') \frac{\xi_1}{|\xi'|^2 + \xi_1^2}\, d\xi_1 \cdot \int_0^\infty \overline{\widehat{\psi}}(\eta_1, \xi') \frac{\eta_1}{|\xi'|^2 + \eta_1^2}\, d\eta_1\, d\xi' \right| \\
&\leqslant C_2^2 \int_{\{|\xi'|>1\}} |\xi'|^{\frac{2-2r_2}{r_2}-1} \left(\int_0^\infty \left|\widehat{\varphi}(\xi_1, \xi')\right|^{r_2'} d\xi_1 \right)^{\frac{1}{r_2'}} \left(\int_0^\infty \left|\widehat{\psi}(\eta_1, \xi')\right|^{r_2'} d\eta_1 \right)^{\frac{1}{r_2'}} d\xi' \\
&\leqslant C_2^2 \left(\int_{\{|\xi'|>1\}} |\xi'|^{\frac{q_1(2-3r_2)}{r_2}}\, d\xi' \right)^{\frac{1}{q_2}} \left(\int_{\{|\xi'|>1\}} \int_0^\infty \left|\widehat{\varphi}(\xi_1, \xi')\right|^{r_2'} d\xi_1\, d\xi' \right)^{\frac{1}{r_2'}} \\
&\quad \times \left(\int_{\{|\xi'|>1\}} \int_0^\infty \left|\widehat{\psi}(\eta_1, \xi')\right|^{r_2'} d\eta_1\, d\xi' \right)^{\frac{1}{r_2'}} \\
&\leqslant C_2^2 A_2^2 \left(\int_{\{|\xi'|>1\}} |\xi'|^{\frac{q_2(2-3r_2)}{r_2}}\, d\xi' \right)^{\frac{1}{q_2}} \|\varphi\|_{L^{r_2}} \|\psi\|_{L^{r_2}}.
\end{aligned} \tag{4.15}$$

Since $1 < r_1 < \frac{2N}{N+2} < r_2 < 2$, a direct computation shows that $\int_{B_{\mathbf{R}^{N-1}}(0,1)} |\xi'|^{\frac{q_1(2-3r_1)}{r_1}}\, d\xi'$ and $\int_{\{|\xi'|>1\}} |\xi'|^{\frac{q_2(2-3r_2)}{r_2}}\, d\xi'$ are finite. From (4.14) and (4.15) we have

$$\left|R(\varphi, \psi)\right| \leqslant K \left(\|\varphi\|_{L^{r_1}} \|\psi\|_{L^{r_1}} + \|\varphi\|_{L^{r_2}} \|\psi\|_{L^{r_2}} \right)$$

and Lemma 4.7 is proved. □

Let r_1 and r_2 be as in Lemma 4.7. Since the maps $\varphi \mapsto T_1\varphi$ and $\varphi \mapsto T_2\varphi$ are obviously continuous from $L^{r_1}(\mathbf{R}^N) \cap L^{r_2}(\mathbf{R}^N)$ into itself and we have shown in the proof of Lemma 4.5 that the bilinear form $P_1(\varphi, \psi) = \int_{\mathbf{R}^N} \frac{1}{|\xi|^2} \widehat{\varphi}(\xi) \overline{\widehat{\psi}(\xi)}\, d\xi$ is continuous on this space, it follows that the left-hand side of (4.12) is continuous on $L^{r_1}(\mathbf{R}^N) \cap L^{r_2}(\mathbf{R}^N)$. By Lemma 4.7, the right-hand side of (4.12) also defines a continuous functional on $L^{r_1}(\mathbf{R}^N) \cap L^{r_2}(\mathbf{R}^N)$. Since (4.12) is valid for any $\varphi \in C_c^\infty(\mathbf{R}^N)$, by density we infer that (4.12) holds for any $\varphi \in L^{r_1}(\mathbf{R}^N) \cap L^{r_2}(\mathbf{R}^N)$. Recall that $u \in H^1(\mathbf{R}^N)$ and by the Sobolev embedding and assumption (a) we have $F(u) \in L^q(\mathbf{R}^N)$ for any $q \in [\max(1, \frac{2}{1+\sigma}), \frac{2N}{(N-2)(1+\sigma)}]$; hence (4.12) is valid for $F(u)$.

Since u is a minimizer, we must have $E(u_1) + E(u_2) - 2E(u) \geqslant 0$. From (4.11) and (4.12) we infer that necessarily

$$\int_{\mathbf{R}^{N-1}} \frac{1}{|\xi'|} \left| \int_0^\infty \mathcal{F}\big(A\big(F(u)\big)\big)(\xi_1, \xi') \frac{\xi_1}{|\xi'|^2 + \xi_1^2}\, d\xi_1 \right|^2 d\xi' = 0. \tag{4.16}$$

Contrary to our previous examples, (4.16) *does not* imply directly $AF(u) \equiv 0$. To see this, consider a function $\psi \in C_c^\infty(0, \infty)$ such that $\operatorname{supp}(\psi) \subset [1, \infty)$, $\psi \not\equiv 0$ and $\int_0^\infty \frac{t}{1+t^2} \psi(t)\, dt = 0$.

(Such a function exists: for example, take two nonnegative functions $\psi_0, \psi_1 \in C_c^\infty(1,\infty)$ with disjoint supports and put $\psi_\tau = (1-\tau)\psi_0 - \tau\psi_1$. There is some $\tau \in (0,1)$ such that $\int_0^\infty \frac{t}{1+t^2}\psi_\tau(t)\,dt = 0$.) Extend ψ to an odd function defined on $\mathbf{R}$. Take $\alpha \in C_c^\infty(\mathbf{R}^{N-1})$ such that $\alpha \not\equiv 0$ and $\operatorname{supp}(\alpha) \subset \mathbf{R}^{N-1} \setminus B(0,1)$ and put $\widehat{f}(\xi_1,\xi') = \alpha(\xi')\psi(\frac{\xi_1}{|\xi'|})$. Then $\widehat{f} \in C_c^\infty(\mathbf{R}^N)$ (hence $f \in \mathcal{S}$), $f \not\equiv 0$ and f is odd with respect to the first variable. However, we have $\int_0^\infty \widehat{f}(\xi_1,\xi')\frac{\xi_1}{|\xi'|^2+\xi_1^2}\,d\xi_1 = 0$ for any $\xi' \neq 0$ and consequently

$$\int_{\mathbf{R}^{N-1}} \frac{1}{|\xi'|}\left|\int_0^\infty \widehat{f}(\xi_1,\xi')\frac{\xi_1}{|\xi'|^2+\xi_1^2}\,d\xi_1\right|^2 d\xi' = 0.$$

To show that u is symmetric with respect to x_1, we argue as follows: since u and u_1 minimize E under the constraint $Q = \lambda$, these functions satisfy the Euler–Lagrange equations $E'(u) + \alpha Q'(u) = 0$, respectively $E'(u_1) + \beta Q'(u_1) = 0$ for some constants α and β, that is

$$-\Delta u - 2I\big(F(u)\big)F'(u) + H'(u) + \alpha G'(u) = 0 \quad \text{in } \mathbf{R}^N, \tag{4.17}$$

$$-\Delta u_1 - 2I\big(F(u_1)\big)F'(u_1) + H'(u_1) + \beta G'(u_1) = 0 \quad \text{in } \mathbf{R}^N. \tag{4.18}$$

We will show in the next lemma that u and u_1 are smooth functions. Then we prove that $I(F(u))(x) = I(F(u_1))(x)$ in the half-space $\{x_1 < 0\}$. Together with assumption (c), this implies that $\alpha = \beta$ in (4.17)–(4.18). Then we will be able to apply the Unique Continuation Principle to prove that $u = u_1$.

Lemma 4.8. *Let $u \in H^1(\mathbf{R}^N)$ be a solution of* (4.17)*, where $F, G, H \in C^2(\mathbf{R})$ satisfy the assumptions* (a) *and* (b) *in Theorem* 4.6*. Then $u \in W^{3,p}(\mathbf{R}^N)$ for any $p \in [2,\infty)$. In particular, $u \in C^2(\mathbf{R}^N)$ and $D^\alpha u$ are continuous and bounded on $\mathbf{R}^N$ if $\alpha \in \mathbf{N}^N$, $|\alpha| \leqslant 2$.*

Proof. The proof relies on a classical boot-strap argument. We show first that $u \in L^\infty(\mathbf{R}^N)$. By the Sobolev embedding we have $u \in L^q(\mathbf{R}^N)$ for $q \in [2, \frac{2N}{N-2}]$. We will improve this estimate by an inductive argument to get the desired conclusion.

We consider only the case $N \geqslant 4$, the proof in the case $N = 3$ being similar. Assume that $u \in L^q(\mathbf{R}^N)$ for any $q \in [2,\beta]$, where $\beta \geqslant \frac{2N}{N-2}$. It is clear that $G'(u), H'(u) \in L^q(\mathbf{R}^N)$ for $q \in [\max(1, \frac{2}{\sigma_1}), \frac{\beta}{\sigma_1}]$ and $F(u) \in L^q(\mathbf{R}^N)$ for $q \in [1, \frac{\beta}{1+\sigma}]$. We distinguish two cases:

Case A. If $\frac{\beta}{1+\sigma} > \frac{N}{2}$, then $I(F(u)) \in L^q(\mathbf{R}^N)$ for any $q \in (\frac{N}{N-2}, \infty]$. We have $F'(u)\chi_{\{|u| \leqslant 1\}} \in L^q(\mathbf{R}^N)$ for $q \in [2,\infty]$, hence $I(F(u))F'(u)\chi_{\{|u| \leqslant 1\}} \in L^q(\mathbf{R}^N)$ for $q \in (1,\infty]$ if $N = 4$, respectively for $q \in [1,\infty]$ if $N \geqslant 5$ and $F'(u)\chi_{\{|u|>1\}} \in L^q(\mathbf{R}^N)$ for $q \in [1, \frac{\beta}{\sigma}]$, hence $I(F(u))F'(u)\chi_{\{|u|>1\}} \in L^q(\mathbf{R}^N)$ if $q \in [1, \frac{\beta}{\sigma}]$. Consequently $I(F(u))F'(u) \in L^q(\mathbf{R}^N)$ for $q \in (1, \frac{\beta}{\sigma}]$ if $N = 4$, respectively for $q \in [1, \frac{\beta}{\sigma}]$ if $N \geqslant 5$. Notice that $\beta \geqslant \frac{2N}{N-2}$ and the second part of assumption (b) imply $\frac{\beta}{\sigma} \geqslant \frac{2}{\sigma_1}$. Using Eq. (4.17) we infer that $\Delta u \in L^q(\mathbf{R}^N)$ for $q \in [\max(1, \frac{2}{\sigma_1}), \min(\frac{\beta}{\sigma_1}, \frac{\beta}{\sigma})]$, $q \neq 1$ if $N = 4$. Let $q_3 = \min(\frac{\beta}{\sigma_1}, \frac{\beta}{\sigma})$. Notice that $q_3 \leqslant \beta$ because $\sigma_1 \geqslant 1$ and $\Delta u \in L^{q_3}(\mathbf{R}^N)$. If $q_3 > \frac{N}{2} \geqslant 2$, then $u \in L^{q_3}(\mathbf{R}^N)$, hence $u \in W^{2,q_3}(\mathbf{R}^N)$

and by the Sobolev embedding we get $u \in L^\infty(\mathbf{R}^N)$. If $q_3 = \frac{N}{2}$, then $u \in W^{2,\frac{N}{2}}(\mathbf{R}^N)$, consequently $u \in L^q(\mathbf{R}^N)$ for any $q \in [2,\infty)$ and repeating the above proof with $\widetilde{\beta} > \beta$ we find $u \in L^\infty(\mathbf{R}^N)$. If $q_3 < \frac{N}{2}$, then necessarily $q_3 = \frac{\beta}{\sigma_1}$ (recall that $\frac{\beta}{\sigma} > \frac{\beta}{1+\sigma} > \frac{N}{2}$ because we are in Case A). By the Sobolev embedding we get $u \in L^{\beta_1}(\mathbf{R}^N)$, where $\frac{1}{\beta_1} = \frac{1}{q_3} - \frac{2}{N} = \frac{\sigma_1}{\beta} - \frac{2}{N}$, thus $\frac{1}{\beta_1} - \frac{1}{\beta} = \frac{\sigma_1 - 1}{\beta} - \frac{2}{N} \leqslant \frac{(\sigma_1-1)(N-2)-4}{2N} < 0$ by (b). Repeating the previous arguments with β replaced by β_1, we find that either $u \in L^\infty(\mathbf{R}^N)$ or $u \in L^{\beta_2}(\mathbf{R}^N)$, where $\beta_2 > \beta_1$ and $\frac{1}{\beta_2} - \frac{1}{\beta_1} \leqslant \frac{(\sigma_1-1)(N-2)-4}{2N}$, and so on. After a finite number of steps we get $u \in L^\infty(\mathbf{R}^N)$.

Case B. If $\frac{\beta}{1+\sigma} \leqslant \frac{N}{2}$, we may suppose that $\frac{\beta}{1+\sigma} < \frac{N}{2}$. By Lemma 4.5(i), $I(F(u)) \in L^q(\mathbf{R}^N)$ for $q \in (\frac{N}{N-2}, (\frac{1+\sigma}{\beta} - \frac{2}{N})^{-1}]$. As in Case A we get $I(F(u))F'(u) \in L^q(\mathbf{R}^N)$ for $q \in [1, (\frac{1+2\sigma}{\beta} - \frac{2}{N})^{-1}]$, $q \neq 1$ if $N = 4$. By (a), (b) and the fact that $\beta \geqslant \frac{2N}{N-2}$ we have $(\frac{1+2\sigma}{\beta} - \frac{2}{N})^{-1} \geqslant \frac{2}{\sigma_1}$. Since $G'(u), H'(u) \in L^q(\mathbf{R}^N)$ for $q \in [\max(1, \frac{2}{\sigma_1}), \frac{\beta}{\sigma_1}]$, using (4.17) we get $\Delta u \in L^q(\mathbf{R}^N)$ for $q \in [\max(1, \frac{2}{\sigma_1}), q_4]$, $q \neq 1$ if $N = 4$, where $q_4 = \min(\frac{\beta}{\sigma_1}, (\frac{1+2\sigma}{\beta} - \frac{2}{N})^{-1})$. If $q_4 \geqslant \frac{N}{2}$ then, as above, we obtain $u \in L^\infty(\mathbf{R}^N)$. Otherwise by the Sobolev embedding we find $u \in L^{\beta_1}(\mathbf{R}^N)$, where $\frac{1}{\beta_1} = \frac{1}{q_4} - \frac{2}{N}$, thus $\frac{1}{\beta_1} - \frac{1}{\beta} \leqslant \max(\frac{(\sigma_1-1)(N-2)-4}{2N}, \frac{\sigma(N-2)-4}{N}) < 0$. Then we restart the process with β_1 instead of β. Continuing in this way, after a finite number of steps we obtain $u \in L^\infty(\mathbf{R}^N)$.

We have proved that $u \in L^q(\mathbf{R}^N)$ for any $q \in [2,\infty]$. Thus $F(u) \in L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$, $I(F(u)) \in L^q(\mathbf{R}^N)$ for $q \in (\frac{N}{N-2}, \infty]$, $F'(u) \in L^2(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$, hence $I(F(u))F'(u) \in L^2(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$. Clearly $G'(u), H'(u) \in L^q(\mathbf{R}^N)$ for $q \in [\max(1, \frac{2}{\sigma_1}), \infty]$. Using (4.17) we have $\Delta u \in L^2(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$, thus $u \in W^{2,p}(\mathbf{R}^N)$ for any $p \in [2,\infty)$. In particular, $\frac{\partial u}{\partial x_i}$ are continuous and bounded on $\mathbf{R}^N$. Differentiating (4.17) with respect to x_i we get

$$-\Delta\left(\frac{\partial u}{\partial x_i}\right) - 2I\left(F'(u)\frac{\partial u}{\partial x_i}\right)F'(u) - 2I\big(F(u)\big)F''(u)\frac{\partial u}{\partial x_i} + G''(u)\frac{\partial u}{\partial x_i} + \alpha H''(u)\frac{\partial u}{\partial x_i} = 0 \quad \text{in } \mathbf{R}^N.$$

It follows that $-\Delta(\frac{\partial u}{\partial x_i}) \in L^2(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$. Since obviously $\frac{\partial u}{\partial x_i} \in L^2(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$, we get $\frac{\partial u}{\partial x_i} \in W^{2,p}(\mathbf{R}^N)$, which implies $u \in W^{3,p}(\mathbf{R}^N)$ for any $p \in [2,\infty)$. $\square$

It follows from Lemma 4.8 that $F(u) \in C^2(\mathbf{R}^N)$ and $F(u) \in W^{2,p}(\mathbf{R}^N)$ for $p \in [1,\infty]$. Using Lemma 4.5(i) and (ii), it is easy to check that $I(F(u)) \in C^2(\mathbf{R}^N)$ and $I(F(u)) \in W^{2,p}(\mathbf{R}^N)$ for $p \in (\frac{N}{N-2}, \infty]$. In particular, $I(F(u)) \in \mathcal{S}'(\mathbf{R}^N)$ and Lemma 4.5(iii) implies $\mathcal{F}(I(F(u)))(\xi) = d_N \frac{1}{|\xi|^2}\widehat{F(u)}(\xi)$, where $d_N = \frac{4\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2}-1)}$. Setting $U = I(F(u))$ we have $-\Delta U = d_N F(u)$.

Next we show that $\frac{\partial U}{\partial x_1}(0,x') = \frac{\partial}{\partial x_1} I(F(u))(0,x') = 0$ for any $x' \in \mathbf{R}^{N-1}$. From (4.16) we infer that $\int_0^\infty \mathcal{F}(A(F(u)))(\xi_1,\xi')\frac{\xi_1}{|\xi'|^2+\xi_1^2}\,d\xi_1 = 0$ for almost every $\xi' \in \mathbf{R}^{N-1}$, that is $\int_{-\infty}^\infty \widehat{F(u)}(\xi_1,\xi')\frac{\xi_1}{|\xi'|^2+\xi_1^2}\,d\xi_1 = 0$ a.e. $\xi' \in \mathbf{R}^{N-1}$, or equivalently

$$\int_{-\infty}^{\infty} \xi_1 \mathcal{F}\big(I\big(F(u)\big)\big)(\xi_1, \xi')\, d\xi_1 = 0 \quad \text{for almost every } \xi' \in \mathbf{R}^{N-1}. \tag{4.19}$$

If $\frac{\partial}{\partial x_1} I(F(u))$ and $\mathcal{F}(\frac{\partial}{\partial x_1} I(F(u)))$ are in $L^1(\mathbf{R}^N)$, by the Fourier inversion theorem (4.19) is equivalent to $\frac{\partial}{\partial x_1} I(F(u))(0, x') = 0$, as desired.

Since we do not know whether $\frac{\partial}{\partial x_1} I(F(u)) \in L^1(\mathbf{R}^N)$ and $\mathcal{F}(\frac{\partial}{\partial x_1} I(F(u))) \in L^1(\mathbf{R}^N)$, we argue as follows: we take an arbitrary test function $\psi \in \mathcal{S}(\mathbf{R}^{N-1})$ and we put $\varphi_n(x_1) = \frac{n}{\sqrt{2\pi}} e^{-\frac{n^2 x_1^2}{2}}$. Clearly, $\varphi_n(x_1) = n\varphi_1(nx_1)$, $\|\varphi_n\|_{L^1(\mathbf{R})} = 1$ and $\widehat{\varphi_n}(\xi_1) = e^{-\frac{\xi_1^2}{2n^2}}$. On one hand, we have, by using Lebesgue's Dominated Convergence Theorem,

$$\begin{aligned}
&\lim_{n\to\infty} \int_{\mathbf{R}^N} \varphi_n(x_1)\psi(x') \left[\frac{\partial}{\partial x_1} I\big(F(u)\big)\right](x_1, x')\, dx \\
&\quad = \lim_{n\to\infty} \int_{\mathbf{R}^N} \varphi_1(y_1)\psi(x') \left[\frac{\partial}{\partial x_1} I\big(F(u)\big)\right]\left(\frac{y_1}{n}, x'\right) dy_1\, dx' \\
&\quad = \int_{\mathbf{R}^{N-1}} \psi(x') \frac{\partial}{\partial x_1}\big(I\big(F(u)\big)\big)(0, x')\, dx'.
\end{aligned} \tag{4.20}$$

On the other hand, we have

$$\begin{aligned}
&\int_{\mathbf{R}^N} \varphi_n(x_1)\psi(x') \left[\frac{\partial}{\partial x_1} I\big(F(u)\big)\right](x_1, x')\, dx \\
&\quad = \left\langle \frac{\partial}{\partial x_1}\big(I\big(F(u)\big)\big), \varphi_n(x_1)\psi(x') \right\rangle_{\mathcal{S}', \mathcal{S}} \\
&\quad = \left\langle \mathcal{F}\left(\frac{\partial}{\partial x_1} I\big(F(u)\big)\right), \mathcal{F}^{-1}\big(\varphi_n(x_1)\psi(x')\big) \right\rangle_{\mathcal{S}', \mathcal{S}} \\
&\quad = \frac{1}{(2\pi)^N} \int_{\mathbf{R}^N} \frac{i d_N \xi_1}{|\xi|^2} \widehat{F(u)}(\xi) e^{-\frac{\xi_1^2}{2n^2}} \widehat{\psi}(-\xi')\, d\xi_1\, d\xi'.
\end{aligned} \tag{4.21}$$

Since $F(u) \in L^2(\mathbf{R}^N)$, for almost every $\xi' \in \mathbf{R}^{N-1}$ we have $\widehat{F(u)}(\cdot, \xi') \in L^2(\mathbf{R})$. For any such ξ', arguing as in (4.13) we get

$$\int_{\mathbf{R}} \left| e^{-\frac{\xi_1^2}{2n^2}} \cdot \frac{\xi_1}{|\xi|^2} \widehat{F(u)}(\xi_1, \xi') \right| d\xi_1 \leqslant \int_{\mathbf{R}} \left| \frac{\xi_1}{\xi_1^2 + |\xi'|^2} \widehat{F(u)}(\xi_1, \xi') \right| d\xi_1 \leqslant \frac{C}{|\xi'|^{\frac{1}{2}}} \big\| \widehat{F(u)}(\cdot, \xi') \big\|_{L^2(\mathbf{R})},$$

where C does not depend on ξ'. Moreover, the Cauchy–Schwarz inequality gives

$$\int_{\mathbf{R}^{N-1}} \frac{C|\widehat{\psi}(-\xi')|}{|\xi'|^{\frac{1}{2}}} \big\|\widehat{F(u)}(\cdot,\xi')\big\|_{L^2(\mathbf{R})}\,d\xi' \leqslant C\bigg(\int_{\mathbf{R}^{N-1}} \frac{|\widehat{\psi}(-\xi')|^2}{|\xi'|}\,d\xi'\bigg)^{\frac{1}{2}} \big\|\widehat{F(u)}\big\|_{L^2(\mathbf{R}^N)} < \infty.$$

By the Dominated Convergence Theorem, we have for almost any $\xi' \in \mathbf{R}^{N-1}$

$$\int_{\mathbf{R}} \frac{\xi_1}{\xi_1^2+|\xi'|^2}\widehat{F(u)}(\xi_1,\xi')e^{-\frac{\xi_1^2}{2n^2}}\,d\xi_1 \to \int_{\mathbf{R}} \frac{\xi_1}{\xi_1^2+|\xi'|^2}\widehat{F(u)}(\xi_1,\xi')\,d\xi_1 = 0 \quad \text{as } n\to\infty.$$

Thus we may use Fubini's theorem, then the Dominated Convergence Theorem on $\mathbf{R}^{N-1}$ to obtain

$$\begin{aligned}
&\int_{\mathbf{R}^N} \frac{\xi_1}{|\xi|^2}\widehat{F(u)}(\xi_1,\xi')e^{-\frac{\xi_1^2}{2n^2}}\psi(-\xi')\,d\xi_1\,d\xi' \\
&\quad = \int_{\mathbf{R}^{N-1}} \psi(-\xi')\int_{\mathbf{R}} \frac{\xi_1}{\xi_1^2+|\xi'|^2}\widehat{F(u)}(\xi_1,\xi')e^{-\frac{\xi_1^2}{2n^2}}\,d\xi_1\,d\xi' \\
&\quad \to \int_{\mathbf{R}^{N-1}} \psi(-\xi')\cdot 0\,d\xi' = 0 \quad \text{as } n\to\infty.
\end{aligned} \tag{4.22}$$

From (4.20)–(4.22) we infer that $\int_{\mathbf{R}^{N-1}} \psi(x')\frac{\partial}{\partial x_1}(I(F(u)))(0,x')\,dx' = 0$. Since $\psi \in \mathcal{S}(\mathbf{R}^{N-1})$ was arbitrary, we have $\frac{\partial}{\partial x_1}(I(F(u)))(0,\cdot) = 0$ in $\mathcal{S}'(\mathbf{R}^{N-1})$, hence $\frac{\partial}{\partial x_1}(I(F(u)))(0,x') = 0$ for any $x' \in \mathbf{R}^{N-1}$ because $\frac{\partial}{\partial x_1}(I(F(u)))$ is a continuous function.

We know that $F(u_1)$ is symmetric with respect to x_1 and a simple change of variables shows that the function $U_1 := I(F(u_1))$ is also symmetric with respect to x_1. Clearly U_1 also belongs to $C^2(\mathbf{R}^N)$ and satisfies $-\Delta U_1 = -\Delta(I(F(u_1))) = d_N F(u_1)$. By symmetry we have $\frac{\partial U_1}{\partial x_1}(0,x') = 0$ for any $x' \in \mathbf{R}^{N-1}$. Since $u_1(x_1,x') = u(x_1,x')$ if $x_1 < 0$, we have proved that the functions U and U_1 are both solutions of the problem

$$\begin{cases} -\Delta W = d_N F(u) & \text{in } \{(x_1,x') \in \mathbf{R}^N \mid x_1 < 0\}, \\ W \in C^2(\mathbf{R}^N) \cap W^{2,p}(\mathbf{R}^N) & \text{for } p > \dfrac{N}{N-2}, \\ \dfrac{\partial W}{\partial x_1}(0,x') = 0 & \text{for any } x' \in \mathbf{R}^{N-1}. \end{cases} \tag{4.23}$$

It is not hard to see that the solution of (4.23) is unique. Consequently, $U(x_1,x') = U_1(x_1,x')$ if $x_1 < 0$. From (4.17) and (4.18) it is obvious that (u,U) and (u_1,U_1) solve the systems

$$\begin{cases} -\Delta u - 2UF'(u) + H'(u) + \alpha G'(u) = 0, \\ -\Delta U - d_N F(u) = 0 \end{cases} \quad \text{in } \mathbf{R}^N, \tag{4.24}$$

respectively

$$\begin{cases} -\Delta u_1 - 2U_1 F'(u_1) + H'(u_1) + \beta G'(u_1) = 0, \\ -\Delta U_1 - d_N F(u_1) = 0 \end{cases} \quad \text{in } \mathbf{R}^N. \tag{4.25}$$

We cannot have $u \equiv 0$ in the half-space $\{x_1 < 0\}$ because this would imply $\lambda = Q(u) = Q(u_1) = 0$. Since u is continuous, necessarily $u((-\infty, 0) \times \mathbf{R}^{N-1}) = u_1((-\infty, 0) \times \mathbf{R}^{N-1})$ contains an interval of the form $(-\varepsilon, 0)$ or $(0, \varepsilon)$ for some $\varepsilon > 0$. Now assumption (c), (4.24), (4.25) and the fact that $(u, U) = (u_1, U_1)$ on $(-\infty, 0) \times \mathbf{R}^{N-1}$ imply that $\alpha = \beta$ in (4.24)–(4.25). As a consequence, we see that $(u - u_1, U - U_1)$ solves a linear system whose coefficients belong to $L^\infty(\mathbf{R}^N)$. Since $(u, U) = (u_1, U_1)$ for $x_1 < 0$ and $(u, U), (u_1, U_1) \in W^{2,p}(\mathbf{R}^N, \mathbf{R}^2)$ if $p \geqslant 2$ and $p > \frac{N}{N-2}$, by using the Unique Continuation Principle we infer that $u = u_1$ (and $U = U_1$) in $\mathbf{R}^N$, that is u is symmetric with respect to x_1.

Similarly we show that u is symmetric with respect to any other hyperplane Π which has the property that $\int_{\Pi_-} G(u(x))\,dx = \int_{\Pi_+} G(u(x))\,dx$, where Π_- and Π_+ are the two half-spaces determined by Π. As in the proof of Theorem 4.1 it follows that after a translation, u is radially symmetric. The proof of Theorem 4.6 is complete. □

4.3. *Standing waves for the Davey–Stewartson equation*

We consider the Davey–Stewartson system

$$\begin{cases} iu_t + \Delta u = f(|u|^2)u - uv_{x_1}, \\ \Delta v = (|u|^2)_{x_1} \end{cases} \quad \text{in } \mathbf{R}^3, \tag{4.26}$$

which can be written as

$$iu_t = -\Delta u + f(|u|^2)u + R_1^2(|u|^2)u, \tag{4.27}$$

where R_1 is the Riesz transform defined by $\widehat{R_1\varphi} = \frac{i\xi_1}{|\xi|}\widehat{\varphi}(\xi)$. Let $F_1(t) = \int_0^t f(\tau)\,d\tau$. It is easy to check that

$$\widetilde{E}(u) = \frac{1}{2}\int_{\mathbf{R}^3} |\nabla u|^2\,dx + \frac{1}{2}\int_{\mathbf{R}^3} F_1(|u|^2)\,dx - \frac{1}{4}\int_{\mathbf{R}^3} \left|R_1(|u|^2)\right|^2 dx$$

is a Hamiltonian for (4.27) and $\widetilde{Q}(u) = \int_{\mathbf{R}^3} |u(x)|^2\,dx$ is a conserved quantity for the same equation. The standing waves for (4.27) are precisely the critical points of $\widetilde{E} + \omega\widetilde{Q}$. As in the previous example, when we minimize $\widetilde{E}(u)$ subject to $\widetilde{Q}(u) = constant$, we may restrict ourselves to real functions u and to the real version of $\widetilde{E}$,

$$E(u) = \frac{1}{2}\int_{\mathbf{R}^3} |\nabla u|^2\,dx + \int_{\mathbf{R}^3} F(u)\,dx - \frac{1}{4}\int_{\mathbf{R}^3} \left|R_1(u^2)\right|^2 dx.$$

We will consider a more general functional than $\widetilde{Q}$, namely $Q(u) = \int_{\mathbf{R}^3} G(u)\,dx$. If $G(u) = u^2$, in order to guarantee the boundedness from below of the functional E on the set

of functions satisfying $Q(u) = \lambda$, the function $F(u)$ is required to behave as $a|u|^\gamma$ for u large, with $a > 0$ and $\gamma > 4$. In the case $F(u) = a|u|^\gamma$, the Cauchy problem for the evolution equation (4.27) has been analysed in [12]. The global existence of solutions was proved in the case $a > 0$ and $\gamma > 4$, while in the case $\gamma = 4$ the global existence was proved if a is sufficiently large.

Still in the case of pure power $F(u) = a|u|^\gamma$, with $a > 0$ and $\gamma > 4$, the existence of minimizers of E subject to the constraint $Q(u) = \int_{\mathbf{R}^3} |u|^2\,dx = \lambda$ can be proved by using the Concentration–Compactness Principle (see [17]) if λ is large enough (this assumption is needed to prevent vanishing).

In [10] the existence of ground states related to the problem (4.26) has been studied. However, our method cannot be used to prove the symmetry of these ground states because the nonlocal term appears in the constraint.

It is well known that R_1 is a linear continuous map from $L^p(\mathbf{R}^3)$ to $L^p(\mathbf{R}^3)$ for $1 < p < \infty$ (see [23]). If $u^2 \in L^2(\mathbf{R}^3)$, then $R_1(u^2) \in L^2(\mathbf{R}^3)$ and by Plancherel's theorem we get

$$\int_{\mathbf{R}^3} \left|R_1\left(u^2\right)\right|^2 dx = \frac{1}{(2\pi)^3} \int_{\mathbf{R}^3} \left|\widehat{R_1\left(u^2\right)}(\xi)\right|^2 d\xi = \frac{1}{(2\pi)^3} \int_{\mathbf{R}^3} \frac{\xi_1^2}{|\xi|^2} \left|\widehat{u^2}(\xi)\right|^2 d\xi. \tag{4.28}$$

We have the following symmetry result.

Theorem 4.9. *Let $u \in H^1(\mathbf{R}^3)$ be a solution of the minimization problem*

$$\text{minimize} \quad E(u) = \frac{1}{2}\int_{\mathbf{R}^3} |\nabla u|^2\,dx + \int_{\mathbf{R}^3} F(u)\,dx - \frac{1}{4}\int_{\mathbf{R}^3} \left|R_1\left(u^2\right)\right|^2 dx$$

$$\text{subject to} \quad Q(u) = \int_{\mathbf{R}^3} G\left(u(x)\right) dx = \lambda \neq 0$$

under the following assumptions:

(a) $F, G : \mathbf{R} \to \mathbf{R}$ *are* C^2 *functions,* $F(0) = F'(0) = 0$, $G(0) = G'(0) = 0$ *and there exist* $C > 0$, $\sigma < 5$ *such that*

$$\left|F'(u)\right| \leqslant C|u|^\sigma \quad \text{and} \quad \left|G'(u)\right| \leqslant C|u|^\sigma \quad \text{for } |u| \geqslant 1.$$

(b) *For any* $\varepsilon > 0$, $G' \not\equiv 0$ *on* $(-\varepsilon, 0)$ *and on* $(0, \varepsilon)$.

Then, after a translation, u is radially symmetric in the variables (x_2, x_3) (i.e. u is axially symmetric).

Proof. Making a translation in the x_2 direction if necessary, we may assume that

$$\int_{\{x_2<0\}} G\left(u(x)\right) dx = \int_{\{x_2>0\}} G\left(u(x)\right) dx = \frac{\lambda}{2}.$$

As before, we define u_1 and u_2 by

$$u_1(x_1,x_2,x_3)=\begin{cases} u(x_1,x_2,x_3) & \text{if } x_2<0,\\ u(x_1,-x_2,x_3) & \text{if } x_2\geqslant 0,\end{cases}\qquad u_2(x_1,x_2,x_3)=\begin{cases} u(x_1,-x_2,x_3) & \text{if } x_2<0,\\ u(x_1,x_2,x_3) & \text{if } x_2\geqslant 0.\end{cases}$$

It is obvious that $Q(u_1)=Q(u_2)=\lambda$. Moreover, using (4.28) we get

$$E(u_1)+E(u_2)-2E(u)=-\frac{1}{4}\frac{1}{(2\pi)^3}\biggl[\int_{\mathbf{R}^3}\frac{\xi_1^2}{|\xi|^2}\bigl|\widehat{u_1^2}(\xi)\bigr|^2\,d\xi+\int_{\mathbf{R}^3}\frac{\xi_1^2}{|\xi|^2}\bigl|\widehat{u_2^2}(\xi)\bigr|^2\,d\xi$$
$$-2\int_{\mathbf{R}^3}\frac{\xi_1^2}{|\xi|^2}\bigl|\widehat{u^2}(\xi)\bigr|^2\,d\xi\biggr]. \tag{4.29}$$

Recall that by (2.40) and (2.41) we have the equality

$$\int_{\mathbf{R}^N}\frac{\xi_j^2}{|\xi|^2}\bigl|\widehat{T_1\varphi}(\xi)\bigr|^2\,d\xi+\int_{\mathbf{R}^N}\frac{\xi_j^2}{|\xi|^2}\bigl|\widehat{T_2\varphi}(\xi)\bigr|^2\,d\xi-2\int_{\mathbf{R}^N}\frac{\xi_j^2}{|\xi|^2}\bigl|\widehat{\varphi}(\xi)\bigr|^2\,d\xi$$
$$=\frac{8}{\pi}\int_{\mathbf{R}^{N-1}}\frac{\xi_j^2}{|\xi'|}\left|\int_0^\infty\widehat{A\varphi}(\xi_1,\xi')\frac{\xi_1}{\xi_1^2+|\xi'|^2}\,d\xi_1\right|^2 d\xi' \tag{4.30}$$

for any $\varphi\in C_c^\infty(\mathbf{R}^N)$, where $j\in\{2,\dots,N\}$. It is obvious that the left-hand side of (4.30) defines a continuous functional on $L^2(\mathbf{R}^N)$. By the next lemma, it follows that the right-hand side of (4.30) also defines a continuous functional on $L^2(\mathbf{R}^N)$. Then the density of $C_c^\infty(\mathbf{R}^N)$ in $L^2(\mathbf{R}^N)$ implies that (4.30) holds for any $\varphi\in L^2(\mathbf{R}^N)$.

Lemma 4.10. *Let $j\in\{2,\dots,N\}$. The bilinear form*

$$S_1(\varphi,\psi)=\int_{\mathbf{R}^{N-1}}\frac{\xi_j^2}{|\xi'|}\int_0^\infty\widehat{\varphi}(\xi_1,\xi')\frac{\xi_1}{\xi_1^2+|\xi'|^2}\,d\xi_1\cdot\int_0^\infty\overline{\widehat{\psi}(\eta_1,\xi')}\frac{\eta_1}{\eta_1^2+|\xi'|^2}\,d\eta_1\,d\xi'$$

is continuous on $L^2(\mathbf{R}^N)\times L^2(\mathbf{R}^N)$.

Proof. As in (4.13) we have

$$\left|\int_0^\infty\widehat{\varphi}(\xi_1,\xi')\frac{\xi_1}{\xi_1^2+|\xi'|^2}\,d\xi_1\right|\leqslant K\frac{1}{|\xi'|^{\frac12}}\left(\int_0^\infty\bigl|\widehat{\varphi}(\xi_1,\xi')\bigr|^2\,d\xi_1\right)^{\frac12},$$

where $K=(\int_0^\infty\frac{t^2}{(1+t^2)^2}\,dt)^{\frac12}$. Consequently

$$
\begin{aligned}
\left|S_1(\varphi,\psi)\right| &\leqslant K^2 \int\limits_{\mathbf{R}^{N-1}} \frac{\xi_j^2}{|\xi'|^2} \left(\int\limits_0^\infty \left|\widehat{\varphi}(\xi_1,\xi')\right|^2 d\xi_1\right)^{\frac{1}{2}} \left(\int\limits_0^\infty \left|\widehat{\psi}(\eta_1,\xi')\right|^2 d\eta_1\right)^{\frac{1}{2}} d\xi' \\
&\leqslant K^2 \int\limits_{\mathbf{R}^{N-1}} \left(\int\limits_0^\infty \left|\widehat{\varphi}(\xi_1,\xi')\right|^2 d\xi_1\right)^{\frac{1}{2}} \left(\int\limits_0^\infty \left|\widehat{\psi}(\eta_1,\xi')\right|^2 d\eta_1\right)^{\frac{1}{2}} d\xi' \\
&\leqslant K^2 \left(\int\limits_{\mathbf{R}^{N-1}} \int\limits_0^\infty \left|\widehat{\varphi}(\xi_1,\xi')\right|^2 d\xi_1\, d\xi'\right)^{\frac{1}{2}} \cdot \left(\int\limits_{\mathbf{R}^{N-1}} \int\limits_0^\infty \left|\widehat{\psi}(\eta_1,\xi')\right|^2 d\eta_1\, d\xi'\right)^{\frac{1}{2}} \\
&\leqslant K_1 \|\varphi\|_{L^2(\mathbf{R}^N)} \|\psi\|_{L^2(\mathbf{R}^N)}. \qquad \square
\end{aligned}
$$

Since $u^2, u_1^2, u_2^2 \in L^2(\mathbf{R}^3)$ (recall that $H^1(\mathbf{R}^3) \subset L^2(\mathbf{R}^3) \cap L^6(\mathbf{R}^3)$), by exchanging the roles of x_1 and x_2 and using (4.29) and (4.30) we find

$$
\begin{aligned}
&E(u_1) + E(u_2) - 2E(u) \\
&= -\frac{1}{4}\frac{1}{(2\pi)^3}\frac{8}{\pi} \int\limits_{\mathbf{R}^2} \frac{\xi_1^2}{\sqrt{\xi_1^2+\xi_3^2}} \left| \int\limits_0^\infty \widehat{A_2(u^2)}(\xi_1,\xi_2,\xi_3) \frac{\xi_2}{\xi_1^2+\xi_2^2+\xi_3^2}\, d\xi_2 \right|^2 d\xi_1\, d\xi_3, \qquad (4.31)
\end{aligned}
$$

where $A_2\varphi = \frac{1}{2}(\varphi(x_1,x_2,x_3) - \varphi(x_1,-x_2,x_3))$.

Since u is a minimizer, we must have $E(u_1) + E(u_2) - 2E(u) \geqslant 0$, consequently the integral in the right-hand side of (4.31) must be zero, which is equivalent to

$$
\int\limits_0^\infty \widehat{A_2(u^2)}(\xi_1,\xi_2,\xi_3) \frac{\xi_2}{\xi_1^2+\xi_2^2+\xi_3^2}\, d\xi_2 = 0 \quad \text{a.e. } (\xi_1,\xi_3) \in \mathbf{R}^2. \qquad (4.32)
$$

In particular, u_1 and u_2 are also minimizers. However, as in the previous example, (4.32) is not sufficient to prove that $A_2(u^2) = 0$. In order to accomplish this task, we will use the Euler–Lagrange equation of u: since u minimizes E under the constraint $Q(u) = \lambda$, there exists a constant α such that $E'(u) + \alpha Q'(u) = 0$, that is

$$
-\Delta u + F'(u) + R_1^2\big(u^2\big)u + \alpha G'(u) = 0. \qquad (4.33)
$$

Lemma 4.11. *If F and G satisfy assumption* (a) *in Theorem* 4.9 *and $u \in H^1(\mathbf{R}^3)$ is a solution of* (4.33)*, then $u \in W^{3,p}(\mathbf{R}^3)$ for any $p \in [2,\infty)$. In particular, $u \in C^2(\mathbf{R}^3)$.*

Since R_1 and R_1^2 are linear continuous mappings from $L^p(\mathbf{R}^3)$ to $L^p(\mathbf{R}^3)$ for $1 < p < \infty$, the proof of Lemma 4.11 is standard, so we omit it.

Let $I(\varphi)(x) = \int_{\mathbf{R}^3} \frac{\varphi(y)}{|x-y|}\, dy$. Using Lemma 4.5 it is easy to see that $I(u^2) \in W^{2,p}(\mathbf{R}^3)$ for any $p \in (3,\infty]$ and $I(u^2)$ is a C^2 function. Moreover, we have

$$
\mathcal{F}\big(R_1^2\big(u^2\big)\big)(\xi) = -\frac{\xi_1^2}{|\xi|^2}\widehat{u^2}(\xi) = -\frac{1}{d_3}\xi_1^2 \widehat{I\big(u^2\big)}(\xi),
$$

where $d_3 = \frac{4\pi^{\frac{3}{2}}}{\Gamma(\frac{1}{2})}$, thus $R_1^2(u^2) = \frac{1}{d_3}\frac{\partial^2}{\partial x_1^2} I(u^2)$. Equation (4.33) can be written as

$$-\Delta u + F'(u) + \frac{1}{d_3}\frac{\partial^2}{\partial x_1^2}\big(I\big(u^2\big)\big)u + \alpha G'(u) = 0. \tag{4.34}$$

Arguing exactly as in the proof of Theorem 4.6, (4.32) implies that $\frac{\partial}{\partial x_2}(I(u^2))(x_1, 0, x_3) = 0$ for any $(x_1, x_3) \in \mathbf{R}^2$.

Since u_1 is also a minimizer, it satisfies the Euler–Lagrange equation

$$-\Delta u_1 + F'(u_1) + \frac{1}{d_3}\frac{\partial^2}{\partial x_1^2}\big(I\big(u_1^2\big)\big)u_1 + \beta G'(u_1) = 0. \tag{4.35}$$

The conclusion of Lemma 4.11 is obviously valid for u_1. Since u_1 is symmetric with respect to x_2, $I(u_1^2)$ is also symmetric with respect to x_2 and, consequently, $\frac{\partial}{\partial x_2}(I(u_1^2))(x_1, 0, x_3) = 0$ for any $(x_1, x_3) \in \mathbf{R}^2$. We set $U = I(u^2)$ and $U_1 = I(u_1^2)$. Recall that $u(x_1, x_2, x_3) = u_1(x_1, x_2, x_3)$ if $x_2 < 0$; thus U and U_1 are both solutions of

$$\begin{cases} -\Delta W = u^2 & \text{in } \mathbf{R} \times (-\infty, 0) \times \mathbf{R}, \\ W \in C^2\big(\mathbf{R}^3\big) \cap W^{2,p}\big(\mathbf{R}^3\big) & \text{for } 3 < p \leqslant \infty, \\ \dfrac{\partial W}{\partial x_2}(x_1, 0, x_3) = 0 & \text{for any } (x_1, x_3) \in \mathbf{R}^2. \end{cases} \tag{4.36}$$

It is not hard to see that the solution of (4.36) is unique. Hence we must have $I(u^2) = I(u_1^2)$ in $\mathbf{R} \times (-\infty, 0] \times \mathbf{R}$. In the same way we obtain $I(u^2) = I(u_2^2)$ in $\mathbf{R} \times [0, \infty) \times \mathbf{R}$.

Now we focus our attention on u_1. Making a translation in the x_3 direction if necessary, we may assume that $\int_{\{x_3<0\}} G(u_1(x))\,dx = \int_{\{x_3>0\}} G(u_1(x))\,dx = \frac{\lambda}{2}$. We define

$$w_1(x_1, x_2, x_3) = \begin{cases} u_1(x_1, x_2, x_3) & \text{if } x_3 < 0, \\ u_1(x_1, x_2, -x_3) & \text{if } x_3 \geqslant 0, \end{cases}$$

$$w_2(x_1, x_2, x_3) = \begin{cases} u_1(x_1, x_2, -x_3) & \text{if } x_3 < 0, \\ u_1(x_1, x_2, x_3) & \text{if } x_3 \geqslant 0. \end{cases}$$

It is obvious that $Q(w_1) = Q(w_2) = \lambda$. Proceeding as above, we find the identity

$$\begin{aligned} &E(w_1) + E(w_2) - 2E(u_1) \\ &\quad = -\frac{1}{4}\frac{1}{(2\pi)^3}\frac{8}{\pi}\int_{\mathbf{R}^2} \frac{\xi_1^2}{\sqrt{\xi_1^2+\xi_2^2}}\left|\int_0^\infty \widehat{A_3\big(u_1^2\big)}(\xi_1, \xi_2, \xi_3)\frac{\xi_3}{\xi_1^2+\xi_2^2+\xi_3^2}\,d\xi_3\right|^2 d\xi_1\,d\xi_2, \end{aligned} \tag{4.37}$$

where $A_3\varphi = \frac{1}{2}(\varphi(x_1, x_2, x_3) - \varphi(x_1, x_2, -x_3))$. Since u_1 is a minimizer, it follows from (4.37) that w_1 and w_2 are also minimizers of E under the constraint $Q = \lambda$; hence w_1 and w_2 satisfy the conclusion of Lemma 4.11 and $I(w_1), I(w_2) \in C^2(\mathbf{R}^3) \cap W^{2,p}(\mathbf{R}^3)$ for $p \in (3, \infty]$. Moreover, the integral in the right-hand side of (4.37) must be zero. As previously, this gives

$\frac{\partial}{\partial x_3} I(u_1^2)(x_1,x_2,0)=0$ for any $(x_1,x_2)\in\mathbf{R}^2$. Proceeding as above, we find $I(u_1^2)=I(w_1^2)$ in $\mathbf{R}^2\times(-\infty,0]$ and $I(u_1^2)=I(w_2^2)$ in $\mathbf{R}^2\times[0,\infty)$.

Now let us consider the function w_1. It is clear that $w_1(x_1,-x_2,-x_3)=w_1(x_1,-x_2,x_3)=w_1(x_1,x_2,x_3)$, i.e. w_1 is symmetric with respect to x_2 and with respect to x_3. Consider a plane Π in $\mathbf{R}^3$ containing the line $\{(x_1,0,0)\mid x_1\in\mathbf{R}\}$ and let Π_+ and Π_- be the two half-spaces determined by Π. Since $(x_1,x_2,x_3)\mapsto(x_1,-x_2,-x_3)$ maps Π_+ onto Π_-, using the symmetry of w_1 we get $\int_{\Pi_+}G(w_1(x))\,dx=\int_{\Pi_-}G(w_1(x))\,dx=\frac{\lambda}{2}$. Let s_Π denote the symmetry in $\mathbf{R}^3$ with respect to Π. We define

$$r_1(x)=\begin{cases} w_1(x) & \text{if } x\in\Pi_-,\\ w_1(s_\Pi(x)) & \text{if } x\in\Pi_+\end{cases}\quad\text{and}\quad r_2(x)=\begin{cases} w_1(s_\Pi(x)) & \text{if } x\in\Pi_-,\\ w_1(x) & \text{if } x\in\Pi_+.\end{cases}$$

Repeating the above arguments we obtain an integral identity analogous to (4.31) and (4.37) which implies that r_1 and r_2 also minimize E subject to the constraint $Q=\lambda$. Furthermore, using the fact that the integral in the right-hand side of this identity must vanish we find

$$\frac{\partial}{\partial n}I\big(w_1^2\big)(x_1,x_2,x_3)=0\quad\text{whenever } (x_1,x_2,x_3)\in\Pi, \tag{4.38}$$

where n is the unit normal to Π. Passing to cylindrical coordinates we write

$$I\big(w_1^2\big)(x_1,x_2,x_3)=I\big(w_1^2\big)(x_1,r\cos\theta,r\sin\theta)=\Phi(x_1,r,\theta),$$

where $r=\sqrt{x_2^2+x_3^2}$. Since $I(w_1^2)$ is a C^2 function and (4.38) is valid for any plane Π containing $\{(x_1,0,0)\mid x_1\in\mathbf{R}\}$, (4.38) is equivalent to $\frac{\partial\Phi}{\partial\theta}=0$, that is Φ does not depend on θ, i.e. $I(w_1^2)(x_1,x_2,x_3)=\Phi_1(x_1,\sqrt{x_2^2+x_3^2})$ for some function Φ_1. In other words, we have proved that $I(w_1^2)$ is radially symmetric in the variables (x_2,x_3). In the same way we show that $I(w_2^2)(x_1,x_2,x_3)=\Phi_2(x_1,\sqrt{x_2^2+x_3^2})$ for some function Φ_2. Since $I(u_1^2)$ is continuous on $\mathbf{R}^3$, $I(u_1^2)=I(w_1^2)$ in the half-space $\{x_3<0\}$ and $I(u_1^2)=I(w_2^2)$ in the half-space $\{x_3>0\}$, we have necessarily $\Phi_1=\Phi_2$, and then $I(u_1^2)$ is radially symmetric in the variables (x_2,x_3).

Similarly, there exists $k\in\mathbf{R}$ such that $\int_{\{x_3<k\}}G(u_2(x))\,dx=\int_{\{x_3>k\}}G(u_2(x))\,dx=\frac{\lambda}{2}$. (We have fixed the origin in such a way that $\int_{\{x_3<0\}}G(u_1(x))\,dx=\int_{\{x_3>0\}}G(u_1(x))\,dx=\frac{\lambda}{2}$ and nothing guarantees a priori that $k=0$.) Arguing as above, we infer that $I(u_2^2)$ is radially symmetric with respect to the variables (x_2,x_3-k). Thus we have proved that there exist continuous functions η, γ defined on $\mathbf{R}\times[0,\infty)$ such that $I(u_1^2)(x_1,x_2,x_3)=\eta(x_1,\sqrt{x_2^2+x_3^2})$ and $I(u_2^2)(x_1,x_2,x_3)=\gamma(x_1,\sqrt{x_2^2+(x_3-k)^2})$. Since $I(u_1^2)(x_1,0,x_3)=I(u^2)(x_1,0,x_3)=I(u_2^2)(x_1,0,x_3)$, we get $\eta(x_1,|x_3|)=\gamma(x_1,|x_3-k|)$ for any $x_1,x_3\in\mathbf{R}$. In particular, if $k\geqslant 0$, for $t\geqslant 0$ we have $\eta(x_1,t+2k)=\gamma(x_1,t+k)=\eta(x_1,t)$; that is, for any fixed x_1, the function $\eta(x_1,\cdot)$ is periodic of period $2k$. On the other hand, we have $I(u_1^2),I(u_2^2)\in W^{2,p}(\mathbf{R}^N)$ for $p\in(3,\infty]$, thus $I(u_1^2)$ and $I(u_2^2)$ tend to zero at infinity, hence $\eta(x_1,t)\to 0$ and $\gamma(x_1,t)\to 0$ as $x_1^2+t^2\to\infty$. We infer that either $k=0$, or $\eta\equiv 0$ in $\mathbf{R}\times[0,\infty)$. In both cases we get $\eta=\gamma$ on $\mathbf{R}\times[0,\infty)$ and $I(u_1^2)=I(u_2^2)$ in $\mathbf{R}^3$. Thus we have $I(u^2)=I(u_1^2)=I(u_2^2)$ on $\mathbf{R}^3$, and $I(u^2)$ is radially symmetric with respect to (x_2,x_3).

Since $Q(u) = Q(u_1) = \lambda \neq 0$, we cannot have $u \equiv 0$ in the half-space $\{x_2 < 0\}$. Assumption (b) implies that there exists $(x_1, x_2, x_3) \in \mathbf{R}^3$, $x_2 < 0$ such that $G'(u(x_1, x_2, x_3)) \neq 0$. Since $u = u_1$ on $\{x_2 < 0\}$ and $I(u^2) = I(u_1^2)$ on $\mathbf{R}^3$, from (4.34) and (4.35) we infer that $\alpha = \beta$. Let $a(x) = \frac{1}{d_3} \frac{\partial^2}{\partial x_1^2}(I(u^2))(x) = \frac{1}{d_3} \frac{\partial^2}{\partial x_1^2}(I(u_1^2))(x)$. We know that a is a continuous and bounded function on $\mathbf{R}^3$. The functions u and u_1 both satisfy the equation $-\Delta w + F'(w) + a(x)w + \alpha G'(w) = 0$ in $\mathbf{R}^3$ and using the Unique Continuation Principle again we conclude that $u \equiv u_1$ in $\mathbf{R}^3$, i.e. u is symmetric with respect to x_2.

In the same way we prove that u is symmetric with respect to x_3 (after possibly a translation). Proceeding as in the proof of Theorem 4.1 we can show that u is symmetric with respect to any plane containing the line $\{(x_1, 0, 0) \mid x_1 \in \mathbf{R}\}$, consequently u is radially symmetric with respect to (x_2, x_3) variables. $\square$

Remark 4.12. (i) We have stated and proved Theorem 4.9 in dimension $N = 3$ only for simplicity. Replacing the term $\int_{\mathbf{R}^3} |R_1(u^2)|^2(x)\,dx$ in $E(u)$ by $\int_{\mathbf{R}^N} |R_1(H(u))|^2(x)\,dx$ and making suitable assumptions on the function H, this result admits a straightforward generalization to $\mathbf{R}^N$, $N \geqslant 3$.

(ii) We do not know whether the minimizers in Theorem 4.9 are symmetric or not with respect to x_1. Recall that by (2.42) we have

$$\int_{\mathbf{R}^N} \frac{\xi_1^2}{|\xi|^2} \big|\widehat{T_1\varphi}(\xi)\big|^2\,d\xi + \int_{\mathbf{R}^N} \frac{\xi_1^2}{|\xi|^2} \big|\widehat{T_2\varphi}(\xi)\big|^2\,d\xi - 2\int_{\mathbf{R}^N} \frac{\xi_1^2}{|\xi|^2} \big|\widehat{\varphi}(\xi)\big|^2\,d\xi$$
$$= -\frac{8}{\pi} \int_{\mathbf{R}^{N-1}} |\xi'| \left| \int_0^\infty \widehat{A\varphi}(\xi) \frac{\xi_1}{\xi_1^2 + |\xi'|^2}\,d\xi_1 \right|^2 d\xi' \tag{4.39}$$

for any $\varphi \in C_c^\infty(\mathbf{R}^N)$. Clearly, the left-hand side of (4.39) is continuous on $L^2(\mathbf{R}^N)$. Proceeding as in Lemma 4.10, it is easy to see that the right-hand side of (4.39) also defines a continuous functional on $L^2(\mathbf{R}^N)$. Consequently, (4.39) holds for any $\varphi \in L^2(\mathbf{R}^N)$. Using (4.28) and (4.39) we have

$$E(T_1u) + E(T_2u) - 2E(u) = \frac{2}{\pi} \frac{1}{(2\pi)^N} \int_{\mathbf{R}^{N-1}} |\xi'| \left| \int_0^\infty \mathcal{F}\big(A\big(H(u)\big)\big)(\xi) \frac{\xi_1}{|\xi|^2}\,d\xi_1 \right|^2 d\xi'. \tag{4.40}$$

The right-hand side in this integral identity is always nonnegative and (4.40) does not imply the symmetry of minimizers with respect to x_1.

(iii) Let us change the sign of the nonlocal term appearing in Theorem 4.9, i.e. let us consider the minimization problem

$$\text{minimize} \quad E_*(u) = \frac{1}{2}\int_{\mathbf{R}^3} |\nabla u|^2\,dx + \int_{\mathbf{R}^3} F(u)\,dx + \frac{1}{4}\int_{\mathbf{R}^3} \big|R_1\big(u^2\big)\big|^2\,dx$$
$$\text{under the constraint} \quad Q(u) := \int_{\mathbf{R}^3} G\big(u(x)\big)\,dx = \lambda. \tag{4.41}$$

The minimizers of this problem (when they exist) give rise to standing waves for Eq. (4.27) where the sign of the nonlocal term $R_1^2(|u|^2)u$ has been reversed. Clearly, the integral identities that we have do not imply the symmetry of solutions of (4.41) with respect to x_2 and x_3.

The symmetry of minimizers of (4.41) with respect to x_1 is also an open problem. As above, in this case we have the identity

$$E_*(T_1u) + E_*(T_2u) - 2E_*(u) = -\frac{2}{\pi}\frac{1}{(2\pi)^3}\int_{\mathbf{R}^2} |\xi'| \left| \int_0^\infty \mathcal{F}\big(A\big(u^2\big)\big)(\xi)\frac{\xi_1}{|\xi|^2}\,d\xi_1 \right|^2 d\xi_2\,d\xi_3. \tag{4.42}$$

If u is a minimizer, the right-hand side of (4.42) must vanish. As in the proof of Theorem 4.9, this implies $\frac{\partial}{\partial x_1}I(u^2)(0,x_2,x_3) = 0$ for any $(x_2,x_3) \in \mathbf{R}^2$. Repeating the argument already used in Theorem 4.9 we get $I(u^2) = I((T_1u)^2)$ on $\{x_1 \leqslant 0\}$ and $I(u^2) = I((T_2u)^2)$ on $\{x_1 \geqslant 0\}$. Moreover, if $\lambda \neq 0$ then u and $u_1 := T_1u$ satisfy the same Euler–Lagrange equation, namely

$$-\Delta w + F'(w) - \frac{1}{d_3}\frac{\partial^2}{\partial x_1^2}\big(I\big(w^2\big)\big)w + \alpha G'(w) = 0. \tag{4.43}$$

Equivalently, defining $U = I(u^2)$ and $U_1 = I(u_1^2)$, we see that (u,U) and (u_1,U_1) are both solutions to the system

$$\begin{cases} -\Delta w + F'(w) - \dfrac{1}{d_3}\dfrac{\partial^2 W}{\partial x_1^2} w + \alpha G'(w) = 0, \\ -\Delta W = w^2. \end{cases} \tag{4.44}$$

Moreover, $(u,U) = (u_1,U_1)$ on $\{x_1 \leqslant 0\}$ and u, u_1 satisfy the conclusion of Lemma 4.11. We do not know whether this information together with the boundary condition $\frac{\partial U}{\partial x_1}(0,x_2,x_3) = \frac{\partial U_1}{\partial x_1}(0,x_2,x_3) = 0$ imply that $u \equiv u_1$.

Remark 4.13. If $N = 3$, the nonlocal term in Theorem 4.9 can be written as

$$\begin{aligned} \int_{\mathbf{R}^3} \big|R_1\big(u^2\big)\big|^2\,dx &= \frac{1}{(2\pi)^3}\int_{\mathbf{R}^3} \frac{\xi_1^2}{|\xi|^2}\big|\widehat{u^2}(\xi)\big|^2\,d\xi = -\frac{1}{d_3(2\pi)^3}\int_{\mathbf{R}^3} \mathcal{F}\bigg(\frac{\partial^2}{\partial x_1^2}I\big(u^2\big)\bigg)(\xi)\overline{\widehat{u^2}(\xi)}\,d\xi \\ &= -\frac{1}{d_3}\int_{\mathbf{R}^3} \frac{\partial^2}{\partial x_1^2}I\big(u^2\big)(x)\overline{u^2(x)}\,dx = -\frac{1}{d_3}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3} u^2(x)K(x-y)u^2(y)\,dx\,dy, \end{aligned}$$

where $K(x) = \frac{\partial^2}{\partial x_1^2}(\frac{1}{|x|}) = \frac{2x_1^2 - x_2^2 - x_3^2}{(x_1^2+x_2^2+x_3^2)^{\frac{5}{2}}}$. Since this kernel changes sign, spherical rearrangements in the variables (x_2,x_3) combined with Riesz' inequality cannot be used to prove the symmetry of minimizers.

5. Some open problems

We close this paper speaking about several problems for which the methods described above (including ours) seem to fail.

First, let us come back to the two minimization problems considered in Theorem 4.1. As before, if u is a minimizer of any of these problems, we may assume that $\int_{\{x_1<0\}} G(u)\,dx = \int_{\{x_1>0\}} G(u)\,dx$ and we set $u_1 = T_1 u$ and $u_2 = T_2 u$. Assume that $s \in (1, \frac{3}{2})$. Then the identities (3.22) and (3.23) are still valid (see Corollary 3.4) and we get

$$E(u_1) + E(u_2) - 2E(u) = -\frac{16\sin(s\pi)}{\pi^2} N_s^2(Au) \geqslant 0 \quad \text{in Case A,}$$

respectively

$$E(u_1) + E(u_2) - 2E(u) = -\frac{16\sin(s\pi)}{\pi^2} \widetilde{N}_s^2(Au) \geqslant 0 \quad \text{in Case B.}$$

It is easy to see that these integral identities work in the wrong direction. Are the minimizers still radially symmetric for $s \in (1, \frac{3}{2})$?

Another problem is to study the symmetry of minimizers of

$$E(u) = \frac{1}{2}\int_{\mathbf{R}^3} |\nabla u|^2 + \int_{\mathbf{R}^3\times\mathbf{R}^3} \frac{1}{|x-y|} u(x)^2 u(y)^2\,dx\,dy + \int_{\mathbf{R}^3} F\big(u(x)\big)\,dx$$

subject to the constraint

$$\int_{\mathbf{R}^3} u^2(x)\,dx = \lambda > 0.$$

In the particular case $F(u) = -C|u|^{8/3}$, this problem arises in connection with the Schrödinger–Poisson–Slater system [22]. Due to the repulsive effect of the nonlocal term, Riesz' inequality as well as the Reflection method work in the wrong direction.

The last problem concerns the symmetry of minimizers of

$$E(u) = \int_{-\infty}^{+\infty} \big(u_x^2(x) + u^3(x)\big)\,dx - \gamma \int_{-\infty}^{+\infty} |\xi| \big|\widehat{u}(\xi)\big|^2\,d\xi,$$

where $\gamma > 0$, subject to the constraint $\int_{-\infty}^{+\infty} u^2(x)\,dx = \lambda > 0$. These two functionals are conserved quantities for the Benjamin equation (see [1,2]). Symmetrization and reflection cannot be used due to the sign of the nonlocal term. Oscillating travelling waves for this equation have been found numerically; perhaps this is an indication that the minimizers of the problem above may change the sign.

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Improved Gagliardo–Nirenberg–Sobolev inequalities on manifolds with positive curvature

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Abstract

We apply the method of [J. Demange, From porous media equation to generalized Sobolev inequalities on a Riemannian manifold, preprint, http://www.lsp.ups-tlse.fr/Fp/Demange/, 2004] and [J. Demange, Porous Media equation and Sobolev inequalities under negative curvature, preprint, http://www.lsp.ups-tlse.fr/Fp/Demange/, 2004], based on the curvature–dimension criterion and the study of Porous Media equation, to the case of a manifold M with strictly positive Ricci curvature. This gives a new way to prove classical Sobolev inequalities on M. Moreover, this enables to improve non-critical Sobolev inequalities as well. As an application, we study the rate of convergence of the solutions of the Porous Media equation to the equilibrium.

Keywords: Porous Media equation; Sobolev inequalities; Curvature–dimension criterion

1. Introduction

In this paper, we derive inequalities from a family of nonlinear partial differential equations on a general n-dimensional compact manifold M whose Ricci curvature is bounded below by a positive constant ρ. The cases of nonnegative curvature and of strictly negative curvature have already been discussed in [7,8]. In this work we will follow the lines of [7,8] by differentiating

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functions called *Entropy* and *Information* linked to a nonlinear partial differential equation of the form:

$$\frac{\partial u}{\partial t}(t,x) = \Delta\sigma(u)(t,x), \quad t \geqslant 0,\ x \in M.$$

Here σ is a nondecreasing function mapping $\mathbb{R}_+$ onto $\mathbb{R}$. In this paper we use power functions $\sigma(x) = x^\alpha$, with $1 - 1/n \leqslant \alpha \leqslant 1$, but analog results can still be obtained for more general σ. These equations are analogous to the Porous Media equations studied in [7,8]. So let d be a real number greater than n and consider the case $\sigma(x) = x^{1-1/d}$. Denote by μ the normalized Riemannian measure on M. We define the Information as

$$I(t) = \int_M u(t,x)\left|\nabla(d-1)u(t,x)^{-1/d}\right|^2 d\mu.$$

Following [7,8] we will prove that

$$-I'(t) \geqslant K\frac{d}{dt}\int_M u(t,x)^{1-2/d}, \tag{1}$$

where K depends on n, d and ρ. The key to get the inequality is the use the curvature–dimension criterion in M (see [1,3–6]): denote by Δ the Laplace–Beltrami operator on M, $\Gamma(f,g) = \nabla f \cdot \nabla g$ the carré du champ operator and $\Gamma_2(f,g) = (\Delta\Gamma(f,g) - \Gamma(f,\Delta g) - \Gamma(\Delta f,g))/2$ the *iterated* carré du champ operator (see Section 2 for more details). Then for any smooth function g:

$$\Gamma_2(g,g) \geqslant \rho\Gamma(g,g) + \frac{1}{n}(\Delta g)^2. \tag{2}$$

In [7] an integrated version of (2) has been established, and it will be useful here again. Although the paper deals with the Laplace–Beltrami operator of M, we could replace M by a space X equipped with an operator L satisfying the condition $CD(\rho,n)$ (X does not need to have any dimension). In this case we would define the operators:

$$\Gamma(f,g) = \big(L(f\cdot g) - f\cdot Lg - g\cdot Lf\big)/2 \quad \text{and}$$
$$\Gamma_2(f,g) = \big(L\Gamma(f,g) - \Gamma(f,Lg) - \Gamma(g,Lf)\big)/2,$$

and still assume (2) where Δ is replaced by L. We shall also assume that L is a differential operator in the sense that for all functions ϕ, ψ:

$$\Gamma\big(\phi(f),\psi(g)\big) = \phi'(f)\psi'(g)\Gamma(f,g) \quad \text{and} \quad L\phi(f) = \phi'(f)Lf + \phi''(f)\Gamma(f,f),$$

and that an integration by part formula holds for L with respect to a given measure μ on X (here the Riemannian measure). Basic examples of such operators on the real line are the following one. Let a be a (smooth) function and $Lf = f'' + a\cdot f'$. Then under certain differential conditions on a (see Section 2), L satisfies (2). A basic example is

$$Lf(x) = f''(x) + (n-1)\tan(x)f'(x),$$

which satisfies (2) with $\rho = n-1$, just like the Laplace–Beltrami operator of the n-dimensional sphere. See [4] for more details. More generally, if X is a manifold of dimension m, the operator $L = \Delta + \nabla h \cdot \nabla$ satisfies (2) if and only if $n \geqslant m$ and

$$(n-m)[\mathit{Ricci} - \operatorname{Hess} h - \rho g] \geqslant \nabla h \otimes \nabla h.$$

A basic study of (1) implies the classical family of Sobolev inequalities on M, and this paper gives in fact a new way of proving them. However a more precise study of (1) leads to an improvement of the inequality in the case when $d > n$, that is when we are below the critical Sobolev exponent. This improvement was already known for the logarithmic Sobolev inequality (see [1,3]), which corresponds to the (limit) case when $d = \infty$. Indeed we get the following inequality for any smooth positive f:

$$\int_M f\left|\nabla (d-1) f^{-1/d}\right|^2 d\mu \geqslant \frac{\psi(\int_M f\, d\mu)^{1-2/d}] - \psi[\int_M f^{1-2/d}\, d\mu]}{\psi'[(\int_M f\, d\mu)^{1-2/d}]},$$

where ψ is not affine (the affine case corresponds to the classical Sobolev inequality). The expression of ψ can be found in Theorem 1 below. An application of those inequalities is the study of the convergence to the equilibrium of the solution to the former PDE. This is the statement of Theorem 2 (see Section 8).

Theorem 1. *Let $n \geqslant 2$ be an integer and M be a compact connected n-dimensional Riemannian manifold with Ricci curvature bounded below by a positive constant ρ. Denote by μ its normalized Riemannian measure. The following inequality holds for $d \in \mathbb{R}$, $d > n$, $d \geqslant 3$, $\alpha = 1/2 - 1/d$ and f, a smooth function mapping M onto $\mathbb{R}_+^*$:*

$$K(n,d)\left\{\psi\left(I^{2\alpha}, I\right) - \psi\left(\left(\int f^{2\alpha}, I\right)\right)\right\} \leqslant \phi\left(\left(\int f^{2\alpha}, I\right)\right) \int \left|\nabla f^{\alpha}\right|^2 d\mu,$$

where $\partial_x \psi(x,y) = \phi(x,y)$, $K(n,d) = \rho(d-2)/(4(1-1/n))$, $I = \int f\, d\mu$, and

$$\phi(x,y) = \exp\left(L(n,d)\frac{x^{d/(4(d-2))}}{y^{1/4}}\right) \quad \text{and} \quad L(n,d) = \frac{d-n}{n+2}\left(4 - 9\frac{d-n}{d(n+2)}\right).$$

Note that Theorem 1 provides a weaker inequality which looks like the Entropy–Logarithmic energy inequality already known for the case $d = \infty$:

Take the notations of Theorem 1. *Then*:

$$\left(\int f\right)^{2\alpha} - \int f^{2\alpha} \leqslant K_1 \log\left(1 + K_2 \int \left|\nabla f^{\alpha}\right|^2\right),$$

where $K_2 = L(n,d)d/(4(d-2)K(n,d)(\int f)^{2\alpha})$ and *$K_1^{-1} = K_2 K(n,d)$* (*see Corollary* 2).

In the last section we give extensions to our result by studying some modified Porous Media equation.

2. Notations

Let $n \geqslant 2$ and M be an n-dimensional compact and connected Riemannian manifold whose Ricci curvature is bounded below by a positive constant ρ (this hypothesis implies the compactness of M). Note $\cdot$ the scalar product on M, ∇ the gradient operator, Δ the classical Laplace–Beltrami operator. Then define the following operators respectively called *carré du champs operator* and iterated *carré du champs operator*:

$$\begin{aligned} \forall f, g, \quad \Gamma(f,g) &= \nabla f \cdot \nabla g, \\ \Gamma_2(f,g) &= \big(\Delta\Gamma(f,g) - \Gamma(f,\Delta g) - \Gamma(\Delta f, g)\big)/2. \end{aligned}$$

Moreover we will note $\Gamma_2(f,f) = \Gamma_2(f)$ and $\Gamma(f,f) = \Gamma(f)$. We know from the Bochner–Lichnerowicz formula (classical in Riemannian geometry) that for any smooth ξ:

$$\Gamma_2(\xi) \geqslant \rho\Gamma(\xi) + \frac{1}{n}(\Delta\xi)^2, \tag{3}$$

which we will call curvature–dimension criterion. See [1,3–6] for more information on this criterion. The proofs made in this paper are still valid if we replace Δ by an operator L and M by a space X, which does not need to have any dimension. Three main assumptions are needed:

(1) L satisfies the last curvature–dimension criterion which goes as follows. Define Γ_2 and Γ by

$$\begin{aligned} \Gamma(f,g) &= \big(L(f\cdot g) - f\cdot Lg - g\cdot Lf\big)/2, \\ \Gamma_2(f,g) &= \big(L\Gamma(f,g) - \Gamma(f,Lg) - \Gamma(g,Lf)\big)/2. \end{aligned}$$

Then one must assume that

$$\Gamma_2(\xi,\xi) \geqslant \rho\Gamma(\xi,\xi) + \frac{1}{n}(L\xi)^2.$$

(2) On (X, L), an integration by parts formula holds with respect to a given measure μ (in the last case μ is the Riemannian measure); in other words, for functions f, g:

$$\int_X (Lf)g\,d\mu = -\int \Gamma(f,g)\,d\mu.$$

(3) L should be a differential operator in the sense of the introduction. Some basic examples of such operators for $X = \mathbb{R}$ or an interval in $\mathbb{R}$ are the operators L where $Lf = f'' + af$, a being a function satisfying:

$$a' \geqslant \rho + \frac{a^2}{n-1}. \tag{4}$$

Indeed in this case

$$\Gamma(f,g) = f'g' \quad \text{and} \quad \Gamma_2(f,g) = f''g'' + a'f'g',$$

and $\Gamma_2(f,f) - \rho\Gamma(f,f) - (Lf)^2/n$ equals

$$f''^2(1-1/n) + f'2\big(a' - a^2/n - \rho\big) + (2a/n)f'f''.$$

This polynomial in f' and f'' is nonnegative for any f if and only if its discriminant is nonpositive, which is condition (4). Therefore if $a(x) = \tan(x\sqrt{\rho/(n-1)})\sqrt{\rho(n-1)}$ then L satisfies the curvature–dimension criterion. Note also that those operators give models for the cases $\rho < 0$ and $\rho = 0$ which we do not discuss here. For more explanations see [4]. To avoid dealing with questions on the existence of solutions to the considered PDEs, we shall make the proof only for the Laplace–Beltrami operator of a manifold M (in this case the dimension of the operator Δ coincide with the dimension of the space M unlike above).

Let $d \geqslant n$, $d > 2$, be a real number and consider the following PDE with smooth positive initial data f:

$$\text{(E)}\qquad \frac{\partial u}{\partial t}(t,x) = \Delta u(t,x)^{1-1/d}, \quad t \geqslant 0,\ x \in M.$$

Since M is compact one can find $\varepsilon > 0$ such that $f \geqslant \varepsilon$. Then consider the following PDE starting from $f - \varepsilon$:

$$\text{(F)}\qquad \frac{\partial v}{\partial t}(t,x) = \Delta\sigma_\varepsilon\big(v(t,x)\big), \quad t \geqslant 0,\ x \in M,$$

where $\sigma_\varepsilon(x) = (x+\varepsilon)^{1-1/d} - \varepsilon^{1-1/d}$. Since $\sigma'_\varepsilon(0) > 0$, standard parabolic results show that a smooth nonnegative solution v to (F) exists for $t \geqslant 0$. Remark that $u = v + \varepsilon$ is a smooth solution to (E) and remains greater than ε. This proves the existence of a smooth solution to (E). We can give examples of such solutions on models manifolds: the fundamental solutions to (E) for $d = n$, that is the one that start from a Dirac delta distribution, are explicit: this happens for $\mathbb{R}^n$, for the sphere and for the hyperbolic space. Of course those solutions are smooth only for $t > 0$, but they are smooth for any time if you consider $t = t_0 > 0$ as initial condition. For instance on the sphere S_n of dimension n, denoting $\langle,\rangle$ the Euclidean scalar product,

$$u(t,x) = \left(\frac{\operatorname{sh}((n-1)t)}{\operatorname{ch}((n-1)t) - \langle x_0, x\rangle}\right)^n,$$

is the solution to the porous media equation (E) with critical exponent $d = n$, starting from the Dirac delta mass in $x_0 \in S_n$. More generally, on a space X equipped with an operator satisfying the above conditions, if one can find a function T whose generalized hessian is $-Tg$, that is,

$$\Gamma\big(\xi, \Gamma(T,\xi)\big) - \Gamma\big(T, \Gamma(\xi)\big)/2 = -T\Gamma(\xi),$$

and whose minimum is -1 with $\arg\min(T) = x_0$, then the solution has the same form with $-\langle x, x_0\rangle$ replaced by $\langle x, T(x_0)\rangle$. For the solutions on $\mathbb{R}^n$ (called Barenblatt solutions) we refer to [10] and for the hyperbolic we refer to [8].

Let us finally define the *Entropy* $E_2(t)$ and *Information* $I(t)$ of the solution u as follows:

$$E_2(t) = \int\limits_M u(t,x)^{1-2/d}\,d\mu,$$

$$I(t) = \int\limits_M u(t,x)\Gamma\big((d-1)u(t,x)^{-1/d}\big)\,d\mu.$$

Unlike what usually happens in Entropy–Energy inequalities (for instance: logarithmic-Sobolev inequalities), where the left-hand side is the "Entropy" and the right-hand side is the derivative of the Entropy, called information, here there will not be such a differential relation between both sides of the inequalities that will be proved. Thus we denote the Entropy E_2 instead of E: $E_2(t) \neq C\frac{d}{dt}I(t)$, $C \in \mathbb{R}$. In the rest of the paper we shall write for simplicity u or u_t in place of $u(t,x)$ (u_t is not a derivative) and ξ will stand for $-(d-1)u^{-1/d}$. When writing integrals, we shall write sometimes $\int H$ or $\int_M H$ instead of $\int_M H\,d\mu$.

3. Integrated curvature–dimension criterion

As a special case of an inequality stated in [7] and which follows from (3), we have the

Lemma 3.1. *Let ψ, H be functions of the real variable, ψ being bijective, increasing and $u : M \to \mathbb{R}_+$ be smooth. Let $G(x) = 2x^2\psi'(x)$, $x \geqslant 0$ and $\xi = \psi(u)$. Then*

$$\begin{aligned}\int G(u)\Gamma_2(\xi) \geqslant \int &\frac{n}{n-1}G(u)\rho\Gamma(\xi)\\ &+ \left(\frac{3}{2}\frac{d}{d\xi}\left(\frac{G(u)}{n-1}\right) - \frac{n+2}{n-1}G(u)H(u)\right)\Gamma\big(\xi,\Gamma(\xi)\big)\\ &+ \int\left(\frac{d^2}{d\xi^2}\left(\frac{G(u)}{n-1}\right) - 2\frac{d}{d\xi}\left(\frac{G(u)H(u)}{n-1}\right) - G(u)H(u)^2\right)\Gamma(\xi)^2\end{aligned}$$

with the natural convention that for any function K, $\frac{d}{d\xi}(K(u)) = \frac{K'(u)}{\psi'(u)}$.

This inequality is obtained by applying formula (3) to $H(\xi)$ instead of ξ and then doing integrations by parts.

4. Differentiation of $I(t)$ and minoration of $-I'(t)$

In the following we differentiate the Information function $I(t)$. Note $\psi(x) = -(d-1)x^{-1/d}$. Recall that u_t is the solution of equation (E) so that for any function $K(t,x)$

$$\int \frac{\partial u}{\partial t}K(t,x) = -\int u_t\Gamma\big(\xi, K(t,\cdot)\big).$$

Therefore we have:

$$\begin{aligned} I'(t) &= \int \big(\partial_t u \Gamma(\xi) + 2u\Gamma\big(\xi, \psi'(u)\partial_t u\big)\big)\, d\mu \\ &= \int \partial_t u\big[\Gamma(\xi) - 2\psi'(u)\big(\Gamma(u,\xi) + u\Delta\xi\big)\big]\, d\mu, \\ -I'(t) &= \int u\Gamma\big(\xi, -\Gamma(\xi) - 2\psi'(u)u\Delta\xi\big). \end{aligned} \tag{5}$$

Let us first consider the term $u\psi'(u)\Delta\xi$. Note that $-1/d = (x\psi'(x))'\psi'(x)^{-1}$. Therefore

$$\begin{aligned} \Gamma\big(\xi, u\psi'(u)\Delta\xi\big) &= u\psi'(u)\Gamma(\xi, \Delta\xi) - \Delta\xi\, \Gamma(\xi)/d \\ &= u\psi'(u)\left[\frac{1}{2}\Delta\Gamma(\xi) - \Gamma_2(\xi)\right] - \Delta\xi\, \Gamma(\xi)/d. \end{aligned}$$

Firstly, the integration by parts formula implies

$$\begin{aligned} \int u^2\psi'(u)\Delta\Gamma(\xi)\, d\mu &= -\int \Gamma\big(u^2\psi'(u), \Gamma(\xi)\big)\, d\mu \\ &= -(1 - 1/d)\int u\Gamma\big(\xi, \Gamma(\xi)\big)\, d\mu. \end{aligned} \tag{6}$$

Secondly,

$$\begin{aligned} -\int u\Delta\xi\,\Gamma(\xi)\, d\mu &= \int \Gamma\big(\xi, u\Gamma(\xi)\big)\, d\mu = \int u\Gamma\big(\xi, \Gamma(\xi)\big)\, d\mu + \int \Gamma(\xi)\Gamma(\xi, u)\, d\mu \\ &= \int u\Gamma\big(\xi, \Gamma(\xi)\big)\, d\mu + \int \frac{1}{\psi'(u)}\Gamma(\xi)^2\, d\mu. \end{aligned} \tag{7}$$

Finally, after collecting terms (6) and (7) there remains:

$$\begin{aligned} -\int 2u\Gamma\big(\xi, u\psi'(u)\Delta\xi\big)\, d\mu &= \int 2u^2\psi'(u)\Gamma_2(\xi)\, d\mu \\ &\quad + (1 - 3/d)\int u\Gamma\big(\xi, \Gamma(\xi)\big)\, d\mu - \frac{2}{d}\int \frac{1}{\psi'(u)}\Gamma(\xi)^2\, d\mu. \end{aligned} \tag{8}$$

Thanks to (5) this leads to

$$-I'(t) = \int \left[2(1 - 1/d)u^{1-1/d}\Gamma_2(\xi) - \frac{3}{d}u\Gamma\big(\xi, \Gamma(\xi)\big) - \frac{2}{d-1}u^{1+1/d}\Gamma(\xi)^2\right] d\mu. \tag{9}$$

Now apply Section 3 to $\psi(x) = -(d-1)x^{-1/d}$ and to our solution at time t, $u = u_t$. Then if $G(x) = 2(1 - 1/d)x^{1-1/d}$, for any function H

$$-I'(t) \geqslant \int \frac{n}{n-1}G(u)\rho\Gamma(\xi) + \int S_1\Gamma\big(\xi, \Gamma(\xi)\big) + \int S_2\Gamma(\xi)^2, \tag{10}$$

where

$$S_1 = \frac{3}{2}\frac{d}{d\xi}\left(\frac{G(u)}{n-1}\right) - \frac{n+2}{n-1}G(u)H(u) - \frac{3}{d}u,$$
$$S_2 = \frac{d^2}{d\xi^2}\left(\frac{G(u)}{n-1}\right) - 2\frac{d}{d\xi}\left(\frac{G(u)H(u)}{n-1}\right) - G(u)H(u)^2 - \frac{2}{d-1}u^{1+1/d}.$$

Now the idea is to choose H such that $S_1 = 0$. A direct computation shows that we shall take

$$H(u) = 3u\frac{n}{(n+2)G(u)}\left(\frac{1}{n} - \frac{1}{d}\right).$$

This leads to $S_2 = C(n,d)u^{1+1/d}$ where

$$C(n,d) = \frac{n(1/n - 1/d)}{2(n+2)(1-1/d)}\left(4 - 9\frac{n}{n+2}(1/n - 1/d)\right). \tag{11}$$

Since $d \geqslant n$, $C(n,d) \geqslant 0$, with equality only if $d = n$. We finally get the following minoration of $-I'(t)$:

$$-I'(t) \geqslant 2\rho\frac{1-1/d}{1-1/n}\int u^{1-1/d}\Gamma(\xi)\,d\mu + C(n,d)\int u^{1+1/d}\Gamma(\xi)^2\,d\mu. \tag{12}$$

This relation is the key to prove the inequalities of this paper. First abord, we will derive from (12) the classical Sobolev inequality by saying that $C(n,d) \geqslant 0$ and then we will enhance by taking into account all terms of the right-hand side.

5. The classical Sobolev inequality

In this section we give an alternative way to prove Sobolev inequalities. Those inequalities are already known on manifolds with Ricci curvature bounded from below by a positive constant. A good reference is [2,9]. Usually the inequality is obtained by compacity argument, and the constant may not be optimal. Then a discussion on extremal solutions to the inequality gives the best constant. Our method however gives the best known constant directly, without discussion, and proves that it is a consequence of hypothesis about curvature and dimension on the operator Δ. We should also mention that there are other ways to prove that these Sobolev inequalities exist, using curvature–dimension assumptions only. One is explained in [3]: first, the author proves a stronger form of the logarithmic-Sobolev inequality, called "weak Sobolev inequality," and manages to establish a Sobolev inequality, however, the constant might not be optimal. Then again a clever discussion implies that the constant can be chosen the same as in classical Riemannian theory. However in this paper we prove the inequality with the best known constant "directly" by associating to an inequality a partial differential equation.

Lemma 5.1. *Let for* $0 \leqslant i \leqslant 2$:

$$E_i(t) = \int_M u(t,x)^{1-i/d}\,d\mu.$$

Then the following differential inequality holds for $t \geqslant 0$:

$$-E_1''(t) \geqslant A E_2'(t), \tag{13}$$

where the constant A depends on d, n and ρ as follows:

$$A = \rho \frac{(d-1)(1-1/d)}{(1-1/n)(d-2)}.$$

Proof. First let us calculate $E_1'(t)$. Recall that u is the solution to (E). Since M is compact the integrations by parts are valid and

$$\begin{aligned} E_1'(t) &= \int (1-1/d)\frac{\partial u}{\partial t} u^{-1/d} = \int (1-1/d)\Delta u^{1-1/d} u^{-1/d} \\ &= -\int (1-1/d)\Gamma\big(u^{1-1/d}, u^{-1/d}\big)\, d\mu. \end{aligned}$$

Recall that $\xi = -(d-1)u^{-1/d}$. Then

$$E_1'(t) = \frac{1}{d}\int u\Gamma(\xi). \tag{14}$$

In the same way we show that

$$E_2'(t) = 2\frac{d-2}{d(d-1)}\int u^{1-1/d}\Gamma(\xi). \tag{15}$$

Replace $\int u\Gamma(\xi)$ and $\int u^{1-1/d}\Gamma(\xi)$ in formula (12) by their expressions given in formulae (14) and (15) and estimate from below the term with $C(nd)$ by 0. This gives the proposition. $\square$

Corollary 1. *For $\alpha = 1/2 - 1/d$ we have the classical Sobolev inequality*:

$$\left(\int f\right)^{2\alpha} - \int f^{2\alpha} \leqslant K(n,d)^{-1}\int \left|\nabla f^\alpha\right|^2,$$

where $K(n,d)$ is defined in Theorem 1. *Letting $d \to n$ gives the Sobolev inequality with critical exponent.*

Proof. First let us study the behavior of $u(t,\cdot)$ as $t \to \infty$. First note that by Hölder inequality $E_1(t) \leqslant E_0^{1-1/d}$ and recall that $E_0 = \int f$ does not depend on t. Hence E_1 is bounded and therefore there exists a diverging sequence t_k such that $\lim_\infty E_1'(t_k) = 0$ (we leave it to the reader). From formula (14), if $\alpha = 1/2 - 1/d$, this implies that

$$\lim_\infty \int_M \Gamma\big(u(t_k,\cdot)^\alpha\big)\, d\mu = 0,$$

which from the classical Poincaré inequality on M implies that

$$\lim_{\infty} \int_M \left[\left(\int u(t_k, \cdot)^\alpha \right) - u(t_k, \cdot)^\alpha \right]^2 d\mu = 0.$$

Once again the sequence $\int u(t_k, \cdot)^\alpha$ is bounded and therefore, up to a subsequence, we can suppose that it has a limit l^α. Thus, up to a subsequence, $u(t_k, \cdot)$ converges to l a.e. Now thanks to the form of the PDE (E) we also know that $\max u(t_k, \cdot) \leqslant \max f$. Hence the convergence also holds in any L^p, $p > 0$. Hence, for $p = 1$ we see that $l = \int f$ (μ is normalized).

Lemma 5.1 ensures that the function

$$m(t) = E_1'(t) + A E_2(t)$$

is non-increasing. Hence a posteriori $m(0) \geqslant \limsup_\infty m(t_k)$. As we saw $\lim_\infty E_1'(t_k) = 0$, and $\lim_\infty E_2(t_k) = (\int f)^{1-2/d}$ (since $u(t_k, \cdot) \to l$ in $L^{1-2/d}$). Therefore

$$A \left\{ \left[\int f \right]^{2\alpha} - \int f^{2\alpha} \right\} \leqslant E_1'(0). \tag{16}$$

Note that from formula (14),

$$E_1'(0) = \frac{4(d-1)^2}{d(d-2)^2} \int \Gamma(f^\alpha).$$

This proves the proposition. □

Note that we can specify the rate at which the solution converges to $\int f$. A consequence of Lemma 5.1 shows that if $M = 2\rho(1 - 1/d)/(1 - 1/n)$, then

$$-I'(t) \geqslant M \max(f)^{-1/d} I(t),$$

which implies an exponential decay of $I(t)$, and more precisely, thanks to formula (14),

$$\int \left| \nabla u(t, \cdot)^\alpha \right|^2 \leqslant \int \left| \nabla f^\alpha \right|^2 \exp\left(-M \max(f)^{-1/d} t \right).$$

This result can however be slightly enhanced: see Section 8.

6. Differential relation between entropy and information

In this section we prove a differential relation between $E_2(t)$ and $E_1(t)$ (or $I(t)$) which improves Lemma 5.1 and proves Theorem 1. Let us begin with a lemma which is a direct consequence of Cauchy–Schwarz inequality.

Lemma 6.1. *For any smooth $\xi : M \to \mathbb{R}$ and $u : M \to \mathbb{R}_+$, $u \neq 0$, we have*

$$\int\limits_M u^{1+1/d}\Gamma(\xi)^2\,d\mu \geqslant \frac{\int_M u\Gamma(\xi)\,d\mu \int_M u^{1-1/d}\Gamma(\xi)\,d\mu}{\int_M u^{1-1/d}\,d\mu \int_M u^{1-3/d}\,d\mu}.$$

Proof. Note $\theta = 1/2 + 1/(2d)$. Just develop the numerator D of the right-hand side in $M \times M$ and use Cauchy–Schwarz inequality:

$$\begin{aligned} D &= \iint\limits_{M\times M} u(x)^{1-1/d}u(y)\Gamma(\xi)(x)\Gamma(\xi)(y)\,d\mu(x)\,d\mu(y) \\ &= \iint\limits_{M\times M} \big[u(x)^{\theta}u(y)^{1-\theta}\Gamma(\xi)(x)\big]\big[u(y)^{\theta}u(x)^{1-1/d-\theta}\Gamma(\xi)(y)\big]\,d\mu(x)\,d\mu(y) \\ &\leqslant \int\limits_M u^{1+1/d}\Gamma(\xi)^2\,d\mu\Big(\int\limits_M u^{1-1/d}\,d\mu\Big)^{1/2}\Big(\int\limits_M u^{1-3/d}\,d\mu\Big)^{1/2}. \qquad \square \end{aligned}$$

Now Lemma 6.1 and inequality (12) bring about the following.

Proposition 6.1. *Let for $0 \leqslant i \leqslant 3$*

$$E_i(t) = \int\limits_M u(t,x)^{1-i/d}\,d\mu.$$

Then the following differential inequality holds for $t \geqslant 0$:

$$-E_1''(t) \geqslant AE_2'(t) + B\frac{E_1'(t)E_2'(t)}{\sqrt{E_1(t)E_3(t)}}, \tag{17}$$

where the constants A and B depend on d, n and ρ as follows:

$$A = \rho\frac{(d-1)(1-1/d)}{(1-1/n)(d-2)} \quad \textit{and} \quad B = C(n,d)\frac{d(d-1)}{2(d-2)},$$

and $C(n,d)$ is defined by formula (11).

Proof. Replace $\int u^{1+1/d}\Gamma(\xi)^2\,d\mu$ of formula (12) by the minoration given by Lemma 6.1 and then replace $\int u\Gamma(\xi)$ and $\int u^{1-1/d}\Gamma(\xi)$ by their expressions given in formula (14) and (15). This gives the proposition. $\square$

7. Integration of the differential inequality

First note that E_0 does not depend on t since (E) is a mass-preserving equation. Then since $u(0,x) = f(x)$, $E_0 = \int_M f\,d\mu$. In this section we integrate a weaker version of inequality (17). Indeed, the presence of $E_3(t)$ and $E_1(t)$ lead us to use the following Hölder majorations:

$$E_1(t) \leqslant \sqrt{E_0E_2(t)} \quad \text{and} \quad E_3(t) \leqslant E_2(t)^{\frac{d-3}{d-2}}.$$

Therefore inequality (17) becomes:

$$-E_1''(t) \geqslant AE_2'(t) + \frac{B}{E_0^{1/4}} E_1'(t)E_2'(t)E_2(t)^{-\frac{3d-8}{4(d-2)}}, \tag{18}$$

and can be put into the following form.

Proposition 7.1.

$$\frac{d}{dt}\big(E_1'(t)\phi\big(E_2(t)\big) + A\psi\big(E_2(t)\big)\big) \leqslant 0,$$

where ψ is a primitive of ϕ and

$$\phi(x) = \exp\left[\frac{B}{E_0^{1/4}}\frac{4(d-2)}{d}x^{d/(4(d-2))}\right].$$

Proof. Remark that inequality (18) reads as follows:

$$0 \geqslant E_1''(t) + AE_2'(t) + E_1'(t)E_2'(t)\frac{\phi'(E_2(t))}{\phi(E_2(t))},$$

$$0 \geqslant E_1''(t)\phi\big(E_2(t)\big) + E_1'(t)\frac{d}{dt}\phi\big(E_2(t)\big) + A\frac{d}{dt}\psi\big(E_2(t)\big).$$

This is exactly the result mentioned in the proposition. □

We now are able to prove Theorem 1 stated in the introduction.

Proof. Recall from the proof of Corollary 1, that there is a subsequence t_k diverging such $u(t_k,\cdot)$ converges to $l = \int f$ a.e. and in any L^p, $p > 0$.

Proposition 7.1 ensures that the function

$$m(t) = E_1'(t)\phi\big(E_2(t)\big) + A\psi\big(E_2(t)\big)$$

is non-increasing. Hence a posteriori $m(0) \geqslant \limsup_\infty m(t_k)$. As we saw $\lim_\infty E_1'(t_k) = 0$, and $\lim_\infty E_2(t_k) = (\int f)^{1-2/d}$ (since $u(t_k,\cdot) \to l$ in $L^{1-2/d}$). Therefore

$$A\left\{\psi\left(\left[\int f\right]^{2\alpha}\right) - \psi\left(\int f^{2\alpha}\right)\right\} \leqslant \phi\left(\int f^{2\alpha}\right)E_1'(0). \tag{19}$$

Note that from formula (14),

$$E_1'(0) = \frac{4(d-1)^2}{d(d-2)^2}\int \Gamma\big(f^\alpha\big).$$

Now if we recall the definitions of ϕ in Proposition 7.1 and of $C(n,d)$ in formula (11), then we get exactly the inequality of Theorem 1. □

We will now formulate a weaker inequality. This is an Entropy–Logarithmic energy inequality which looks like the one obtained in [1,3]. In fact, letting $d \to \infty$ gives that inequality (since it makes use of the heat equation). These inequalities imply the classical non-critical Sobolev inequality (which we proved by our method in Section 5) by simple use of inequality $\log(1+x) \leqslant x$, $x > 0$. Other classical proofs can be found for instance in [9], they do not however consider the improved form using log function. Here it appears as a weaker form of Theorem 1.

Corollary 2. *Take the notations of Theorem* 1. *Then*

$$\left(\int f\right)^{2\alpha} - \int f^{2\alpha} \leqslant K_1 \log\left(1 + K_2 \int \left|\nabla\left(f^{\alpha}\right)\right|^2\right),$$

where $K_2 = L(n,d)d/(4(d-2)K(n,d)(\int f)^{2\alpha})$ *and* $K_1^{-1} = K_2 K(n,d)$. *Moreover we have the classical Sobolev inequality*:

$$\left(\int f\right)^{2\alpha} - \int f^{2\alpha} \leqslant K(n,d)^{-1} \int \left|\nabla f^{\alpha}\right|^2.$$

Letting $d \to n$ *gives the Sobolev inequality with critical exponent.*

Proof. The second inequality comes from the first one using inequality $\log(1+x) \leqslant x$. For the first one, observe that if $x \leqslant u \leqslant y$ and $\beta = d/(4(d-2))$, then

$$u^{\beta} - x^{\beta} \geqslant \beta y^{\beta-1}(u-x).$$

Now choose $x = \int f^{2\alpha}$ and $y = (\int f)^{2\alpha}$. Then

$$\frac{\psi(y) - \psi(x)}{\phi(x)} = \int_x^y \exp\left(\frac{L(n,d)}{E_0^{1/4}}\left(u^{\beta} - x^{\beta}\right)\right) du \geqslant K_1\left(\exp\left(K_1^{-1}(x-y)\right) - 1\right).$$

Use this inequality and Theorem 1 to get the first inequality of Corollary 2. □

8. Application to the convergence of the solution

As was shown in the last sections there is a sequence t_k such that $u(t_k, \cdot)$ converges to $\int f$ a.e. as $k \to \infty$. However we can be more precise. What is well known that the logarithmic-Sobolev inequalities give informations on the decay of the solutions to heat equations towards equilibrium. In this section, we give estimates on the convergence of the solutions of porous media equations, using the improved Gagliardo–Nirenberg–Sobolev inequalities of this paper instead. The strategy is similar to Section 5 however. Will give an explicit bound on the H^1-norm of $u(t,\cdot)^{\alpha}$ which is more precise than the one given in Section 5. This is easily obtained thanks to formula (17). Note that the functional inequality itself will not give information on the behaviour of the norm of gradients of the solution but only on Hölder norms (the function S in the following). However, the method used in this paper provides such an information, due to the computation of $E_2''(t)$.

Keep notations of Theorem 1 and note $l = \int f^{1-2/d}$. Let $u(t,\cdot)$ be the smooth solution of the Porous Media equation (E) starting from $f > 0$ smooth, and note $\alpha = 1/2 - 1/d$. Let for $t \geqslant 0$, $S(t,\cdot)$ be the nonnegative function,

$$S(t,\cdot) = \left(\int f\right)^{2\alpha} - u(t,\cdot)^{2\alpha} + 2\alpha\left(\int f\right)^{-2/d}\left(u(t,\cdot) - \int f\right).$$

Let

$$K = 2\rho(d-1)/\big(d^2(1-1/n)\max f^{1/d}\big)$$

and

$$L(t) = \big[l - \psi^{-1}\big(\psi(l) - Z(0)\exp - Kt\big)\big]e^{-Kt}.$$

Let $A = 2\rho\frac{1-1/d}{1-1/n}$ and $C(n,d) \geqslant 0$ be given by formula (11) and

$$A' = A\max(f)^{-1/d}, \qquad I = \int |\nabla f^\alpha|^2, \qquad D(n,d) = C(n,d)(d-1)^2/(d-2)^2.$$

We have the following estimates for both Hölder norms and H^1 norms.

Theorem 2.

(i) *For* $t \geqslant 0$

$$\|S(t,\cdot)\|_{L^1(M)} \leqslant L(t)e^{-Kt}.$$

(ii) *Moreover, for* $t \geqslant 0$ *we have*

$$\int |\nabla u(t,\cdot)^\alpha|^2 \leqslant \frac{AIe^{-A't}}{A + D(n,d)I(1-e^{-A't})}. \tag{20}$$

(iii) *Finally we have the following estimates*:

$$\limsup_{t\to\infty} \frac{\log\|S(t,\cdot)\|_{L^1(M)}}{t} \leqslant -2\frac{\rho(d-1)}{d^2(1-1/n)}\left(\int f\right)^{-1/d}, \tag{21}$$

$$\limsup_{t\to\infty} \frac{\log\|\nabla u^\alpha(t,\cdot)\|_2^2}{t} \leqslant -2\rho\frac{1-1/d}{1-1/n}\left(\int f\right)^{-1/d}. \tag{22}$$

Proof. First remark that a simple convexity argument implies that $S(t,\cdot) \geqslant 0$. Let us apply Theorem 1 to $u(t,\cdot)$. Note $l = E_0^{2\alpha}$ and $Z(t) = \psi(l) - \psi(E_2(t))$. Therefore

$$\begin{aligned} Z(t) &\leqslant \phi\big(E_2(t)\big)K(n,d)^{-1}\frac{d(d-2)^2}{4(d-1)^2}\int u\Gamma(\xi) \\ &\leqslant \phi\big(E_2(t)\big)K(n,d)^{-1}\frac{d(d-2)^2}{4(d-1)^2}\max f^{1/d}\int u^{1-1/d}\Gamma(\xi), \end{aligned}$$

since $u(t,\cdot) \leqslant \max f$. By (15)

$$Z(t) \leqslant \phi\big(E_2(t)\big) K^{-1} E_2'(t),$$

where $K = 2\rho(d-1)/(d^2(1-1/n) \max f^{1/d})$. In other words,

$$K Z(t) \leqslant -Z'(t).$$

This yields that $Z(t)\exp(Kt)$ is a non-increasing function of t. Thus

$$Z(t) \leqslant Z(0)\exp(-Kt).$$

Rewriting this in a different form leads to

$$\int S(t,\cdot) = l - E_2(t) \leqslant l - \psi^{-1}\big(\psi(l) - Z(0)\exp(-Kt)\big).$$

With the notations, the right-hand side equals $L(t)\exp(-Kt)$. Then obviously

$$\lim_{t\to\infty} L(t) = \frac{\psi(l) - \psi(E_2(0))}{\phi(l)}.$$

Moreover, a simple convexity argument shows that $L(t) \leqslant l - E_2(0)$.

For the last part of the theorem, by (17), the estimation $u(t,\cdot) \leqslant \max f$ and Hölder inequality we get

$$-I'(t) \geqslant A' I(t) + C(n,d)\max(f)^{-1/d} \frac{I(t)^2}{(\int f)^{1-2/d}}.$$

This autonomous differential inequation is equivalent to

$$\frac{d}{dt}\left(\log\left[\frac{I(t)}{A' + C(n,d)\max(f)^{-1/d} I(t)}\right]\right) \leqslant -A',$$

which leads to formula (20) of the theorem, since

$$I(t) = \frac{(d-1)^2}{(d-2)^2} \int \big|\nabla u(t,\cdot)^\alpha\big|^2.$$

This proves (i) and (ii). (iii) is a direct consequence of (i) and (ii). For instance (i) gives that

$$\limsup_{t\to\infty} \frac{\log\|S(t,\cdot)\|_{L^1(M)}}{t} \leqslant -2\frac{\rho(d-1)}{d^2(1-1/n)}(\max f)^{-1/d}.$$

The semi-group property of the solution, which tells that $u(t+s,\cdot)_{t\geqslant 0}$ is the solution starting from $u(s,\cdot)$ at $t=0$ implies that f can be replaced by $u(s,\cdot)$:

$$\limsup_{t\to\infty} \frac{\log\|S(t,\cdot)\|_{L^1(M)}}{t} \leqslant -2\frac{\rho(d-1)}{d^2(1-1/n)}\big(\max u(s,\cdot)\big)^{-1/d}.$$

The same argument can be derived from (ii). So it is clear that (iii) is obtained as soon as $\lim_{s\to\infty} \max u(s,\cdot) = \int f$. This is done in Lemma 8.1. $\square$

We shall note that the exponential decay of $S(t,\cdot)$ implies the exponential decay of $\|u(t,\cdot) - l)\|_2^2$ since the Taylor formula and the inequality $\min u(t,\cdot) \leqslant \max f$ imply:

$$S(t,x) \geqslant \alpha(1/2+1/d)/2\big(u(t,x)-l\big)^2 \max(f)^{-1/2-1/d}.$$

Note also that we can have precise estimates only if we include the term $\max f$ in the formulae (i) and (ii). However, this is not convenient, since this quantity is not conserved along the path, unlike the quantity $\int f$ which appears in (21) and (22) of (iii). In this sense (iii) gives a candidate to be the precise rate of decay of the solution: we suspect that in (i) and (ii), the rate of decay of (iii) is still valid, even if this implies modifying the function in front of the exponential. However we are not able to prove it yet.

Lemma 8.1. *With the notations of Theorem* 1

$$\lim_{t\to\infty} \max\big\{u(t,\cdot)\big\} = \int_M f\,d\mu.$$

Proof. This proof is just an adaptation of the work made in [3] by D. Bakry: here we deal with the semi-group given by the Porous Media equation, which of course is nonlinear, whereas in [3], the author studies the convergence towards equilibrium for linear semi-groups. However, there is no big difficulty in proving Lemma 8.1. Let $\hat{m}(t)$ and $\hat{p}(t)$ be two functions and f be the smooth strictly positive starting function of $u(t,\cdot)$. The idea is to differentiate

$$U(t) = e^{-\hat{m}(t)}\|u_t\|_{\hat{p}(t)}.$$

As in [4, Théorème 3.3], we have that

$$U'(t) = \frac{U(t)\hat{p}'(t)}{\hat{p}^2(t)\int u_t^{\hat{p}(t)}}\left(E_{\hat{p}(t)}(u_t) - \frac{\hat{p}(t)^2}{\hat{p}'(t)}\left(\mathcal{E}_{\hat{p}(t)}(u_t) + \hat{m}'(t)\int u_t^{\hat{p}(t)}\right)\right),$$

where for any function g, and real number p,

$$E_p(g) = \int_M g^p \log g^p\,d\mu - \left(\int_M g^p\,d\mu\right)\log\int_M g^p\,d\mu,$$

$$\mathcal{E}_p(g) = -\int_M \Delta g^{1-1/n} g^{p-1}.$$

Of course, the generator of the semi-group is $\Delta g^{1-1/n}$ instead of Δg. Therefore, using the inequality

$$\forall g, p\quad \mathcal{E}_p(g) \geqslant (\max g)^{-1/n}(1-1/n)\int_M \Gamma\big(g, g^{p-1}\big)\,d\mu,$$

and reasoning as in [3, Théorème 3.3], we see that $U(t)$ is non-increasing as soon as

$$\frac{\hat{p}(t)^2(1-1/n)}{(\max f)^{1/n}c(\hat{p}(t))} = \hat{p}'(t) \quad \text{and} \quad m\big(\hat{p}(t)\big)(\max f)^{-1/n}(1-1/n) = \hat{m}'(t),$$

and a logarithmic-Sobolev inequality with constants $c(p)$ and $m(p)$ holds for given functions c and m and p lying in an interval $[a,b]$:

$$E_p(g) \leqslant c(p)\left(\mathcal{E}_p(g) + m(p)\int_M g^p\,d\mu\right).$$

Thus, Théorème 3.3 of [3] is true with a change of time $\nu = (\max f)^{1/n}/(1-1/n)$:

$$\big\|u(t\times\nu,\cdot)\big\|_b \leqslant e^{\hat{m}}\|f\|_a,$$

with

$$t = \int_a^b \frac{c(u)}{u^2}\,du \quad \text{and} \quad \hat{m} = \int_a^b m(u)\frac{c(u)}{u^2}\,du.$$

As a result, the subsequent propositions of [3] still hold up to the correct change of time. Hence

$$\limsup_{t\to\infty}\|u_t\|_\infty \leqslant \int_M f\,d\mu.$$

On the other hand, it is obvious that $\|u_t\|_\infty \geqslant \int u_t = \int f$. The lemma then follows. □

Note that by adapting the argument of [3] to our semi-group as in the proof of the previous lemma, we can get the uniform convergence towards equilibrium, and give precise estimates.

9. Another Sobolev inequality

In this section we will not detail the proof, since we use the same arguments as in the previous sections and in [7]. We present a Sobolev inequality, still valid under positive curvature, that generalizes the one of the n-dimensional sphere $\mathbb{S}^n$. The idea is to consider the modified Porous Media equation, as in [7]:

$$\partial_t u = \operatorname{div}\big\{u\nabla\big(-(n-1)u^{-1/n} + T\big)\big\},$$

starting from a smooth function f. Here $T > 0$ is a given function (we will give the assumptions later). Once again the idea is to differentiate the energy function:

$$I(t) = \int_M u_t\Gamma\big(-(n-1)u^{-1/n} + T\big)\,d\mu.$$

The calculus is similar to the one of the previous sections, except that we must consider the terms in which T appears. So if $\xi = -(n-1)u^{-1/n} + T$, and $S = \xi - T$ we get that

$$-I'(t) = 2\frac{n-1}{n}\int_M u_t^{1-1/n}\Gamma_2(\xi_t)\,d\mu + 2\frac{n+1}{n}\int u_t \operatorname{Hess} T(\nabla\xi,\nabla\xi)\,d\mu + R,$$

where

$$R = -\frac{2}{n}\int_M u_t\left\{\operatorname{Hess}\xi(3\nabla\xi - 2\nabla T,\nabla\xi) + \frac{n}{n-1}u^{1/n}\Gamma(S,\xi)^2\right\}d\mu.$$

Now we must use the general curvature–dimension inequality of [7] to give a minoration of the term with Γ_2. This leads to

$$-I'(t) \geqslant 2\int u \operatorname{Hess} T(\nabla\xi,\nabla\xi) + 2\rho\int u^{1-1/n}\Gamma(\xi).$$

The reader sees that if we have conditions on T such as

$$\operatorname{Hess} T \geqslant \left(-\frac{\rho}{n-1}T + \alpha\right)g, \quad \alpha\in\mathbb{R},$$

then if $T = -(n-1)v^{-1/n}$,

$$\begin{aligned}
-I'(t) &\geqslant \int u\Gamma\left(\xi, 2\alpha\xi - \frac{\rho}{n-1}\xi^2\right)d\mu = \int \partial_t u\left(-2\alpha\xi + \frac{\rho}{n-1}\xi^2\right)d\mu \\
&= \frac{d}{dt}\int_M\left(\int_{v(x)}^{u_t(x)}\left[-2\alpha\big(-(n-1)s^{-1/n} + T(x)\big)\right.\right. \\
&\quad \left.\left. + \frac{\rho}{n-1}\big(-(n-1)s^{-1/n} + T(x)\big)^2\right]ds\right)d\mu(x).
\end{aligned}$$

Now we integrate the last inequality between 0 and ∞, assuming that v has the same integral as f (and hence $\lim_\infty u(t;\cdot) = v(\cdot)$). This leads to the theorem below.

Theorem 3. *Let $n > 2$ and M be an n-dimensional Riemannian manifold, with a curvature bounded below by a positive constant ρ, a metric g, and equipped with a function $T > 0$ satisfying*:

$$\operatorname{Hess} T \geqslant \left(-\frac{\rho}{n-1}T + \alpha\right)g, \quad \alpha\in\mathbb{R}.$$

Put $T = (n-1)v^{-1/n}$. Then for any smooth positive function f such that $\int f = \int v$, we have

$$E(f) \leqslant I(f), \quad \textit{where}$$
$$I(f) = \int_M f\Gamma\big(-(n-1)f^{-1/n} + T\big)\,d\mu,$$

$$E(f)=\int_M\left(\int_{f(x)}^{v(x)}\left[-2\alpha\big(-(n-1)s^{-1/n}+T(x)\big)+\frac{\rho}{n-1}\big(-(n-1)s^{-1/n}+T(x)\big)^2\right]ds\right)d\mu(x).$$

On the sphere, the reader can verify that this gives the optimal Sobolev inequality with T being any first eigenvalue of the Laplacian (modulo a constant). Indeed, in this case, the terms in which T appears in I and E cancel each other.

10. Conclusion

This paper, together with [7,8], concludes the study of Sobolev inequalities in manifolds M with Ricci curvatures bounded from below and eventually equipped with special functions as in [7,8], with the help of the porous media equation:

$$\frac{\partial u}{\partial t}=\Delta u^\alpha+\operatorname{div}(u\nabla T),$$

where $1-1/\dim(M)\leqslant\alpha\leqslant 1$ and T is a special function. Now since Sobolev inequalities can be derived on $\mathbb{R}^n$ through mass-transportation techniques (see [11]), we can wonder if this is the case on manifolds such as the sphere or the hyperbolic space, or more general Riemannian structures.

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Values of the Pukánszky invariant in McDuff factors

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Abstract

In 1960 Pukánszky introduced an invariant associating to every masa in a separable II_1 factor a non-empty subset of $\mathbb{N} \cup \{\infty\}$. This invariant examines the multiplicity structure of the von Neumann algebra generated by the left-right action of the masa. In this paper it is shown that any non-empty subset of $\mathbb{N} \cup \{\infty\}$ arises as the Pukánszky invariant of some masa in a separable McDuff II_1 factor containing a masa with Pukánszky invariant $\{1\}$. In particular the hyperfinite II_1 factor and all separable McDuff II_1 factors with a Cartan masa satisfy this hypothesis. In a general separable McDuff II_1 factor we show that every subset of $\mathbb{N} \cup \{\infty\}$ containing ∞ is obtained as a Pukánszky invariant of some masa.

Keywords: Pukánszky invariant; Masa; McDuff factor; II_1 factor

1. Introduction

In [12] Pukánszky introduced an invariant for a maximal abelian self-adjoint subalgebra (masa) inside a separable II_1 factor, which he used to exhibit a countable infinite family of singular masas in the hyperfinite II_1 factor no pair of which are conjugate by an automorphism. The invariant associates a non-empty subset of $\mathbb{N} \cup \{\infty\}$ to each masa A in a separable II_1 factor N as follows. Let $\mathcal{A}$ be the abelian von Neumann subalgebra of $\mathbb{B}(L^2(N))$ generated by A and JAJ, where J denotes the canonical involution operator on $L^2(N)$. The orthogonal projection e_A from $L^2(N)$ onto $L^2(A)$ lies in $\mathcal{A}$ and the algebra $\mathcal{A}'(1 - e_A)$ is type I so decomposes as

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	e	$e^{\perp}$
e	1	2
$e^{\perp}$	2	1

Fig. 1. Symbolic description of the multiplicity structure of $\mathcal{A}_1$.

a direct sum of type I_n-algebras. The Pukánszky invariant of A is the set of those $n \in \mathbb{N} \cup \{\infty\}$ appearing in this decomposition and is denoted $\mathrm{Puk}(A)$. See also [13, Section 2].

There has been recent interest in the range of values of the Pukánszky invariant in various II_1 factors. Nesheyev and Størmer used ergodic constructions to show that any set containing 1 arises as a Pukánszky invariant of a masa in the hyperfinite II_1 factor [7, Corollary 3.3]. Sinclair and Smith produced further subsets using group theoretic properties in [13] and with Dykema in [4], which also examines free group factors. In the other direction Dykema has shown that $\sup \mathrm{Puk}(A) = \infty$, whenever A is a masa in a free group factor [3].

In this paper we show that every non-empty subset of $\mathbb{N} \cup \{\infty\}$ arises as the Pukánszky invariant of a masa in the hyperfinite II_1 factor by means of an approximation argument. More generally we obtain the same result in any separable McDuff II_1 factor containing a simple masa, that is one with Pukánszky invariant $\{1\}$ (Corollary 6.2). These factors are the first for which the range of the Pukánszky invariant has been fully determined. Without assuming the presence of a simple masa we are able to show that every separable McDuff II_1 factor contains a masa with Pukánszky invariant $\{\infty\}$ and hence we obtain every subset of $\mathbb{N} \cup \{\infty\}$ containing ∞ as a Pukánszky invariant of some masa in these factors (Theorem 6.7). In particular, there are uncountably many singular masas in any separable McDuff factor, no pair of which is conjugate by an automorphism of the factor.

Section 4 contains a construction for producing masas in McDuff II_1 factors. Given a McDuff II_1 factor N_0 we shall repeatedly tensor on copies of the hyperfinite II_1 factor—this gives us a chain $(N_s)_{s=0}^{\infty}$ of II_1 factors whose direct limit N is isomorphic to N_0. We shall produce a masa A in N by giving an approximating sequence of masas A_s in each N_s such that $A_s \subset A_{s+1}$ and defining $A = (\bigcup_{s=0}^{\infty} A_s)''$. This idea has its origin in [16] working in the hyperfinite II_1 factor arising as the infinite tensor produce of finite matrix algebras, although using finite matrix algebras can only yield masas with Pukánszky invariant $\{1\}$, [17, Theorem 4.1].

In the remainder of the introduction we outline the construction of a masa with Pukánszky invariant $\{2, 3\}$. Initially we shall produce a masa A_1 in N_1 such that the multiplicity structure of $\mathcal{A}_1$ (the algebra generated by the left-right action of A_1 on $L^2(N_1)$) is represented by Fig. 1. By this we mean that e is a projection of trace $1/2$ in A and that $\mathcal{A}_1' eJeJ$ and $\mathcal{A}_1' e^{\perp} Je^{\perp} J$ are both type I_1, while $\mathcal{A}_1' eJe^{\perp} J$ and $\mathcal{A}_1' e^{\perp} JeJ$ are type I_2.

At the second stage we subdivide e and $e^{\perp}$ to obtain four projections in A_2 and arrange for the multiplicity structure of $\mathcal{A}_2$ to be represented by the left diagram in Fig. 2. We then cut each of these projections in half again and ensure that the multiplicity structure of $\mathcal{A}_3$ is represented by the second diagram in Fig. 2, where 1's appear down the diagonal. It is important to do this in such a way that a limiting argument can be used to obtain the multiplicity structure of $\mathcal{A} = (A \cup JAJ)''$. If this is done successfully, then the multiplicity structure of $\mathcal{A}$ will be represented by Fig. 3, where the diagonal line has multiplicity 1. If we further ensure that the

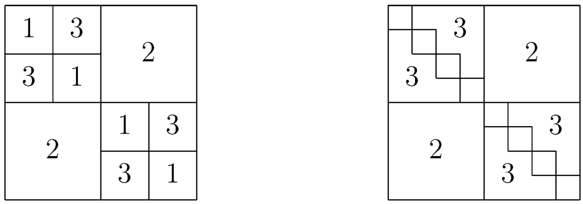

Fig. 2. The multiplicity structures of $\mathcal{A}_2$ and $\mathcal{A}_3$.

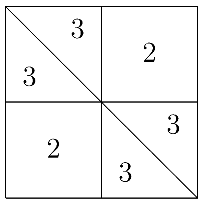

Fig. 3. The multiplicity structure of $\mathcal{A}$.

1	3	1	1
3	1	1	1
1	1	1	3
1	1	3	1

Fig. 4. Mixed Pukánszky invariant structure of the masas D_1, D_2, D_3, D_4.

projections used to cut down the masas A_r in this construction generate A, then the diagonal line in Fig. 3 corresponds to the projection e_A with range $L^2(A)$ and this is the projection explicitly removed in the definition of $\mathrm{Puk}(A)$. The resulting masa A will then have Pukánszky invariant $\{2,3\}$ as required.

To get from Fig. 1 to the left diagram in Fig. 2 in a compatible way, we 'tensor on' the diagram in Fig. 4. This is done by producing masas D_1, D_2, D_3, D_4 in the hyperfinite II_1 factor R such that $(D_i \cup JD_jJ)'$ is type I_1 unless i, j is the unordered pair $\{1,2\}$ or $\{3,4\}$. In these cases $(D_i \cup JD_jJ)'$ is type I_3. Given projections e_1, e_2, e_3, e_4 in A_1 with $e = e_1 + e_2$ and $e^\perp = e_3 + e_4$ and $\mathrm{tr}(e_i) = 1/4$ for each i we shall define A_2 in $N_2 = N_1 \mathbin{\overline{\otimes}} R$ by

$$A_2 = \bigoplus_{i=1}^{4} A_1 e_i \mathbin{\overline{\otimes}} D_i.$$

In this way $\mathcal{A}_2$ has the required multiplicity structure.

In Sections 2 and 3 we develop the concept of mixed Pukánszky invariants of pairs of masas to handle the families (D_i), which we will repeatedly adjoin. The main result is Theorem 3.5, which ensures that the family D_1, D_2, D_3, D_4 above, and other families in this style can indeed be found. In Section 4 we give the details of the inductive construction and in Section 5 we compute

the Pukánszky invariant of the resulting masa. We end in Section 6 by collecting together the main results.

2. Mixed Pukánszky invariants

In this paper all II_1 factors will be separable. In this way we only need one infinite cardinal denoted ∞. We shall write $\mathbb{N}_\infty$ for the set $\mathbb{N} \cup \{\infty\}$ henceforth.

Definition 2.1. Given a type I von Neumann algebra M we shall write $\mathrm{Type}(M)$ for the set of those $m \in \mathbb{N}_\infty$ such that M has a non-zero component of type I_m.

Given a II_1 factor N, write tr for the unique faithful trace on N with $\mathrm{tr}(1) = 1$. For $x \in N$, let $\|x\|_2 = \mathrm{tr}(x^*x)^{1/2}$, a pre-Hilbert space norm on N. The completion of N in this norm is denoted $L^2(N)$. Define a conjugate linear isometry J from $L^2(N)$ into itself by extending $x \mapsto x^*$ by continuity from N.

Definition 2.2. Given two masas A and B in a II_1 factor N define the *mixed Pukánszky invariant* of A and B to be the set $\mathrm{Type}((A \cup JBJ)')$, where the commutant is taken in $\mathbb{B}(L^2(N))$. We denote this set $\mathrm{Puk}(A, B)$ or $\mathrm{Puk}_N(A, B)$ when it is necessary. Note that $\mathrm{Puk}(A, A) = \mathrm{Puk}(A) \cup \{1\}$ for any masa A, the extra 1 arising as the Jones projection e_A is not removed in the definition of $\mathrm{Puk}(A, A)$.

It is immediate that $\mathrm{Puk}(A, B)$ is a conjugacy invariant of a pair of masas (A, B) in a II_1 factor, i.e. that if θ is an automorphism of N we have $\mathrm{Puk}(A, B) = \mathrm{Puk}(\theta(A), \theta(B))$. If we only apply θ to one masa in the pair then we may get different mixed invariants. For an inner automorphism this is not the case.

Proposition 2.3. *Let A and B be masas in a II_1 factor N. For any unitaries $u, v \in N$ we have*

$$\mathrm{Puk}(uAu^*, vBv^*) = \mathrm{Puk}(A, B).$$

Proof. Consider the automorphism $\Theta = \mathrm{Ad}(uJvJ)$ of $B(L^2(N))$, which has $\Theta(A) = uAu^*$ and $\Theta(JBJ) = JvBv^*J$. Therefore $(A \cup JBJ)'$ and $(uAu^* \cup J(vBv^*)J)'$ are isomorphic, so have the same type decomposition. □

The Pukánszky invariant is well behaved with respect to tensor products [13, Lemma 2.1]. So too is the mixed Pukánszky invariant. Given $E, F \subset \mathbb{N}_\infty$ write $E \cdot F = \{mn \mid m \in E, n \in F\}$, where by convention $n\infty = \infty n = \infty$ for any $n \in \mathbb{N}_\infty$.

Lemma 2.4. *Let $(N_i)_{i \in I}$ be a countable family of finite factors. Suppose that we have masas A_i and B_i in N_i for each $i \in I$. Let N be the finite factor obtained as the von Neumann tensor product of the N_i with respect to the product trace and let A and B be the tensor products of the A_i and B_i, respectively. Then A and B are masas in N. When I is finite,*

$$\mathrm{Puk}_N(A, B) = \prod_{i \in I} \mathrm{Puk}_{N_i}(A_i, B_i).$$

If I is infinite, and each $\mathrm{Puk}_{N_i}(A_i, B_i) = \{n_i\}$ *for some* $n_i \in \mathbb{N}_\infty$*, then* $\mathrm{Puk}_N(A, B) = \{n\}$*, where* $n = \prod_I n_i$*, when all but finitely many* $n_i = 1$*, and* $n = \infty$*, otherwise.*

Proof. That A and B are masas follows from Tomita's commutation theorem, see [6, Theorem 11.2.16]. Suppose first that I is finite. For each $i \in I$, let $(p_{i,n})_{n\in\mathbb{N}_\infty}$ be the decomposition of the identity projection into projections in $(A_i \cup JB_iJ)'' \subset B(L^2(N_i))$ such that $(A_i \cup JB_iJ)'p_{i,n}$ is type I_n for each $n \in \mathbb{N}_\infty$ (some of these projections may be zero). Then given any family $(n_i)_i$ in $\mathbb{N}_\infty$, $p = \bigotimes_{i\in I} p_{i,n_i}$ is a central projection in $(A \cup JBJ)'$ and $(A \cup JBJ)'p$ is type I_m where $m = \prod_{i\in I} n_i$. All these projections are mutually orthogonal with sum 1. Therefore $\mathrm{Puk}_N(A, B)$ consists of those m such that $p \neq 0$ and this occurs if and only if all the corresponding p_{i,n_i} appearing in the tensor product are non-zero. These are precisely the m in $\prod_{i\in I} \mathrm{Puk}_{N_i}(A_i, B_i)$.

Suppose I is infinite and each $\mathrm{Puk}_{N_i}(A_i, B_i) = \{n_i\}$, for some $n_i \in \mathbb{N}_\infty$. Let $\mathcal{A}_i = (A_i \cup JB_iJ)'' \subset \mathbb{B}(L^2(N_i))$ and $\mathcal{A}_i'$ the commutant of $\mathcal{A}_i$ in $\mathbb{B}(L^2(N_i))$. Let $\mathcal{A} = (A \cup JBJ)''$ in $\mathbb{B}(L^2(N))$ and $\mathcal{A}'$ the commutant of $\mathcal{A}$ in this algebra. The Tomita commutation theorem gives

$$\mathcal{A}' = \overline{\bigotimes} \mathcal{A}_i' \subseteq \overline{\bigotimes} \mathbb{B}\big(L^2(N_i)\big) \cong \mathbb{B}\big(L^2(N)\big).$$

Since each $\mathcal{A}_i' \cong \mathcal{A}_i \mathbin{\overline{\otimes}} \mathbb{M}_{n_i}$, where $\mathbb{M}_{n_i}$ is the $n_i \times n_i$ matrices (or $\mathbb{B}(H)$ for some separable infinite-dimensional Hilbert space when $n_i = \infty$). Thus

$$\mathcal{A}' \cong \left(\overline{\bigotimes} A_i\right) \mathbin{\overline{\otimes}} \left(\overline{\bigotimes} \mathbb{M}_{n_i}\right) \cong A \mathbin{\overline{\otimes}} \mathbb{M}_n,$$

so $\mathcal{A}'$ is homogeneous of type I_n. $\square$

Given two masas A and B in a II_1 factor N we can form the algebra $M_2(N)$ of 2×2 matrices over N. We can construct a masa in $M_2(N)$

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \,\middle|\, a \in A, b \in B \right\},$$

which we denote $A \oplus B$—the direct sum of A and B. In [13] it is noted that if B is a unitary conjugate of A, then the Pukánszky invariant of $A \oplus B$ can be determined from that of A (and hence B). Indeed we have

$$\mathrm{Puk}(A \oplus uAu^*) = \mathrm{Puk}(A) \cup \{1\},$$

whenever u is a unitary in N. The initial motivation for the introduction of the mixed Pukánszky invariant was to aid in the study of the Pukánszky invariant of these direct sums since

$$\mathrm{Puk}(A \oplus B) = \mathrm{Puk}(A) \cup \mathrm{Puk}(B) \cup \mathrm{Puk}(A, B),$$

whenever A and B are masas in a II_1 factor N. As we shall subsequently see, the Pukánszky invariant behaves badly with respect to the direct sum construction. In the next section we shall give Cartan masas A and B in the hyperfinite II_1 factor such that $\mathrm{Puk}(A \oplus B) = \{1, n\}$ for any $n \in \mathbb{N}_\infty$, and given non-empty sets $E, F, G \subset \mathbb{N}_\infty$ we shall construct, in Theorem 6.4, masas A and B in the hyperfinite II_1 factor such that $\mathrm{Puk}(A) = E$, $\mathrm{Puk}(B) = F$ and $\mathrm{Puk}(A, B) = G$.

Hence it is not possible to make a more general statement about the Pukánszky invariant of a direct sum than

$$\mathrm{Puk}(A \oplus B) \supset \mathrm{Puk}(A) \cup \mathrm{Puk}(B).$$

3. Mixed invariants of Cartan masas in R

In this section we shall construct large families of Cartan masas in the hyperfinite II_1 factor, each masa will have Pukánszky invariant $\{1\}$ by virtue of being Cartan [11, Section 3]. Our objective will be to control the mixed Pukánszky invariant of any two elements from the family. We start by constructing a family of three Cartan masas in the hyperfinite II_1 factor and then use Lemma 2.4 to produce the desired result.

Lemma 3.1. *For each $n \in \mathbb{N}_\infty$ there exists Cartan masas A, B, C in the hyperfinite* II_1 *factor such that* $\mathrm{Puk}(A, B) = \{n\}$ *while* $\mathrm{Puk}(A, C) = \mathrm{Puk}(B, C) = \{1\}$.

We shall first establish Lemma 3.1 when n is finite. The lemma is immediate for $n = 1$, take $A = B = C$ to be any Cartan masa in the hyperfinite II_1 factor. Let $n \geqslant 2$ be a fixed integer until further notice. Since any two Cartan masas in the hyperfinite II_1 factor are conjugate by an automorphism [2], we shall fix a Cartan masa A arising as the diagonals in an infinite tensor product and then construct $B = \theta(A)$ and $C = \phi(A)$ by exhibiting appropriate automorphisms θ and ϕ of R. Let M denote the $n \times n$ matrices and D_0 denote the diagonal $n \times n$ matrices, a masa in M. Write $(e_i)_{i=0}^{n-1}$ for the minimal projections of D_0 so e_i has 1 in the (i,i)th entry and 0, elsewhere. Let

$$w = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 1 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}$$

a unitary in M, which, in its action by conjugation, cyclically permutes the minimal projections of D_0. That is $we_iw^* = e_{i-1}$ with the subtraction taken mod n. The abelian algebra generated by w is a masa D_1 in M, which is orthogonal to D_0 [10, Section 3]. Write $(f_i)_{i=0}^{n-1}$ for the minimal projections of D_1. Define

$$v = \sum_{i=0}^{n-1} w^i \otimes f_i \tag{1}$$

a unitary in $D_1 \otimes D_1 \subset M \otimes M$.

We shall produce A, B and C in the hyperfinite II_1 factor R realised as $(\bigotimes_{r=1}^{\infty} M)''$. Let $A = (\bigotimes_{r=1}^{\infty} D_0)''$. For each r consider the unitary $u_r = 1^{\otimes(r-1)} \otimes v$, which lies in $M^{\otimes(r+1)} \subset R$. All of these unitaries commute (as they lie in the masa $(\bigotimes_{r=1}^{\infty} D_1)''$ in R) and satisfy $u_r^n = 1$. We are able to define automorphisms

$$\theta = \lim_{r\to\infty} \mathrm{Ad}(u_1 u_2 \dots u_r), \qquad \phi = \lim_{r\to\infty} \mathrm{Ad}(u_1 u_3 u_5 \dots u_{2r+1})$$

of R with the limit taken pointwise in $\|.\|_2$. Convergence follows, since for $x \in M^{\otimes r}$ we have $u_s x u_s^* = x$ whenever $s > r$ and such x are $\|.\|_2$-dense in R. In this way θ and ϕ define *-isomorphisms of R into R. As $\theta^n = I$ and $\phi^n = I$ (since the u_rs commute and each $u_r^n = 1$), we see that θ and ϕ are onto and so automorphisms of R. Define Cartan masas $B = \theta(A)$ and $C = \phi(A)$ in R. The calculations of $\mathrm{Puk}(A, C)$ and $\mathrm{Puk}(B, C)$ are straightforward.

Lemma 3.2. *With the notation above, we have* $\mathrm{Puk}(A, C) = \mathrm{Puk}(B, C) = \{1\}$.

Proof. We re-bracket the infinite tensor product defining R as

$$R = (M \otimes M)\,\overline{\otimes}\,(M \otimes M)\,\overline{\otimes}\cdots$$

so that R is the infinite tensor product of copies of $M \otimes M$. Since u_{2r+1} lies in $1^{\otimes 2r} \otimes (M \otimes M)$ we see that ϕ factorises as $\prod_{s=1}^{\infty} \mathrm{Ad}(v)$ with respect to this decomposition. Lemma 2.4 then tells us that $\mathrm{Puk}(A, C)$ is the set product of infinitely many copies of $\mathrm{Puk}_{M\otimes M}(D_0 \otimes D_0, v(D_0 \otimes D_0)v^*)$. Since $D_0 \otimes D_0$ and $v(D_0 \otimes D_0)v^*$ are masas in $M_0 \otimes M_0$ a simple dimension check ensures that $\mathrm{Puk}_{M\otimes M}(D_0 \otimes D_0, v(D_0 \otimes D_0)v^*) = \{1\}$ and hence $\mathrm{Puk}(A, C) = \{1\}$.

Observe that $\mathrm{Puk}(B, C) = \mathrm{Puk}(\theta(A), \phi(A)) = \mathrm{Puk}(\phi^{-1}\theta(A), A)$. As all the u_r commute, we have

$$\phi^{-1} \circ \theta = \lim_{r\to\infty} \mathrm{Ad}(u_2 u_4 \ldots u_{2r})$$

with pointwise $\|.\|_2$ convergence. This time we re-bracket the tensor product defining R as

$$R = M\,\overline{\otimes}\,(M \otimes M)\,\overline{\otimes}\,(M \otimes M)\,\overline{\otimes}\cdots,$$

and since $u_{2r} = 1^{\otimes 2r-1} \otimes v \in 1 \otimes 1^{\otimes 2(r-1)} \otimes (M \otimes M)$, we obtain $\mathrm{Puk}(B, C) = \{1\}$ in the same way. □

The key tool in establishing that $\mathrm{Puk}(A, B) = \{n\}$ is the following calculation, which we shall use to produce n equivalent abelian projections for the commutant of the left-right action.

Lemma 3.3. *Use the notation preceding Lemma* 3.2. *For $r = 0, 1, \ldots, n-1$ let ξ_r denote f_r taken in the first copy of M in the tensor product making up R, thought of as a vector in $L^2(R)$. For any $m \geqslant 0$, $i_1, i_2, \ldots, j_m, j_1, j_2, \ldots, j_m = 0, 1, \ldots, n-1$ and $r, s = 0, 1, \ldots, n-1$ we have*

$$\big\langle (e_{i_1} \otimes \cdots \otimes e_{i_m})\xi_r \theta(e_{j_1} \otimes \cdots \otimes e_{j_m}), \xi_s \big\rangle_{L^2(R)} = \delta_{r,s} n^{-(2m+1)}. \tag{2}$$

Proof. We proceed by induction. When $m = 0$, (2) reduces to $\langle \xi_r, \xi_s\rangle = \delta_{r,s} n^{-1}$, which follows as $\langle \xi_r, \xi_s\rangle = \mathrm{tr}(f_r f_s^*)$ and $(f_r)_{r=0}^{n-1}$ are the minimal projections of a masa in the $n \times n$ matrices.

For $m > 0$ observe that $\theta(e_{j_1} \otimes \cdots \otimes e_{j_m}) = u_1 \ldots u_m (e_{j_1} \otimes \cdots \otimes e_{j_m}) u_m^* \ldots u_1^*$. With the subtraction in the subscript taken mod n, we have

$$u_m (e_{j_1} \otimes \cdots \otimes e_{j_m}) u_m^* = e_{j_1} \otimes \cdots \otimes e_{j_{m-1}} \otimes \left(\sum_{k=0}^{n-1} e_{j_m - k} \otimes f_k \right)$$

from (1) and $w e_{j_m} w^* = e_{j_m - 1}$. Therefore

$$\begin{aligned}
&\big\langle (e_{i_1}\otimes\cdots\otimes e_{i_m})\xi_r\theta(e_{j_1}\otimes\cdots\otimes e_{j_m}),\xi_s\big\rangle\\
&\quad=\left\langle (e_{i_1}\otimes\cdots\otimes e_{i_m})\xi_r u_1\dots u_{m-1}\left(e_{j_1}\otimes\cdots\otimes e_{j_m-1}\otimes\sum_{k=0}^{n-1}e_{j_m-k}\otimes f_k\right)u_{m-1}^*\dots u_1^*,\xi_s\right\rangle\\
&\quad=\mathrm{tr}\left(\sum_{k=0}^{n-1}\big((e_{i_1}\otimes\cdots\otimes e_{i_m})f_r u_1\dots u_{m-1}(e_{j_1}\otimes\cdots\otimes e_{j_{m-1}}\otimes e_{j_m-k})u_{m-1}^*\dots u_1^* f_s^*\big)\otimes f_k\right)\\
&\quad=n^{-1}\mathrm{tr}\left((e_{i_1}\otimes\cdots\otimes e_{i_m})f_r u_1\dots u_{m-1}\left(e_{j_1}\otimes\cdots\otimes e_{j_{m-1}}\otimes\sum_{k=0}^{n-1}e_{j_m-k}\right)u_{m-1}^*\dots u_1^* f_s^*\right)\\
&\quad=n^{-1}\mathrm{tr}\big((e_{i_1}\otimes\cdots\otimes e_{i_m})f_r\theta(e_{j_1}\otimes\cdots\otimes e_{j_{m-1}}\otimes 1)f_s^*\big)
\end{aligned}\tag{3}$$

as the f_k in the third line is the only object appearing in the $(m+1)$-tensor position and tr is a product trace. This produces the factor $n^{-1}=\mathrm{tr}(f_k)$. We obtain (3) as $e_{j_1}\otimes\cdots\otimes e_{j_{m-1}}\otimes 1$ lies in $M^{\otimes(m-1)}$ so $\theta(e_{j_1}\otimes\cdots\otimes e_{j_{m-1}}\otimes 1)=u_1\dots u_{m-1}(e_{j_1}\otimes\cdots\otimes e_{j_{m-1}}\otimes 1)u_{m-1}^*\dots u_1^*$.

Now $\theta(f_r)=f_r$ for all r (since each u_m commutes with f_r) and θ is trace preserving. In this way we obtain

$$\begin{aligned}
&\big\langle (e_{i_1}\otimes\cdots\otimes e_{i_m})\xi_r\theta(e_{j_1}\otimes\cdots\otimes e_{j_m}),\xi_s\big\rangle\\
&\quad=n^{-1}\mathrm{tr}\big(\theta^{-1}(e_{i_1}\otimes\cdots\otimes e_{i_m})f_r(e_{j_1}\otimes\cdots\otimes e_{j_{m-1}}\otimes 1)f_s^*\big).
\end{aligned}$$

We now apply the same argument again giving us

$$\begin{aligned}
&\big\langle (e_{i_1}\otimes\cdots\otimes e_{i_m})\xi_r\theta(e_{j_1}\otimes\cdots\otimes e_{j_m}),\xi_s\big\rangle\\
&\quad=n^{-2}\mathrm{tr}\big(\theta^{-1}(e_{i_1}\otimes\cdots\otimes e_{i_{m-1}}\otimes 1)f_r(e_{j_1}\otimes\cdots\otimes e_{j_{m-1}}\otimes 1)f_s^*\big)\\
&\quad=n^{-2}\mathrm{tr}\big((e_{i_1}\otimes\cdots\otimes e_{i_{m-1}})f_r\theta(e_{j_1}\otimes\cdots\otimes e_{j_{m-1}})f_s^*\big)\\
&\quad=n^{-2}\big\langle (e_{i_1}\otimes\cdots\otimes e_{i_{m-1}})\xi_r\theta(e_{j_1}\otimes\cdots\otimes e_{j_{m-1}}),\xi_s\big\rangle.
\end{aligned}$$

The lemma now follows by induction. □

We can now complete the proof of Lemma 3.1.

Proof of Lemma 3.1. We continue to let $n\geqslant 2$ be a fixed integer and let A and B be the masas introduced before Lemma 3.2. Let $\mathcal{C}$ be the abelian algebra $(A\cup JBJ)''$ in $\mathbb{B}(L^2(R))$. We continue to write ξ_r for f_r (in the first tensor position) thought of as a vector in $L^2(R)$. For each r, let P_r be the orthogonal projection in $\mathbb{B}(L^2(R))$ onto $\overline{\mathcal{C}\xi_r}$, an abelian projection in $\mathcal{C}'$.

Since elements $(e_{i_1}\otimes\cdots\otimes e_{i_m})f_r\theta(e_{j_1}\otimes\cdots\otimes e_{j_m})$, where $m\geqslant 0$ and $i_1,\dots,i_m,j_1,\dots,j_m=0,1,\dots,n-1$, have dense linear span in $\overline{\mathcal{C}\xi_r}$, Lemma 3.3 implies that P_r is orthogonal to P_s when $r\neq s$. Furthermore, for each m, the elements

$$(e_{i_1}\otimes\cdots\otimes e_{i_m})f_r\theta(e_{j_1}\otimes\cdots\otimes e_{j_{m-1}}\otimes 1)$$

indexed by $i_1,\dots,i_m, j_1,\dots,j_{m-1}, r=0,1,\dots,n-1$ are n^{2m} pairwise orthogonal non-zero elements of $M^{\otimes m}$, the $n^m\times n^m$ matrices. Therefore, $M^{\otimes m}$ is contained in the range of $P_0+P_1+\dots+P_{n-1}$ for each m so that $\sum_{r=0}^{n-1}P_r=1$.

It remains to show that all the P_r are equivalent in $\mathcal{C}'$, from which it follows that $\mathcal{C}'$ is homogeneous of type I_n. Given $r\neq s$ we must define a partial isometry $v_{r,s}\in\mathcal{C}'$ with $v_{r,s}v_{r,s}^*=P_s$ and $v_{r,s}^*v_{r,s}=P_r$. Lemma 3.3 allows us to define $v_{r,s}$ by extending the map $\xi_r\mapsto\xi_s$ by (A,B)-modularity. More precisely define linear maps

$$v_{r,s}^{(m)}:\mathrm{Span}\big(D_0^{\otimes m}f_r\theta\big(D_0^{\otimes m}\big)\big)\to\mathrm{Span}\big(D_0^{\otimes m}f_s\theta\big(D_0^{\otimes m}\big)\big)$$

by extending

$$v_{r,s}^{(m)}\big((e_{i_1}\otimes\dots\otimes e_{i_m})f_r\theta(e_{j_1}\otimes\dots\otimes e_{j_m})\big)=(e_{i_1}\otimes\dots\otimes e_{i_m})f_s\theta(e_{j_1}\otimes\dots\otimes e_{j_m})$$

by linearity. Lemma 3.3 shows that these maps preserve $\|.\|_2$ and that $v_{r,s}^{(m+1)}$ extends $v_{r,s}^{(m)}$. Let $v_{r,s}$ be the closure of the union of the $v_{r,s}^{(m)}$. This is patently a partial isometry in $\mathcal{C}'$ with domain projection P_r and range projection P_s. Hence $\mathrm{Puk}(A,B)=\{n\}$ and combining this with Lemma 3.2 establishes Lemma 3.1 when n is finite.

When the n of Lemma 3.1 is ∞ we take a tensor product. More precisely find Cartan masas A_0,B_0,C_0 in the hyperfinite II_1 factor R_0 such that $\mathrm{Puk}(A_0,B_0)=\{2\}$ and $\mathrm{Puk}(A_0,C_0)=\mathrm{Puk}(B_0,C_0)=\{1\}$. Now form the hyperfinite II_1 factor R by taking the infinite tensor product of copies of R_0. The Cartan masas A, B and C in R obtained from the infinite tensor product of copies of A_0, B_0 and C_0 have $\mathrm{Puk}(A,B)=\{\infty\}$, and $\mathrm{Puk}(A,C)=\mathrm{Puk}(B,C)=\{1\}$ by Lemma 2.4. □

Remark 3.4. By fixing a Cartan masa D in a II_1 factor N we could consider the map $\theta\mapsto\mathrm{Puk}(D,\theta(D))$, which (by Proposition 2.3) induces a map on $\mathrm{Out}\,N$. This map is not necessarily constant on outer conjugacy classes, as the automorphisms θ and ϕ of the hyperfinite II_1 factor above have outer order n and obstruction to lifting 1 so are outer conjugate by [1].

Let us now give the main result of this section.

Theorem 3.5. *Let I be a countable set and let Λ be a symmetric matrix over $\mathbb{N}_\infty$ indexed by I, with $\Lambda_{i,i}=1$ for all $i\in I$. There exist Cartan masas $(D_i)_{i\in I}$ in the hyperfinite II_1 factor such that $\mathrm{Puk}(D_i,D_j)=\{\Lambda_{i,j}\}$ for all $i,j\in I$.*

Proof. Let I and Λ be as in the statement of Theorem 3.5. For each unordered pair $\{i,j\}$ of distinct elements of I, use Lemma 3.1 to find Cartan masas $(D_r^{\{i,j\}})_{r\in I}$ in the copy of the hyperfinite II_1 factor denoted $R^{\{i,j\}}$ such that

$$\mathrm{Puk}\big(D_r^{\{i,j\}},D_s^{\{i,j\}}\big)=\begin{cases}\{\Lambda_{i,j}\} & \{r,s\}=\{i,j\},\\ \{1\} & \text{otherwise.}\end{cases}$$

This is achieved by taking $D_i^{(i,j)}=A$, $D_j^{(i,j)}=B$ and $D_r^{(i,j)}=C$ for $r\neq i, r\neq j$ where A,B,C are the masas resulting from taking $n=\Lambda_{i,j}$ in Lemma 3.1. Now form the copy of the hyperfinite

II_1 factor $R=\overline{\bigotimes}_{\{i,j\}}R^{\{i,j\}}$ and masas $D_r=\overline{\bigotimes}_{\{i,j\}}D_r^{\{i,j\}}$ for $r\in I$. Lemma 2.4 ensures these masas have

$$\mathrm{Puk}(D_i,D_j)=\{\Lambda_{i,j}\}$$

for all $i,j\in I$. $\square$

We can immediately deduce the existence of masas with certain Pukánszky invariants. The subsets below where first found in [7] using ergodic methods.

Corollary 3.6. *Let E be a finite subset of $\mathbb{N}_\infty$ with $1\in E$. Then there exists a masa in the hyperfinite II_1 factor whose Pukánszky invariant is E.*

Proof. If we work in the $n\times n$ matrices $M_n(R)$ over the hyperfinite II_1 factor, and form the direct sum $A=D_1\oplus D_2\oplus\cdots\oplus D_n$ of n Cartan masas, then

$$\mathrm{Puk}(A)=\{1\}\cup\bigcup_{i<j}\mathrm{Puk}(D_i,D_j).$$

The corollary then follows from Theorem 3.5 by choosing a large but finite I and appropriate values of $\Lambda_{i,j}$ depending on the set E. $\square$

All the pairs of Cartan masas we have produced have had a singleton for their mixed Pukánszky invariant. What are the possible values of $\mathrm{Puk}(A,B)$ when A and B are Cartan masas in a II_1 factor?

4. The main construction

In this section we give a construction of masas in McDuff II_1 factors, which we use to establish the main results of the paper in Section 6. We need to introduce a not insubstantial amount of notation. Let N_0 be a fixed separable McDuff II_1 factor and for each $r\in\mathbb{N}$, let $R^{(r)}$ be a copy of the hyperfinite II_1 factor. Let $N_r=N_0\mathbin{\overline{\otimes}}R^{(1)}\mathbin{\overline{\otimes}}\cdots\mathbin{\overline{\otimes}}R^{(r)}$ so that with the inclusion map $x\mapsto x\otimes 1_{R^{(r+1)}}$ we can regard N_r as a von Neumann subalgebra of N_{r+1}. We let N be the direct limit of this chain, so that

$$N=\left(N_0\mathbin{\overline{\otimes}}\bigotimes_{r=1}^{\infty}R^{(r)}\right)''$$

acting on $L^2(N_0)\otimes\bigotimes_{r=1}^{\infty}L^2(R^{(r)})$. The II_1 factor N is isomorphic to N_0 and we shall regard all the N_r as subalgebras of N.

Whenever we have a masa D inside a II_1 factor, we are able to use the isomorphism between D and $L^\infty[0,1]$ to choose families of projections $e_i^{(m)}(D)$ in D for $m\in\mathbb{N}$ and $i=(i_1,\dots,i_m)\in\{0,1\}^m$, which satisfy:

(1) For each m the 2^m projections $e_i^{(m)}(D)$ are pairwise orthogonal and each projection has trace 2^{-m};

(2) For each m and $i=(i_1,\dots,i_m)\in\{0,1\}^m$ we have

$$e_i^{(m)}(D)=e_{i\vee 0}^{(m+1)}(D)+e_{i\vee 1}^{(m+1)}(D),$$

where $i\vee 0=(i_1,\dots,i_m,0)$ and $i\vee 1=(i_1,\dots,i_m,1)$;

(3) The projections $e_i^{(m)}(D)$ generate D.

In the procedure that follows we shall assume that masas come with these projections when needed.

For $m\in\mathbb{N}$ and $r\geqslant 0$, let $I(r,m)$ denote the set of all $i=(i^{(0)},i^{(1)},\dots,i^{(r)})$ where $i^{(r-s)}=(i_1^{(r-s)},i_2^{(r-s)},\dots,i_{m+s}^{(r-s)})\in\{0,1\}^{m+s}$ is a sequence of zeros and ones of length $m+s$. In this way the last sequence, $i^{(r)}$, has length m and each earlier sequence is one element longer than the following sequence. We have restriction maps from $I(r,m)$ to $I(r-1,m+1)$ obtained by forgetting about the last sequence $i^{(r)}$. Note that $i^{(r-1)}$ has length $m+1$ so that this restriction does lie in $I(r-1,m+1)$. We can also restrict by shortening the length of all the sequences. In full generality we have restriction maps from $I(r,m)$ into $I(s,l)$ whenever $s\leqslant r$ and $l\leqslant m+r-s$. Given $i\in I(r,m)$ and $k\in I(s,l)$ (for $s\leqslant r$ and $l\leqslant m+r-s$) write $i\geqslant k$ if the restriction of i to $I(s,l)$ is precisely k. When $i\in I(r,m)$ for some r, we write $i|_s$ for the restriction of i to $I(s,1)$ for $s\leqslant r$. We take $i|_{-1}=j|_{-1}$ as a convention for all $i,j\in I(r,m)$.

The inputs to our construction are a masa A_0 in N_0 and values $\Lambda_{i,j}^{(r)}=\Lambda_{j,i}^{(r)}\in\mathbb{N}_\infty$ for all $r=0,1,2,\dots$ and $i,j\in I(r,1)$ with $i\neq j$ and $i|_{r-1}=j|_{r-1}$. We regard these as fixed henceforth. For $i\in I(0,m)$, define $f_i^{(0,m)}=e_{i^{(0)}}^{(m)}(A_0)$. Suppose inductively that we have produced masas $A_s\subset N_s$ for each $s\leqslant r$ and that, for each $m\in\mathbb{N}$, projections $(f_i^{(s,m)})_{i\in I(s,m)}$ in A_s have been specified such that:

(i) For each $m\in\mathbb{N}$ and $s\leqslant r$, the $|I(s,m)|$ projections $(f_i^{(s,m)})_{i\in I(s,m)}$ are pairwise orthogonal and each has trace $|I(m,s)|^{-1}$;

(ii) For each $m\in\mathbb{N}$, $s\leqslant r$ and $i\in I(s,m)$ we have

$$f_i^{(s,m)}=\sum_{\substack{j\in I(s,m+1)\\ j\geqslant i}} f_j^{(s,m+1)};$$

(iii) For any $s\leqslant t\leqslant r$ and $i\in I(s,m+t-s)$ we have

$$f_i^{(s,m+t-s)}=\sum_{\substack{j\in I(t,m)\\ j\geqslant i}} f_j^{(t,m)},$$

noting that in this statement we regard the $f^{(s,m+t-s)}$ as lying inside N_t;

(iv) For each $s\leqslant r$ the projections $\{f_i^{(s,m)}\mid m\in\mathbb{N},\ i\in I(s,m)\}$ generate A_s.

Note that conditions (iii) and (iv) ensure that $A_s\subset A_t$.

To define A_{r+1}, use Theorem 3.5 to produce Cartan masas $(D_i^{(r+1)})_{i\in I(r,1)}$ in $R^{(r+1)}$ such that when $i \neq j$ we have

$$\mathrm{Puk}\big(D_i^{(r+1)}, D_j^{(r+1)}\big) = \begin{cases} \{\Lambda_{i,j}^{(r)}\} & i|_{r-1} = j|_{r-1}, \\ \{1\} & \text{otherwise.} \end{cases} \tag{4}$$

Let A_{r+1} be given by

$$A_{r+1} = \bigoplus_{i\in I(r,1)} A_r f_i^{(r,1)} \otimes D_i^{(r+1)} \tag{5}$$

a masa in $N_r \mathbin{\overline{\otimes}} R^{(r+1)} = N_{r+1}$, which has $A_r \subset A_{r+1}$. To complete the inductive construction we must define $f_i^{(r+1,m)}$ for $i \in I(r+1,m)$ in a manner which satisfies conditions (i)–(iv) above. Given $m \in \mathbb{N}$ and $i \in I(r+1,m)$, let i' be the restriction of i to $I(r,m+1)$ and recall that $i|_r$ is the restriction of i to $I(r,1)$. Now define

$$f_i^{(r+1,m)} = f_{i'}^{(r,m+1)} \otimes e_{i^{(r+1)}}^{(m)}\big(D_{i|_r}^{(r+1)}\big). \tag{6}$$

Since $f_{i'}^{(r,m+1)} \leqslant f_{i|_r}^{(r,1)}$, this does define a projection in A_{r+1}. That the $f_i^{(r+1,m)}$ satisfy the required conditions is routine. We give the details as Lemma 4.1 below for completeness.

Lemma 4.1. *The projections* $(f_i^{(r+1,m)})_{i\in I(r+1,m)}$ *defined in* (6) *satisfy the conditions* (i)–(iv) *above.*

Proof. For $m \in \mathbb{N}$ fixed, the projections $(f_i^{(r+1,m)})_{i\in I(r+1,m)}$ are pairwise orthogonal and have trace $|I(r+1,m)|^{-1}$ as the projections $(f_{i'}^{(r,m+1)})_{i'\in I(r,m+1)}$ are pairwise orthogonal with trace $|I(r,m+1)|^{-1}$ and the projections $(e_j^{(m)}(D_{i|_r}^{(r+1)}))_{j\in\{0,1\}^m}$ are also pairwise orthogonal and each have trace 2^{-m}. In this way the projections for A_{r+1} satisfy condition (i).

For condition (ii), fix $i \in I(r+1,m)$ for some $m \in \mathbb{N}$ and let i' be as in the definition of $f_i^{(r+1,m)}$. Now

$$\begin{aligned} f_i^{(r+1,m)} &= f_{i'}^{(r,m+1)} \otimes e_{i^{(r+1)}}^{(m)}\big(D_{i|_r}^{(r+1)}\big) \\ &= \sum_{\substack{j'\in I(r,m+2)\\ j'\geqslant i'}} f_{j'}^{(r,m+2)} \otimes \big(e_{i^{(r+1)}\vee 0}^{(m+1)}\big(D_{i|_r}^{(r+1)}\big) + e_{i^{(r+1)}\vee 1}^{(m+1)}\big(D_{i|_r}^{(r+1)}\big)\big) \\ &= \sum_{\substack{j\in I(r+1,m+1)\\ j\geqslant i}} f_j^{(r+1,m+1)} \end{aligned}$$

from condition (ii) for the $f_{i'}^{(r,m+1)}$ and the second condition in the definition of the $e_k^{(m)}(D)$. This is precisely condition (ii).

We only need to check condition (iii) when $t = r+1$, so take $s \leqslant r$, $m \in \mathbb{N}$ and $i \in I(s, m+r+1-s)$. By the inductive version of (iii) we have

$$f_i^{(s,m+r+1-s)} = \sum_{\substack{j\in I(r,m+1)\\ j\geqslant i}} f_j^{(r,m+1)}.$$

For each $j \in I(r, m+1)$ with $j \geqslant i$ we have

$$\begin{aligned} f_j^{(r,m+1)} \otimes 1_{R^{(r+1)}} &= f_j^{(r,m+1)} \otimes \sum_{j^{(r+1)}\in\{0,1\}^m} e^{(m)}_{j^{(r+1)}}\big(D^{(r+1)}_{j|_r}\big) \\ &= \sum_{\substack{k\in I(r+1,m)\\ k\geqslant j}} f_k^{(r+1,m+1)}, \end{aligned}$$

where $j|_r$ is the restriction of j to $I(r, 1)$. Therefore,

$$f_i^{(s,m+r+1-s)} = \sum_{\substack{k\in I(r+1,m)\\ k\geqslant i}} f_k^{(r+1,m+1)},$$

which is condition (iii).

For $j \in I(r, 1)$, the projections $f_k^{(r,m)}$ indexed by $k \in I(r,m)$ with $k \geqslant j$ generate the cut-down $A_r f_j^{(r,1)}$. Hence the projections $f_i^{(r+1,m)}$, for $i \in I(r+1,m)$ with $i \geqslant j$ generate $A_r f_j^{(r,1)} \otimes D_j^{(r+1)}$. In this way we see that the projections $f_i^{(r+1,m)}$ for $i \in I(r+1,m)$ generate A_{r+1}, which is condition (iv). □

This completes the inductive stage of the construction. We have masas A_r in N_r for each r such that $A_r \otimes 1_{R^{(r+1)}} \subset A_{r+1}$. We shall regard all these masas as subalgebras of the infinite tensor product II$_1$ factor N, where they are no longer maximal abelian. Define $A = (\bigcup_{r=0}^{\infty} A_r)''$, an abelian subalgebra of R. For $r \geqslant 0$ we have

$$A_r' \cap N = A_r \mathbin{\overline{\otimes}} R^{(r+1)} \mathbin{\overline{\otimes}} R^{(r+2)} \mathbin{\overline{\otimes}} \cdots$$

so that for $x \in N_r \subset N$ we have $\mathbb{E}_{A_r'\cap N}(x) = \mathbb{E}_{A_r}(x)$, where $\mathbb{E}_M$ denotes the unique trace-preserving conditional expectation onto the von Neumann subalgebra M. As $A_r \subset A \subset A' \cap N \subset A_r' \cap N$ we obtain $\mathbb{E}_A(x) = \mathbb{E}_{A'\cap N}(x)$ for any $x \in \bigcup_{r=0}^{\infty} N_r$. These x are weakly dense in N so $A = A' \cap N$ is a masa in N, see [9, Lemma 2.1].

5. The Pukánszky invariant of A

Our objective here is to compute the Pukánszky invariant of the masas of Section 4 in terms of the masa A_0 and the specified values $\Lambda_{i,j}^{(r)}$. Following the usual convention, we shall write $\mathcal{A}$ for the algebra $(A \cup JAJ)''$, an abelian subalgebra of $\mathbb{B}(L^2(N))$.

Lemma 5.1. *Let A be a masa produced by means of the construction described in Section* 4. *Then*

$$\operatorname{Puk}(A)=\bigcup_{r=0}^{\infty}\bigcup_{\substack{i,j\in I(r,1)\\ i\neq j\\ i|_{r-1}=j|_{r-1}}}\operatorname{Type}\bigl(\mathcal{A}'f_i^{(r,1)}Jf_j^{(r,1)}J\bigr).$$

Proof. Fix $s\geqslant 0$, $m\in\mathbb{N}$ and $i\in I(s,m)$. Let $r=s+m-1$, so that condition (iii) gives

$$f_i^{(s,m)}=\sum_{\substack{j\in I(r,1)\\ j\geqslant i}}f_j^{(r,1)}.$$

Condition (iv) shows that the projections $f_i^{(s,m)}$, for $m\in\mathbb{N}$ and $i\in I(s,m)$, generate A_s. Hence every A_s is contained in the abelian von Neumann algebra generated by all the $f_i^{(r,1)}$ for $i\in I(r,1)$ and $r\geqslant 0$, so these projections generate $A=(\bigcup_{s=1}^{\infty}A_s)''$.

Writing B_r for the abelian von Neumann subalgebra of N generated by the projections $(f_i^{(r,1)})_{i\in I(r,1)}$, Lemma 2.1 of [9] shows us that

$$\lim_{r\to\infty}\bigl\|\mathbb{E}_{B_r'\cap N}(x)-\mathbb{E}_A(x)\bigr\|_2=0$$

for all $x\in N$, where $\mathbb{E}_M$ denotes the trace-preserving conditional expectation onto the von Neumann subalgebra M of N. It is well known that $\mathbb{E}_{B_r'\cap N}=\sum_{i\in I(r,1)}f_i^{(r,1)}Jf_i^{(r,1)}J$ in this case, so

$$e_A=\lim_{r\to\infty}\sum_{i\in I(r,1)}f_i^{(r,1)}Jf_i^{(r,1)}J,$$

with strong-operator convergence. Hence

$$1-e_A=\sum_{r=0}^{\infty}\sum_{\substack{i,j\in I(r,1)\\ i\neq j\\ i|_{r-1}=j|_{r-1}}}f_i^{(r,1)}Jf_j^{(r,1)}J$$

so the only contributions to the Pukánszky invariant of A come from the central cutdowns $\mathcal{A}'f_i^{(r,1)}Jf_j^{(r,1)}J$ for $r\geqslant 0$, $i,j\in I(r,1)$ with $i\neq j$ and $i|_{r-1}=j|_{r-1}$. $\square$

For $s\geqslant 0$, write $\mathcal{A}_s$ for the abelian von Neumann algebra $(A_s\cup JA_sJ)''\subset\mathbb{B}(L^2(N_s))$. For the rest of this section we shall denote operators in $\mathbb{B}(L^2(N_s))$ with a superscript ${}^{(s)}$. Since

$$\mathbb{B}\bigl(L^2(N_{s+1})\bigr)=\mathbb{B}\bigl(L^2(N_s)\bigr)\mathbin{\overline{\otimes}}\mathbb{B}\bigl(L^2\bigl(R^{(s+1)}\bigr)\bigr)$$

we have $T^{(s)}\otimes I_{L^2(R^{(s+1)})}\in\mathbb{B}(L^2(N_{s+1}))$ for all $T^{(s)}\in\mathbb{B}(L^2(N_s))$. We shall write $T^{(s+1)}$ for this operator, and

$$T=T^{(s)}\otimes I_{L^2(R^{(s+1)})}\otimes I_{L^2(R^{(s+2)})}\otimes\cdots$$

for this extension of $T^{(s)}$ to $L^2(N)$. We refer to these operators as the canonical extensions of $T^{(s)}$. For $T^{(s)} \in \mathcal{A}_s$, we have $T^{(s+1)} \in \mathcal{A}_{s+1}$ and $T \in \mathcal{A}$, since $A_s \subset A_{s+1} \subset A$. Let p_s denote the orthogonal projection from $L^2(N)$ onto $L^2(N_s)$.

Proposition 5.2. *Let $s \geqslant 0$ and $T^{(s)} \in \mathbb{B}(L^2(N_s))$. Then $T^{(s)} \in \mathcal{A}_s'$ if and only if the extension T lies in $\mathcal{A}'$. Also $p_s\mathcal{A}'p_s = \mathcal{A}_s'$.*

Proof. Let $T \in \mathbb{B}(L^2(N))$ lie in $\mathcal{A}'$. For each s and $x \in A_s$, we have $p_sxp_s = xp_s = p_sx$ and $p_sJxJp_s = JxJp_s = p_sJxJ$. Then p_sTp_s commutes with both x and JxJ and hence lies in $\mathcal{A}_s'$. This gives $p_s\mathcal{A}'p_s \subset \mathcal{A}_s'$ and shows that if T is the canonical extension of some $T^{(s)} \in \mathbb{B}(L^2(N_s))$, then $T^{(s)} \in \mathcal{A}_s'$.

For the converse, consider $T^{(s)} \in \mathcal{A}_s'$ and take $x \in A_{s+1}$ so that

$$x = \sum_{i \in I(s,1)} x_i f_i^{(s,1)} \otimes y_i$$

for some $x_i \in A_s$ and $y_i \in D_i^{(s+1)}$ by the inductive definition of A_{s+1} in Eq. (5). Then $T^{(s+1)}$ commutes with x since $T^{(s)}$ commutes with each $x_i f_i^{(s,1)}$. Similarly $T^{(s+1)}$ commutes with JxJ, so $T^{(s+1)} \in \mathcal{A}_{s+1}'$. Proceeding by induction, we see that $T^{(r)} \in \mathcal{A}_r'$ for all $r \geqslant s$. Hence, the canonical extension T commutes with x and JxJ for all $x \in \bigcup_{r=0}^{\infty} A_r$ and these elements are weakly dense in $\mathcal{A}$, so $T \in \mathcal{A}'$. For $T^{(s)} \in \mathbb{B}(L^2(N_s))$ the canonical extension T has $p_sTp_s = T^{(s)}$, so $\mathcal{A}_s' \subset p_s\mathcal{A}'p_s$. $\square$

Our objective is to determine the type decomposition of the $\mathcal{A}' f_i^{(r,1)} J f_j^{(r,1)} J$ appearing in Lemma 5.1. For $r \geqslant 0$ and $i \in I(r,1)$, the inductive definition (6) ensures that

$$f_i^{(r,1)} = e_{i^{(0)}}^{(r+1)}(A_0) \otimes e_{i^{(1)}}^{(r)}\big(D_{i|_0}^{(1)}\big) \otimes \cdots \otimes e_{i^{(r)}}^{(1)}\big(D_{i|_{r-1}}^{(r)}\big)$$

recalling that $i|_s$ is the restriction of i to $I(s,1)$.

Lemma 5.3. *Let $r \geqslant 0$ and $i, j \in I(r,1)$ have $i \neq j$ and $i|_{r-1} = j|_{r-1}$. Let $Q^{(0)} \in \mathcal{A}_0 e_{i^{(0)}}^{(r+1)}(A_0) J e_{j^{(0)}}^{(r+1)}(A_0) J$ be a non-zero projection such that $\mathcal{A}_0' Q^{(0)}$ is homogeneous of type I_m for some $m \in \mathbb{N}_\infty$. Then, writing Q for the canonical extension of $Q^{(0)}$ to $L^2(N)$, $\mathcal{A}' f_i^{(r,1)} J f_j^{(r,1)} J Q$ is homogeneous of type $\mathrm{I}_{m\Lambda_{i,j}^{(r)}}$.*

Proof. Fix $m \in \mathbb{N}_\infty$ and $Q^{(0)} \neq 0$ as in the statement of the lemma. Observe that

$$\begin{aligned} A_{r+1} f_i^{(r,1)} &= A^{(r)} f_i^{(r,1)} \,\overline{\otimes}\, D_i^{(r+1)} \\ &= A_0 e_{i^{(0)}}^{(r+1)}(A_0) \,\overline{\otimes}\, D_{i|_0}^{(1)} e_{i^{(1)}}^{(r)}\big(D_{i|_0}^{(1)}\big) \,\overline{\otimes} \cdots \overline{\otimes}\, D_{i|_{r-1}}^{(r)} e_{i^{(r)}}^{(1)}\big(D_{i|_{r-1}}^{(r)}\big) \,\overline{\otimes}\, D_i^{(r+1)} \end{aligned}$$

so that

$$\begin{aligned} \mathcal{A}_{r+1} f_i^{(r,1)} J f_j^{(r,1)} J Q^{(r+1)} = \mathcal{A}_0 Q^{(0)} \,\overline{\otimes}\, \big(D_{i|_0}^{(1)} \cup JD_{i|_0}^{(1)}J\big)'' e_{i^{(1)}}^{(r)}\big(D_{i^{(0)}}^{(1)}\big) J e_{j^{(1)}}^{(r)}\big(D_{i^{(0)}}^{(1)}\big) J \\ \overline{\otimes} \cdots \overline{\otimes}\, \big(D_{i|_{r-1}}^{(r)} \cup JD_{i|_{r-1}}^{(r)}J\big)'' e_{i^{(r)}}^{(1)}\big(D_{i|_{r-1}}^{(r)}\big) J e_{j^{(r)}}^{(1)}\big(D_{i|_{r-1}}^{(r)}\big) J \,\overline{\otimes}\, \big(D_i^{(r+1)} \cup JD_j^{(r+1)}J\big)'', \end{aligned}$$

using $i|_s = j|_s$ for $s = 0, \dots, r-1$. We are also abusing notation by writing J for the modular conjugation operator regardless of the space on which it operates. Taking commutants gives

$$\begin{aligned}
&\mathcal{A}'_{r+1} f_i^{(r,1)} J f_j^{(r,1)} J Q^{(r+1)} \\
&\quad = \mathcal{A}'_0 Q^{(0)} \mathbin{\overline{\otimes}} \big(D_{i|_0}^{(1)} \cup J D_{i|_0}^{(1)} J\big)' e_{i^{(1)}}^{(r)} \big(D_{i^{(0)}}^{(1)}\big) J e_{j^{(1)}}^{(r)} \big(D_{i^{(0)}}^{(1)}\big) J \\
&\qquad \mathbin{\overline{\otimes}} \cdots \mathbin{\overline{\otimes}} \big(D_{i|_{r-1}}^{(r)} \cup J D_{i|_{r-1}}^{(r)} J\big)' e_{i^{(r)}}^{(1)} \big(D_{i|_{r-1}}^{(r)}\big) J e_{j^{(r)}}^{(1)} \big(D_{i|_{r-1}}^{(r)}\big) J \mathbin{\overline{\otimes}} \big(D_i^{(r+1)} \cup J D_j^{(r+1)} J\big)'.
\end{aligned}$$

For $s \leqslant r$, each $(D_{i|_{s-1}}^{(s)} \cup J D_{i|_{s-1}}^{(s)} J)''$ is maximal abelian in $\mathbb{B}(L^2(R^{(s)}))$ since $D_{i|_{s-1}}^{(s)}$ is a Cartan masa so has Pukánszky invariant $\{1\}$. The masas $D_k^{(r+1)}$ where defined in (4) so that $(D_i^{(r+1)} \cup J D_j^{(r+1)} J)'$ is homogeneous of type $\mathrm{I}_{\Lambda_{i,j}^{(r)}}$. We learn that $\mathcal{A}'_{r+1} f_i^{(r,1)} J f_j^{(r,1)} J Q^{(r+1)}$ is homogeneous of type $\mathrm{I}_{m'}$, where $m' = m\Lambda_{i,j}^{(r)}$.

Find a family of pairwise orthogonal projections $(Q_q^{(r+1)})_{0 \leqslant q < m'}$ with sum $Q^{(r+1)}$ and which are equivalent abelian projections in $\mathcal{A}'_{r+1} f_i^{(r,1)} J f_j^{(r,1)} J Q^{(r+1)}$. The canonical extensions $(Q_q)_{0 \leqslant q < m'}$ to $L^2(N)$ form a family of pairwise orthogonal projections in $\mathcal{A}'Q$ (by Proposition 5.2) with sum Q. These projections are equivalent in $\mathcal{A}'Q$ as if $V^{(r+1)}$ is a partial isometry in $\mathcal{A}'_r Q^{(r+1)}$ with $V^{(r+1)} V^{(r+1)*} = Q_q$ and $V^{(r+1)*} V^{(r+1)} = Q_{q'}$, then Proposition 5.2 ensures that the canonical extension V lies in $\mathcal{A}'$. It is immediate that $VV^* = Q_q$ and $V^*V = Q_{q'}$. We shall show that these projections are abelian projections in $\mathcal{A}'$. It will then follow that $\mathcal{A}'Q$ is homogeneous of type $\mathrm{I}_{m'}$.

For $s \geqslant r+1$ and $k, l \in I(s,1)$ with $k \geqslant i$ and $l \geqslant j$, we have

$$A_{s+1} f_k^{(s,1)} = A_s f_k^{(s,1)} \mathbin{\overline{\otimes}} D_k^{(s+1)}$$

so that

$$\mathcal{A}_{s+1}\big(f_k^{(s,1)} J f_l^{(s,1)} J\big) Q^{(s+1)} \cong \mathcal{A}_s\big(f_k^{(s,1)} J f_l^{(s,1)} J\big) Q^{(s)} \mathbin{\overline{\otimes}} \big(D_k^{(s+1)} \cup J D_l^{(s+1)} J\big)''.$$

Again we take commutants to obtain

$$\mathcal{A}'_{s+1}\big(f_k^{(s,1)} J f_l^{(s,1)} J\big) Q^{(s+1)} \cong \mathcal{A}'_s\big(f_k^{(s,1)} J f_l^{(s,1)} J\big) Q^{(s)} \mathbin{\overline{\otimes}} \big(D_k^{(s+1)} \cup J D_l^{(s+1)} J\big)'.$$

Since $i \neq j$ it is not possible for $k|_{s-1}$ to equal $l|_{s-1}$, so (4) shows us that $(D_k^{(s+1)} \cup J D_l^{(s+1)} J)'$ is abelian. Therefore, if $Q_q^{(s)} f_k^{(s,1)} J f_l^{(s,1)} J$ (some $q = 1, \dots, m'$) is an abelian projection in $\mathcal{A}'_s$, then $Q_q^{(s+1)} f_k^{(s+1,1)} J f_l^{(s+1,1)} J$ is abelian in $\mathcal{A}'_{s+1}$. The projections $f_k^{(s,1)} J f_l^{(s,1)} J$ are central and satisfy

$$\sum_{\substack{k,l \in I(s,1) \\ k \geqslant i \\ l \geqslant j}} f_k^{(s,1)} J f_l^{(s,1)} J = f_i^{(r,1)} J f_j^{(r,1)} J.$$

By induction and summing over all $k \geqslant i$ and $l \geqslant j$, we learn that $(Q_q^{(s)})_{0 \leqslant q < m'}$ form a family of equivalent abelian projections in $\mathcal{A}'Q^{(s)}$ with sum s for every $s \geqslant r+1$.

For $s \geqslant r+1$ and each q, the algebras $\mathcal{A}'_s Q_q^{(s)} = p_s \mathcal{A}' Q_q p_s$ are abelian. Since the projections p_s tend strongly to the identity, we see that each $\mathcal{A}' Q_q$ is abelian too. $\square$

We can now describe the Pukánszky invariant of the masas in Section 4.

Theorem 5.4. *Let A be a masa in a separable McDuff* II_1 *factor produced via the construction of Section* 4. *That is we are given a masa $A_0 \subset N_0$ and values $\Lambda_{i,j}^{(r)} = \Lambda_{j,i}^{(r)} \in \mathbb{N}_\infty$ for $r \geqslant 0$, $i, j \in I(r,1)$ with $i \neq j$ and $i|_{r-1} = j|_{r-1}$. Then*

$$\operatorname{Puk}(A) = \bigcup_{r=0}^{\infty} \bigcup_{\substack{i,j \in I(r,1) \\ i \neq j \\ i|_{r-1} = j|_{r-1}}} \Lambda_{i,j}^{(r)} \cdot \operatorname{Type}\big(\mathcal{A}'_0 e_{i^{(0)}}^{(r+1)}(A_0) J e_{j^{(0)}}^{(r+1)}(A_0) J\big). \tag{7}$$

Proof. For $r \geqslant 0$, $i, j \in I(r,1)$ with $i \neq j$ and $i|_{r-1} = j|_{r-1}$, it follows from Lemma 5.3 that

$$\operatorname{Type}\big(\mathcal{A}' f_i^{(r,1)} J f_j^{(r,1)} J\big) = \Lambda_{i,j}^{(r)} \cdot \operatorname{Type}\big(\mathcal{A}'_0 e_{i^{(0)}}^{(r+1)}(A_0) \cup J e_{j^{(0)}}^{(r+1)}(A_0) J\big).$$

The theorem then follows from Lemma 5.1. $\square$

6. Main results

We start by applying Theorem 5.4 when $\operatorname{Puk}(A_0)$ is a singleton.

Theorem 6.1. *For $n \in \mathbb{N}$, suppose that N_0 is a separable McDuff* II_1 *factor containing a masa with Pukánszky invariant $\{n\}$. For every non-empty set $E \subset \mathbb{N}_\infty$, there exists a masa A in N_0 with* $\operatorname{Puk}(A) = \{n\} \cdot E$.

Proof. Let A_0 be a masa in N_0 with $\operatorname{Puk}(A) = \{n\}$ and choose the values $\Lambda_{i,j}^{(r)} = \Lambda_{j,i}^{(r)}$ for $r \geqslant 0$ and $i, j \in I(r,1)$ with $i \neq j$ and $i|_{r-1} = j|_{r-1}$ so that

$$E = \big\{ \Lambda_{i,j}^{(r)} \bigm| r \geqslant 0,\ i, j \in I(r,1),\ i \neq j,\ i|_{r-1} = j|_{r-1} \big\}.$$

The resulting masa A in $N \cong N_0$ produced by the main construction has Pukánszky invariant $\{n\} \cdot E$ by Theorem 5.4. $\square$

Since Cartan masas have Pukánszky invariant $\{1\}$, we obtain the following corollary immediately.

Corollary 6.2. *Let N be a McDuff* II_1 *factor containing a simple masa, for example a Cartan masa. Every non-empty subset of $\mathbb{N}_\infty$ arises as the Pukánszky invariant of a masa in N.*

A little more care enables us to address the question of the range of the Pukánszky invariant on singular masas in the hyperfinite II_1 factor and other McDuff II_1 factors containing a simple singular masa. Pukánszky's original work [12] exhibits a simple singular masa in the hyperfinite II_1 factor.

Corollary 6.3. *Let N be a separable McDuff factor containing a simple singular masa, such as the hyperfinite* II_1 *factor. Given any non-empty $E \subset \mathbb{N}_\infty$ there is a singular masa A in N with* $\mathrm{Puk}(A) = E$.

Proof. If $1 \notin E$, a masa in N with Pukánszky invariant E is automatically singular by [11, Remark 3.4]. We have already produced these masas in Corollary 6.2. The hypothesis ensures us a simple singular masa in N. For the remaining case of some $E \neq \{1\}$ with $1 \in E$, let A_1 be a singular masa in N with $\mathrm{Puk}_{N_1}(A_1) = \{1\}$ and A_2 be a singular masa in the hyperfinite II_1 factor R with $\mathrm{Puk}_R(A_2) = E \setminus \{1\}$. Then $A = A_1 \mathbin{\overline{\otimes}} A_2$ is a masa in $N \mathbin{\overline{\otimes}} R \cong N$. Lemma 2.1 of [13] ensures that

$$\mathrm{Puk}(A) = \{1\} \cup \big(E \setminus \{1\}\big) \cup 1 \cdot \big(E \setminus \{1\}\big) = E.$$

The singularity of A is Corollary 2.4 of [15]. $\square$

Next we justify the claims made at the end of Section 2.

Theorem 6.4. *Let $E, F, G \subset \mathbb{N}_\infty$ be non-empty. Then there exist masas B and C in the hyperfinite* II_1 *factor with* $\mathrm{Puk}(B) = E$, $\mathrm{Puk}(C) = F$ *and* $\mathrm{Puk}(B, C) = G$.

Proof. Let R_0 be a copy of the hyperfinite II_1 factor and A_0 a Cartan masa in R_0. An element k of $I(0,1)$ is of the form $(k^{(0)})$ where $k^{(0)}$ is a 1-tuple—either 0 or 1. Write $\mathbf{0}$ and $\mathbf{1}$ for these two elements and let $e_0 = f_{\mathbf{0}}^{(1)}$ and $e_1 = f_{\mathbf{1}}^{(1)}$ so that e_0 and e_1 are orthogonal projections in A with $\mathrm{tr}(e_0) = \mathrm{tr}(e_1) = 1/2$. Choose the $\Lambda_{i,j}^{(r)} = \Lambda_{j,i}^{(r)}$ such that:

$$\begin{aligned}
E &= \big\{\Lambda_{i,j}^{(r)} \bigm| r \geqslant 1,\ i, j \in I(r,1), i \neq j, i|_{r-1} = j|_{r-1}, i, j \geqslant \mathbf{0}\big\},\\
F &= \big\{\Lambda_{i,j}^{(r)} \bigm| r \geqslant 1,\ i, j \in I(r,1), i \neq j, i|_{r-1} = j|_{r-1}, i, j \geqslant \mathbf{1}\big\},\\
G &= \big\{\Lambda_{i,j}^{(r)} \bigm| r \geqslant 0,\ i, j \in I(r,1), i \neq j, i|_{r-1} = j|_{r-1}, i \geqslant \mathbf{0}, j \geqslant \mathbf{1}\big\},\\
&= \big\{\Lambda_{i,j}^{(r)} \bigm| r \geqslant 0,\ i, j \in I(r,1), i \neq j, i|_{r-1} = j|_{r-1}, i \geqslant \mathbf{1}, j \geqslant \mathbf{0}\big\}.
\end{aligned}$$

For $r, s = 0, 1$, let $Q_{r,s} = (1 - e_A)e_r J e_s J$ a projection in $\mathcal{A}$. Now Lemmas 5.3 and 5.1 ensure that $\mathcal{A}' Q_{0,0}$ has a non-zero I_m cutdown if and only if $m \in E$, $\mathcal{A}' Q_{1,1}$ has a non-zero I_m cutdown if and only if $m \in F$, $\mathcal{A}'(Q_{0,1} + Q_{1,0})$ has a non-zero I_m cutdown if and only if $m \in G$.

We now regard A as a direct sum. Consider the copy of the hyperfinite II_1 factor $S = e_0 R e_0$ so that choosing a partial isometry $v \in R$ with $v^*v = e_0$ and $vv^* = e_1$ gives rise to an isomorphism between R and $M_2(S)$—the 2×2 matrices over S. Define masas in S by $B = Ae_0$ and $C = v^*(Ae_1)v$. The discussion above ensures that $\mathrm{Puk}(B) = E$, $\mathrm{Puk}(C) = F$ and $\mathrm{Puk}(B, C) = G$. Note that $\mathrm{Puk}(B, C)$ is independent of v by Proposition 2.3. $\square$

Remark 6.5. If $E \subset \mathbb{N}_\infty$ contains at least two elements then we can modify the construction in Section 4 to produce uncountably many pairwise non-conjugate masas in the hyperfinite II_1 factor R each with Pukánszky invariant E. The idea is to control the supremum of the trace of a projection in the masa A such that $\mathrm{Puk}_{eRe}(Ae) = \{n\}$ for some fixed $n \in E$. For each $t \in (0,1)$, we can produce masas A in R and a projection $e \in A$ with $\mathrm{tr}(e) = t$ such that (with the intuitive

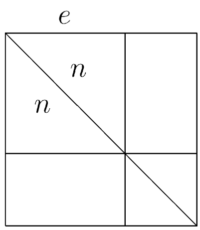

Fig. 5. The multiplicity structure of $\mathcal{A}$.

diagrams of the introduction) the multiplicity structure of $\mathcal{A}$ is represented by Fig. 5, with 1 down the diagonal and $E \setminus \{n\}$ occurring in the unmarked areas. All these masas must be pairwise non-conjugate.

No modifications are required to obtain any diadic rational for t, we follow Theorem 6.4 to control the multiplicity structure of $\mathcal{A}$. For general t we can approximate the required structure using diadic rationals, leaving the area we are unable to handle at each stage with multiplicity 1 so it can be adjusted at a subsequent stage.

Remark 6.6. For a masa A in a property Γ-factor N, the property that A contains non-trivial centralising sequences for N has been used to distinguish between non-conjugate masas, see for example [5,7,14]. We can easily adjust the construction of Section 4 to ensure that all the masas produced have this property. Suppose that we identify each $R^{(r)}$ with $R^{(r)} \mathbin{\overline{\otimes}} R^{(r)}$ and we replace the masas $D_i^{(r)}$ in $R^{(r)}$ by $D_i^{(r)} \mathbin{\overline{\otimes}} E^{(r)}$ where $E^{(r)}$ is a fixed Cartan masa in $R^{(r)}$. By Lemma 2.4 this does not alter the mixed Pukánszky invariants of the family, so the Pukánszky invariant of the masa resulting from the construction remains unchanged. This masa now contains non-trivial centralising sequences for N. By way of contrast, the examples in [4,13] arise from inclusions $H \subset G$ of a an abelian group inside a discrete I.C.C. group G with $gHg^{-1} \cap H = \{1\}$ for all $g \in G \setminus H$. The resulting masa $\mathcal{L}(H)$ cannot contain non-trivial centralising sequences for the II_1 factor $\mathcal{L}(G)$, [10].

Very recently Ozawa and Popa have shown that not every McDuff II_1 factor contains a Cartan masa. Indeed in [8] they show that there are no Cartan masas in $\mathcal{L}\mathbb{F}_2 \mathbin{\overline{\otimes}} R$. It is not known whether every McDuff factor must contain a simple masa (one with Pukánszky invariant $\{1\}$) or a masa whose Pukánszky invariant is a finite subset of $\mathbb{N}$. We can however obtain subsets containing ∞ as Pukánszky invariants of masas in a general separable McDuff II_1 factor.

Theorem 6.7. *Let N be a separable McDuff II_1 factor. For every set $E \subset \mathbb{N}_\infty$ with $\infty \in E$ there is a singular masa B in N with* $\mathrm{Puk}(B) = E$.

Proof. Taking all the $\Lambda_{i,j}^{(r)} = \infty$, gives us a masa A in N with $\mathrm{Puk}(A) = \{\infty\}$ by Theorem 5.4 (regardless of the masa A_0). Now use the isomorphism $N \cong N \mathbin{\overline{\otimes}} R$, where R is the hyperfinite II_1 factor. Let $B = A \mathbin{\overline{\otimes}} A_1$, where A_1 is a singular masa in R with $\mathrm{Puk}_R(A_1) = E$. Lemma 2.1 of [13] gives

$$\mathrm{Puk}(B) = \{\infty\} \cup E \cup \{\infty\} \cdot E = E. \qquad \square$$

In particular every separable McDuff II_1 factor contains uncountably many pairwise non-conjugate singular masas.

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On the subspaces of *JF* and *JT* with non-separable dual ☆

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Abstract

It is proved that every subspace of James Tree space (*JT*) with non-separable dual contains an isomorph of James Tree complemented in *JT*. This yields that every complemented subspace of *JT* with non-separable dual is isomorphic to *JT*. A new *JT* like space denoted as *TF* is defined. It is shown that every subspace of James Function space (*JF*) with non-separable dual contains an isomorph of *TF*. The later yields that every subspace of *JF* with non-separable dual contains isomorphs of c_0 and ℓ_p for $2 \leqslant p < \infty$. The analogues of the above results for bounded linear operators are also proved.

Keywords: Banach spaces with non-separable dual; James Tree space; James Function space

1. Introduction

The aim of the present work is to study the structure of the subspaces with non-separable dual, as well as conservation properties of operators for the two fundamental examples of separable Banach spaces not containing ℓ_1 with non-separable dual. Both examples appeared in the 70s and actually established the aforementioned class. The first one is the well known and extensively studied James Tree space (*JT*) invented by R.C. James [8]. The second, due to J. Lin-

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denstrauss [10] is a function space so-called James Function space (*JF*). J. Lindenstrauss defined *JF* as the completion of $L_1[0,1]$ endowed with an appropriate norm. He also pointed out that the Voltera operator defines an isometric embedding of *JF* into the space V_2 of the functions with square (quadratic) bounded variation. At the beginning of 80s S.V. Kisliakov [9] observed that the Voltera image of *JF* coincides with the subspace V_2^0 of the square absolutely continuous functions. He also used this intrinsic representation to provide easier proofs of the basic properties of *JF*.

Throughout the paper we shall consider the representation of *JF* as V_2^0. Notice that the subspaces of *JT* and V_2^0 have quite divergent structure. Indeed as is well known *JT* is ℓ_2 saturated. On the other hand, V_2^0, rather unexpectedly, contains c_0. Furthermore unpublished results of H.P. Rosenthal, G. Schechtman and also G.D. Berg, E. Odell yield that V_2^0 contains all ℓ_p for $2 \leqslant p < \infty$. These results show that V_2^0 is not isomorphic to a subspace of *JT*. On the other side, S. Buechler and E. Odell [5] have shown that *JT* is not isomorphic to a subspace of V_2^0. Related to the subspaces of *JT* and V_2^0 we prove the following. We start with the results concerning the subspaces of *JT*.

Theorem 1. *Every subspace X of JT with non-separable dual contains an isomorph of JT complemented in JT.*

The above yields the following.

Corollary 2. *Every complemented subspace of JT with non-separable dual is isomorphic to JT. In particular JT is primary.*

The primary property of *JT* has been proved by A.D. Andrew [1]. He derived that from a result concerning operators in $\mathcal{B}(JT)$. More precisely he has proved that for every $T \in \mathcal{B}(JT)$ either T or $I - T$ preserves a copy of *JT* with its image complemented in *JT*. We extend this in the following manner.

Theorem 3. *Let $T \in \mathcal{B}(JT)$ with $T^*[JT^*]$ non-separable. Then there exists a subspace X of JT isomorphic to JT such that $T|_X$ is an isomorphism and $T[X]$ is complemented in JT.*

The reader would compare the above to the classical H.P. Rosenthal's theorem concerning operators with domain $C([0,1])$ [13].

In order to describe the structure of the subspaces with non-separable dual of V_2^0 we introduce a new Tree-like space denoted as *TF*. The space *TF* is the completion of $c_{00}(\mathcal{D})$ under the norm $\|x\|_{TF} = \sup(\sum_{i=1}^k (\sum_{\alpha \in \mathcal{I}_i} x(\alpha))^2)^{1/2}$ where the supremum is taken over all "non-separated" (ns) families of disjoint finite segments of $\mathcal{D}$. This space has non-separable dual and does not contain ℓ_1. Also the bases $(e_\alpha)_{\alpha \in \mathcal{D}}$ of *TF* and *JT* share similar properties. Namely for every pairwise incomparable sequence $(\alpha_n)_n$ of $\mathcal{D}$, the sequence $(e_{\alpha_n})_n$ is equivalent to ℓ_2 basis and for every chain $(\alpha_n)_n$ the sequence $(e_{\alpha_n})_n$ generates James quasi-reflexive space. On the other hand, *TF* contains isomorphic copies of the "stopping time" space S^2 and hence it contains c_0 and ℓ_p for $2 \leqslant p < \infty$. The embedding of S^2 into *TF* follows from deep and intricate arguments included in the aforementioned work of Buechler and Odell and the embedding of ℓ_p, $2 \leqslant p < \infty$ into S^2 is due to Rosenthal and Schechtman. Let us point out that the corresponding of Theorem 1, Corollary 2 and Theorem 3, remain valid for the space *TF*. In particular *TF* is

primary and also does not contain isomorphs of *JT*. Our main goal is to establish that *TF* is the fundamental prototype for the subspaces of V_2^0 with non-separable dual. Namely the following holds.

Theorem 4. *Let X be a subspace of V_2^0 with non-separable dual. Then TF is isomorphic to a subspace of X. In particular X contains isomorphs of c_0 and ℓ_p for $2 \leqslant p < \infty$.*

As consequence we obtain Buechler and Odell's unpublished result namely that *JT* is not isomorphic to a subspace of V_2^0. Moreover every reflexive saturated subspace of V_2^0 has separable dual. As is shown in [4] every non-reflexive subspace of V_2^0 contains either c_0 or l_2. It is not known if every reflexive subspace of V_2^0 contains some l_p for $2 \leqslant p < \infty$. A positive answer to this problem would led to the conclusion that every subspace X of V_2^0 with non-separable dual is comparable with the arbitrary infinite dimensional subspace Y of V_2^0.

We also prove a similar to Theorem 3 for the space V_2^0 stated as follows.

Theorem 5. *Let $T \in \mathcal{B}(V_2^0)$ with $T^*[(V_2^0)^*]$ non-separable. Then there exists a subspace X of V_2^0 isomorphic to TF such that $T|_X$ is an isomorphism.*

Our approach in proving Theorem 1 for *JT* and Theorem 4 for V_2^0 is based on the understanding of the structure of X^{**} for X as in their statements. It is worth mentioning that the main advantage of the consideration of *JF* as V_2^0 is the complete understanding of JF^{**} which naturally coincides with the space V_2 and moreover the w^*-topology on bounded subsets of V_2 is the pointwise one. The proof of the above results relies on the following theorem concerning the structure of V_2.

Theorem 6. *The space V_2 is the direct topological sum of the subspaces V_2^{rc} and V_2^d.*

In the above statement V_2^{rc} denotes the subspace of V_2 consisting of all right continuous functions and V_2^d the subspace generated by the set $\{\chi_t\colon t \in (0,1)\}$, where χ_t is the characteristic function of $\{t\}$. The later is isomorphic to $\ell_2(0,1)$ under the natural correspondence. Let us recall that $JT^{**} \approx JT \oplus \ell_2(\{0,1\}^{\mathbb{N}})$. Among the differences of *JF* and *JT* is that *JF* is not complemented in its second dual. The same property also holds for the space *TF*. Concerning the subspaces of V_2^0 with non-separable dual we prove the following.

Proposition 7. *Let X be a subspace of V_2^0 with non-separable dual. Then $X^{**} \cap V_2^d$ is isomorphic to $\ell_2(\mathfrak{c})$.*

The above proposition is based on Theorem 6 and on the following recent structural result ([3], see also [2]) which is a key ingredient for our proof.

Theorem 8. *Let X be a separable Banach space not containing ℓ_1 with non-separable dual. Then there exists an 1-unconditional family $(x_\tau^{**})_{\tau\in T}$ in B_X^{**} of size of the continuum which is w^*-discrete and accumulating to zero.*

It is notable that it does not seem easy to show that $X^{**} \cap V_2^d \neq \{0\}$ without the use of Theorem 8. The proof of Theorem 4 lifts data from the space $X^{**} \cap V_2^d$ into X itself by defining a family $(f_\alpha)_{\alpha\in\mathcal{D}}$ equivalent to the standard Schauder basis of *TF*. Note also that in this process the topology of $[0, 1]$ plays a key role.

The paper is organized into eight sections. Section 2 is devoted to the proof of Theorem 6 and Proposition 7. In Section 3 we study the key concept of tree families of functions. Their definition goes as follows.

Definition 9. For every $\alpha \in \mathcal{D}$, let f_α be a function in V_2^0, (I_α, J_α) be a pair of closed intervals in $[0, 1]$. The family $\mathcal{F} = (f_\alpha, (I_\alpha, J_\alpha))_{\alpha\in\mathcal{D}}$ will be called a tree family of functions if the following are satisfied.

(1) $\sup_{\alpha\in\mathcal{D}} \|f_\alpha\|_{V_2^0} < +\infty$.
(2) For every $\alpha \in \mathcal{D}$, $J_\alpha \subseteq \mathring{I}_\alpha$, $I_{\alpha^\frown 0} \cup I_{\alpha^\frown 1} \subseteq \mathring{J}_\alpha$.
(3) For every $\alpha \in \mathcal{D}$, $f_\alpha|_{J_\alpha} = 1$ and $\operatorname{supp} f_\alpha \subseteq \mathring{I}_\alpha$.

Here by $\mathcal{D}$ we denote the dyadic tree, for $\alpha \in \mathcal{D}$ and $\varepsilon \in \{0, 1\}$ by $\alpha^\frown\varepsilon$ we denote the immediate successors of α in $\mathcal{D}$, and $\mathring{I}_\alpha$, $\mathring{J}_\alpha$ denote the interiors of the corresponding intervals. In Section 3 we shall present an extension of the above definition, concerning tree families of functions with coefficients.

Definition 10. A tree family $\mathcal{F} = (f_\alpha, (I_\alpha, J_\alpha))_{\alpha\in\mathcal{D}}$ will be called increasing if for every $\alpha \in \mathcal{D}$, $I_{\alpha^\frown 0} < I_{\alpha^\frown 1}$ (i.e. $\max I_{\alpha^\frown 0} < \min I_{\alpha^\frown 1}$), decreasing if for every $\alpha \in \mathcal{D}$, $I_{\alpha^\frown 0} > I_{\alpha^\frown 1}$ and monotone if it is either increasing or decreasing.

Increasing tree families of functions have been used by S. Buechler and E. Odell [5].

The following theorem, proved in Section 3, connects monotone tree families with the standard basis of the space *TF*.

Theorem 11. *Every monotone tree family of functions $(f_\alpha, (I_\alpha, J_\alpha))_{\alpha\in\mathcal{D}}$ in V_2^0 is equivalent, under the natural correspondence, to the standard basis $(e_\alpha)_{\alpha\in\mathcal{D}}$ of TF.*

Beyond this result tree families have a decisive presence in the proof of Theorems 4 and 5. In Section 4 we state and prove some auxiliary results concerning condensation points of uncountable subsets of $\mathbb{R}^\mathbb{N}$ and an approximation lemma for elements of V_2^d. The simplest form of this result is the following.

Lemma 12 *(Approximation lemma). Let $\varepsilon > 0$, I_0 be an open interval of $(0, 1)$, $t_0 \in I_0$ and $(g_n)_n$ be a sequence in V_2^0 w^*-converging to χ_{t_0}. Then there exists a triplet (f, I, J) where $f \in V_2^0$ and I, J are intervals in $(0, 1)$ and such that $t_0 \in \mathring{J} \subseteq J \subseteq \mathring{I} \subseteq I \subseteq I_0$, $0 \leqslant f \leqslant 1$, $\operatorname{supp} f \subseteq \mathring{I}$, $J = \{t \in [0, 1]\colon f(t) = 1\}$, f is piecewise linear and $\|f\|_{V_2^0} = \sqrt{2}$. Moreover there exists a finite convex combination h of $(g_n)_n$ such that $\|f - h\|_{V_2^0} < \varepsilon$.*

In Section 5 we present the proof of Theorem 5. It is based on the aforementioned approximation lemma and the following.

Lemma 13. *Let $T : V_2^0 \to V_2^0$ be a bounded linear operator such that the dual operator $T^* : (V_2^0)^* \to (V_2^0)^*$ has non-separable range. Then the set $A = \{t \in (0,1)\colon T^{**}(\chi_t) \in V_2^d \setminus \{0\}\}$ is uncountable.*

Sections 6 and 7 are devoted to the proof of Theorem 4. Its proof, compared to the one of Theorem 5 is more involved. To handle this case we introduce the concept of a forest of tree families which is a countable collection of tree families with a precise relation between them. In Section 4 we prove that the diagonal family of functions, corresponding to a forest of tree families, is equivalent to *TF* basis. In Section 5, for a given X with non-separable dual, we establish the existence of a forest of tree families such that the diagonal family of functions is almost contained in X, yielding the proof of Theorem 4.

The final section is devoted to the proofs of Theorem 1, Corollary 2 and Theorem 3. We follow the same scheme (i.e. tree families, forests of tree families) as in the proofs of Theorems 4 and 5. We also discuss the analogous results for the space *TF*.

We close this section by recalling some notation and definitions mainly concerning the dyadic tree $\mathcal{D}$, that is the set of all finite sequences α in $\{0,1\}$ included the empty sequence denoted by $\varnothing$. The *length* $|\alpha|$ of $\alpha \in \mathcal{D}$, is defined to be 0 if $\alpha = \varnothing$ while if $\alpha \in \{0,1\}^n$, $n \in \mathbb{N}$, $|\alpha| = n$. The initial segment partial ordering on $\mathcal{D}$ will be denoted by $\sqsubseteq$ and we will write $\alpha \sqsubset \beta$ if $\alpha \sqsubseteq \beta$ and $\alpha \neq \beta$. By $\alpha \perp \beta$ we mean that α, β are $\sqsubseteq$-incomparable (that is neither $\alpha \sqsubseteq \beta$ nor $\beta \sqsubseteq \alpha$). If A, B are subsets of $\mathcal{D}$ then we write $A \perp B$ if for all $\alpha \in A$ and for all $\beta \in B$, $\alpha \perp \beta$.

A subset $\mathcal{I}$ of $\mathcal{D}$ is called a *segment* if $(\mathcal{I}, \sqsubseteq)$ is linearly ordered by $\sqsubseteq$ and moreover for every $\alpha \sqsubset \gamma \sqsubset \beta$, γ is contained in $\mathcal{I}$ provided that α, β belong to $\mathcal{I}$. For a segment $\mathcal{I}$ of $\mathcal{D}$, by $\min \mathcal{I}$ we denote the $\sqsubseteq$-least element of $\mathcal{I}$. If $\mathcal{I}$ is finite then $\max \mathcal{I}$ denotes the $\sqsubseteq$-greatest element of $\mathcal{I}$. Notice that for two segments $\mathcal{I}_1, \mathcal{I}_2$ of $\mathcal{D}$, $\mathcal{I}_1 \perp \mathcal{I}_2$ if and only if $\min \mathcal{I}_1 \perp \min \mathcal{I}_2$.

A segment $\mathcal{I}$ is called *initial* if the empty sequence $\varnothing$ belongs to $\mathcal{I}$. For any $\alpha \in \mathcal{D}$, let $\mathcal{I}(\alpha) = \{\gamma \in \mathcal{D}\colon \gamma \sqsubseteq \alpha\}$. Then clearly $\mathcal{I}(\alpha)$ is an initial segment of $\mathcal{D}$. For $\alpha, \beta \in \mathcal{D}$, the *infimum* of $\{\alpha, \beta\}$ is defined by $\alpha \wedge \beta = \max(\mathcal{I}(\alpha) \cap \mathcal{I}(\beta))$.

The *lexicographical ordering* of $\mathcal{D}$, denoted by $\leqslant_{\mathrm{lex}}$ is defined as follows. For every $\alpha, \beta \in \mathcal{D}$, $\alpha \leqslant_{\mathrm{lex}} \beta$ if either $\alpha \sqsubseteq \beta$ or $\alpha \perp \beta$, $\delta^\frown 0 \sqsubseteq \alpha$ and $\delta^\frown 1 \sqsubseteq \beta$ where $\delta = \alpha \wedge \beta$. Also we write $\alpha <_{\mathrm{lex}} \beta$ if $\alpha \leqslant_{\mathrm{lex}} \beta$ and $\alpha \neq \beta$. The lexicographical ordering is a total ordering of $\mathcal{D}$.

The set $\mathcal{D}$ takes the form of a sequence $(\alpha_n)_{n\in\mathbb{N}}$ where $n < m$ if either $|\alpha_n| < |\alpha_m|$ or $|\alpha_n| = |\alpha_m|$ and $\alpha_n <_{\mathrm{lex}} \alpha_m$. This ordering of $\mathcal{D}$ (which identifies $\mathcal{D}$ with $\mathbb{N}$) will be called the *natural ordering* of $\mathcal{D}$.

Let us point out that the lexicographical ordering $\leqslant_{\mathrm{lex}}$ restricted on the incomparable pairs (α, β) of $\mathcal{D}$ is actually induced by the corresponding one of $\{0,1\}^{\mathbb{N}}$ when the later is naturally identified with the Cantor set in $[0,1]$. Thus $\leqslant_{\mathrm{lex}}$ is a linear but not a well ordering of $\mathcal{D}$. Let us also note that other authors use the term "lexicographical ordering" to denote what we call the natural ordering.

For a set $I \subseteq \mathbb{R}$ the interior of I will be denoted by $\mathring{I}$ and the boundary of I by ∂I. For two closed and disjoint intervals I, J of $[0,1]$ we will write $I < J$ if $\max I < \min J$.

Finally the symbol $|\cdot|$ is also used to denote the cardinality $|A|$ of a set A and the length $|I|$ of an interval I in $\mathbb{R}$.

2. The space V_2

In this section we provide a decomposition of the space V_2 and we use it for studying the structure of the uncountable w^*-discrete and accumulating to zero subsets of V_2. We start by recalling the definitions of V_2, V_2^0 and JF from [9] and [10].

2.1. Definition and basic properties

Let $f:[0,1]\to\mathbb{R}$ and for every $\mathcal{P}=\{t_0<\cdots<t_p\}\subseteq[0,1]$ let $\alpha_2(f,\mathcal{P})=(\sum_{i=0}^{p-1}(f(t_{i+1})-f(t_i))^2)^{1/2}$. Then $V_2=\{f:[0,1]\to\mathbb{R}\colon f(0)=0 \text{ and } \|f\|_{V_2}<\infty\}$ where $\|f\|_{V_2}=\sup\alpha_2(f,\mathcal{P})$ and the supremum is taken over all finite subsets $\mathcal{P}$ of $[0,1]$. The space V_2^0 is the subspace of V_2 consisting of all $f\in V_2$ satisfying that $\lim_{\varepsilon\to 0}\sup\{\alpha_2(f,\mathcal{P})\colon \delta(\mathcal{P})\leqslant\varepsilon\}=0$ where $\delta(\mathcal{P})=\max_{0\leqslant i\leqslant p-1}|t_{i+1}-t_i|$. The completion of $L_1[0,1]$ under the norm $\|f\|_{JF}=\sup(\sum_{j=1}^m(\int_{I_j}f\,d\mu)^2)^{1/2}$, where the supremum is taken over all finite families $\{I_j\}_{j=1}^m$ of disjoint intervals in $[0,1]$, is the space JF. Notice that V_2^0 is isometric to JF via the Voltera operator $V(f)(t)=\int_0^t f(x)\,d\mu$. It is also well known (see [9]) that $(V_2^0)^{**}=V_2$ and in addition the weak* and the pointwise topology coincide on bounded subsets of V_2. Moreover (see [4]) every $f\in C[0,1]\cap V_2$ is a difference of bounded semicontinuous functions defined on $K=(B_{(V_2^0)^*},w^*)$. This actually yields that for every $f\in C[0,1]\cap V_2$ and every bounded sequence $(f_n)_n$ in V_2^0 weak*-converging to f there exists a convex block sequence $(g_n)_n$ of $(f_n)_n$ equivalent to the summing basis of c_0 (see [7]).

We pass now to present some properties related to the points of discontinuity of the functions of V_2. We shall need the following notation.

Notation 1. For every $f\in V_2$, by $D(f)$ we denote the set of all points of discontinuity of f. For all $t\in[0,1]$ let $f(t^+)=\lim_{s\to t^+}f(s)$ and $f(t^-)=\lim_{s\to t^-}f(s)$, where by convention we set $f(0^-)=f(0)$ and $f(1^+)=f(1)$.

Fact 14. *For every $f\in V_2$ the following hold.*

(i) *For every $t\in(0,1]$, $f(t^-)=\lim_{s\to t^-}f(s)$ and $f(t^+)=\lim_{s\to t^+}f(s)$ exist. Hence $D(f)$ is at most countable.*
(ii) $\sum_{t\in D(f)}|f(t)-f(t^-)|^2\leqslant\|f\|_{V_2}^2$ *and* $\sum_{t\in D(f)}|f(t)-f(t^+)|^2\leqslant\|f\|_{V_2}^2$.

2.2. The decomposition of the space V_2

Let us define

$$V_2^c=C[0,1]\cap V_2,\qquad V_2^{rc}=\overline{V_2^c+\langle\{\chi_{[t,1]}\colon 0<t\leqslant 1\}\rangle}\quad\text{and}\quad V_2^d=\overline{\langle\{\chi_t\colon 0<t<1\}\rangle},$$

where χ_t is the characteristic function of $\{t\}$ and $\chi_{[t,1]}$ is the characteristic function of $[t,1]$. The following properties of the above spaces are easily shown.

(1) V_2^c is a closed subspace of V_2.
(2) Every $g\in V_2^{rc}$ is right continuous.

(3) The space V_2^d is isomorphic to $\ell_2(0,1)$. In particular every $H \in V_2^d$ is of the form $H = \sum_{t\in A} \lambda_t \chi_t$ where A is a countable subset of $(0,1)$ and $\sum_{t\in A} |\lambda_t|^2 \leqslant \|H\|_{V_2}^2 \leqslant 4\sum_{t\in A} |\lambda_t|^2$.
(4) The space $\overline{\langle\{\chi_{[t,1]}\colon 0<t\leqslant 1\}\rangle}$ is isometric to $J(0,1]$, where $J(0,1]$ is the James space on the interval $(0,1]$.

The main result of this subsection is the following.

Theorem 15. *The space V_2 is the topological direct sum of V_2^{rc} and V_2^d, namely $V_2 = V_2^{rc} \oplus V_2^d$.*

The proof of Theorem 15 is based on a series of lemmas stated below.

Lemma 16. *For every $f \in V_2$ there exist $r \in V_2$ and $H \in V_2^d$ such that $f = r + H$, r is right continuous and $\|H\|_{V_2} \leqslant 2\|f\|_{V_2}$.*

Proof. Let $f \in V_2$ and let $D(f)$ be the set of all points of discontinuity of f. We set $H = \sum_{t\in D(f)}(f(t) - f(t^+))\chi_t$ and $r = f - H$. By Fact 14(ii) and (3) above we have that $\|H\|_{V_2}^2 \leqslant 4\|f\|_{V_2}^2$. Also Fact 14(i) easily yields that r is right continuous. □

Lemma 17. *For every right continuous $r \in V_2$ and every finite subset F of $D(r)$ there exist a right continuous $g \in V_2$ and $h \in \langle\{\chi_{[t,1]}\colon 0<t\leqslant 1\}\rangle$ such that $r = g + h$, $D(g) = D(r) \setminus F$ and $\sum_{t\in D(g)} |g(t) - g(t^-)|^2 = \sum_{t\in D(r)\setminus F} |r(t) - r(t^-)|^2$.*

Proof. Let $r \in V_2$ be right continuous. We set $h = \sum_{t\in F}(r(t) - r(t^-))\chi_{[t,1]}$ and $g = r - h$. Then h and so g are right continuous functions on $[0,1]$. It is easily checked that $D(g) = D(r) \setminus F$ and that for all $t \in [0,1] \setminus F$, $g(t) - g(t^-) = r(t) - r(t^-)$. □

Lemma 18. *For every $g \in V_2$, every $t \in (0,1]$ and every $\varepsilon > 0$ there exists $0 < \delta < t$ such that $\sup\{\alpha_2(g,\mathcal{P})\colon \mathcal{P}$ finite subset of $[t-\delta,t)\} \leqslant \varepsilon$.*

Proof. Suppose that for some $g \in V_2$, $t \in (0,1]$ and $\varepsilon > 0$ the conclusion fails. This yields that there exist a strictly decreasing sequence $(\delta_k)_{k=1}^\infty$ of positive real numbers and a sequence $(\mathcal{P}_k)_{k=1}^\infty$ of finite subsets of $[0,1]$ such that for all $k \geqslant 1$, $\alpha_2(g,\mathcal{P}_k) > \varepsilon$ and $\mathcal{P}_k \subseteq [t-\delta_k, t-\delta_{k+1})$. But then for every $n \geqslant 1$, $\|g\|_{V_2}^2 \geqslant \sum_{k=1}^n \alpha_2^2(g,\mathcal{P}_k) > n\varepsilon^2$, which is impossible since $f \in V_2$. □

Lemma 19. *Let $g \in V_2$ be right continuous such that the set $D(g)$ of the points of discontinuity of g is infinite. Then for every $\varepsilon > 0$ there exists $\phi \in V_2^c$ such that $\|\phi - g\|_{V_2}^2 \leqslant 64\sum_{t\in D(g)} |g(t) - g(t^-)|^2 + \varepsilon$.*

Proof. Let $D(g) = \{t_n\}_{n=1}^\infty$ be an enumeration of $D(g)$. We set $g_0 = \phi$ and $I_0 = \varnothing$. By induction on $n \geqslant 1$ we will construct a sequence $(g_n)_{n=1}^\infty$ of functions in V_2 and a sequence $(I_n)_{n=1}^\infty$ of closed intervals in $[0,1]$ such that for all $n \geqslant 1$ the following are satisfied.

(P1) $t_n \in \bigcup_{k=1}^n I_k$.
(P2) Either $I_n = I_{n-1}$ and $g_n = g_{n-1}$, or $I_n \cap (\bigcup_{k=0}^{n-1} I_k) = \varnothing$ and $\varnothing \neq \{t \in [0,1]\colon g_n(t) \neq g_{n-1}(t)\} \subseteq \mathring{I}_n$.

(P3) $D(g_n) \subseteq \{t_{n+1}, t_{n+2}, \ldots\}$.
(P4) $\|g_n - g_{n-1}\|_{V_2}^2 \leqslant 16|g(t_n) - g(t_n^-)|^2 + \varepsilon/2^{n+2}$.

Assume that the above construction has been carried out. By induction on $m \geqslant 1$ and using (P2), we have that $\|g_{k+m} - g_k\|_{V_2}^2 \leqslant 4\sum_{n=k+1}^{k+m} \|(g_n - g_{n-1})\|_{V_2}^2$, for all $k \geqslant 0$, and so by (P4) we get that

$$\|g_{k+m} - g_k\|_{V_2}^2 \leqslant 64 \sum_{n=k+1}^{k+m} \bigl|g(t_n) - g\bigl(t_{n-1}^-\bigr)\bigr|^2 + \sum_{n=k+1}^{k+m} \varepsilon/2^n \tag{1}$$

for all $k \geqslant 0$ and all $m \geqslant 1$. By Fact 14 and (1) we get that $(g_n)_n$ is a Cauchy sequence in V_2. Let $\phi = \lim_n g_n$. Since $\|g_n - \phi\|_\infty \leqslant \|g_n - \phi\|_{V_2}$ we have that $g_n \xrightarrow{\|\cdot\|_\infty} \phi$ and so by (P3), we see that ϕ is continuous. Moreover setting $k = 0$ in (1) and taking limits the conclusion of the lemma follows.

Let us pass now to the construction. Assume that for some $n \geqslant 1$, $g_0, \ldots, g_{n-1}$ and $I_0, \ldots, I_{n-1}$ have been defined so that $g_0 = g$, $I_0 = \varnothing$ and (P1)–(P4) are satisfied. If $t_n \in \bigcup_{k=0}^{n-1} I_k$ then we set $I_n = I_{n-1}$ and $g_n = g_{n-1}$ and so (P1)–(P4) trivially hold. Otherwise by Lemma 18 and Fact 14, we may select $\delta_n > 0$ such that the following hold.

(a) $[t_n - \delta_n, t_n] \cap (\bigcup_{k=0}^{n-1} I_k) = \varnothing$.
(b) For every finite subset $\mathcal{P}$ of $[t_n - \delta_n, t_n)$, $\alpha_2(g, \mathcal{P}) \leqslant \sqrt{\varepsilon/2^{n+3}}$.
(c) g is continuous at $t_n - \delta_n$ and $|g(t) - g(t_n)| \leqslant 2|g(t_n) - g(t_n^-)|$, for all $t \in [t_n - \delta_n, t_n)$.

We set $I_n = [t_n - \delta_n, t_n]$. We define $g_n : [0, 1] \to \mathbb{R}$ to be linear on I_n and equal to g_{n-1} on $I \setminus \mathring{I}_n$.

It is easy to see that I_n and g_n satisfy (P1)–(P3). To verify (P4) choose a subset $\mathcal{P} = \{s_0 < \cdots < s_p\}$ of $[0, 1]$. Since $g_n(t) - g_{n-1}(t) = 0$ for every $t \notin (t_n - \delta_n, t_n)$, in order to estimate $\alpha_2(g_n - g_{n-1}, \mathcal{P})$ we may assume that $s_0 = t_n - \delta_n$, and $s_p = t_n$. Since $g_n|_{I_n}$ is a linear function, by (c) above we obtain that

$$\sum_{i=0}^{p-1} \bigl|g_n(s_{i+1}) - g_n(s_i)\bigr|^2 \leqslant 4\bigl|g(t_n) - g\bigl(t_n^-\bigr)\bigr|^2. \tag{2}$$

Moreover by (b) and (c) we get that

$$\sum_{i=0}^{p-1} \bigl|g(s_{i+1}) - g(s_i)\bigr|^2 \leqslant \frac{\varepsilon}{2^{n+3}} + \bigl|g(t_{n-1}) - g(t_n)\bigr|^2 \leqslant \frac{\varepsilon}{2^{n+3}} + 4\bigl|g(t_n) - g\bigl(t_n^-\bigr)\bigr|^2. \tag{3}$$

Since $g_{n-1}|_{I_n} = g|_{I_n}$, $\alpha_2(g_n - g_{n-1}, \mathcal{P}) = \alpha_2(g_n - g, \mathcal{P})$ and so by (2) and (3) we have that

$$\begin{aligned}\alpha_2(g_n - g_{n-1}, \mathcal{P}) &\leqslant 2\sum_{i=0}^{p-1} \bigl|g_n(s_{i+1}) - g_n(s_i)\bigr|^2 + 2\sum_{i=0}^{p-1} \bigl|g(s_{i+1}) - g(s_i)\bigr|^2 \\ &\leqslant 16\bigl|g(t_n) - g\bigl(t_n^-\bigr)\bigr|^2 + \varepsilon/2^{2n+2},\end{aligned}$$

and so (P4) is satisfied. The proof of the lemma is complete. □

As we have already mentioned for every $r \in V_2$ if $r \in V_2^{rc}$ then r is right continuous. The next lemma states that the converse is also true.

Lemma 20. *Let $r \in V_2$ be right continuous. Then $r \in V_2^{rc}$.*

Proof. We distinguish two cases.

Case 1. The set $D(r)$ is finite. Then by Lemma 17 for $F = D(r)$ there exist $h \in < \{\chi_{[t,1]}\colon t \in (0,1]\} >$ and $g \in V_2^c$ such that $r = g + h$. Hence $r \in V_2^{rc}$.

Case 2. The set $D(r)$ is infinite. Then by Fact 14, $\sum_{t\in D(r)} |r(t) - r(t^-)|^2 < \infty$. Therefore for every $\varepsilon > 0$ there exists a finite $F \subseteq D(r)$, such that $\sum_{t\in D(r)\setminus F} |r(t) - r(t^-)|^2 < \varepsilon/128$. By Lemma 17 there exists $h \in \langle\{\chi_{[t,1]}\colon t \in (0,1]\}\rangle$ such that $g = r - h$ is a right continuous function in V_2 and $\sum_{t\in D(g)} |g(t) - g(t^-)|^2 = \sum_{t\in D(r)\setminus F} |r(t) - r(t^-)|^2$. Finally by Lemma 19 there exists $\phi \in V_2^c$ such that $\|g - \phi\|_{V_2} \leqslant 64 \sum_{t\in D(g)} |g(t) - g(t^-)|^2 + \varepsilon/2$. Hence $\|r - (\phi + h)\|_{V_2} = \|g - \phi\|_{V_2} < \varepsilon$ and so $r \in \overline{V_2^c + \langle\{\chi_{[t,1]}\colon t \in (0,1]\}\rangle} = V_2^{rc}$. □

Proof of Theorem 15. First notice that $V_2^{rc} \cap V_2^d = \{0\}$. Moreover by Lemma 16 and Lemma 20 for every $f \in V_2$ there exist $H \in V_2^d$ with $\|H\|_{V_2} \leqslant 2\|f\|_{V_2}$ and $r \in V_2^{rc}$ such that $f = r + H$. Hence $V_2 = V_2^{rc} \oplus V_2^d$.

We close by showing that V_2^{rc} is not isomorphic to the direct sum of V_2^c and

$$\overline{\langle\{\chi_{[t,1]}\colon 0 < t \leqslant 1\}\rangle}. \qquad \square$$

Proposition 21. *The sum $V_2^c + \overline{\langle\{\chi_{[t,1]}\colon 0 < t \leqslant 1\}\rangle}$ is not closed in V_2.*

Proof. It is clear that $V_2^c \cap \overline{\langle\{\chi_{[t,1]}\colon 0 < t \leqslant 1\}\rangle} = \{0\}$. So it suffices to show that the distance of the unit spheres of V_2^c and $\overline{\langle\{\chi_{[t,1]}\colon 0 < t \leqslant 1\}\rangle}$ is zero. Let $\varepsilon > 0$ and $n \geqslant 1$ be such that $\frac{2}{\sqrt{n}} < \varepsilon$. Choose $0 < t_1 < \cdots < t_n < 1$ and set $f = \sum_{i=1}^n \frac{1}{n}\chi_{[t_i,1]}$. Let $\delta > 0$ be such that $t_1 - \delta > 0$ and for every $1 \leqslant i \leqslant n-1$, $[t_i - \delta, t_i] \cap [t_{i+1} - \delta, t_{i+1}] = \varnothing$. Now define $g : [0,1] \to \mathbb{R}$ as follows.

(1) $g(t) = f(t)$ if $t \notin \bigcup_{i=1}^n (t_i - \delta, t_i)$.
(2) For every $1 \leqslant i \leqslant n-1$ and every $t \in [t_i - \delta, t_i]$ $g(t) = f(t_i) + \lambda_i(t - t_i)$, where $\lambda_i = (f(t_i) - f(t_i - \delta))/\delta$.

Clearly g is continuous. It is easy to check that $\|f\|_{V_2} = \|g\|_{V_2} = 1$ and that $\|g - f\|_{V_2} \leqslant 2(\sum_{i=1}^n \frac{1}{n^2})^{1/2} = \frac{2}{\sqrt{n}} < \varepsilon$. □

Remark 1. (a) Notice that the function g above belongs to V_2^0. Therefore by the same proof we obtain that the sum $V_2^0 + \overline{\langle\{\chi_{[t,1]}\colon 0 < t \leqslant 1\}\rangle}$ is also not closed in V_2.

(b) Let us also note that the natural embedding of V_2^0 into V_2 is not complemented. This follows from the fact that c_0 is isomorphic to a subspace of V_2^0 and the observation that the canonical embedding of any separable Banach space X containing c_0 into X^{**} is not complemented.

2.3. On subspaces of V_2^0 with non-separable dual

In this subsection we show that for every subspace X of V_2^0 with non-separable dual, the space $X^{**} \cap V_2^d$ is isomorphic to $\ell_2(0,1)$.

Lemma 22. *Let $\mathcal{G}$ be a bounded and uncountable subset of V_2 such that the constant zero function is the unique w^*-accumulation point of $\mathcal{G}$. Then the set $\mathcal{G} \setminus V_2^d$ is at most countable.*

Proof. Let $\mathcal{G} = \{G_\tau \colon \tau \in T\}$ be an enumeration of $\mathcal{G}$ where T is an uncountable set. By Theorem 15, for every $\tau \in T$ there exist $g_\tau \in V_2^{rc}$ and $H_\tau \in V_2^d$ such that $G_\tau = g_\tau + H_\tau$. Hence $\mathcal{G} \setminus V_2^d = \{G_\tau \colon \tau \in T_0\}$, where $T_0 = \{\tau \in T \colon g_\tau \neq 0\}$.

Suppose that T_0 is uncountable and for every $\tau \in T_0$ choose $t_\tau \in (0,1]$ such that $G_\tau(t_\tau) \neq 0$. Our assumption for the family $\mathcal{G}$ yields that for every $t \in [0,1]$ and every $\varepsilon > 0$ the set $\{\tau \in T \colon |G_\tau(t)| \geqslant \varepsilon\}$ is finite and so the set $\{\tau \in T \colon G_\tau(t) \neq 0\}$ is at most countable. Hence we may suppose that for all $\tau \in T_0$, $G_\tau(1) = 0$. As $1 \notin \operatorname{supp} H$ for all $H \in V_2^d$, we have that for every $\tau \in T_0$, $0 = G_\tau(1) = g_\tau(1)$ and so $t_\tau \in (0,1)$.

Using the right continuity of g_τ and standard cardinality arguments, we conclude that there exist a closed interval $[a,b]$ with $0 < a < b < 1$, a positive real number $\varepsilon > 0$ and an uncountable subset Γ_1 of T_0, such that for every $t \in [a,b]$ and every $\tau \in T_1$, $|g_\tau(t)| > \varepsilon$. Now let A be an infinite countable subset of T_1. Since the support of each H_τ is at most countable we can choose $t_0 \in [a,b] \setminus \bigcup_{\tau \in A} \operatorname{supp} H_\tau$. But then $|G_\tau(t_0)| = |g_\tau(t_0)| > \varepsilon$ for all $\tau \in A$ which is a contradiction. Therefore the set T_0 is countable and the proof is complete. □

Proposition 23. *Let X be a subspace of V_2^0 with non-separable dual. Then there exists a normalized family $\mathcal{H} \subseteq V_2^d \cap X^{**}$ equivalent to the standard basis of $\ell_2(0,1)$. Therefore $X^{**} \cap V_2^d$ is isomorphic to $\ell_2(\mathfrak{c})$.*

Proof. We have that X is a separable Banach space with non-separable dual and ℓ_1 does not embed in X. Hence by Theorem 8 the unit ball of X^{**} contains an 1-unconditional family $\mathcal{G}$ of size of the continuum such that $0 \in V_2$ is the unique w^*-accumulation point of $\mathcal{G}$. We set $\mathcal{H} = \{G/\|G\|_{V_2} \colon G \in \mathcal{G} \cap V_2^d\}$. Then $\mathcal{H}$ is a normalized unconditional family in $V_2^d \cap X^{**}$ and by Lemma 22 we get that $|\mathcal{H}| = \mathfrak{c}$. Since V_2^d is isomorphic to $\ell_2(0,1)$, $\mathcal{H}$ is equivalent to the standard basis of $\ell_2(0,1)$ and so $X^{**} \cap V_2^d$ is isomorphic to $\ell_2(\mathfrak{c})$. □

3. Tree families and the space *TF*

In this section we define tree families of functions with coefficients and we introduce the space *TF*. The relation of the later space with tree families is also investigated.

3.1. Tree families of functions with coefficients

In this subsection we introduce the concept of the tree family of functions with coefficients in V_2^0 extending Definition 9 in the introduction.

Definition 24. For every $\alpha \in \mathcal{D}$, let f_α be a function in V_2^0, (I_α, J_α) be a pair of closed intervals in $[0,1]$ and λ_α be a real number.

The family $\mathcal{F} = (f_\alpha, (I_\alpha, J_\alpha), \lambda_\alpha)_{\alpha\in\mathcal{D}}$ will be called a tree family of functions with coefficients if the following are satisfied.

(TF1) $\sup_{\alpha\in\mathcal{D}} \|f_\alpha\|_{V_2^0} < +\infty$.
(TF2) For every $\alpha \in \mathcal{D}$, $J_\alpha \subseteq \mathring{I}_\alpha$, $I_{\alpha^\frown 0} \cup I_{\alpha^\frown 1} \subseteq \mathring{J}_\alpha$ and $I_{\alpha^\frown 0} \cap I_{\alpha^\frown 1} = \varnothing$.
(TF3) For every $\alpha \in \mathcal{D}$, $f_\alpha|_{J_\alpha} = 1$ and $\operatorname{supp} f_\alpha \subseteq \mathring{I}_\alpha$.
(TF4) $\lambda_\varnothing \neq 0$ and $|\lambda_{\alpha^\frown\varepsilon} - \lambda_\alpha| < |\lambda_\alpha|/2^{2|\alpha|+2}$, for all $\alpha \in \mathcal{D}$ and every $\varepsilon \in \{0, 1\}$.

By (TF4) it can be easily shown that for every $\alpha \sqsubset \beta$,

$$\frac{|\lambda_\varnothing|}{2} < |\lambda_\alpha| < 2|\lambda_\varnothing| \quad \text{and} \quad |\lambda_\beta - \lambda_\alpha| < \frac{|\lambda_\alpha|}{2^{2|\alpha|+1}} < \frac{|\lambda_\varnothing|}{2^{2|\alpha|}}. \tag{4}$$

Also by (TF2) we have that for every $\alpha, \beta \in \mathcal{D}$,

$$\alpha \sqsubset \beta \quad \Leftrightarrow \quad I_\beta \subseteq \mathring{J}_\alpha \subseteq I_\alpha \quad \text{and} \quad \alpha \perp \beta \quad \Leftrightarrow \quad I_\alpha \cap I_\beta = \varnothing. \tag{5}$$

A tree family $\mathcal{F} = (f_\alpha, (I_\alpha, J_\alpha), \lambda_\alpha)_{\alpha\in\mathcal{D}}$ will be called (a) *increasing* if for every $\alpha \in \mathcal{D}$, $\max I_{\alpha^\frown 0} < \min I_{\alpha^\frown 1}$, (b) *decreasing* if for every $\alpha \in \mathcal{D}$, $\max I_{\alpha^\frown 1} < \min I_{\alpha^\frown 0}$ and (c) *monotone* if it is either increasing or decreasing. It is clear that every tree family can be reordered so as to become a monotone tree family. Moreover note that if $\mathcal{F}$ is increasing (respectively decreasing) then for every $\alpha \perp \beta$, $\alpha <_{\mathrm{lex}} \beta$ if and only if $\max I_\alpha < \min I_\beta$ (respectively $\max I_\beta < \min I_\alpha$) in $[0, 1]$. It is also convenient to use the following notation for monotone tree families. We will say that a tree family is 0-*monotone* if it is increasing and 1-*monotone* if it is decreasing. Similarly for a pair (x, y) of real numbers we will write $x <_0 y$ if $x < y$ and $x <_1 y$ if $y < x$.

In the case $\lambda_\alpha = 1$, for all $\alpha \in \mathcal{D}$ the tree family of functions will be denoted simply by $\mathcal{F} = (f_\alpha, (I_\alpha, J_\alpha))_{\alpha\in\mathcal{D}}$ and the above definition coincides with Definition 9. In the sequel tree families of functions with coefficients will be also called tree families letting the triplet $\mathcal{F} = (f_\alpha, (I_\alpha, J_\alpha), \lambda_\alpha)_{\alpha\in\mathcal{D}}$ to give the precise meaning.

3.2. The space TF

In this subsection we will define the space *TF* as the completion of $c_{00}(\mathcal{D})$ under a certain norm. First let us make the following definition.

Definition 25. Let $\mathcal{I}$, $\mathcal{I}_1, \mathcal{I}_2$ be non-empty finite segments of $\mathcal{D}$. Let $\alpha = \min\mathcal{I}$, $\beta = \max\mathcal{I}$, $\alpha_1 = \min\mathcal{I}_1$ and $\alpha_2 = \min\mathcal{I}_2$. We say that $\mathcal{I}$ separates $\mathcal{I}_1$ and $\mathcal{I}_2$ if the following are satisfied.

(i) $\beta \perp \alpha_1$, $\beta \perp \alpha_2$ and either $\alpha_1 <_{\mathrm{lex}} \beta <_{\mathrm{lex}} \alpha_2$ or $\alpha_2 <_{\mathrm{lex}} \beta <_{\mathrm{lex}} \alpha_1$.
(ii) $\alpha \sqsubseteq \alpha_1 \wedge \alpha_2$.

A family $\mathcal{S}$ of finite segments of $\mathcal{D}$ will be called a (ns)-family if for every $\mathcal{I}, \mathcal{I}_1, \mathcal{I}_2$ in $\mathcal{S}$, $\mathcal{I}$ does not separate $\mathcal{I}_1$ and $\mathcal{I}_2$.

Let $(e_\alpha)_{\alpha\in\mathcal{D}}$ be the algebraic basis of $c_{00}(\mathcal{D})$. For any segment $\mathcal{I}$ of $\mathcal{D}$, define $I^*: c_{00}(\mathcal{D}) \to \mathbb{R}$ by $\mathcal{I}^*(x) = \sum_{\alpha\in\mathcal{I}} \mu_\alpha$, for every $x = \sum_{\alpha\in\mathcal{D}} \mu_a e_\alpha \in c_{00}(\mathcal{D})$. Then

$$\|x\|_{TF} = \sup\left(\sum_{i=1}^{k} \big(\mathcal{I}_i^*(x)\big)^2\right)^{1/2}$$

where the supremum is taken over all finite (ns)-families $\{\mathcal{I}_i\}_{i=1}^k$ consisting of finite and pairwise disjoint segments of $\mathcal{D}$.

Observe that for every finite subset S of $\mathcal{D}$, the family $\mathcal{S} = \{\{\alpha\}: \alpha \in S\}$ is a (ns)-family of pairwise disjoint segments of $\mathcal{D}$. Hence for every $(\mu_\alpha)_{\alpha\in\mathcal{D}} \in c_{00}(\mathcal{D})$,

$$\left(\sum_{\alpha\in\mathcal{D}} \mu_\alpha^2\right)^{1/2} \leqslant \left\|\sum_{\alpha\in\mathcal{D}} \mu_\alpha e_\alpha\right\|_{TF}. \tag{6}$$

Let $\mathcal{S}$ be a (ns)-family of finite segments of $\mathcal{D}$ and $\mathcal{S}'$ be a family of segments such that for every $\mathcal{I}' \in \mathcal{S}'$ there is $\mathcal{I} \in \mathcal{S}$ with $\mathcal{I}' \subseteq \mathcal{I}$. Then it is easily shown that $\mathcal{S}'$ is also a (ns)-family. So we have the following.

(1) The set $\{e_\alpha\}_{\alpha\in\mathcal{D}}$, under the natural ordering of $\mathcal{D}$, is a bimonotone Schauder basis for *TF*.
(2) For every (ns)-family $\{\mathcal{I}_i\}_{i=1}^k$ of pairwise disjoint finite segments of $\mathcal{D}$ and every $x = \sum_{\alpha\in\mathcal{D}} \mu_\alpha e_\alpha \in c_{00}(\mathcal{D})$,

$$\sum_{i=1}^{k} \left\|\sum_{\alpha\in\mathcal{I}_i} \mu_\alpha e_\alpha\right\|_{TF}^2 \leqslant \left\|\sum_{\alpha\in\mathcal{D}} \mu_\alpha e_\alpha\right\|_{TF}^2. \tag{7}$$

Moreover let $s : \mathcal{D} \to \mathcal{D}$ be the *mirror map* defined as follows. For any $\alpha \in \mathcal{D}$, if $\alpha = \varnothing$ then $s(\varnothing) = \varnothing$ and if $\alpha \in \{0,1\}^n$ for some $n \geqslant 1$, then $s(\alpha)$ is the unique element of $\mathcal{D}$ such that $s(\alpha) \in \{0,1\}^n$ and $s(\alpha)(i) = 1$ if and only if $\alpha(i) = 0$, for all $1 \leqslant i \leqslant n$. Notice that for every (ns)-family $\mathcal{S}$, the family $s(\mathcal{S}) = \{s(\mathcal{I}): \mathcal{I} \in \mathcal{S}\}$ is also a (ns)-family. This easily yields that

$$\left\|\sum_{\alpha\in\mathcal{D}} \mu_{s(\alpha)} e_\alpha\right\|_{TF} = \left\|\sum_{\alpha\in\mathcal{D}} \mu_\alpha e_\alpha\right\|_{TF} \tag{8}$$

for every $(\mu_\alpha)_{\alpha\in\mathcal{D}} \in c_{00}(\mathcal{D})$.

Remark 2. Let $\phi: \mathcal{D} \to \mathcal{D}$ be an one to one and onto map which preserves $\sqsubseteq$, that is $\alpha \sqsubseteq \beta$ if and only if $\phi(\alpha) \sqsubseteq \phi(\beta)$. It is obvious that ϕ preserves disjoint families of segments and thus any such ϕ induces an isometry of the space *JT* onto itself. In the case of *TF* this is not in general true, since the arbitrary such a ϕ does not preserve the (ns)-property of disjoint families. As we have already mentioned for ϕ the identity or the mirror map, the later property remains valid and thus both induce isometries of *TF*. This observation will be used in the sequel.

The following theorem shows that the space *TF* shares with the classical James Tree space similar properties.

Theorem 26.

(1) *For any infinite antichain* $(\alpha_n)_{n=1}^{\infty}$ *of* $\mathcal{D}$ *the sequence* $(e_{\alpha_n})_n$ *is equivalent to the usual basis of* ℓ_2.
(2) *For any infinite chain* $(\alpha_n)_{n=1}^{\infty}$ *of* $\mathcal{D}$ *the sequence* $(e_{\alpha_n})_n$ *is equivalent to the basis of the James quasi-reflexive space.*
(3) *The space TF does not contain an isomorphic copy of* ℓ_1 *and the dual of TF is non-separable.*

Parts (1) and (2) of the above theorem are easily obtained by the definition of the norm of *TF*. The last part can be shown as the corresponding statement in *JT* (the fact that *TF* does not contain a copy of ℓ_1 is also a consequence of Theorem 30 below).

The structure of *TF* is richer than that of *JT*. This is described in Corollary 29. We start by recalling that the stopping time space S^2 is the completion of $c_{00}(\mathcal{D})$ under the norm $\|x\|_{S^2} = \sup(\sum_{\alpha\in A} x(\alpha)^2)^{1/2}$ where the supremum is taken over all antichains A of $\mathcal{D}$. Related to the space S^2 we have the following unpublished result which is due to H. Rosenthal and G. Schechtman.

Theorem 27. *The space* S^2 *contains isomorphs of* c_0 *and* ℓ_p *for* $2 \leqslant p < \infty$.

A proof of the corresponding result for S^1, i.e. S^1 contains c_0 and ℓ_p for $1 \leqslant p < \infty$, is included in N. Dew's PhD thesis [6]. Also S. Buechler and E. Odell in [5] have shown that every subspace of V_2^0 generated by an increasing tree family $(f_\alpha, (I_\alpha, J_\alpha))_{\alpha\in\mathcal{D}}$, contains c_0. A slight modification of their delicate method yields the following.

Theorem 28. *The space* S^2 *embeds into TF.*

By Theorems 27 and 28 we obtain the following.

Corollary 29. *The space TF contains isomorphs of* c_0 *and* ℓ_p *for* $2 \leqslant p < \infty$.

3.3. The embedding of TF in V_2^0

Theorem 30. *Let* $\mathcal{F} = (f_\alpha, (I_\alpha, J_\alpha))_{\alpha\in\mathcal{D}}$ *be a monotone tree family. Set* $M = \sup_{\alpha\in\mathcal{D}} \|f_\alpha\|_{V_2^0}$ *and* $C = (25M^2 + 48M + 32)^{1/2}$. *Then for every* $n \geqslant 0$ *and every sequence of scalars* $(\mu_\alpha)_{|\alpha|\leqslant n}$, *we have that*

$$\Big\|\sum_{|\alpha|\leqslant n} \mu_\alpha e_\alpha\Big\|_{TF} \leqslant \Big\|\sum_{|\alpha|\leqslant n} \mu_\alpha f_\alpha\Big\|_{V_2^0} \leqslant C\Big\|\sum_{|\alpha|\leqslant n} \mu_\alpha e_\alpha\Big\|_{TF}. \tag{9}$$

The following is an immediate consequence of the above theorem.

Corollary 31. *The space* V_2^0 *contains a copy of TF.*

Notice that it suffices to prove Theorem 30 only in the case of an increasing tree family. Indeed let $\mathcal{F}' = (f'_\alpha, (I'_\alpha, J'_\alpha))_{\alpha\in\mathcal{D}}$ be a decreasing tree family. Set $f_\alpha = f'_{s(\alpha)}$, $I_\alpha = I'_{s(\alpha)}$ and $J_\alpha = J'_{s(\alpha)}$ where s is the mirror map on $\mathcal{D}$ defined in the preceding subsection. Then $\mathcal{F} = (f_\alpha, (I_\alpha, J_\alpha))_{\alpha\in\mathcal{D}}$

is an increasing tree family. Moreover for every $n \geqslant 0$ and every sequence of scalars $(\mu_\alpha)_{|\alpha|\leqslant n}$, we have that

$$\Big\| \sum_{|\alpha|\leqslant n} \mu_\alpha f'_\alpha \Big\|_{V_2^0} = \Big\| \sum_{|\alpha|\leqslant n} \mu_{s(\alpha)} f'_{s(\alpha)} \Big\|_{V_2^0} = \Big\| \sum_{|\alpha|\leqslant n} \mu_{s(\alpha)} f_\alpha \Big\|_{V_2^0}. \tag{10}$$

Suppose now that Theorem 30 holds for every increasing tree family. Then by (9) and (10) we have that

$$\Big\| \sum_{|\alpha|\leqslant n} \mu_{s(\alpha)} e_\alpha \Big\|_{TF} \leqslant \Big\| \sum_{|\alpha|\leqslant n} \mu_\alpha f'_\alpha \Big\|_{V_2^0} \leqslant C \Big\| \sum_{|\alpha|\leqslant n} \mu_{s(\alpha)} e_\alpha \Big\|_{TF}$$

and using (8) the result follows.

Fix for the sequel an increasing tree family $\mathcal{F} = (f_\alpha, (I_\alpha, J_\alpha))_{\alpha\in\mathcal{D}}$, an integer $n \geqslant 0$ and a sequence of scalars $(\mu_\alpha)_{|\alpha|\leqslant n}$. Theorem 30 will follow by Lemmas 32 and 33 stated below.

Lemma 32. $\| \sum_{|\alpha|\leqslant n} \mu_\alpha e_\alpha \|_{TF} \leqslant \| \sum_{|\alpha|\leqslant n} \mu_\alpha f_\alpha \|_{V_2^0}$.

Proof. Let $I_\alpha = [l_0^\alpha, l_1^\alpha]$ and $J_\alpha = [m_0^\alpha, m_1^\alpha]$. By the definition of the tree family, for every $\alpha, \beta \in \mathcal{D}$ and for every $\varepsilon \in \{0, 1\}$ we have that

(a) $f_\alpha(l_\varepsilon^\beta) \in \{0, 1\}$ and $f_\alpha(l_\varepsilon^\beta) = 1 \Leftrightarrow \alpha \sqsubset \beta$.
(b) $f_\alpha(m_\varepsilon^\beta) \in \{0, 1\}$ and $f_\alpha(m_\varepsilon^\beta) = 1 \Leftrightarrow \alpha \sqsubseteq \beta$.

Let $f = \sum_{|\alpha|\leqslant n} \mu_\alpha f_\alpha$ and $\tilde{f} = \sum_{|\alpha|\leqslant n} \mu_\alpha e_\alpha$.

Let $\mathcal{S} = \{\mathcal{I}_1, \dots, \mathcal{I}_k\}$ be a (ns)-family of pairwise disjoint finite segments of $\mathcal{D}$ such that $\|\tilde{f}\|_{TF}^2 = \sum_{i=1}^k (\mathcal{I}_i^*(\tilde{f}))^2$. Setting $\alpha_i = \min \mathcal{I}_i$ and $\beta_i = \max \mathcal{I}_i$, by (a) and (b) above we get that for any $\varepsilon \in \{0, 1\}$ and any $1 \leqslant i \leqslant k$,

$$\mathcal{I}_i^*(\tilde{f}) = \sum_{\alpha_i \sqsubseteq \alpha \sqsubseteq \beta_i} \mu_\alpha = f\big(m_\varepsilon^{\beta_i}\big) - f\big(l_\varepsilon^{\alpha_i}\big). \tag{11}$$

For each $1 \leqslant i \leqslant k$ we will associate an $\varepsilon_i \in \{0, 1\}$ as follows. Let

$$[k]_i^1 = \big\{ j \in \{1, \dots, k\}\colon\ \beta_i \perp \alpha_j,\ \beta_i <_{\mathrm{lex}} \alpha_j \text{ and } \alpha_i \sqsubset \alpha_j \big\}$$

and

$$[k]_i^0 = \big\{ j \in \{1, \dots, k\}\colon\ \beta_i \perp \alpha_j,\ \alpha_j <_{\mathrm{lex}} \beta_i \text{ and } \alpha_i \sqsubset \alpha_j \big\}.$$

Since the family $\mathcal{S}$ is a (ns)-family, we have that for any $1 \leqslant i \leqslant k$ and any $\varepsilon \in \{0, 1\}$, if $[k]_i^\varepsilon \neq \varnothing$ then $[k]_i^{1-\varepsilon} = \varnothing$. Now we set $\varepsilon_i = 1 - \varepsilon$ if $[k]_i^\varepsilon \neq \varnothing$ and $\varepsilon_i = 0$ if $[k]_i^0 = [k]_i^1 = \varnothing$. Let also I^i be the interval of $[0, 1]$ with endpoints $l_{\varepsilon_i}^{\alpha_i}$ and $m_{\varepsilon_i}^{\beta_i}$. Since $\alpha_i \sqsubseteq \beta_i$, by (5) we have that $l_0^{\alpha_i} < m_0^{\beta_i}$ and $m_1^{\beta_i} < l_1^{\alpha_i}$ and so if $\varepsilon_i = 0$ then $I^i = [l_0^{\alpha_i}, m_0^{\beta_i}]$ while if $\varepsilon_i = 1$ then $I^i = [m_1^{\beta_i}, l_1^{\alpha_i}]$. Notice that in both cases $I^i \subseteq I_{\alpha_i}$.

Claim 1. *For every $i, j \in \{1, \dots, k\}$ with $i \neq j$, $I^i \cap I^j = \varnothing$.*

Given the above claim and by reordering we may assume that for any $1 \leqslant i < j \leqslant k$, $\max I^i < \min I^j$. So setting $\mathcal{P} = \{t_0 < \cdots < t_{2k-1}\} = \bigcup_{i=1}^k \{\min I^i, \max I^i\}$, we have that for every $0 \leqslant i \leqslant k-1$, $I^{i+1} = [t_{2i}, t_{2i+1}]$.

Therefore by (11) we have that

$$\|\tilde{f}\|_{TF} = \left\{ \sum_{i=0}^{k-1} \bigl(f(t_{2i+1}) - f(t_{2i})\bigr)^2 \right\}^{1/2} \leqslant \alpha_2(f, \mathcal{P}) \leqslant \|f\|_{V_2^0}. \tag{12}$$

So it remains to prove the claim. To this end let $i \neq j$ in $\{1, \dots, k\}$. We distinguish the following cases.

Case 1. $\alpha_i \perp \alpha_j$. Then as $I^i \subseteq I_{\alpha_i}$ and $I^j \subseteq I_{\alpha_j}$ the result follows by (5).

Case 2. α_i, α_j are comparable. Then without loss of generality we may suppose that $\alpha_i \sqsubset \alpha_j$. We have the following subcases.

Subcase 2.1. β_i, α_j are also comparable. Then as the segments $\mathcal{I}_i$ are pairwise disjoint and $\alpha_i \sqsubset \alpha_j$, we must have $\beta_i \sqsubset \alpha_j$.

Hence either $\beta_i^\frown 0 \sqsubseteq \alpha_j$ or $\beta_i^\frown 1 \sqsubseteq \alpha_j$. If $\beta_i^\frown 0 \sqsubseteq \alpha_j$ then by (5) we have that $l_0^{\alpha_i} < m_0^{\beta_i} < l_0^{\beta_i^\frown 0} < l_1^{\beta_i^\frown 0} < m_1^{\beta_i} < l_1^{\beta_i}$ and $I^j \subseteq I_{\beta_i^\frown 0} = [l_0^{\beta_i^\frown 0}, l_1^{\beta_i^\frown 0}]$. If $\varepsilon_i = 0$ then $I^i = [l_0^{\alpha_i}, m_0^{\beta_i}]$ and so $I^i \cap I^j \subseteq [l_0^{\alpha_i}, m_0^{\beta_i}] \cap [l_0^{\beta_i^\frown 0}, l_1^{\beta_i^\frown 0}] = \varnothing$. If $\varepsilon_i = 1$ then $I^i = [m_1^{\beta_i}, l_1^{\alpha_i}]$ and again $I^i \cap I^j \subseteq [m_1^{\beta_i}, l_1^{\alpha_i}] \cap [l_0^{\beta_i^\frown 0}, l_1^{\beta_i^\frown 0}] = \varnothing$. The case $\beta_i^\frown 1 \sqsubseteq \alpha_j$ is similarly treated.

Subcase 2.2. $\beta_i \perp \alpha_j$. Then either $\beta_i <_{\text{lex}} \alpha_j$ or $\alpha_j <_{\text{lex}} \beta_i$. Suppose that $\beta_i <_{\text{lex}} \alpha_j$ and let $\delta = \beta_i \wedge \alpha_j$. Then $\delta^\frown 0 \sqsubseteq \beta_i$, $\delta^\frown 1 \sqsubseteq \alpha_j$ and $j \in [k]_i^1$. Hence $\varepsilon_i = 0$ and $I^i = [l_0^{\alpha_i}, m_0^{\beta_i}]$. Since the tree family is increasing we have that $l_1^{\delta^\frown 0} < l_0^{\delta^\frown 1}$ and so by (5), $l_0^{\alpha_i} < m_0^{\beta_i} < l_1^{\delta^\frown 0} < l_0^{\delta^\frown 1} < l_1^{\delta^\frown 1}$ and $I^j \subseteq I_{\delta^\frown 1} = [l_0^{\delta^\frown 1}, l_1^{\delta^\frown 1}]$. Therefore $I^i \cap I^j \subseteq [l_0^{\alpha_i}, m_0^{\beta_i}] \cap [l_0^{\delta^\frown 1}, l_1^{\delta^\frown 1}] = \varnothing$. The case $\alpha_j <_{\text{lex}} \beta_i$ is similar.

By the above the proof of the claim as well as of Lemma 32 is complete. □

Lemma 33. $\|\sum_{|\alpha| \leqslant n} \mu_\alpha f_\alpha\|_{V_2^0} \leqslant (25M^2 + 48M + 32)^{1/2} \|\sum_{|\alpha| \leqslant n} \mu_\alpha e_\alpha\|_{TF}$.

Proof. Fix for the following a finite subset $\mathcal{P} = \{t_0 < \cdots < t_p\}$ of $I_\varnothing$. For every $0 \leqslant i \leqslant p$, set $\mathcal{I}_i = \{\alpha \in \mathcal{D}\colon |\alpha| \leqslant n \text{ and } t_i \in I_\alpha\}$. It is clear that $\mathcal{I}_i$ is a non-empty initial segment of $\mathcal{D}$ and let $\beta_i = \max \mathcal{I}_i$. Notice that for every $\alpha \in \mathcal{D}$, if $\alpha \sqsubset \beta_i$ then $f_\alpha(t_i) = 1$ and if $\alpha \notin \mathcal{I}_i$, $f_\alpha(t_i) = 0$.

Let $f = \sum_{|\alpha| \leqslant n} \mu_\alpha f_\alpha$ and $\tilde{f} = \sum_{|\alpha| \leqslant n} \mu_\alpha e_\alpha$. Then by the above we have that $f(t_i) = \sum_{|\alpha| \leqslant n} \mu_\alpha f_\alpha(t_i) = \sum_{\alpha \in I_i} \mu_\alpha f_\alpha(t_i) = (\mathcal{I}_i^* + (f_{\beta_i}(t_i) - 1)e_{\beta_i}^*)(\tilde{f})$, for every $0 \leqslant i \leqslant p$, and so setting $r_i = f_{\beta_i}(t_i) - 1$ we get that

$$f(t_{i+1}) - f(t_i) = \bigl(\mathcal{I}_{i+1}^* - \mathcal{I}_i^*\bigr)(\tilde{f}) + \bigl(r_{i+1} e_{\beta_{i+1}}^*(\tilde{f}) - r_i e_{\beta_i}^*(\tilde{f})\bigr). \tag{13}$$

Let $\mathcal{Q}_0 = \{i \in \{0, \dots, p-1\}\colon \beta_i = \beta_{i+1}\}$ and $\mathcal{Q}_1 = \{0, \dots, p-1\} \setminus \mathcal{Q}_0$. Then

$$\sum_{i=0}^{p-1} \bigl|f(t_{i+1}) - f(t_i)\bigr|^2 = \sum_{i \in \mathcal{Q}_0} \bigl|f(t_{i+1}) - f(t_i)\bigr|^2 + \sum_{i \in \mathcal{Q}_1} \bigl|f(t_{i+1}) - f(t_i)\bigr|^2. \tag{14}$$

Claim 2. $\sum_{i \in \mathcal{Q}_0} |f(t_{i+1}) - f(t_i)|^2 \leqslant M^2 \cdot \|\tilde{f}\|_{TF}^2$.

Proof. Indeed if $\beta_i = \beta_{i+1} = \alpha$ then $\mathcal{I}_i = \mathcal{I}_{i+1}$ and so by (13) we have that $|f(t_{i+1}) - f(t_i)| = |f_\alpha(t_{i+1}) - f_\alpha(t_i)| e_\alpha^*(\tilde{f})$. Therefore

$$\begin{aligned}
\sum_{i \in \mathcal{Q}_0} \bigl|f(t_{i+1}) - f(t_i)\bigr|^2 &= \sum_{|a| \leqslant n} \sum_{\{i \in \mathcal{Q}_0\colon \beta_i = \alpha\}} \bigl|f(t_{i+1}) - f(t_i)\bigr|^2 \\
&\leqslant \sum_{|a| \leqslant n} \sum_{i=0}^{p-1} \bigl|f_a(t_{i+1}) - f_a(t_i)\bigr|^2 \bigl(e_a^*(\tilde{f})\bigr)^2 \\
&\leqslant \sum_{|a| \leqslant n} \|f_a\|_{V_2^0}^2 \bigl(e_\alpha^*(\tilde{f})\bigr)^2 \leqslant M^2 \|\tilde{f}\|_{TF}^2
\end{aligned}$$

and the proof of the claim is complete. □

We continue by giving an upper estimation for $\sum_{i \in \mathcal{Q}_1} |f(t_{i+1}) - f(t_i)|^2$. By (13) we have that

$$\sum_{i \in \mathcal{Q}_1} \bigl|f(t_{i+1}) - f(t_i)\bigr|^2 \leqslant 2 \sum_{i \in \mathcal{Q}_1} \Bigl(\bigl|\bigl(\mathcal{I}_{i+1}^* - \mathcal{I}_i^*\bigr)(\tilde{f})\bigr|^2 + \bigl|r_{i+1} e_{\beta_{i+1}}^*(\tilde{f}) - r_i e_{\beta_i}^*(\tilde{f})\bigr|^2\Bigr). \tag{15}$$

Claim 3. $\sum_{i \in \mathcal{Q}_1} |(\mathcal{I}_{i+1}^* - \mathcal{I}_i^*)(\tilde{f})|^2 \leqslant 4 \|\tilde{f}\|_{TF}^2$.

Proof. For every $i \in \mathcal{Q}_1$ set $\mathcal{I}_{i,i+1} = \mathcal{I}_i \setminus \mathcal{I}_{i+1}$ and $\mathcal{I}_{i+1,i} = \mathcal{I}_{i+1} \setminus \mathcal{I}_i$. It is easy to see that for every $i \in \mathcal{Q}_1$, $\mathcal{I}_{i,i+1}$ and $\mathcal{I}_{i+1,i}$ are segments of $\mathcal{D}$ and that $\mathcal{I}_{i+1}^* - \mathcal{I}_i^* = \mathcal{I}_{i+1,i}^* - \mathcal{I}_{i,i+1}^*$. Hence

$$\sum_{i \in \mathcal{Q}_1} \bigl|\bigl(\mathcal{I}_{i+1}^* - \mathcal{I}_i^*\bigr)(\tilde{f})\bigr|^2 \leqslant 2 \sum_{i \in \mathcal{Q}_1} \bigl(\mathcal{I}_{i,i+1}^*(\tilde{f})\bigr)^2 + 2 \sum_{i \in \mathcal{Q}_1} \bigl(\mathcal{I}_{i+1,i}^*(\tilde{f})\bigr)^2. \tag{16}$$

Notice that the family $\mathcal{S}_1 = \{\mathcal{I}_{i,i+1}\colon i \in \mathcal{Q}_1\}$ consists of pairwise disjoint segments. Indeed assume that there exist $i, j \in \mathcal{Q}_1$ with $i < j$ and $\alpha \in \mathcal{D}$ such that $\alpha \in \mathcal{I}_{i,i+1} \cap \mathcal{I}_{j,j+1}$. Then $\alpha \in \mathcal{I}_i \cap \mathcal{I}_j$ which means that $t_i, t_j \in I_\alpha$. As $t_i < t_{i+1} \leqslant t_j$ we obtain that $t_{i+1} \in I_\alpha$ and so $\alpha \in \mathcal{I}_{i+1}$. But then $\alpha \notin \mathcal{I}_i \setminus \mathcal{I}_{i+1} = \mathcal{I}_{i,i+1}$, which is a contradiction.

We proceed by showing that $\mathcal{S}_1$ is a (ns)-family. So let $i, j, k \in \mathcal{Q}_1$, $\alpha_i = \min \mathcal{I}_{i,i+1}$, $\alpha_j = \min \mathcal{I}_{j,j+1}$ and $\alpha_k = \min \mathcal{I}_{k,k+1}$. Since $\mathcal{I}_{i,i+1} = \mathcal{I}_i \setminus \mathcal{I}_{i+1}$ we have that $\max \mathcal{I}_{i,i+1} = \max \mathcal{I}_i = \beta_i$. Suppose that $\mathcal{I}_{i,i+1}$ separates $\mathcal{I}_{j,j+1}$ and $\mathcal{I}_{k,k+1}$. Then without loss of generality we may assume that $\alpha_j <_{\mathrm{lex}} \beta_i <_{\mathrm{lex}} \alpha_k$. Since the tree family $\mathcal{F} = (f_\alpha, (I_\alpha, J_\alpha))_{\alpha \in \mathcal{D}}$ is increasing we have that $\max I_{\alpha_j} < \min I_{\beta_i} < \max I_{\beta_i} < \min I_{\alpha_k}$. Hence $t_j < t_i < t_{i+1} \leqslant t_k$ in $[0, 1]$ and since $t_j, t_k \in I_{\alpha_j \wedge \alpha_k}$, we conclude that $t_i, t_{i+1} \in I_{\alpha_j \wedge \alpha_k}$. This means that $\alpha_j \wedge \alpha_k \in \mathcal{I}_i \cap \mathcal{I}_{i+1}$ and

so $\alpha_j \wedge \alpha_k \sqsubseteq \beta_i \wedge \beta_{i+1}$. Now since $\mathcal{I}_{i,i+1} = \mathcal{I}_i \setminus (\mathcal{I}_i \cap \mathcal{I}_{i+1}) = \{\alpha \in \mathcal{I}_i\colon \beta_i \wedge \beta_{i+1} \sqsubset \alpha\}$ we have that $\alpha_j \wedge \alpha_k \sqsubset \alpha_i$ a contradiction.

By similar arguments it is shown that the family $\mathcal{S}_2 = \{\mathcal{I}_{i+1,i}\colon i \in \mathcal{Q}_1\}$ is also a (ns)-family of pairwise disjoint finite segments of $\mathcal{D}$. So

$$\sum_{i\in\mathcal{Q}_1} \big(\mathcal{I}^*_{i,i+1}(\tilde{f})\big)^2 \leqslant \|\tilde{f}\|^2_{TF} \quad \text{and} \quad \sum_{i\in\mathcal{Q}_1} \big(\mathcal{I}^*_{i+1,i}(\tilde{f})\big)^2 \leqslant \|\tilde{f}\|^2_{TF}. \tag{17}$$

By (16) and (17) the proof of Claim 3 is complete. □

Claim 4. $\sum_{i\in\mathcal{Q}_1} |r_{i+1}e^*_{\beta_{i+1}}(\tilde{f}) - r_i e^*_{\beta_i}(\tilde{f})|^2 \leqslant 12(1+M)^2 \|\tilde{f}\|^2_{TF}$.

Proof. Notice that

$$\sum_{i\in\mathcal{Q}_1} \big|r_{i+1}e^*_{\beta_{i+1}}(\tilde{f}) - r_i e^*_{\beta_i}(\tilde{f})\big|^2 \leqslant 2\sum_{i\in\mathcal{Q}_1} \big|r_i e^*_{\beta_i}(\tilde{f})\big|^2 + 2\sum_{i\in\mathcal{Q}_1} \big|r_{i+1}e^*_{\beta_{i+1}}(\tilde{f})\big|^2. \tag{18}$$

For every $\alpha \in \mathcal{D}$ set $\mathcal{Q}_1^\alpha = \{i \in \mathcal{Q}_1\colon \beta_i = \alpha\}$. It is clear that

$$\sum_{i\in\mathcal{Q}_1} \big|r_i e^*_{\beta_i}(\tilde{f})\big|^2 \leqslant \sum_{|\alpha|\leqslant n} \sum_{i\in\mathcal{Q}_1^\alpha} \big|r_i e^*_{\beta_i}(\tilde{f})\big|^2 \leqslant (1+M)^2 \sum_{|\alpha|\leqslant n} \sum_{i\in\mathcal{Q}_1^\alpha} \big|e^*_{\beta_i}(\tilde{f})\big|^2.$$

Observe that for every $\alpha \in \mathcal{D}$, $i \in \mathcal{Q}_1^\alpha$ if and only if $t_i \in I_\alpha \setminus (I_{\alpha^\frown 0} \cup I_{\alpha^\frown 1})$ and moreover $I_\alpha \setminus (I_{\alpha^\frown 0} \cup I_{\alpha^\frown 1}) = I^1_\alpha \cup I^2_\alpha \cup I^3_\alpha$, where $I^1_\alpha, I^2_\alpha, I^3_\alpha$ are pairwise disjoint intervals. So if $i, j \in \mathcal{Q}_1^\alpha$ and $i < j$ we have that there exist $\varepsilon_1, \varepsilon_2 \in \{1, 2, 3\}$ such that $t_i \in I^{\varepsilon_1}_\alpha$ and $t_j \in I^{\varepsilon_2}_\alpha$. Notice that $\varepsilon_1 \neq \varepsilon_2$. Indeed otherwise $t_{i+1} \in I^{\varepsilon_1}_\alpha$ and so $\beta_{i+1} = \alpha = \beta_i$, which is a contradiction since $i \in \mathcal{Q}_1$. Hence for every $\alpha \in \mathcal{D}$ with $|\alpha| \leqslant n$, $|\mathcal{Q}_1^\alpha| \leqslant 3$ and so

$$\sum_{i\in\mathcal{Q}_1} \big|r_i e^*_{\beta_i}(\tilde{f})\big|^2 \leqslant 3(1+M)^2 \sum_{|\alpha|\leqslant n} \big|e^*_\alpha(\tilde{f})\big|^2 \leqslant 3(1+M)^2 \|\tilde{f}\|^2_{TF}.$$

By similar arguments we get that $\sum_{i\in\mathcal{Q}_1} |r_{i+1}e^*_{\beta_{i+1}}(\tilde{f})|^2 \leqslant 3(1+M)^2\|\tilde{f}\|^2_{TF}$. Substituting in (18) the proof of Claim 4 is complete. □

By (14), (15) and the above claims we obtain that

$$\sum_{i=0}^{p-1} \big|f(t_{i+1}) - f(t_i)\big|^2 \leqslant \big(25M^2 + 48M + 32\big)\|\tilde{f}\|^2_{TF}.$$

Since the above holds for every finite subset $\mathcal{P} = \{t_0 < \cdots < t_p\}$ of $I_\varnothing$ and $f(t) = 0$ for all $t \in [0,1] \setminus \mathring{I}_\varnothing$, we conclude that $\|f\|_{V_2^0} \leqslant C\|\tilde{f}\|_{TF}$ and the proof of the lemma is complete. □

3.4. Some upper TF estimates

In this subsection we will present some upper *TF* estimates for monotone tree families with a non-constant sequence $(\lambda_\alpha)_{\alpha\in\mathcal{D}}$ which will be used in the next section.

Lemma 34. *Let $(\lambda_\alpha)_{\alpha\in\mathcal{D}}$ be a family of scalars satisfying* (TF4) *of Definition* 24. *Then for every sequence of scalars $(\mu_\alpha)_{\alpha\in\mathcal{D}}$ and every finite segment $\mathcal{I}$ of $\mathcal{D}$ we have that $|\sum_{\alpha\in\mathcal{I}}\mu_\alpha\lambda_\alpha| \leqslant 4|\lambda_\varnothing|\cdot\|\sum_{\alpha\in\mathcal{I}}\mu_\alpha e_\alpha\|_{TF}$.*

Proof. Let $\beta=\max\mathcal{I}$. Then

$$\left|\sum_{\alpha\in\mathcal{I}}\mu_\alpha\lambda_\alpha\right| = \left|\sum_{\alpha\in\mathcal{I}}\mu_\alpha(\lambda_\alpha-\lambda_\beta)+\sum_{\alpha\in\mathcal{I}}\mu_\alpha\lambda_\beta\right| \leqslant \left(\sum_{\alpha\in\mathcal{I}}|\mu_\alpha|^2\right)^{1/2}\cdot\left(\sum_{\alpha\in\mathcal{I}}|\lambda_\alpha-\lambda_\beta|^2\right)^{1/2}+|\lambda_\beta|\left|\sum_{\alpha\in\mathcal{I}}\mu_\alpha\right|.$$

Let $x=\sum_{\alpha\in\mathcal{I}}\mu_\alpha e_\alpha$ and notice that $\{\sum_{\alpha\in\mathcal{I}}|\mu_\alpha|^2\}^{1/2}=\{\sum_{\alpha\in\mathcal{I}}(e^*_\alpha(x))^2\}^{1/2}$ and $|\sum_{\alpha\in\mathcal{I}}\mu_\alpha| = |\mathcal{I}^*(x)|$. Since the families $\{\{\alpha\}\colon \alpha\in\mathcal{I}\}$ and $\{\mathcal{I}\}$ are obviously (ns)-families of pairwise disjoint segments of $\mathcal{D}$ we have that $\|x\|_{TF}$ is an upper bound for $|\sum_{\alpha\in\mathcal{I}}\mu_\alpha|$ and $\{\sum_{\alpha\in\mathcal{I}}|\mu_\alpha|^2\}^{1/2}$. Moreover by (4), $(\sum_{\alpha\in\mathcal{I}}|\lambda_\alpha-\lambda_\beta|^2)^{1/2}\leqslant\sqrt{2}|\lambda_\varnothing|$ and the conclusion follows. □

Lemma 35. *Let $(f_\alpha,(I_\alpha,J_\alpha),\lambda_\alpha)_{\alpha\in\mathcal{D}}$ be a monotone tree family. Then for every $n\in\mathbb{N}$ and every sequence of scalars $(\mu_\alpha)_{\alpha\leqslant n}$ we have that*

$$\left\|\sum_{|\alpha|\leqslant n}\mu_\alpha\lambda_\alpha f_\alpha\right\|_{V_2^0}\leqslant 4C|\lambda_\varnothing|\left\|\sum_{|\alpha|\leqslant n}\mu_\alpha e_\alpha\right\|_{TF}.$$

Proof. By Theorem 30, $\|\sum_{|\alpha|\leqslant n}\mu_\alpha\lambda_\alpha f_\alpha\|_{V_2^0}\leqslant C\|\sum_{|\alpha|\leqslant n}\mu_\alpha\lambda_\alpha e_\alpha\|_{TF}$. Let $\mathcal{F}$ be a (ns)-family of pairwise disjoint segments of $\mathcal{D}$ such that $\|\sum_{|\alpha|\leqslant n}\mu_\alpha\lambda_\alpha e_\alpha\|_{TF}=\{\sum_{\mathcal{I}\in\mathcal{F}}|\sum_{\alpha\in\mathcal{I}}\mu_\alpha\lambda_\alpha|^2\}^{1/2}$ and so by Lemma 34 and (7),

$$\left\|\sum_{|\alpha|\leqslant n}\mu_\alpha\lambda_\alpha f_\alpha\right\|_{V_2^0}\leqslant 4C|\lambda_\varnothing|\left\{\sum_{\mathcal{I}\in\mathcal{F}}\left\|\sum_{\alpha\in\mathcal{I}}\mu_\alpha e_\alpha\right\|_{TF}^2\right\}^{1/2}\leqslant 4C|\lambda_\varnothing|\left\|\sum_{|\alpha|\leqslant n}\mu_\alpha e_\alpha\right\|_{TF}. \qquad \square$$

For the next lemma we need some notation already used in the proof of Theorem 30. For convenience we reproduce it here.

Let $(f_\alpha,(I_\alpha,J_\alpha),\lambda_\alpha)_{\alpha\in\mathcal{D}}$ be a monotone tree family, $\mathcal{P}=\{t_0<\cdots<t_p\}$ be a finite subset of $[0,1]$ and $n\geqslant 0$. For $0\leqslant i\leqslant p$, let $\mathcal{I}_i=\{\alpha\colon |\alpha|\leqslant n \text{ and } t_i\in I_\alpha\}$ and for $0\leqslant i\leqslant p-1$, let $\mathcal{I}_{i,i+1}=\mathcal{I}_i\setminus\mathcal{I}_{i+1}$ and $\mathcal{I}_{i+1,i}=\mathcal{I}_{i+1}\setminus\mathcal{I}_i$. Finally $M=\sup_{\alpha\in\mathcal{D}}\|f_\alpha\|_{V_2^0}$.

Lemma 36. *Under the above notation the following hold.*

(1) *For all $0\leqslant i\leqslant p$, $|\sum_{\alpha\in\mathcal{I}_i}\mu_\alpha\lambda_\alpha f_\alpha(t_i)|\leqslant 2|\lambda_\varnothing|(M+3)\|\sum_{\alpha\in\mathcal{I}_i}\mu_\alpha e_\alpha\|_{TF}$.*

(2) $\sum_{i=0}^{p-1}|\sum_{\alpha\in\mathcal{I}_{i,i+1}}\mu_\alpha\lambda_\alpha f_\alpha(t_i)|^2\leqslant 16|\lambda_\varnothing|^2(M^2+3)\|\sum_{|\alpha|\leqslant n}\mu_\alpha e_\alpha\|_{TF}^2$.

(3) $\sum_{i=0}^{p-1} |\sum_{\alpha\in\mathcal{I}_{i+1,i}} \mu_\alpha\lambda_\alpha f_\alpha(t_{i+1})|^2 \leqslant 16|\lambda_\varnothing|^2(M^2+3)\|\sum_{|\alpha|\leqslant n}\mu_\alpha e_\alpha\|_{TF}^2$.

Proof. (1) Let $0\leqslant i\leqslant p-1$ and let $\beta_i = \max\mathcal{I}_i$. Then $|\sum_{\alpha\in\mathcal{I}_i}\mu_\alpha\lambda_\alpha f_\alpha(t_i)| \leqslant |\sum_{\alpha\in\mathcal{I}_i}\mu_\alpha\lambda_\alpha| + |\lambda_{\beta_i}||\mu_{\beta_i}||f_{\beta_i}(t_i)-1|$. It is clear that $|f_{\beta_i}(t_i)-1|\leqslant M+1$, $|\mu_{\beta_i}|\leqslant \|\sum_{\alpha\in\mathcal{I}_i}\mu_\alpha e_\alpha\|_{TF}$ and by (4), $|\lambda_{\beta_i}|\leqslant 2|\lambda_\varnothing|$.

Hence $|\lambda_{\beta_i}||\mu_{\beta_i}||f_{\beta_i}(t_i)-1| \leqslant 2|\lambda_\varnothing|(M+1)\cdot\|\sum_{\alpha\in\mathcal{I}_i}\mu_\alpha e_\alpha\|_{TF}$. Finally by Lemma 34, $|\sum_{\alpha\in\mathcal{I}_i}\mu_\alpha\lambda_\alpha|\leqslant 4|\lambda_\varnothing|\cdot\|\sum_{\alpha\in\mathcal{I}_i}\mu_\alpha e_\alpha\|_{TF}$ and the result follows.

(2) For $0\leqslant i\leqslant p-1$ let $\beta_i=\max\mathcal{I}_i$, $\mathcal{Q}_1=\{i\in\{0,\dots,p-1\}\colon \beta_i\neq\beta_{i+1}\}$. It is easy to see that $\mathcal{Q}_1=\{i\in\{0,\dots,p-1\}\colon \mathcal{I}_{i,i+1}\neq\varnothing\}$ and $\beta_i=\max\mathcal{I}_{i,i+1}$. Hence

$$\sum_{i=0}^{p-1}\Big|\sum_{\alpha\in\mathcal{I}_{i,i+1}}\mu_\alpha\lambda_\alpha f_\alpha(t_i)\Big|^2 = \sum_{i\in\mathcal{Q}_1}\Big|\sum_{\alpha\in\mathcal{I}_{i,i+1}}\mu_\alpha\lambda_\alpha f_\alpha(t_i)\Big|^2$$

$$\leqslant 2\Big(\sum_{i\in\mathcal{Q}_1}\Big|\sum_{\alpha\in\mathcal{I}_{i,i+1}}\mu_\alpha\lambda_\alpha\Big|^2 + \sum_{i\in\mathcal{Q}_1}|\mu_{\beta_i}\lambda_{\beta_i}|^2\big(f_{\beta_i}(t_i)-1\big)^2\Big). \tag{19}$$

By Lemma 34,

$$\sum_{i\in\mathcal{Q}_1}\Big|\sum_{\alpha\in\mathcal{I}_{i,i+1}}\mu_\alpha\lambda_\alpha\Big|^2 \leqslant 16|\lambda_\varnothing|^2\Big(\sum_{i\in\mathcal{Q}_1}\Big\|\sum_{\alpha\in\mathcal{I}_{i,i+1}}\mu_\alpha e_\alpha\Big\|_{TF}^2\Big). \tag{20}$$

As we have already shown (see the proof of Theorem 30) the family $\{\mathcal{I}_{i,i+1}\colon i\in\mathcal{Q}_1\}$ is a (ns)-family of pairwise disjoint segments of $\mathcal{D}$. Hence using (7),

$$\sum_{i\in\mathcal{Q}_1}\Big\|\sum_{\alpha\in\mathcal{I}_{i,i+1}}\mu_\alpha e_\alpha\Big\|_{TF}^2 \leqslant \Big\|\sum_{|\alpha|\leqslant n}\mu_\alpha e_\alpha\Big\|_{TF}^2. \tag{21}$$

Moreover it is easy to see that

$$\sum_{i\in\mathcal{Q}_1}|\mu_{\beta_i}\lambda_{\beta_i}|^2\big(f_{\beta_i}(t_i)-1\big)^2 \leqslant 8|\lambda_\varnothing|^2\big(M^2+1\big)\Big\|\sum_{|\alpha|\leqslant n}\mu_\alpha e_\alpha\Big\|_{TF}^2. \tag{22}$$

Substituting (20)–(22) in (19) the conclusion follows.

(3) The proof is identical to the above. □

4. Approximation lemmas

In the first subsection we present some auxiliary results concerning condensation points of uncountable subsets of $\mathbb{R}^{\mathbb{N}}$. To illustrate the reason of introducing the concept and proving the results let us point out the following. Assume that S is an uncountable subset of $\mathbb{R}^2$ such that for each $(t,s)\neq(t',s')$ in S we have that $t\neq t'$, $s\neq s'$. Then it is easy to see that there exist two tree families $(I_\alpha)_{\alpha\in\mathcal{D}}$, $(J_\alpha)_{\alpha\in\mathcal{D}}$ of intervals such that for every $\alpha\in\mathcal{D}$, $(I_\alpha\times J_\alpha)\cap S$ is uncountable. However it does not seem obvious that the two families can be chosen in a monotone manner. The aim of this subsection is to provide tools for selecting monotone tree families.

In the second subsection we state and prove a lemma permitting us to lift information from V_2^d into subspaces of V_2^0. This also a key ingredient for constructing tree families.

4.1. On condensation points of uncountable subsets of $\mathbb{R}^{\mathbb{N}}$

Notation 2. Let V be an open subset of $\mathbb{R}$ and let $x_0 \in V$. We set $V^{(x_0,0)} = \{x \in V\colon x \leqslant x_0\}$ and $V^{(x_0,1)} = \{x \in V\colon x \geqslant x_0\}$. Let now $\mathbf{x} = (x_j)_{j=1}^{\infty} \in (0,1)^{\mathbb{N}}$ and let $\mathbf{V} = \prod_{j=1}^{\infty} V_j$ be a basic open nbd of $\mathbf{x}$. Given $k \geqslant 1$ and $\theta = (\theta_j)_{j=1}^{k} \in \{0,1\}^k$, we define the θ-part of $\mathbf{V}$ with respect to $\mathbf{x}$ to be the set $\mathbf{V}^{(\mathbf{x},\theta,k)} = \prod_{j=1}^{k} V_j^{(x_j,\theta_j)} \times \prod_{j=k+1}^{\infty} V_j$. By convention for the empty sequence $\varnothing$, we set $\mathbf{V}^{(\mathbf{x},\varnothing,0)} = \mathbf{V}$.

Definition 37. Let $S \subseteq \mathbb{R}^{\mathbb{N}}$, $k \geqslant 0$ and $\theta \in \{0,1\}^k$. A point $\mathbf{x} \in \mathbb{R}^{\mathbb{N}}$ will be called (θ,k)-condensation point of S if for every basic open nbd $\mathbf{V}$ of $\mathbf{x}$ in $\mathbb{R}^{\mathbb{N}}$, $\mathbf{V}^{(\mathbf{x},\theta,k)} \cap S$ is uncountable.

An uncountable subset S of $\mathbb{R}^{\mathbb{N}}$ will be called (θ,k)-almost condensed if all but countably many points of S are (θ,k)-condensation points of S. In particular S will be called hereditarily (θ,k)-almost condensed if every uncountable subset S' of S is (θ,k)-almost condensed.

Notice that for $k=0$ and $\theta=\varnothing$ a point $\mathbf{x} \in \mathbb{R}^{\mathbb{N}}$ is $(\varnothing,0)$-condensation point of $S \subseteq \mathbb{R}^{\mathbb{N}}$ if and only if $\mathbf{x}$ is a condensation point of S. Hence every uncountable $S \subseteq \mathbb{R}^{\mathbb{N}}$ is hereditarily $(\varnothing,0)$-almost condensed.

Lemma 38. *Let S_0 be an uncountable subset of $\mathbb{R}^{\mathbb{N}}$, $k_0 \geqslant 0$, $\theta_0 \in \{0,1\}^{k_0}$ and suppose that S_0 is hereditarily (θ_0,k_0)-almost condensed. Then for all $k > k_0$ there exist $S \subseteq S_0$ uncountable and $\theta \in \{0,1\}^k$ with $\theta_0 \sqsubset \theta$ such that S is hereditarily (θ,k)-almost condensed. In particular for every $k>0$ there exist $S \subseteq S_0$ and $\theta \in \{0,1\}^k$ such that S is hereditarily (θ,k)-almost condensed.*

Proof. Assume on the contrary that there exists $k > k_0$ such that for every uncountable $S \subseteq S_0$ and every $\theta \in \{0,1\}^k$ with $\theta_0 \sqsubset \theta$, S is not hereditarily (θ,k)-almost condensed. Let $\theta_1,\dots,\theta_d$ be an enumeration of the set $\{\theta \in \{0,1\}^k\colon \theta_0 \sqsubset \theta\}$. By our assumption S_0 is not hereditarily (θ_1,k)-almost condensed. Hence there exists an uncountable subset $S_0' \subseteq S_0$ which is not (θ_1,k)-almost condensed, namely the set $S_1 = \{\mathbf{x} \in S_0'\colon \mathbf{x}$ is not (θ_1,k)-condensation point of $S_0'\}$ is uncountable. Observe that every $\mathbf{x}$ of S_1 is not (θ_1,k)-condensation point of S_1. Continuing in the same way we construct a decreasing sequence $S_0 \supseteq S_1 \supseteq \cdots \supseteq S_d$ of uncountable subsets of S_0 such that for every $\mathbf{x} \in S_i$ and every $1 \leqslant i \leqslant d$, $\mathbf{x}$ is not (θ_i,k)-condensation point of S_i.

Let $S = S_d$. Since $S \subseteq S_i$, every point of S is not (θ_i,k)-condensation point of S for all $1 \leqslant i \leqslant d$. Therefore for every $\mathbf{x} \in S$ and for every $1 \leqslant i \leqslant d$ there exists an open nbd $\mathbf{V}_i$ of $\mathbf{x} \in \mathbb{R}^{\mathbb{N}}$ such that $\mathbf{V}_i^{(\mathbf{x},\theta_i,k)} \cap S$ is countable. Let $\mathbf{V} = \bigcap_{i=1}^d \mathbf{V}_i$. Then $\mathbf{V}$ is an open nbd of $\mathbf{x}$. Notice that $\mathbf{V}^{(\mathbf{x},\theta_0,k_0)} = \bigcup_{i=1}^d \mathbf{V}^{(\mathbf{x},\theta_i,k)} \subseteq \bigcup_{i=1}^d \mathbf{V}_i^{(\mathbf{x},\theta_i,k)}$ and so $\mathbf{V}^{(\mathbf{x},\theta_0,k_0)} \cap S$ is countable. Thus every $\mathbf{x} \in S$ is not (θ_0,k_0)-condensation point of S, a contradiction since S is an uncountable subset of S_0 and S_0 is hereditarily (θ_0,k_0)-almost condensed. $\square$

4.2. Approximations of the elements of V_2^d

Lemma 39. *Let $(g_n)_n$ be a sequence in V_2^0 w^*-converging to a function $H = \sum_{j=1}^{\infty} \lambda_j \chi_{t_j}$ in V_2^d. Let $n_0 \in \mathbb{N}$ and let $\{\Delta_j\colon 1 \leqslant j \leqslant n_0\}$ be a family of disjoint open intervals of $[0,1]$ with $t_j \in \Delta_j$, for all $1 \leqslant j \leqslant n_0$.*

Then for every $\varepsilon > 0$ *there exist a finite convex combination* h *of* $(g_n)_n$, *a family* $\{f_j\}_{j=1}^{n_0}$ *of functions of* V_2^0 *and a family* $\{(I_j, J_j)\}_{j=1}^{n_0}$ *of pairs of closed intervals of* $[0,1]$ *such that the following are satisfied.*

(i) *For every* $1 \leqslant j \leqslant n_0$, $t_j \in \mathring{J}_j \subseteq J_j \subseteq \mathring{I}_j \subseteq I_j \subseteq \Delta_j$, $0 \leqslant f_j \leqslant 1$, $\operatorname{supp} f_j \subset \mathring{I}_j$, $J_j = \{t \in [0,1]\colon f_j(t) = 1\}$, f_j *is piecewise linear and* $\|f_j\|_{V_2^0} = \sqrt{2}$.

(ii) $\|\sum_{j=1}^{n_0} \lambda_j f_j - h\|_{V_2^0} < \varepsilon + 2(\sum_{j>n_0} |\lambda_j|^2)^{1/2}$.

Proof. Let $\varepsilon > 0$. Since V_2^0 contains no copy of ℓ_1, by [12] there exist a sequence $(g'_n)_n$ in V_2^0 such that $g'_n \xrightarrow{w^*} \sum_{j>n_0} \lambda_j \chi_{t_j}$ and $\|g'_n\|_{V_2^0} \leqslant \|\sum_{j>n_0} \lambda_j \chi_{t_j}\|_{V_2^0} \leqslant 2 \cdot (\sum_{j>n_0} |\lambda_j|^2)^{1/2}$.

Let $\delta > 0$ be such that $\{(t_j - \delta, t_j + \delta)\}_{j=1}^{n_0}$ is a family of pairwise disjoint open intervals in $(0,1)$ with $(t_j - \delta, t_j + \delta) \subseteq \Delta_j$.

For each $1 \leqslant j \leqslant n_0$ we define a sequence of trapezoid functions $(f_{(n,j)})_{n\in\mathbb{N}}$ as follows

$$f_{(n,j)}(x) = \begin{cases} 0 & \text{if } x \notin [t_j - \frac{\delta}{2^n}, t_j + \frac{\delta}{2^n}], \\ 1 & \text{if } x \in [t_j - \frac{\delta}{2^{n+1}}, t_j + \frac{\delta}{2^{n+1}}], \\ \frac{2^{n+1}}{\delta}(x - t_j) + 2 & \text{if } x \in [t_j - \frac{\delta}{2^n}, t_j - \frac{\delta}{2^{n+1}}], \\ \frac{2^{n+1}}{\delta}(t_j - x) + 2 & \text{if } x \in [t_j + \frac{\delta}{2^{n+1}}, t_j + \frac{\delta}{2^n}]. \end{cases}$$

Since on bounded subsets of V_2 the pointwise and the weak*-topology coincide, it is clear that $\sum_{j\leqslant n_0} \lambda_j f_{(n,j)} \xrightarrow{w^*} \sum_{j\leqslant n_0} \lambda_j \chi_{t_j}$. Hence

$$\sum_{j\leqslant n_0} \lambda_j f_{(n,j)} - g_n + g'_n \xrightarrow{w} 0$$

and therefore there exist a convex combination $\sum_{n=1}^m r_n(\sum_{j\leqslant n_0} \lambda_j f_{(n,j)} - g_n + g'_n) = \sum_{j\leqslant n_0} \lambda_j(\sum_{n=1}^m r_n f_{(n,j)}) - \sum_{n=1}^m r_n g_n + \sum_{n=1}^m r_n g'_n$ such that

$$\left\| \sum_{j\leqslant n_0} \lambda_j \left(\sum_{n=1}^m r_n f_{(n,j)} \right) - \sum_{n=1}^m r_n g_n + \sum_{n=1}^m r_n g'_n \right\|_{V_2^0} < \varepsilon. \tag{23}$$

We set $h = \sum_{n=1}^m r_n g_n$ and for $1 \leqslant j \leqslant n_0$, let $J_j = [t_j - \frac{\delta}{2^{m+1}}, t_j + \frac{\delta}{2^{m+1}}]$, $I_j = [t_j - \frac{\delta}{2^m}, t_j + \frac{\delta}{2^m}]$ and $f_j = \sum_{n=1}^m r_n f_{(n,j)}$. It is easily checked that h, $\{f_j\}_{j=1}^{n_0}$ and $\{(I_j, J_j)\}_{j=1}^{n_0}$ satisfy the conclusion of the lemma. $\square$

5. Operators on V_2^0 preserving a copy of *TF*

The aim of this section is to prove the following theorem.

Theorem 40. *Let* $T : V_2^0 \to V_2^0$ *be a bounded linear operator such that the dual operator* $T^* : (V_2^0)^* \to (V_2^0)^*$ *has non-separable range. Then there exists a subspace* Y *of* V_2^0 *isomorphic to TF such that the restriction of* T *on* Y *is an isomorphism.*

In the following we fix an operator $T : V_2^0 \to V_2^0$ such that $T^*[(V_2^0)^*]$ is non-separable.

Lemma 41. *The set* $A = \{t \in (0,1)\colon\ T^{**}(\chi_t) \in V_2^d \setminus \{0\}\}$ *is uncountable.*

Proof. By Theorem 3.2 of [4] there exists a bounded onto linear map $Q : (V_2^0)^* \to l_2([0,1])$ with $\operatorname{Ker} Q$ separable. So, since T^* has non-separable range we obtain that the same holds for $Q \circ T^* : (V_2^0)^* \to l_2([0,1])$. Therefore denoting by $(e_t^*)_{t\in[0,1]}$ the bi-orthogonal functionals of the usual basis of $l_2([0,1])$, we have that the set

$$B = \bigl\{t \in [0,1]\colon \text{ there exists } x^* \in \bigl(V_2^0\bigr)^* \text{ such that } e_t^*\bigl(QT^*\bigl(x^*\bigr)\bigr) \neq 0\bigr\}$$

is uncountable.

The above yields that for every $t \in B$, $T^{**}Q^*(e_t^*) \neq 0$. Since $\{e_t^*\}_{t\in B}$ is weakly discrete set accumulating to zero we conclude that there exists uncountable $B' \subseteq B$ such that the sets

$$\bigl\{Q^*\bigl(e_t^*\bigr)\colon t \in B'\bigr\},\ \bigl\{T^{**}Q^*\bigl(e_t^*\bigr)\colon t \in B'\bigr\}$$

are uncountable weakly discrete and contained in V_2^d (Lemma 22). In particular $T^{**}[V_2^d] \cap V_2^d$ is non-separable. Since $\{\chi_t\colon t \in (0,1)\}$ generates V_2^d the result follows. $\square$

Remark 3. It is easy to see that by passing if necessary to a further uncountable subset of A we may assume that the supports of $T^{**}(\chi_t)$, $t \in A$, are pairwise disjoint. This is due to the fact that the constant function zero is the unique weak* (and so pointwise) accumulation point of the family $\{T^{**}(\chi_t)\}_{t\in A}$.

Lemma 42. *Let* $(\varepsilon_\alpha)_{\alpha\in\mathcal{D}}$ *be a sequence of positive real numbers. Then there exist two monotone tree families* $\mathcal{F} = (f_\alpha, (I_\alpha, J_\alpha), \lambda_\alpha)_{\alpha\in\mathcal{D}}$, $\mathcal{F}' = (f'_\alpha, (I'_\alpha, J'_\alpha))_{\alpha\in\mathcal{D}}$ *and a family of functions* $(r_\alpha)_{\alpha\in\mathcal{D}}$ *in* V_2^0 *with the following properties.*

(P1) *For every* $\alpha \in \mathcal{D}$, $\|T(\lambda_\alpha f_\alpha) - (f'_\alpha + r_\alpha)\| \leqslant \varepsilon_\alpha$.
(P2) *For every* $\alpha, \beta \in \mathcal{D}$, *if* $I'_\alpha = [l_0^\alpha, l_1^\alpha]$ *and* $J'_\alpha = [m_0^\alpha, m_1^\alpha]$ *then* $r_\beta(l_0^\alpha) = r_\beta(l_1^\alpha) = r_\beta(m_0^\alpha) = r_\beta(m_0^\alpha) = 0$.

Granting Lemma 42, we proceed to the proof of Theorem 40 as follows.

Proof of Theorem 40. Let $g_\alpha = \lambda_\alpha f_\alpha$, $\alpha \in \mathcal{D}$. We claim that $(g_\alpha)_{\alpha\in\mathcal{D}}$ and $(T(g_\alpha))_{\alpha\in\mathcal{D}}$ are both equivalent to the basis $(e_\alpha)_{\alpha\in\mathcal{D}}$ of the space *TF*. Indeed by Lemma 35 there exists $C > 0$ such that for every $n \in \mathbb{N}$ and every sequence of scalars $(\mu_\alpha)_{|\alpha|\leqslant n}$,

$$\Bigl\|\sum_{|\alpha|\leqslant n} \mu_\alpha g_\alpha\Bigr\|_{V_2^0} \leqslant 4C|\lambda_\varnothing|\Bigl\|\sum_{|\alpha|\leqslant n} \mu_\alpha e_\alpha\Bigr\|_{TF}. \tag{24}$$

On the other hand,

$$\frac{1}{\|T\|}\Bigl\|\sum_{|\alpha|\leqslant n} \mu_\alpha T(g_\alpha)\Bigr\|_{V_2^0} \leqslant \Bigl\|\sum_{|\alpha|\leqslant n} \mu_\alpha g_\alpha\Bigr\|_{V_2^0}. \tag{25}$$

By (P1) of Lemma 42 for a family $(\varepsilon_\alpha)_\alpha$ with $\sum_{\alpha\in\mathcal{D}}\varepsilon_\alpha$ sufficiently small, we have that

$$\frac{1}{2}\Big\|\sum_{|\alpha|\leqslant n}\mu_\alpha\big(f'_\alpha+r_\alpha\big)\Big\|_{V_2^0}\leqslant\Big\|\sum_{|\alpha|\leqslant n}\mu_\alpha T(g_\alpha)\Big\|_{V_2^0}. \tag{26}$$

By the proof of Lemma 32, there exists a finite subset $\mathcal{P}$ of $\bigcup_{\alpha\in\mathcal{D}}\{l_0^\alpha, l_1^\alpha, m_0^\alpha, m_1^\alpha\}$ such that

$$\Big\|\sum_{|\alpha|\leqslant n}\mu_\alpha e_\alpha\Big\|_{TF}\leqslant\alpha_2\Big(\sum_{|\alpha|\leqslant n}\mu_\alpha f'_\alpha,\mathcal{P}\Big). \tag{27}$$

Then by (P2) of Lemma 42, we obtain that

$$\Big\|\sum_{|\alpha|\leqslant n}\mu_\alpha\big(f'_\alpha+r_\alpha\big)\Big\|_{V_2^0}\geqslant\alpha_2\Big(\sum_{|\alpha|\leqslant n}\mu_\alpha\big(f'_\alpha+r_\alpha\big),\mathcal{P}\Big)=\alpha_2\Big(\sum_{|\alpha|\leqslant n}\mu_\alpha f'_\alpha,\mathcal{P}\Big). \tag{28}$$

By all the above the result follows. □

It remains to prove Lemma 42. We recall that a tree family is 0-*monotone* if it is increasing and 1-*monotone* if it is decreasing. Similarly for a pair (x, y) of real numbers we will write $x <_0 y$ if $x < y$ and $x <_1 y$ if $y < x$.

Proof of Lemma 42. By Lemma 41 and Remark 3, there exists an uncountable set $A \subseteq (0, 1)$ such that the following are satisfied.

(a) For every $t \in A$, there exists a (finite or infinite) initial segment F_t of $\mathbb{N}$ such that $T^{**}(\chi_t) = \sum_{j\in F_t}\mu_j^t\chi_{s_j^t} \in V_2^d \setminus \{0\}$.
(b) The supports of $T^{**}(\chi_t)$, $S_t = \{s_j^t\colon\ j \in F_t\}$, $t \in A$, are pairwise disjoint.
(c) There exists $\delta > 0$ such that $|\mu_1^t| > \delta$, for all $t \in A$.

We set $\lambda_t = 1/\mu_1^t$, $t \in A$ and we observe that $T^{**}(\lambda_t\chi_t) = \chi_{s_t} + \sum_{j\in F_t, j\geqslant 2}\lambda_j^t\chi_{s_j^t}$ where $s_t = s_t^1$ and $\lambda_j^t = \mu_j^t/\mu_1^t$, for all $j \geqslant 2$, $j \in F_t$. For every $t \in A$ we set $\mathbf{x}_t = (t, s_t, \lambda_t, 0, 0, \ldots)$ and let $S = \{\mathbf{x}_t\colon\ t \in A\}$. By Lemma 38 and by passing, if necessary, to a further uncountable subset of A, we may suppose that there exists $\theta = (\theta(1), \theta(2)) \in \{0, 1\}^2$ such that S is hereditarily $(\theta, 2)$-almost condensed. By recursion we construct a $\theta(1)$-monotone tree family $(f_\alpha, (I_\alpha, J_\alpha), \lambda_\alpha)_{\alpha\in\mathcal{D}}$, a $\theta(2)$-monotone tree family $(f'_\alpha, (I'_\alpha, J'_\alpha))_{\alpha\in\mathcal{D}}$ and a family $(r_\alpha)_{\alpha\in\mathcal{D}}$ of functions of V_2^0 such that for every $\alpha \in \mathcal{D}$ the following are satisfied.

(i) If

$$V_{\lambda_\alpha}=\Big(\Big(1-\frac{\operatorname{sgn}(\lambda_\alpha)}{2^{2|\alpha|+2}}\Big)\lambda_\alpha,\Big(1+\frac{\operatorname{sgn}(\lambda_\alpha)}{2^{2|\alpha|+2}}\Big)\lambda_\alpha\Big)$$

then the set $S \cap (\mathring{J}_\alpha \times \mathring{J}_{\alpha'} \times V_{\lambda_\alpha})$ is uncountable.
(ii) For every $\beta \in \mathcal{D}$ with $|\beta| < |\alpha|$, $\{l_\beta^0, l_\beta^1, m_\beta^0, m_\beta^1\} \cap \operatorname{supp} r_\alpha = \varnothing$.

(iii) For every $\beta \in \mathcal{D}$ with $|\beta| = |\alpha|$, $\operatorname{supp} r_\alpha \cap \operatorname{supp} f'_\beta = \varnothing$.
(iv) $\|T(\lambda_\alpha f_\alpha) - (f'_\alpha + r_\alpha)\|_{V_2} < \varepsilon_\alpha$.

Let us point out that conditions (ii) and (iii) yield property (P2) of the lemma. Suppose that for some $n \geqslant 0$, the construction has been carried out up to all $\alpha \in \mathcal{D}$ with $|\alpha| \leqslant n$. The initial inductive step, namely $n = 0$, is similar, and in fact simpler, to the general one and so we omit it. By condition (i) we have that for all α with $|\alpha| = n$ and $\varepsilon \in \{0, 1\}$, we can choose $t_{\alpha^\frown\varepsilon}$ such that $\mathbf{x}_{t_{\alpha^\frown\varepsilon}} \in \mathring{J}_\alpha \times \mathring{J}_{\alpha'} \times V_{\lambda_\alpha}$ and $\mathbf{x}_{t_{\alpha^\frown\varepsilon}}$ be a $(\theta, 2)$-condensation point of S. Moreover $\mathbf{x}_{t_{\alpha^\frown 0}}, \mathbf{x}_{t_{\alpha^\frown 1}}$ are selected such that $t_{\alpha^\frown 0} <_{\theta(1)} t_{\alpha^\frown 1}$ and $s_{t_{\alpha^\frown 0}} <_{\theta(2)} s_{t_{\alpha^\frown 1}}$. We also choose $n_0 \in \mathbb{N}$ such that $\|\sum_{j>n_0, j\in F_t} \lambda_j^{t_\beta} \chi_{s_j^{t_\beta}}\|_{V_2} \leqslant 2(\sum_{j>n_0, j\in F_t} |\lambda_j^{t_\beta}|^2)^{1/2} < \varepsilon_\beta/2$, for every β with $|\beta| = n+1$. By condition (b) above, we have that $S_{t_\beta} \cap S_{t_{\beta'}} = \varnothing$ for $\beta \neq \beta'$ and so we can select pairwise disjoint open intervals $\Delta_{(j,\beta)}$, $j = 1, \ldots, n_0$, $|\beta| = n+1$ with $s_j^{t_\beta} \in \Delta_{(j,\beta)}$, $\Delta_{(j,\beta)} \cap \{l_\alpha^0, l_\alpha^1, m_\alpha^0, m_\alpha^1\} = \varnothing$ and for every α with $|\alpha| \leqslant n$ and every $\varepsilon \in \{0, 1\}$, $\Delta_{(1,\alpha^\frown\varepsilon)} \subseteq \mathring{J}_{\alpha'}$. For every α with $|\alpha| \leqslant n$ and $\varepsilon \in \{0, 1\}$, we choose pairwise disjoint open intervals $\Delta_{\alpha^\frown\varepsilon} \subseteq \mathring{J}_\alpha$ with $t_{\alpha^\frown\varepsilon} \in \Delta_{\alpha^\frown\varepsilon}$.

For every β with $|\beta| = n+1$ we consider a sequence $(f_n^\beta)_n$ of trapezoid functions (see the proof of Lemma 39), converging pointwise to χ_{t_β} and $\operatorname{supp} f_n^\beta \subseteq \Delta_\beta$. We set $g_n^\beta = T(\lambda_{t_\beta} f_n^\beta)$ and we observe that $(g_n^\beta)_n$ converges pointwise to $T^{**}(\lambda_{t_\beta}\chi_{t_\beta})$. By Lemma 39, for $\varepsilon = \varepsilon_\beta/2$ there exist a finite convex combination f_β of $(f_n^\beta)_n$, f'_β, (I'_β, J'_β), with $\operatorname{supp} f'_\beta \subseteq \mathring{I}'_\beta$, $J'_\beta = \{t \in [0,1]:\ f'_\beta(t) = 1\}$, and $f_j^{t_\beta}$, $j \in \{1, \ldots, n_0\} \cap F_{t_\beta}$ such that

$$\left\| T(\lambda_{t_\beta} f_\beta) - \left(f'_\beta + \sum_{j=2, j\in F_t}^{n_0} \lambda_j^{t_\beta} f_j^{t_\beta} \right) \right\|_{V_2} < \varepsilon_\beta.$$

Setting $\lambda_\beta = \lambda_{t_\beta}$ and $r_\beta = \sum_{j=2, j\in F_t}^{n_0} \lambda_j^{t_\beta} f_j^{t_\beta}$ it is easily checked that the proof of the inductive step is complete. $\square$

Remark 4. It is worth pointing out that the proof of Theorem 40 is considerably simpler than the proof of Theorem 4 which we shall present in the next two sections. This is due to the presence of the operator T. Indeed it is proved that there exists a tree family with coefficients $\mathcal{F} = (f_\alpha, (I_\alpha, J_\alpha), \lambda_\alpha)_{\alpha\in\mathcal{D}}$ such that $(\lambda_\alpha f_\alpha)_{\alpha\in\mathcal{D}}$, $(T(\lambda_\alpha f_\alpha))_{\alpha\in\mathcal{D}}$ are both equivalent to the basis $(e_\alpha)_{\alpha\in\mathcal{D}}$ of *TF*. Note that for $(\lambda_\alpha f_\alpha)_{\alpha\in\mathcal{D}}$ we are able to establish only that is dominated by the basis of *TF* (see Lemma 35). On the other hand, for the family $(T(\lambda_\alpha f_\alpha))_{\alpha\in\mathcal{D}}$ it is proved that dominates the basis $(e_\alpha)_{\alpha\in\mathcal{D}}$ of *TF*. The remaining inequalities are derived from the previous ones with the use of the operator T.

6. Forests of tree families

In this section we will introduce the notion of the *forest* of tree families. Roughly speaking a forest is an infinite sequence of tree families connected in a precise manner.

6.1. The definition of the forest of tree families

Let $(E_n)_{n=0}^\infty$ be a sequence of non-empty finite intervals of $\mathbb{N}$, with $\min E_0 = 1$ and $\min E_{n+1} = \max E_n + 1$ and let $\mathcal{Z} = \{(\beta, j) \in \mathcal{D} \times \mathbb{N}:\ j \in E_{|\beta|}\} = \bigcup_{k=0}^\infty (\{0,1\}^k \times E_k)$.

For every $(\beta, j) \in \mathcal{Z}$ set $\mathcal{T}_{(\beta,j)} = \{(\alpha, j)\colon \alpha \in \mathcal{D}$ and $\beta \sqsubseteq \alpha\}$ and define

$$\mathcal{T} = \bigcup_{(\beta,j)\in\mathcal{Z}} \mathcal{T}_{(\beta,j)} = \left\{(\alpha, j) \in \mathcal{D} \times \mathbb{N}\colon\ j \in \bigcup_{n=0}^{|\alpha|} E_n\right\}.$$

Definition 43. Let $(E_n)_{n=0}^{\infty}$, $\mathcal{Z}$ and $\mathcal{T}$ be as above.
For every $(\beta, j) \in \mathcal{Z}$ let

$$\mathcal{F}_{(\beta,j)} = \big(f_{(\alpha,j)}, (I_{(\alpha,j)}, J_{(\alpha,j)}), \lambda_{(\alpha,j)}\big)_{(\alpha,j)\in\mathcal{T}_{(\beta,j)}}$$

be a tree family (where we naturally identify $\mathcal{T}_{(\beta,j)}$ with $\mathcal{D}$ through the map: $\mathcal{T}_{(\beta,j)} \ni (\beta^\frown\alpha, j) \to \alpha \in \mathcal{D}$).

The family $\mathcal{F} = (\mathcal{F}_{(\beta,j)})_{(\beta,j)\in\mathcal{Z}} = (f_{(\alpha,j)}, (I_{(\alpha,j)}, J_{(\alpha,j)}), \lambda_{(\alpha,j)})_{(\alpha,j)\in\mathcal{T}}$ will be called a forest of tree families (determined by the sequence $(E_n)_n$), if the following conditions are satisfied.

(F1) $\sup\{\|f_{(\alpha,j)}\|_{V_2^0}\colon (\alpha, j) \in \mathcal{T}\} < +\infty$.
(F2) For every $n \geqslant 0$ and $(\alpha_1, j_1) \neq (\alpha_2, j_2)$ in $\mathcal{T}$ with $|\alpha_1| = |\alpha_2| = n$, $I_{(\alpha_1,j_1)} \cap I_{(\alpha_2,j_2)} = \varnothing$.
(F3) For every $n \geqslant 1$, every $(\beta, k) \in \mathcal{Z}$ with $|\beta| = n$ and every $(\alpha, j) \in \mathcal{T}$ with $|\alpha| \leqslant n-1$, either $I_{(\beta,k)} \cap I_{(\alpha,j)} = \varnothing$ or $I_{(\beta,k)} \subseteq \mathring{I}_{(\alpha,j)} \setminus \partial J_{(\alpha,j)}$.
(F4) For every $n \geqslant 1$, every $(\gamma, k) \in \mathcal{T}$ with $|\gamma| = n$ and every $\mathcal{P} = \{t_0 < \cdots < t_p\} \subseteq I_{(\gamma,k)}$,

$$\sum_{(\alpha,j)\in\mathcal{T},\ |\alpha|<n} \left(\sum_{i=0}^{p-1} \big|f_{(\alpha,j)}(t_i) - f_{(\alpha,j)}(t_{i+1})\big|^2\right) \leqslant \frac{1}{2^{2n} \cdot |\bigcup_{m=0}^{n} E_m|}.$$

(F5) For every $\alpha \in \mathcal{D}$, $\lambda_{(\alpha,1)} = 1$.
(F6) For every $n \geqslant 1$ and every $\beta \in \mathcal{D}$ with $|\beta| = n$, $\sum_{j\in E_n} |\lambda_{(\beta,j)}|^2 \leqslant 1/2^{2n}$.
(F7) For every $n \geqslant 3$,

$$\min\big\{|\lambda_{(\beta,j)}|\colon\ |\beta| = n-2,\ j \in E_{n-2}\big\} \geqslant 2\max\big\{|\lambda_{(\beta',j')}|\colon\ |\beta'| = n,\ j \in E_n\big\}.$$

6.2. The tree $(\mathcal{T}, \preccurlyeq)$

On the set $\mathcal{T}$ we define the following strict partial ordering

$$(\alpha_1, j_1) \prec (\alpha_2, j_2) \quad \text{if } I_{(\alpha_2,j_2)} \subseteq \mathring{I}_{(\alpha_1,j_1)}$$

and we write $(\alpha_1, j_1) \preccurlyeq (\alpha_2, j_2)$ if either $(\alpha_1, j_1) = (\alpha_2, j_2)$ or $(\alpha_1, j_1) \prec (\alpha_2, j_2)$.

Using (F2) and (F3) of Definition 43 it is easy to show the following properties of $(\mathcal{T}, \preccurlyeq)$.

(T1) For every $(\beta, j) \in \mathcal{Z}$, the partially ordered set $(\mathcal{T}_{(\beta,j)}, \preccurlyeq)$ is naturally ordered isomorphic to $(\mathcal{D}_\beta, \sqsubseteq)$ where $D_\beta = \{\alpha \in \mathcal{D}\colon\ \beta \sqsubseteq \alpha\}$. That is $(\beta, j) \preccurlyeq (\alpha_1, j) \preccurlyeq (\alpha_2, j)$ if and only if $\beta \sqsubseteq \alpha_1 \sqsubseteq \alpha_2$.
(T2) $(\alpha_1, j_1), (\alpha_2, j_2) \in \mathcal{T}$ are incomparable if and only if $I_{(\alpha_1,j_1)} \cap I_{(\alpha_2,j_2)} = \varnothing$.
(T3) Let $(\beta_1, j_1), (\beta_2, j_2) \in \mathcal{Z}$, $(\alpha_1, j_1) \in \mathcal{T}_{(\beta_1,j_1)}$, $(\alpha_2, j_2) \in \mathcal{T}_{(\beta_2,j_2)}$ and suppose that $(\alpha_1, j_1) \prec (\alpha_2, j_2)$. Then the following hold.

(i) $j_1 \in E_{n_1}$, $j_2 \in E_{n_2}$ and $n_1 < n_2$.
(ii) $|\alpha_1| < |\alpha_2|$.
(iii) $(\alpha_1, j_1) \prec (\beta_2, j_2) \preccurlyeq (\alpha_2, j_2)$.

The above properties yield that for every $(\alpha, j) \in \mathcal{T}$ the set of all its predecessors is finite and linearly ordered by $\preccurlyeq$. Therefore $(\mathcal{T}, \preccurlyeq)$ is a tree. It is also easy to see that $\mathcal{T}$ has at most countably many roots and every element of $\mathcal{T}$ has at least two and at most countably many immediate successors.

An interesting feature of the tree $\mathcal{T}$ is the structure of its finite segments which is described by the following fact.

Fact 44. *Let $\mathcal{I}$ be a finite non-empty segment of $\mathcal{T}$ and let $\{(\beta_1, j_1) \prec \cdots \prec (\beta_l, j_l)\}$ be the $\prec$-increasing enumeration of the set $\{(\beta, j) \in \mathcal{Z}\colon \mathcal{I} \cap \mathcal{T}_{(\beta,j)} \neq \varnothing\}$. For each $1 \leqslant k \leqslant l$, we set $\mathcal{I}^{(k)} = \mathcal{I} \cap \mathcal{T}_{(\beta_k, j_k)}$. Then the following are satisfied.*

(1) $\mathcal{I} = \bigcup_{k=1}^{l} \mathcal{I}^{(k)}$.
(2) *If $l \geqslant 2$ then for all $2 \leqslant k \leqslant l$, $\mathcal{I}^{(k)}$ is an initial segment of $\mathcal{T}_{(\beta_k, j_k)}$, that is $(\beta_k, j_k) = \min \mathcal{I}^{(k)}$.*
(3) *For $1 \leqslant k \leqslant l$ let $\mathrm{pr}_{\mathcal{D}}(\mathcal{I}^{(k)}) = \{\alpha \in \mathcal{D}\colon (\alpha, j_k) \in \mathcal{I}^{(k)}\}$ be the projection of $\mathcal{I}^{(k)}$ on $\mathcal{D}$. Then the family $\{\mathrm{pr}_{\mathcal{D}}(\mathcal{I}^{(k)})\colon 1 \leqslant k \leqslant l\}$ consists of pairwise disjoint segments of $\mathcal{D}$. In particular $|\max \mathrm{pr}_{\mathcal{D}}(\mathcal{I}^{(k)})| < |\min \mathrm{pr}_{\mathcal{D}}(\mathcal{I}^{(k+1)})|$, for all $1 \leqslant k < l$.*

By the above fact we have that every finite segment $\mathcal{I}$ of $\mathcal{T}$ admits a decomposition into a finite number of segments $\mathcal{I}^{(1)}, \dots, \mathcal{I}^{(l)}$ of the trees $\mathcal{T}_{(\beta,j)}$ for $(\beta, j) \in \mathcal{Z}$. The ordered sequence $(\mathcal{I}^{(1)}, \dots, \mathcal{I}^{(l)})$ will be called the *analysis* of $\mathcal{I}$. Moreover we will call $\mathcal{I}$ *vertical* if $\mathcal{I} \subseteq \mathcal{T}_{(\beta,j)}$ for some unique $(\beta, j) \in \mathcal{Z}$ and *diagonal* otherwise.

6.3. Forests of monotone tree families and the space TF

Given a forest $\mathcal{F} = (\mathcal{F}_{(\beta,j)})_{(\beta,j)\in\mathcal{Z}} = (f_{(\alpha,j)}, (I_{(\alpha,j)}, J_{(\alpha,j)}), \lambda_{(\alpha,j)})_{(\alpha,j)\in\mathcal{T}}$ of tree families, for every $\alpha \in \mathcal{D}$, let $G_\alpha = \sum_{j \in \bigcup_{n=0}^{|\alpha|} E_n} \lambda_{(\alpha,j)} f_{(\alpha,j)}$. The family $(G_\alpha)_{\alpha\in\mathcal{D}}$ will be called the *diagonal family of functions corresponding to the forest $\mathcal{F}$*. We are now ready to state the main result of this section.

Theorem 45. *Let $\mathcal{F} = (\mathcal{F}_{(\beta,j)})_{(\beta,j)\in\mathcal{Z}} = (f_{(\alpha,j)}, (I_{(\alpha,j)}, J_{(\alpha,j)}), \lambda_{(\alpha,j)})_{(\alpha,j)\in\mathcal{T}}$ be a forest of tree families and let $(G_\alpha)_{\alpha\in\mathcal{D}}$ be the diagonal family of functions corresponding to $\mathcal{F}$. Suppose also that for every $(\beta, j) \in \mathcal{Z}$ the tree family $\mathcal{F}_{(\beta,j)}$ is monotone. Then for every $n \geqslant 0$ and for every sequence of scalars $(\mu_\alpha)_{|\alpha|\leqslant n}$ we have that*

$$\Big\| \sum_{|\alpha|\leqslant n} \mu_\alpha e_\alpha \Big\|_{TF} \leqslant \Big\| \sum_{|\alpha|\leqslant n} \mu_\alpha G_\alpha \Big\|_{V_2^0} \leqslant C \Big\| \sum_{|\alpha|\leqslant n} \mu_\alpha e_\alpha \Big\|_{TF}$$

where $C = 32(\Lambda_0 + 1)^{1/2}(M^2 + 9)^{1/2}$, $M = \sup\{\|f_{(\alpha,j)}\|_{V_2^0}\colon (\alpha, j) \in \mathcal{T}\}$ and $\Lambda_0 = \sum_{j\in E_0} |\lambda_{(\varnothing,j)}|^2$.

The proof of Theorem 45 follows by Lemma 46 and Proposition 47 stated below which verify the lower and the upper *TF* estimates for the family $(G_\alpha)_{\alpha\in\mathcal{D}}$.

Lemma 46. *Under the assumptions of Theorem* 45, *for every* $n \geqslant 0$ *and every sequence of scalars* $(\mu_\alpha)_{|\alpha|\leqslant n}$ *we have that*

$$\Big\| \sum_{|\alpha|\leqslant n} \mu_\alpha e_\alpha \Big\|_{TF} \leqslant \Big\| \sum_{|\alpha|\leqslant n} \mu_\alpha G_\alpha \Big\|_{V_2^0}.$$

Proof. Let $n \geqslant 0$ and $(\mu_\alpha)_{|\alpha|\leqslant n}$ be a sequence of scalars. By (F5) of Definition 43, $\mathcal{F}_{(\varnothing,1)}$ is a monotone tree family of the form $\mathcal{F}_{(\varnothing,1)} = (f_{(\alpha,1)}, (I_{(\alpha,1)}, J_{(\alpha,1)}))_{\alpha\in\mathcal{D}}$. By the proof of Lemma 32, we have that there exists $\mathcal{P} \subseteq \bigcup_{\alpha\in\mathcal{D}}(\partial I_{(\alpha,j)} \cup \partial J_{(\alpha,j)})$ such that

$$\Big\| \sum_{|\alpha|\leqslant n} \mu_\alpha e_\alpha \Big\|_{TF} \leqslant \alpha_2\Big(\sum_{|\alpha|\leqslant n} \mu_\alpha f_{(\alpha,1)}, \mathcal{P}\Big). \tag{29}$$

By conditions (F2) and (F3) of the definition of the forest of tree families, we conclude that for every $t \in \mathcal{P}$ and every $(\alpha, j) \in \mathcal{T}$ with $j > 1$, $f(\alpha, j)(t) = 0$. Hence for all $\alpha \in \mathcal{D}$ and all $t \in \mathcal{P}$, $G_\alpha(t) = \sum_{j\in\bigcup_{n=0}^{|\alpha|} E_n} \lambda_{(\alpha,j)} f_{(\alpha,j)}(t) = f_{(\alpha,1)}(t)$. So for all $t \in \mathcal{P}$, $\sum_{|\alpha|\leqslant n} \mu_\alpha G_\alpha(t) = \sum_{|\alpha|\leqslant n} \mu_\alpha f_{(\alpha,1)}(t)$ which gives that

$$\alpha_2\Big(\sum_{|\alpha|\leqslant n} \mu_\alpha f_{(\alpha,1)}, \mathcal{P}\Big) = \alpha_2\Big(\sum_{|\alpha|\leqslant n} \mu_\alpha G_\alpha, \mathcal{P}\Big). \tag{30}$$

By (29) and (30), we immediately have

$$\Big\| \sum_{|\alpha|\leqslant n} \mu_\alpha e_\alpha \Big\|_{TF} \leqslant \alpha_2\Big(\sum_{|\alpha|\leqslant n} \mu_\alpha G_\alpha, \mathcal{P}\Big) \leqslant \Big\| \sum_{|\alpha|\leqslant n} \mu_\alpha G_\alpha \Big\|_{V_2^0}. \qquad \square \tag{31}$$

We pass now to the upper *TF* estimate.

Proposition 47. *For every* $n \geqslant 0$ *and every sequence of scalars* $(\mu_\alpha)_{|\alpha|\leqslant n}$,

$$\Big\| \sum_{|\alpha|\leqslant n} \mu_\alpha G_\alpha \Big\|_{V_2^0} \leqslant 32(\Lambda_0 + 1)^{1/2}\big(M^2 + 9\big)^{1/2} \Big\| \sum_{\alpha\leqslant n} \mu_\alpha e_\alpha \Big\|_{TF}.$$

For the proof of Proposition 47 we fix for the following a non-negative integer $n \geqslant 0$, a sequence of scalars $(\mu_\alpha)_{|\alpha|\leqslant n}$ and a finite subset $\mathcal{P} = \{t_0 < \cdots < t_p\}$ of [0, 1].

For every $0 \leqslant i \leqslant p - 1$ let $\mathcal{I}_i = \{(\alpha, j) \in \mathcal{T}\colon\ |\alpha| \leqslant n \text{ and } t_i \in I_{(\alpha,j)}\}$. It is clear that $\mathcal{I}_i$ is an initial segment of $\mathcal{T} = (\mathcal{T}, \preccurlyeq)$. Since $f_{(\alpha,j)}(t_i) = 0$ for all $(\alpha, j) \notin \mathcal{I}_i$, we have that $\sum_{|\alpha|\leqslant n} \mu_\alpha G_\alpha(t_i) = \sum_{(\alpha,j)\in\mathcal{I}_i} \mu_\alpha \lambda_{(\alpha,j)} f_{(\alpha,j)}(t_i)$ and so by simple calculations we get that

$$\begin{aligned}
\alpha_2^2\Big(\sum_{|\alpha|\leqslant n} \mu_\alpha G_\alpha, \mathcal{P}\Big) &= \sum_{i=0}^{p-1} \Big| \sum_{|\alpha|\leqslant n} \mu_\alpha G_\alpha(t_{i+1}) - \sum_{|\alpha|\leqslant n} \mu_\alpha G_\alpha(t_i)\Big|^2 \\
&\leqslant 3(S_{(-)} + S_0 + S_{(+)})
\end{aligned} \tag{32}$$

where

$$S_{(-)}=\sum_{i=0}^{p-1}\Bigl|\sum_{(\alpha,j)\in\mathcal{I}_{i+1}\setminus\mathcal{I}_i}\mu_\alpha\lambda_{(\alpha,j)}f_{(\alpha,j)}(t_{i+1})\Bigr|^2,$$

$$S_0=\sum_{i=0}^{p-1}\Bigl|\sum_{(\alpha,j)\in\mathcal{I}_i\cap\mathcal{I}_{i+1}}\mu_\alpha\lambda_{(\alpha,j)}\bigl(f_{(\alpha,j)}(t_{i+1})-f_{(\alpha,j)}(t_i)\bigr)\Bigr|^2,$$

$$S_{(+)}=\sum_{i=0}^{p-1}\Bigl|\sum_{(\alpha,j)\in\mathcal{I}_i\setminus\mathcal{I}_{i+1}}\mu_\alpha\lambda_{(\alpha,j)}f_{(\alpha,j)}(t_i)\Bigr|^2.$$

Lemma 48. *Under the above notations*

$$S_0\leqslant 8(\Lambda_0+1)\bigl(M^2+1\bigr)\Bigl\|\sum_{|\alpha|\leqslant n}\mu_\alpha e_\alpha\Bigr\|_{TF}^2.$$

Proof. For every $(\gamma,k)\in\mathcal{T}$ with $|\gamma|\leqslant n$, let us set

$$\mathcal{Q}_{(\gamma,k)}=\bigl\{i\in\{0,\dots,p-1\}\colon \max(\mathcal{I}_i\cap\mathcal{I}_{i+1})=(\gamma,k)\bigr\}$$

$$\text{and}\quad S_{(\gamma,k)}=\sum_{i\in\mathcal{Q}_{(\gamma,k)}}\Bigl|\sum_{(\alpha,j)\preccurlyeq(\gamma,k)}\mu_\alpha\lambda_{(\alpha,j)}\bigl(f_{(\alpha,j)}(t_{i+1})-f_{(\alpha,j)}(t_i)\bigr)\Bigr|^2.$$

Notice that

$$S_0=\sum_{l=0}^{n}\sum_{|\gamma|=l}\sum_{k\in\bigcup_{m=0}^{l}E_m}S_{(\gamma,k)}.\tag{33}$$

Let $(\gamma,k)\in\mathcal{T}$, with $|\gamma|\leqslant n$. If $\gamma=\varnothing$ then for every $k\in E_0$ we have that

$$S_{(\varnothing,k)}=|\mu_\varnothing|^2|\lambda_{(\varnothing,k)}|^2\sum_{i\in\mathcal{Q}_{(\varnothing,k)}}\bigl|f_{(\varnothing,k)}(t_{i+1})-f_{(\varnothing,k)}(t_i)\bigr|^2\leqslant|\mu_\varnothing|^2|\lambda_{(\varnothing,k)}|^2M^2$$

and so

$$\sum_{k\in E_0}S_{(\varnothing,k)}\leqslant|\mu_\varnothing|^2\Lambda_0M^2.\tag{34}$$

If $|\gamma|=l$ with $1\leqslant l\leqslant n$ and $k\in\bigcup_{m=0}^{l}E_m$, then by the definition of $\mathcal{Q}_{(\gamma,k)}$ for every $i\in\mathcal{Q}_{(\gamma,k)}$, $\{t_i,t_{i+1}\}\subseteq I_{(\gamma,k)}$. Hence by property (F4) of Definition 43,

$$\sum_{(\alpha,j)\prec(\gamma,k)}\sum_{i\in\mathcal{Q}_{(\gamma,k)}}\bigl|f_{(\alpha,j)}(t_{i+1})-f_{(\alpha,j)}(t_i)\bigr|^2\leqslant\frac{1}{2^{2l}|\bigcup_{m=0}^{l}E_m|}.\tag{35}$$

Therefore

$$\begin{aligned}
S_{(\gamma,k)} &\leqslant 2 \sum_{i\in\mathcal{Q}_{(\gamma,k)}} \Bigl| \sum_{(\alpha,j)\prec(\gamma,k)} \mu_\alpha \lambda_{(\alpha,j)} \bigl(f_{(\alpha,j)}(t_{i+1}) - f_{(\alpha,j)}(t_i)\bigr)\Bigr|^2 \\
&\quad + 2|\mu_\gamma|^2 |\lambda_{(\gamma,k)}|^2 \sum_{i\in\mathcal{Q}_{(\gamma,k)}} \bigl|f_{(\gamma,k)}(t_{i+1}) - f_{(\gamma,k)}(t_i)\bigr|^2 \\
&\leqslant 2 \sum_{i\in\mathcal{Q}_{(\gamma,k)}} \Bigl(\sum_{(\alpha,j)\prec(\gamma,k)} |\mu_\alpha|^2 |\lambda_{(\alpha,j)}|^2 \Bigr) \Bigl(\sum_{(\alpha,j)\prec(\gamma,k)} \bigl|f_{(\alpha,j)}(t_{i+1}) - f_{(\alpha,j)}(t_i)\bigr|^2 \Bigr) \\
&\quad + 2|\mu_\gamma|^2 |\lambda_{(\gamma,k)}|^2 \|f_{(\gamma,k)}\|^2_{V_2^0} \\
&\leqslant 2 \Bigl(\sum_{(\alpha,j)\prec(\gamma,k)} |\mu_\alpha|^2 |\lambda_{(\alpha,j)}|^2 \Bigr) \Bigl(\sum_{(\alpha,j)\prec(\gamma,k)} \sum_{i\in\mathcal{Q}_{(\gamma,k)}} \bigl|f_{(\alpha,j)}(t_{i+1}) - f_{(\alpha,j)}(t_i)\bigr|^2 \Bigr) \\
&\quad + 2|\mu_\gamma|^2 |\lambda_{(\gamma,k)}|^2 M^2 \\
&\leqslant 2 \Bigl(\sum_{(\alpha,j)\prec(\gamma,k)} |\mu_\alpha|^2 |\lambda_{(\alpha,j)}|^2 \Bigr) \frac{1}{2^{2l} |\bigcup_{m=0}^{l} E_m|} + 2|\mu_\gamma|^2 |\lambda_{(\gamma,k)}|^2 M^2 \\
&\leqslant 2 \Lambda^2_{(\gamma,k)} \Bigl(\sum_{|\alpha|\leqslant n} |\mu_\alpha|^2 \Bigr) \frac{1}{2^{2l} |\bigcup_{m=0}^{l} E_m|} + 2|\mu_\gamma|^2 |\lambda_{(\gamma,k)}|^2 M^2
\end{aligned}$$

where $\Lambda_{(\gamma,k)} = \max\{|\lambda_{(\alpha,j)}|\colon (\alpha, j) \prec (\gamma, k)\}$.

Hence for every $\gamma \in \mathcal{D}$ with $|\gamma| = l$ and $1 \leqslant l \leqslant n$,

$$\sum_{k\in\bigcup_{m=0}^{l} E_m} S_{(\gamma,k)} \leqslant 2\Lambda^2_{(\gamma,k)} \frac{1}{2^{2l}} \sum_{|\alpha|\leqslant n} |\mu_\alpha|^2 + 2M^2 |\mu_\gamma|^2 \sum_{k\in\bigcup_{m=0}^{l} E_m} |\lambda_{(\gamma,k)}|^2. \tag{36}$$

By (4) we have that $|\lambda_{(\alpha,j)}| < 2|\lambda_{(\beta,j)}|$, for every $(\beta, j) \in \mathcal{Z}$ and every $(\alpha, j) \in \mathcal{T}_{(\beta,j)}$. Therefore by Fact 44 and property (F6) of Definition 43, we easily obtain that $\Lambda^2_{(\gamma,k)} \leqslant 4(\Lambda_0 + 1)$ and $\sum_{k\in\bigcup_{m=0}^{l} E_m} |\lambda_{(\gamma,k)}|^2 \leqslant 4(\Lambda_0 + 1)$. So for every $1 \leqslant l \leqslant n$,

$$\sum_{|\gamma|=l} \sum_{k\in\bigcup_{m=0}^{l} E_m} S_{(\gamma,k)} \leqslant 8(\Lambda_0 + 1) \Bigl(\frac{1}{2^l} \sum_{|\alpha|\leqslant n} |\mu_\alpha|^2 + M^2 \sum_{|\gamma|=l} |\mu_\gamma|^2 \Bigr). \tag{37}$$

By (34) and (37), we conclude that

$$\begin{aligned}
\sum_{l=0}^{n} \sum_{|\gamma|=l} \sum_{k\in\bigcup_{m=0}^{l} E_m} S_{(\gamma,k)} &\leqslant 8(\Lambda_0 + 1) \Bigl(\sum_{|\alpha|\leqslant n} |\mu_\alpha|^2 + M^2 \sum_{|\gamma|\leqslant n} |\mu_\gamma|^2 \Bigr) \\
&\leqslant 8(\Lambda_0 + 1)\bigl(M^2 + 1\bigr) \Bigl\| \sum_{|\alpha|\leqslant n} \mu_\alpha e_\alpha \Bigr\|^2_{TF}
\end{aligned}$$

and the proof of the lemma is complete. □

Next we provide upper *TF* estimates for the quantities $S_{(-)}$ and $S_{(+)}$.

Lemma 49. $\max\{S_{(-)}, S_{(+)}\} \leqslant 160(\Lambda_0+1)(M^2+9)\|\sum_{|\alpha|\leqslant n}\mu_\alpha e_\alpha\|_{TF}^2$.

Proof. By symmetry it suffices to show the above for $S_{(+)}$. We set $\mathcal{I}_{i,i+1}=\mathcal{I}_i\setminus\mathcal{I}_{i+1}$ and let $\mathcal{Q}^{(1)}=\{i\in\{0,\dots,p-1\}\colon \mathcal{I}_{i,i+1}\neq\varnothing\}$. By Fact 44, for every $i\in\mathcal{Q}^{(1)}$ there exists $l_i\geqslant 1$ such that $\mathcal{I}_{i,i+1}=\bigcup_{k=1}^{l_i}\mathcal{I}_{i,i+1}^{(k)}$, where $\{\mathcal{I}_{(i,i+1}^{(k)}\colon 1\leqslant k\leqslant l_i\}$ is the analysis of $\mathcal{I}_{i,i+1}$ into vertical segments of $(\mathcal{T},\preccurlyeq)$. We also set $\mathcal{Q}^{(2)}=\{i\in\mathcal{Q}^{(1)}\colon l_i>1\}$. Then

$$S_{(+)}\leqslant 2\big(S_{(+)}^1+S_{(+)}^2\big). \tag{38}$$

where

$$S_{(+)}^1=\sum_{i\in\mathcal{Q}^{(1)}}\Bigg|\sum_{(\alpha,j)\in\mathcal{I}_{i,i+1}^{(1)}}\mu_\alpha\lambda_{(\alpha,j)}f_{(\alpha,j)}(t_i)\Bigg|^2$$

and

$$S_{(+)}^2=\sum_{i\in\mathcal{Q}^{(2)}}\Bigg|\sum_{(\alpha,j)\in\bigcup_{k=2}^{l_i}\mathcal{I}_{i,i+1}^{(k)}}\mu_\alpha\lambda_{(\alpha,j)}f_{(\alpha,j)}(t_i)\Bigg|^2.$$

Claim 5. $S_{(+)}^1\leqslant 16(\Lambda_0+1)(M^2+3)\|\sum_{|\alpha|\leqslant n}\mu_\alpha e_\alpha\|_{TF}^2$.

Proof. For every $(\beta,j)\in\mathcal{Z}$ we set $\mathcal{Q}_{(\beta,j)}^{(1)}=\{i\in\mathcal{Q}^{(1)}\colon \mathcal{I}_{i,i+1}^{(1)}\subseteq\mathcal{T}_{(\beta,j)}\}$. Then

$$S_{(+)}^1=\sum_{(\beta,j)\in\mathcal{Z}_n}\sum_{i\in\mathcal{Q}_{(\beta,j)}^{(1)}}\Bigg|\sum_{(\alpha,j)\in\mathcal{I}_{i,i+1}^{(1)}}\mu_\alpha\lambda_{(\alpha,j)}f_{(\alpha,j)}(t_i)\Bigg|^2 \tag{39}$$

where $\mathcal{Z}_n=\{(\beta,j)\in\mathcal{Z}\colon |\beta|\leqslant n \text{ and } j\in E_{|\beta|}\}$.

Fix a $(\beta,j)\in\mathcal{Z}_n$ and for $0\leqslant i\leqslant p$, set $\mathcal{I}_i^{(\beta,j)}=\mathcal{I}_i\cap\mathcal{T}_{(\beta,j)}$ and for $0\leqslant i\leqslant p-1$, let $\mathcal{I}_{i,i+1}^{(\beta,j)}=\mathcal{I}_i^{(\beta,j)}\setminus\mathcal{I}_{i+1}^{(\beta,j)}$.

It is easy to see that for every $i\in\mathcal{Q}_{(\beta,j)}^{(1)}$, $\mathcal{I}_{i,i+1}^{(\beta,j)}=\mathcal{I}_{i,i+1}^{(1)}$ and therefore

$$\sum_{i\in\mathcal{Q}_{(\beta,j)}^{(1)}}\Bigg|\sum_{(\alpha,j)\in\mathcal{I}_{i,i+1}^{(1)}}\mu_\alpha\lambda_{(\alpha,j)}f_{(\alpha,j)}(t_i)\Bigg|^2=\sum_{i\in\mathcal{Q}_{(\beta,j)}^{(1)}}\Bigg|\sum_{(\alpha,j)\in\mathcal{I}_{i,i+1}^{(\beta,j)}}\mu_\alpha\lambda_{(\alpha,j)}f_{(\alpha,j)}(t_i)\Bigg|^2. \tag{40}$$

Since for every $0\leqslant i\leqslant p$, it is clear that $\mathcal{I}_i^{(\beta,j)}=\{(\alpha,j)\in\mathcal{T}_{(\beta,j)}\colon t_i\in I_{(\alpha,j)}\}$ and $\mathcal{F}_{(\beta,j)}=(f_{(\alpha,j)},(I_{(\alpha,j)},J_{(\alpha,j)}),\lambda_{(\alpha,j)})_{(\alpha,j)\in\mathcal{T}_{(\beta,j)}}$ is a monotone tree family, we can apply Lemma 36(ii) and so

$$\sum_{i\in\mathcal{Q}_{(\beta,j)}^{(1)}}\Bigg|\sum_{(\alpha,j)\in\mathcal{I}_{i,i+1}^{(\beta,j)}}\mu_\alpha\lambda_{(\alpha,j)}f_{(\alpha,j)}(t_i)\Bigg|^2\leqslant 16|\lambda_{(\beta,j)}|^2\big(M^2+3\big)\Bigg\|\sum_{|\alpha|\leqslant n}\mu_\alpha e_\alpha\Bigg\|_{TF}^2. \tag{41}$$

By (39)–(41) we get that

$$S^1_{(+)} \leqslant 16\Bigl(\sum_{(\beta,j)\in\mathcal{Z}_n} |\lambda_{(\beta,j)}|^2\Bigr)\bigl(M^2+3\bigr)\Bigl\|\sum_{|\alpha|\leqslant n} \mu_\alpha e_\alpha\Bigr\|^2_{TF}. \tag{42}$$

Finally by (F6), $\sum_{(\beta,j)\in\mathcal{Z}_n} |\lambda_{(\beta,j)}|^2 \leqslant \Lambda_0 + 1$ and the result follows. □

Claim 6. *Let $i \in \mathcal{Q}^{(2)}$ and let $S^{(2,i)}_{(+)} = |\sum_{(\alpha,j)\in\bigcup_{k=2}^{l_i}\mathcal{I}^{(k)}_{i,i+1}} \mu_\alpha \lambda_{(\alpha,j)} f_{(\alpha,j)}(t_i)|^2$. Then*

$$S^{(2,i)}_{(+)} \leqslant 32\Bigl(\sum_{k=2}^{l_i} |\lambda_{(\beta_k,j_k)}|^2\Bigr)(M+3)^2\Bigl\|\sum_{|\alpha|\leqslant n} \mu_\alpha e_\alpha\Bigr\|^2_{TF}$$

where $(\beta_2, j_2) \prec \cdots \prec (\beta_{l_i}, j_{l_i})$ is the $\prec$-increasing sequence in $\mathcal{Z}$ with $\mathcal{I}^{(k)}_{i,i+1} \subseteq \mathcal{T}_{(\beta_k,j_k)}$ for all $2 \leqslant k \leqslant l_i$.

Proof. It is clear that

$$S^{(2,i)}_{(+)} \leqslant \Biggl(\sum_{k=2}^{l_i}\Biggl|\sum_{(\alpha,j)\in\mathcal{I}^{(k)}_{i,i+1}} \mu_\alpha \lambda_{(\alpha,j)} f_{(\alpha,j)}(t_i)\Biggr|\Biggr)^2. \tag{43}$$

For every $2 \leqslant k \leqslant l_i$ set $\mathcal{I}^{(\beta_k,j_k)}_i = \mathcal{I}_i \cap \mathcal{T}_{(\beta_k,j_k)}$ and notice that $\mathcal{I}^{(\beta_k,j_k)}_i = \mathcal{I}^{(k)}_{i,i+1}$. Hence for every $2 \leqslant k \leqslant l_i$,

$$\Biggl|\sum_{(\alpha,j)\in\mathcal{I}^{(k)}_{i,i+1}} \mu_\alpha \lambda_{(\alpha,j)} f_{(\alpha,j)}(t_i)\Biggr| = \Biggl|\sum_{(\alpha,j)\in\mathcal{I}^{(\beta_k,j_k)}_i} \mu_\alpha \lambda_{(\alpha,j)} f_{(\alpha,j)}(t_i)\Biggr|. \tag{44}$$

Since $\mathcal{I}^{(\beta_k,j_k)}_i = \{(\alpha, j_k) \in \mathcal{T}_{(\beta_k,j_k)}\colon\ t_i \in I_{(\alpha,j_k)}\}$ and $\mathcal{F}_{(\beta_k,j_k)}$ is a monotone tree family, Lemma 36(i) yields that

$$\begin{aligned}\Biggl|\sum_{(\alpha,j)\in\mathcal{I}^{(\beta_k,j_k)}_i} \mu_\alpha \lambda_{(\alpha,j)} f_{(\alpha,j)}(t_i)\Biggr| &\leqslant 2|\lambda_{(\beta_k,j_k)}|(M+3)\Biggl\|\sum_{\mathrm{pr}_{\mathcal{D}}(\mathcal{I}^{(\beta_k,j_k)}_i)} \mu_\alpha e_\alpha\Biggr\|_{TF}\\ &\leqslant 2|\lambda_{(\beta_k,j_k)}|(M+3)\Biggl\|\sum_{\alpha\in\mathcal{D}} \mu_\alpha e_\alpha\Biggr\|_{TF}\end{aligned} \tag{45}$$

where $\mathrm{pr}_{\mathcal{D}}(\mathcal{I}^{(\beta_k,j_k)}_i) = \{\alpha \in \mathcal{D}\colon\ (\alpha, j_k) \in \mathcal{I}^{(k)}\}$. Hence by (43)–(45) we obtain that

$$S^{(2,i)}_{(+)} \leqslant 4\Biggl(\sum_{k=2}^{l_i} |\lambda_{(\beta_k,j_k)}|\Biggr)^2 (M+3)^2\Biggl\|\sum_{|\alpha|\leqslant n} \mu_\alpha e_\alpha\Biggr\|^2_{TF}. \tag{46}$$

For each $2 \leqslant k \leqslant l_i$ let n_k be the unique positive integer with $j_k \in E_{n_k}$. By property (T3) of the tree $(\mathcal{T}, \preccurlyeq)$, we have that $(n_k)_{k=2}^{l_i}$ is a strictly increasing sequence. We set $F_0 = \{k \in \{2, \dots, l_i\}\colon n_k \text{ is even}\}$ and $F_1 = \{k \in \{2, \dots, l_i\}\colon n_k \text{ is odd}\}$. For $\varepsilon \in \{0, 1\}$, if $F_\varepsilon \neq \varnothing$ let $k_\varepsilon = \min F_\varepsilon$. If $F_0 \neq \varnothing$ then by condition (F7), $\sum_{k \in F_0} |\lambda_{(\beta_k, j_k)}| \leqslant \sum_{k \in F_0} |\lambda_{(\beta_{k_0}, j_{k_0})}|/2^{m_k}$, where $m_k = (n_k - n_{k_0})/2$. Hence $\sum_{k \in F_0} |\lambda_{(\beta_k, j_k)}| \leqslant 2|\lambda_{(\beta_{k_0}, j_{k_0})}|$. Similarly if $F_1 \neq \varnothing$, $\sum_{k \in F_1} |\lambda_{(\beta_k, j_k)}| \leqslant 2|\lambda_{(\beta_{k_1}, j_{k_1})}|$. Therefore in the general case

$$\left(\sum_{k=2}^{l_i} |\lambda_{(\beta_k, j_k)}|\right)^2 \leqslant 2\left(\sum_{k \in F_0} |\lambda_{(\beta_k, j_k)}|\right)^2 + 2\left(\sum_{k \in F_1} |\lambda_{(\beta_k, j_k)}|\right)^2$$
$$\leqslant 8\big(|\lambda_{(\beta_{k_0}, j_{k_0})}|^2 + |\lambda_{(\beta_{k_1}, j_{k_1})}|^2\big) \leqslant 8 \sum_{k=2}^{l_i} |\lambda_{(\beta_k, j_k)}|^2. \tag{47}$$

Substituting (47) in (46) the proof of the claim is complete. □

Claim 7. $S_{(+)}^2 \leqslant 64(M^2 + 9) \| \sum_{|\alpha| \leqslant n} \mu_\alpha e_\alpha \|_{TF}^2$.

Proof. For each $i \in \mathcal{Q}^{(2)}$, let $F_i = \{(\beta, j) \in \mathcal{Z}\colon \mathcal{I}_{i,i+1} \cap \mathcal{T}_{(\beta, j)} \neq \varnothing\}$. By Claim 6 we have that

$$S_{(+)}^2 = \sum_{i \in \mathcal{Q}^{(2)}} S_{(+)}^{(2,i)} \leqslant 32(M + 3)^2 \left(\sum_{i \in \mathcal{Q}^{(2)}} \sum_{(\beta, j) \in {}_*F_i} |\lambda_{(\beta, j)}|^2\right) \left\| \sum_{|\alpha| \leqslant n} \mu_\alpha e_\alpha \right\|_{TF}^2 \tag{48}$$

where ${}_*F_i = F_i \setminus \{\min F_i\}$. It is easy to see (using similar arguments as those in the proof of Theorem 30), that $\mathcal{I}_{i,i+1} \cap \mathcal{I}_{j,j+1} = \varnothing$, for all $i, j \in \mathcal{Q}^{(2)}$ with $i \neq j$. Since ${}_*F_i \subseteq \mathcal{I}_{i,i+1}$ for all $i \in \mathcal{Q}^{(2)}$, we have that ${}_*F_i \cap_* F_j = \varnothing$ for every $i, j \in \mathcal{Q}^{(2)}$ with $i \neq j$. Moreover $\bigcup_{i \in \mathcal{Q}^{(2)}} {}_*F_i \subseteq \{(\beta, j) \in \mathcal{Z}\colon 1 \leqslant |\beta| \leqslant n\}$. Therefore by (F6),

$$\sum_{i \in \mathcal{Q}^{(2)}} \sum_{(\beta, j) \in {}_*F_i} |\lambda_{(\beta, j)}|^2 \leqslant \sum_{k=1}^{n} \sum_{|\beta| = k} \sum_{j \in E_k} |\lambda_{(\beta, j)}|^2 \leqslant \sum_{k=1}^{n} \frac{1}{2^k} \leqslant 1$$

and so by (48) we get that

$$S_{(+)}^2 \leqslant 32(M + 3)^2 \left\| \sum_{|\alpha| \leqslant n} \mu_\alpha e_\alpha \right\|_{TF}^2 \leqslant 64\big(M^2 + 9\big) \left\| \sum_{|\alpha| \leqslant n} \mu_\alpha e_\alpha \right\|_{TF}^2 \tag{49}$$

and the proof of Claim 7 is complete. □

By (38), Claim 5 and Claim 7 the proof of Lemma 49 is also complete. □

Proof of Proposition 47. It follows easily by (32), Lemma 48 and Lemma 49. □

7. The embedding of *TF* into subspaces of V_2^0

This section is devoted to the proof of the following.

Theorem 50. *Every subspace of V_2^0 with non-separable dual contains an isomorph of TF.*

The above theorem extends Corollary 31 and moreover yields that every subspace of *TF* with non-separable dual contains a copy of *TF*.

In order to prove Theorem 50 we will show that for every $\delta > 0$ and every subspace X of V_2^0 with non-separable dual there exists a forest $\mathcal{F}$ of monotone tree families such that the diagonal family of functions corresponding to $\mathcal{F}$ is δ-almost contained in X. If this has been achieved then the result will follow by Theorem 45.

Lemma 51. *Let X be a subspace of V_2^0 with non-separable dual. Then there exists a constant $M_0 > 0$ such that for every $\delta > 0$ there exist a sequence $(E_n)_{n=0}^{\infty}$ of successive intervals of $\mathbb{N}$, a forest $\mathcal{F} = (f_{(\alpha,j)}, (I_{(\alpha,j)}, J_{(\alpha,j)}), \lambda_{(\alpha,j)})_{(\alpha,j)\in\mathcal{T}}$ of monotone tree families determined by $(E_n)_{n=0}^{\infty}$ and a family $(h_\alpha)_{\alpha\in\mathcal{D}}$ of elements of X, satisfying the following properties.*

(1) *$\sum_{j\in E_0} |\lambda_{(\varnothing,j)}|^2 \leqslant M_0^2$ and for every $(\alpha, j) \in \mathcal{T}$, $\|f_{(\alpha,j)}\|_{V_2^0} = \sqrt{2}$.*
(2) *$\sum_{\alpha\in\mathcal{D}} \|G_\alpha - h_\alpha\|_{V_2^0} \leqslant \delta$, where $(G_\alpha)_{\alpha\in\mathcal{D}}$ is the diagonal family of functions corresponding to $\mathcal{F}$.*

Proof. By Proposition 23 and arguing as in the proof of Theorem 40 (see also Remark 3) we obtain an uncountable bounded family $\mathcal{H}_0 = \{H_\tau\colon \tau \in T_0\} \subseteq X^{**} \cap V_2^d$ weakly accumulating to zero such that

(a) for every $\tau \neq \tau'$ in T_0, $\operatorname{supp} H_\tau \cap \operatorname{supp} H_{\tau'} = \varnothing$.
(b) For every $\tau \in T_0$, there exists $t \in (0, 1)$ in the support of H_τ with $H_\tau(t) = 1$.

Moreover by taking sums of the form $\sum_{n=1}^{\infty} \frac{1}{2^n} H_n$ with $H_n \in \mathcal{H}_0$ for all $n \in \mathbb{N}$, if it is necessary, we may also assume that

(c) For every $\tau \in T_0$, $\operatorname{supp} H_\tau$ is infinite.

Let $M_0 > 0$ be such that $\|H_\tau\|_{V_2} \leqslant M_0$ and for every $\tau \in T_0$, let $\mathbf{t}_\tau = (t_j^\tau)_{j=1}^{\infty} \subseteq (0, 1)$ be an enumeration of the support of H_τ such that $H_\tau(t_1^\tau) = 1$. Hence for each $\tau \in T_0$, $H_\tau = \sum_{j=1}^{\infty} \lambda_j^\tau \chi_{t_j^\tau}$ where $\lambda_1^\tau = 1$. For every $\tau \in T_0$ define $\mathbf{x}_\tau = (\lambda_1^\tau, t_1^\tau, \lambda_2^\tau, t_2^\tau, \ldots) \in \mathbb{R}^{\mathbb{N}}$ and set $S_0 = \{\mathbf{x}_\tau\colon \tau \in T_0\}$. Notice that for every $\tau \in T_0$, $(\mathbf{x}_\tau(2j))_{j=1}^{\infty}$ is the enumeration of the support of H_τ.

Let $0 < \delta < 1$. By recursion on $n \geqslant 0$ and for $\alpha \in \mathcal{D}$ with $|\alpha| = n$ we will construct the following.

(C1) A sequence of uncountable subsets $(T_\alpha)_{\alpha\in\mathcal{D}}$ with $T_\varnothing \subseteq T_0$ and such that for all $\alpha \in \mathcal{D}$, $T_{\alpha^\frown 0} \cup T_{\alpha^\frown 1} \subseteq T_\alpha$ and $T_{\alpha^\frown 0} \cap T_{\alpha^\frown 1} = \varnothing$.
(C2) A strictly increasing sequence $(k_n)_{n=0}^{\infty}$ of positive integers.
(C3) A family $(\theta_\alpha)_{\alpha\in\mathcal{D}}$ in $\mathcal{D}$ with $\theta_\alpha \in \{0, 1\}^{2k_{|\alpha|}}$ and such that $\theta_\alpha \sqsubset \theta_\beta$, for every $\alpha \sqsubset \beta$ in $\mathcal{D}$.

(C4) A forest $\mathcal{F}=(f_{(\alpha,j)},(I_{(\alpha,j)},J_{(\alpha,j)}),\lambda_{(\alpha,j)})_{(\alpha,j)\in\mathcal{T}}$ of tree families determined by $(E_n)_n$ where $E_0=\{1,\dots,k_0\}$ and $E_n=\{k_{n-1}+1,\dots,k_n\}$ for $n\geqslant 1$.
(C5) A family $(h_\alpha)_{\alpha\in\mathcal{D}}$ of functions in X,

such that the following are satisfied.

(P1) For every $\alpha\in\mathcal{D}$, the set $S_\alpha=\{\mathbf{x}_\tau\colon\ \tau\in T_\alpha\}$ is hereditarily $(\theta_\alpha,2k_{|\alpha|})$-almost condensed.
(P2) For every $\alpha\in\mathcal{D}$ and every $j=1,\dots,k_{|\alpha|}$, let

$$V_{\lambda_{(\alpha,j)}}=\left(\left(1-\frac{\operatorname{sgn}(\lambda_{(\alpha,j)})}{2^{2|\alpha|+2}}\right)\lambda_{(\alpha,j)},\left(1+\frac{\operatorname{sgn}(\lambda_{(\alpha,j)})}{2^{2|\alpha|+2}}\right)\lambda_{(\alpha,j)}\right)$$

and let

$$V_\alpha=\prod_{j=1}^{k_{|\alpha|}}(V_{\lambda_{(\alpha,j)}}\times \mathring{J}_{(\alpha,j)})\times\prod_{j=k_{|\alpha|}+1}^{\infty}\mathbb{R}.$$

Then $S_\alpha\subseteq V_\alpha$.
(P3) The tree family $\mathcal{F}_{(\beta,j)}=(f_{(\alpha,j)},(I_{(\alpha,j)},J_{(\alpha,j)}),\lambda_{(\alpha,j)})_{\beta\sqsubseteq\alpha}$, is $\theta_\beta(2j)$-monotone, for all $\beta\in\mathcal{D}$ and all $j\in E_{|\beta|}$.
(P4) For every $(\alpha,j)\in\mathcal{T}$, $\|f_{(\alpha,j)}\|_{V_2^0}=\sqrt{2}$ and there exists $\tau_\alpha\in T_\alpha$, such that $\lambda_{(\alpha,j)}=\lambda_j^{\tau_\alpha}$ for all $j=1,\dots,k_{|\alpha|}$.
(P5) For every $\alpha\in\mathcal{D}$, if $G_\alpha=\sum_{j=1}^{k_{|\alpha|}}\lambda_{(\alpha,j)}f_{(\alpha,j)}$ then $\|G_\alpha-h_\alpha\|_{V_2^0}<\delta/2^{2|\alpha|+1}$.

Notice that by (P5) and (P4),

$$\sum_{\alpha\in\mathcal{D}}\|G_\alpha-h_\alpha\|_{V_2^0}<\delta$$

and

$$\sum_{j\in E_0}|\lambda_{(\varnothing,j)}|^2=\sum_{j\in E_0}\left|\lambda_j^{\tau_\varnothing}\right|^2\leqslant\|H_{\tau_\varnothing}\|_{V_2}^2\leqslant M_0^2.$$

The initial step (i.e. $n=0$) of the construction goes as follows. Since for all $\tau\in T_0$, $\sum_{j=1}^{\infty}|\lambda_j^\tau|^2\leqslant\|H_\tau\|_{V_2}^2\leqslant M_0^2$ there exist an integer $k_0\geqslant 1$ and an uncountable $T_0'\subseteq T_0$ such that for all $\tau\in T_0'$, $\sum_{j>k_0}|\lambda_j^\tau|^2<\delta^2/2^6$. Set $S_0'=\{\mathbf{x}_\tau\colon\ \tau\in T_0'\}$. Let $\tau_\varnothing\in T_0'$ be such that $\mathbf{x}_{\tau_\varnothing}$ is a condensation point of S_0'. Then $H_{\tau_\varnothing}=\sum_{j=1}^{\infty}\lambda_j^{\tau_\varnothing}\chi_{t_j^{\tau_\varnothing}}$ and $\operatorname{supp}H_{\tau_\varnothing}=\{t_j^{\tau_\varnothing}\colon\ j\in\mathbb{N}\}\subseteq(0,1)$. For every $1\leqslant j\leqslant k_0$ we pick pairwise disjoint open interval $\Delta_{(\varnothing,j)}$ in $(0,1)$ such that $t_j^{\tau_\varnothing}\in\Delta_{(\varnothing,j)}$. Let $(g_n)_n$ be a sequence in X weak*-converging to $H_{\tau_\varnothing}$. Applying Lemma 39 for $n_0=k_0$, $H=H_{\tau_\varnothing}$, $\Delta_j=\Delta_{(\varnothing,j)}$, $1\leqslant j\leqslant k_0$, and $\varepsilon=\delta/2^2$ we obtain a convex combination $h=h_\varnothing$ of $(g_n)_n$, and for $1\leqslant j\leqslant k_0$, functions $f_j=f_{(\varnothing,j)}$ in V_2^0, and pairs of intervals $(I_j,J_j)=(I_{(\varnothing,j)},J_{(\varnothing,j)})$ satisfying (i) and (ii) of the lemma. We set $\lambda_{(\varnothing,j)}=\lambda_j^{\tau_\varnothing}$ for all $j=1,\dots,k_0$. Notice that $\lambda_{(\varnothing,1)}=1$, $\|f_{(\varnothing,j)}\|_{V_2^0}=\sqrt{2}$, for all $1\leqslant j\leqslant k_0$ and $\|G_\varnothing-h_\varnothing\|_{V_2^0}<\delta/2$.

Let $T_0'' = \{t \in T_0'\colon \mathbf{x}_\tau \in V_\varnothing\}$ and $S_0'' = \{\mathbf{x}_\tau\colon \tau \in T_0''\}$. Since $\mathbf{x}_{\tau_\varnothing} \in V_\varnothing$, T_0'' and S_0'' are uncountable and so by Lemma 38 there exists an uncountable subset $T_\varnothing$ of T_0'' and $\theta_\varnothing \in \{0,1\}^{2k_0}$ such that $S_\varnothing = \{\mathbf{x}_\tau\colon \tau \in T_\varnothing\}$ is hereditarily $(\theta_\varnothing, 2k_0)$-almost condensed. The initial step of the construction is complete.

Suppose that the construction has been carried out up to some $n \geqslant 0$. Consider the sets T_α, $|\alpha| = n$. It is easy to see that there exist an integer $k_{n+1} > k_n$ and uncountable subsets $T_\alpha' \subseteq T_\alpha$ such that for all α with $|\alpha| = n$ the following are satisfied.

(i) $\sum_{j>k_{n+1}} |\lambda_j^\tau|^2 < \delta^2/2^{4n+10}$.
(ii) If $n+1 \geqslant 3$,

$$\min\bigl\{|\lambda_{(\beta,j)}|\colon |\beta| = n-1,\ j \in E_{n-1}\bigr\} \geqslant 2\max\bigl\{\bigl|\lambda_j^\tau\bigr|\colon \tau \in T_\alpha',\ j > k_{n+1}\bigr\}.$$

Let $S_\alpha' = \{\mathbf{x}_\tau\colon \tau \in T_\alpha'\}$, $|\alpha| = n$. Since by our inductive assumption S_α is hereditarily $(\theta_\alpha, 2k_n)$-almost condensed and $S_\alpha' \subseteq S_\alpha \subseteq V_\alpha$ we may choose $\tau_{\alpha^\frown 0}$ and $\tau_{\alpha^\frown 1}$ in T_α' such that for all α with $|\alpha| = n$ the following hold.

(iii) $\mathbf{x}_{\tau_{\alpha^\frown 0}}, \mathbf{x}_{\tau_{\alpha^\frown 1}}$ are $(\theta_\alpha, 2k_n)$-condensation points of S_α'.
(iv) For all $j = 1, \ldots, k_n$, $t_j^{\tau_{\alpha^\frown 0}} <_{\theta_\alpha(2j)} t_j^{\tau_{\alpha^\frown 1}}$.

Moreover using the fact that the family $\{H_\tau\colon \tau \in T_\alpha'\}$ weakly accumulates to zero we may also assume that for every β with $|\beta| = n+1$,

(v) $\{t_k^{\tau_\beta}\colon k_n < k \leqslant k_{n+1}\} \cap (\partial I_{(\alpha,j)} \cup \partial J_{(\alpha,j)}) = \varnothing$, for all $|\alpha| \leqslant n$ and all $1 \leqslant j \leqslant k_{|\alpha|}$.

Since $\mathbf{x}_{\tau_{\alpha^\frown\varepsilon}} \in V_\alpha$, using (v) above we can choose pairwise disjoint open intervals $\Delta_{(\beta,j)}$, for all $|\beta| = n+1$ and all $1 \leqslant j \leqslant k_{n+1}$ such that the following are satisfied.

(vi) For every $|\alpha| = n$, $\varepsilon \in \{0,1\}$, $1 \leqslant j \leqslant k_{n+1}$, $t_j^{\tau_{\alpha^\frown\varepsilon}} \in \Delta_{(\alpha^\frown\varepsilon,j)} \subseteq \mathring{J}_{(\alpha,j)}$.
(vii) For every $|\beta| = n+1$, $k_n < k \leqslant k_{n+1}$, $1 \leqslant j \leqslant k_n$ and $|\alpha| \leqslant n$, either $\Delta_{(\beta,k)} \cap I_{(\alpha,j)} = \varnothing$ or $\Delta_{(\beta,k)} \subseteq \mathring{I}_{(\alpha,j)} \setminus \partial J_{(\alpha,j)}$.

We may also assume that the length of each $\Delta_{(\beta,k)}$, $|\beta| = n+1$, $1 \leqslant k \leqslant k_{n+1}$, is small enough to ensure that

(viii) For every $\mathcal{P} = \{t_0 < \cdots < t_p\} \subseteq \Delta_{(\beta,k)}$,

$$\sum_{\{(\alpha,j)\colon |\alpha| \leqslant n,\, 1 \leqslant j \leqslant k_{|\alpha|}\}} \left(\sum_{i=0}^{p-1} \bigl|f_{(\alpha,j)}(t_i) - f_{(\alpha,j)}(t_{i+1})\bigr|^2\right) \leqslant \frac{1}{2^{2n+2}k_{n+1}}.$$

Now for each β with $|\beta| = n+1$, we choose a sequence $(g_n^\beta)_n$ in X weak*-converging to H_{τ_β}. Applying Lemma 39 for $n_0 = k_{n+1}$, $H = H_{\tau_\beta}$, $\Delta_j = \Delta_{(\beta,j)}$, $1 \leqslant j \leqslant k_{n+1}$, and $\varepsilon = \delta/2^{2(n+2)}$ we obtain a convex combination $h = h_\beta$ of $(g_n^\beta)_n$, functions $f_j = f_{(\beta,j)}$ in V_2^0, and pairs of intervals $(I_j, J_j) = (I_{(\beta,j)}, J_{(\beta,j)})$ satisfying the following.

(ix) For every β with $|\beta| = n+1$ and $1 \leqslant j \leqslant k_{n+1}$, $t_j^{\tau_\beta} \in \mathring{J}_{(\beta,j)} \subseteq I_{(\beta,j)} \subseteq \Delta_{(\beta,j)}$, $\operatorname{supp} f_{(\beta,j)} \subseteq \mathring{J}_{(\beta,j)}$, $J_{(\beta,j)} = \{t \in [0,1]\colon f_{(\beta,j)}(t) = 1\}$, $f_{(\beta,j)}$ is piecewise linear and $\| f_{(\beta,j)} \|_{V_2^0} = \sqrt{2}$.

(x) For every β with $|\beta| = n+1$ and $1 \leqslant j \leqslant k_{n+1}$, setting $\lambda_{(\beta,j)} = \lambda_j^{\tau_\beta}$ and $G_\beta = \sum_{j=1}^{k_{n+1}} \lambda_{(\beta,j)} f_{(\alpha,j)}$, we have that $\lambda_{(\beta,1)} = \lambda_1^{\tau_\beta} = 1$ and $\|G_\beta - h_\beta\|_{V_2^0} < \delta/2^{2n+3}$.

For all $|\alpha| = n$ and $\varepsilon \in \{0,1\}$, we set

$$T'_{\alpha^\frown\varepsilon} = \{\tau \in T'_\alpha\colon \mathbf{x}_\tau \in V_{\alpha^\frown\varepsilon}\} \quad \text{and} \quad S'_{\alpha^\frown\varepsilon} = \{\mathbf{x}_\tau\colon \tau \in T'_{\alpha^\frown\varepsilon}\}.$$

Since $\tau_{\alpha^\frown\varepsilon} \in T'_{\alpha^\frown\varepsilon}$, by condition (iii) above, the sets $T'_{\alpha^\frown\varepsilon}$ and $S'_{\alpha^\frown\varepsilon}$ are uncountable. By Lemma 38 there exist an uncountable subset $T_{\alpha^\frown\varepsilon} \subseteq T'_{\alpha^\frown\varepsilon}$ and $\theta_{\alpha^\frown\varepsilon} \in \{0,1\}^{2k_{n+1}}$ with $\theta_\alpha \sqsubset \theta_{\alpha^\frown\varepsilon}$ such that the set $S_{\alpha^\frown\varepsilon} = \{\mathbf{x}_\tau\colon \tau \in T_{\alpha^\frown\varepsilon}\}$ is a hereditarily $(\theta_{\alpha^\frown\varepsilon}, 2k_{n+1})$-almost condensed subset of $V_{\alpha^\frown\varepsilon}$.

It is easily verified that conditions (i) and (ii) above yield properties (F6) and (F7) of the definition of the forest of tree families (Definition 43). Since $I_{(\beta,j)} \subseteq \Delta_{(\beta,j)}$ and the intervals $\Delta_{(\beta,j)}$ are pairwise disjoint we have that for every $(\beta_1, j_1) \neq (\beta_2, j_2)$ with $|\beta_1| = |\beta_2| = n+1$ and $1 \leqslant j_1, j_2 \leqslant k_{n+1}$, $I_{(\beta_1,j_1)} \cap I_{(\beta_2,j_2)} = \varnothing$, which gives (F2). Also conditions (vii) and (viii) remain true for $I_{(\beta,j)}$ in place of $\Delta_{(\beta,j)}$ and so we obtain (F3) and (F4). Moreover (F1) and (F5) follow immediately by (ix) and (x). Finally notice that condition (iv) and the fact that $\theta_\alpha \sqsubset \theta_\beta$ for all $\alpha \sqsubset \beta$, yield that the tree family $\mathcal{F}_{(\beta,j)}$ is $\theta_\beta(2j)$-monotone, for all $\beta \in \mathcal{D}$ and $j \in E_{|\beta|}$. The proof of the inductive step as well as of the lemma is complete. □

Proof of Theorem 50. Let X be a subspace of V_2^0 with non-separable dual. By Lemma 51 and Theorem 45, we obtain that there exists $M_0 > 0$ such that for every $\delta > 0$ there exist a forest $\mathcal{F}$ of monotone tree families and a family $(h_\alpha)_{\alpha\in\mathcal{D}} \subseteq X$ such that the diagonal family of functions $(G_\alpha)_{\alpha\in\mathcal{D}}$ corresponding to $\mathcal{F}$ satisfies the following.

(i) $\sum_{\alpha\in\mathcal{D}} \|G_\alpha - h_\alpha\|_{V_2^0} \leqslant \delta$.

(ii) For all $n \geqslant 0$ and all sequences of scalars $(\mu_\alpha)_{|\alpha|\leqslant n}$,

$$\Big\| \sum_{|\alpha|\leqslant n} \mu_\alpha e_\alpha \Big\|_{TF} \leqslant \Big\| \sum_{|\alpha|\leqslant n} \mu_\alpha G_\alpha \Big\|_{V_2^0} \leqslant C \Big\| \sum_{|\alpha|\leqslant n} \mu_\alpha e_\alpha \Big\|_{TF},$$

where $C = 32\sqrt{11}(M_0^2 + 1)^{1/2}$.

By (i) and (ii) above we obtain that the family $(h_\alpha)_{\alpha\in\mathcal{D}}$ is equivalent to the Schauder basis of *TF* provided that δ is sufficiently small. □

8. Subspaces of *JT*

This section is devoted to the proofs of Theorem 1, Corollary 2 and Theorem 3 stated in the introduction. Recall that the James Tree space is the completion of $c_{00}(\mathcal{D})$ under the norm $\|x\|_{JT} = \sup(\sum_{i=1}^m (\mathcal{I}_i^*(x))^2)^{1/2}$ where the supremum is taken over all finite families $\{\mathcal{I}_i\}_{i=1}^m$ of pairwise disjoint segments of $\mathcal{D}$. (As usual for $x = \sum_{\alpha\in\mathcal{D}} \mu_\alpha e_\alpha \in c_{00}(\mathcal{D})$ and for a segment $\mathcal{I}$ of $\mathcal{D}$, we set $\mathcal{I}^*(x) = \sum_{\alpha\in\mathcal{D}} \mu_\alpha$.) We will first introduce the analogues of the notions of the tree

families and of the forest of tree families in the space *JT*. The definitions and the proofs here are much simpler than the corresponding ones in the space V_2^0.

8.1. Tree families in JT

Definition 52. For every $\alpha \in \mathcal{D}$, let $x_\alpha \in JT$, $\mathcal{I}_\alpha$ be a segment of $\mathcal{D}$ and $\lambda_\alpha \in \mathbb{R}$ with $\lambda_\varnothing \neq 0$. The family $\mathcal{F} = (x_\alpha, \mathcal{I}_\alpha, \lambda_\alpha)_{\alpha\in\mathcal{D}}$ is called a tree family in *JT* provided that for all $\alpha \in \mathcal{D}$ the following are satisfied.

(1) $x_\alpha \in \operatorname{conv}\{e_\beta\colon \beta \in \mathcal{I}_\alpha\}$, that is $x_\alpha = \sum_{\beta\in\mathcal{I}_\alpha} r_\beta e_\beta$ where $\sum_{\beta\in\mathcal{I}_\alpha} r_\beta = 1$ and for all $\beta \in \mathcal{I}_\alpha$, $0 \leqslant r_\beta \leqslant 1$.
(2) $\mathcal{I}_{\alpha^\frown 0} \perp \mathcal{I}_{\alpha^\frown 1}$ and $\max \mathcal{I}_\alpha \sqsubseteq \min \mathcal{I}_{\alpha^\frown 0} \wedge \min \mathcal{I}_{\alpha^\frown 1}$.
(3) For every $\varepsilon \in \{0,1\}$, $|\lambda_{\alpha^\frown\varepsilon} - \lambda_\alpha| < |\lambda_\alpha|/2^{2|\alpha|+2}$.

Notation 3. Let $\mathcal{F} = (x_\alpha, \mathcal{I}_\alpha, \lambda_\alpha)_{\alpha\in\mathcal{D}}$ be a tree family in *JT*.

(1) For $\alpha \in \mathcal{D}$ let $\beta_\alpha = \min \mathcal{I}_{\alpha^\frown 0} \wedge \min \mathcal{I}_{\alpha^\frown 1}$. Let $\mathcal{J}_\varnothing = \{\beta \in \mathcal{D}\colon \min \mathcal{I}_\varnothing \sqsubseteq \beta \sqsubseteq \beta_\varnothing\}$ and for every $\alpha \in \mathcal{D}$ with $|\alpha| \geqslant 1$, let $\mathcal{J}_\alpha = \{\beta \in \mathcal{D}\colon \beta_{\alpha^-} \sqsubset \beta \sqsubseteq \beta_\alpha\}$ where α^- denotes the unique immediate predecessor of α, that is $\alpha^- = \max\{\beta \in \mathcal{D}\colon \beta \sqsubset \alpha\}$. Notice that $\mathcal{I}_\alpha \subseteq J_\alpha$ for all $\alpha \in \mathcal{D}$.

(2) Given the family of segments $(\mathcal{J}_\alpha)_{\alpha\in\mathcal{D}}$ defined above, we set $T_\mathcal{F} = \bigcup_{\alpha\in\mathcal{D}} \mathcal{J}_\alpha \subseteq \mathcal{D}$. It is clear that $T_\mathcal{F}$ with the induced by $\mathcal{D}$ partial ordering $\sqsubseteq$ is a subtree of $\mathcal{D}$ with root $\min \mathcal{I}_\varnothing$. Moreover $T_\mathcal{F}$ is a complete subtree of $\mathcal{D}$, that is every segment of $T_\mathcal{F}$ is a segment of $\mathcal{D}$.

(3) For a segment $\mathcal{I}$ of $\mathcal{D}$ we set $\mathcal{I}^\mathcal{F} = \{\alpha \in \mathcal{D}\colon \mathcal{I}_\alpha \cap \mathcal{I} \neq \varnothing\}$. It is easy to see that $\mathcal{I}^\mathcal{F}$ is a segment of $\mathcal{D}$. We also set $\mathring{\mathcal{I}}^\mathcal{F} = \mathcal{I}^\mathcal{F} \setminus \{\min \mathcal{I}^\mathcal{F}, \max \mathcal{I}^\mathcal{F}\}$. Notice that for all $\alpha \in \mathring{\mathcal{I}}^\mathcal{F}$, $\mathcal{I}_\alpha \subseteq \mathcal{J}_\alpha \subseteq \mathcal{I}$. Also if $\mathcal{I}_1 \cap \mathcal{I}_2 = \varnothing$ then $\mathring{\mathcal{I}}_1^\mathcal{F} \cap \mathring{\mathcal{I}}_2^\mathcal{F} = \varnothing$. Generally if $\mathcal{I}_1 \cap \mathcal{I}_2 = \varnothing$ then either $\mathcal{I}_1^\mathcal{F} \cap \mathcal{I}_2^\mathcal{F} = \varnothing$, or $\min \mathcal{I}_1^\mathcal{F} = \max \mathcal{I}_2^\mathcal{F}$, or $\max \mathcal{I}_1^\mathcal{F} = \min \mathcal{I}_2^\mathcal{F}$.

The next lemma is straightforward.

Lemma 53. *Let $\mathcal{F} = (x_\alpha, \mathcal{I}_\alpha, \lambda_\alpha)_{\alpha\in\mathcal{D}}$ be a tree family in JT, $\mathcal{I}$ be a finite segment of $\mathcal{D}$. Then for every $(\mu_\alpha)_{\alpha\in\mathcal{D}} \in c_{00}(\mathcal{D})$ the following hold.*

(i) $\mathcal{I}^*(\sum_{\alpha\in\mathcal{D}} \mu_\alpha x_\alpha) = \mathcal{I}^*(\sum_{\alpha\in\mathcal{I}^\mathcal{F}} \mu_\alpha x_\alpha)$.
(ii) *If $\mathcal{I}^\mathcal{F} \neq \varnothing$, $\alpha_0 = \min \mathcal{I}^\mathcal{F}$ and $\beta_0 = \max \mathcal{I}^\mathcal{F}$ then*

$$\left|\mathcal{I}^*\left(\sum_{\alpha\in\mathcal{D}} \mu_\alpha x_\alpha\right)\right| \leqslant |\mu_{\alpha_0}| \left|\sum_{\alpha\in\mathcal{I}\cap\mathcal{I}_{\alpha_0}} r_\alpha\right| + \left|\sum_{\alpha\in\mathring{\mathcal{I}}^\mathcal{F}} \mu_\alpha\right| + |\mu_{\beta_0}| \left|\sum_{\alpha\in\mathcal{I}\cap\mathcal{I}_{\beta_0}} r_\alpha\right|. \tag{50}$$

Proposition 54. *Let $\mathcal{F} = (x_\alpha, \mathcal{I}_\alpha, \lambda_\alpha)_{\alpha\in\mathcal{D}}$ be a tree family in JT and let $X_\mathcal{F} = \overline{\langle\{x_\alpha\}_{\alpha\in\mathcal{D}}\rangle}$ be the closed subspace of JT generated by $\{x_\alpha\}_{\alpha\in\mathcal{D}}$. Then the following hold.*

(1) *The sequence $(x_\alpha)_{\alpha\in\mathcal{D}}$ is equivalent under the natural ordering of $\mathcal{D}$ to the usual basis $(e_\alpha)_{\alpha\in\mathcal{D}}$ of JT. In particular for every $n \geqslant 0$ and every sequence of scalars $(\mu_\alpha)_{|\alpha|\leqslant n}$,*

$$\left\|\sum_{|\alpha|\leqslant n} \mu_\alpha e_\alpha\right\|_{JT} \leqslant \left\|\sum_{|\alpha|\leqslant n} \mu_\alpha x_\alpha\right\|_{JT} \leqslant 3\left\|\sum_{|\alpha|\leqslant n} \mu_\alpha e_\alpha\right\|_{JT}. \tag{51}$$

(2) *There exists a projection $P: JT \to X_\mathcal{F}$ such that $\|P\| \leqslant 3$.*

Proof. (1) We start with the lower estimate. For every segment $\mathcal{I}$ of $\mathcal{D}$, let $\phi_{\mathcal{F}}(\mathcal{I}) = \bigcup_{\alpha\in\mathcal{I}} \mathcal{J}_\alpha \subseteq T_{\mathcal{F}}$. By the properties of $\mathcal{J}_\alpha$ we get that for every family $\{\mathcal{I}_i\}_{i=1}^k$ of disjoint segments of $\mathcal{D}$, $\{\phi_{\mathcal{F}}(\mathcal{I}_i)\}_{i=1}^k$ is again a family of disjoint segments of $\mathcal{D}$. Moreover for every sequence of scalars $(\mu_\alpha)_{\alpha\in\mathcal{D}} \in c_{00}(\mathcal{D})$ and every segment $\mathcal{I}$ of $\mathcal{D}$, we have $\mathcal{I}^*(\sum_{\alpha\in\mathcal{D}} \mu_\alpha e_\alpha) = \sum_{\alpha\in\mathcal{I}} \mu_\alpha = \phi_{\mathcal{F}}(\mathcal{I})^*(\sum_{\alpha\in\mathcal{D}} \mu_\alpha x_\alpha)$ and so $\| \sum_{\alpha\in\mathcal{D}} \mu_\alpha e_\alpha\|_{JT} \leqslant \| \sum_{\alpha\in\mathcal{D}} \mu_\alpha x_\alpha\|_{JT}$.

We pass now to the upper estimate. Let $x = \sum_{\alpha\in\mathcal{D}} \mu_\alpha x_\alpha$, where $(\mu_\alpha)_{\alpha\in\mathcal{D}} \in c_{00}(\mathcal{D})$ and let $\mathcal{I}_1, \dots, \mathcal{I}_k$ be disjoint segments of $\mathcal{D}$ such that $\sum_{i=1}^k (\mathcal{I}_i^*(x))^2 = \|x\|_{JT}^2$. Let $\mathcal{I}_i^{\mathcal{F}} = \{\alpha \in \mathcal{D}\colon \mathcal{I}_i \cap \mathcal{I}_\alpha \neq \varnothing\}$. We may suppose that $\mathcal{I}_i^{\mathcal{F}} \neq \varnothing$ for all $1 \leqslant i \leqslant k$. Let $\alpha_i = \min \mathcal{I}_i^{\mathcal{F}}$, $\mathring{\mathcal{I}}_i^{\mathcal{F}} = \{\alpha \in \mathcal{D}\colon \alpha_i \sqsubset \alpha \sqsubset \beta_i\}$ and $\beta_i = \max \mathcal{I}_i^{\mathcal{F}}$. Then by (50),

$$\sum_{i=1}^k \big(\mathcal{I}_i^*(x)\big)^2 \leqslant 3 \sum_{\alpha\in\mathcal{D}} |\mu_\alpha|^2 \sum_{i\in F_1^\alpha} \Big(\sum_{\beta\in\mathcal{I}_i\cap\mathcal{I}_\alpha} r_\beta \Big)^2 + 3 \sum_{i=1}^k \Big(\sum_{\alpha\in\mathring{\mathcal{I}}_i^{\mathcal{F}}} \mu_\alpha \Big)^2 + 3 \sum_{\alpha\in\mathcal{D}} |\mu_\alpha|^2 \sum_{i\in F_2^\alpha} \Big(\sum_{\beta\in\mathcal{I}_i\cap\mathcal{I}_\alpha} r_\beta \Big)^2 \tag{52}$$

where $F_1^\alpha = \{i \in \{1, \dots, k\}\colon \alpha_i = \alpha\}$ and $F_2^\alpha = \{i \in \{1, \dots, k\}\colon \beta_i = \alpha\}$.

It is easy to see that every summand on the right-hand side of (52) is bounded above by $\| \sum_{\alpha\in\mathcal{D}} \mu_\alpha e_\alpha\|_{JT}^2$ (for the second summand notice that the family $\{\mathring{\mathcal{I}}_i^{\mathcal{F}}\colon 1 \leqslant i \leqslant k\}$ consists of pairwise disjoint segments) and the result follows.

(2) Define $P : JT \to X_{\mathcal{F}}$ by $P(x) = \sum_{\alpha\in\mathcal{D}} \mathcal{J}_\alpha^*(x) x_\alpha$, $x \in JT$. Obviously for every $\alpha \in \mathcal{D}$, $P(x_\alpha) = x_\alpha$. Let $x = \sum_{\alpha\in\mathcal{D}} \mu_\alpha e_\alpha \in c_{00}(\mathcal{D})$. Then by part (1) of the proposition, $\|P(x)\|_{JT} = \| \sum_{\alpha\in\mathcal{D}} \mathcal{J}_\alpha^*(x) x_\alpha\|_{JT} \leqslant 3 \| \sum_{\alpha\in\mathcal{D}} \mathcal{J}_\alpha^*(x) e_\alpha\|_{JT}$. Using similar arguments as those of the proof of the lower JT-estimate in part (1) above, it is easy to see that $\| \sum_{\alpha\in\mathcal{D}} \mathcal{J}_\alpha^*(x) e_\alpha\|_{JT} = \| \sum_{\alpha\in\mathcal{D}} (\sum_{\beta\in\mathcal{J}_\alpha} \mu_\beta) e_\alpha\|_{JT} \leqslant \| \sum_{\alpha\in\mathcal{D}} \mu_\alpha e_\alpha\|_{JT} = \|x\|_{JT}$ and therefore $\|P\| \leqslant 3$. $\square$

Lemma 55. *Let $\mathcal{F} = (x_\alpha, \mathcal{I}_\alpha, \lambda_\alpha)_{\alpha\in\mathcal{D}}$ be a tree family in JT and $\mathcal{I}$ be a finite segment of $\mathcal{D}$. Then for every $(\mu_\alpha)_{\alpha\in\mathcal{D}} \in c_{00}(\mathcal{D})$, $|\mathcal{I}^*(\sum_{\alpha\in\mathcal{D}} \mu_\alpha \lambda_\alpha x_\alpha)| \leqslant 8|\lambda_\varnothing| \| \sum_{\alpha\in\mathcal{I}^{\mathcal{F}}} \mu_\alpha e_\alpha\|_{JT}$.*

Proof. Let $\alpha_0 = \min \mathcal{I}^{\mathcal{F}}$ and $\beta_0 = \max \mathcal{I}^{\mathcal{F}}$. As in Lemma 34 it is shown that

$$\Big| \sum_{\alpha_0\sqsubset\alpha\sqsubset\beta_0} \mu_\alpha \lambda_\alpha \Big| \leqslant 4|\lambda_\varnothing| \Big\| \sum_{\alpha_0\sqsubset\alpha\sqsubset\beta_0} \mu_\alpha e_\alpha \Big\|_{JT}. \tag{53}$$

Also, since $|\lambda_\alpha| \leqslant 2|\lambda_\varnothing|$ for all $\alpha \in \mathcal{D}$, by (53) and (50), $|\mathcal{I}^*(\sum_{\alpha\in\mathcal{D}} \mu_\alpha \lambda_\alpha x_\alpha)| \leqslant |\mu_{\alpha_0}||\lambda_{\alpha_0}| + 4|\lambda_\varnothing| \| \sum_{\alpha_0\sqsubset\alpha\sqsubset\beta_0} \mu_\alpha e_\alpha\|_{JT} + |\mu_{\beta_0}||\lambda_{\beta_0}| \leqslant 8|\lambda_\varnothing| \| \sum_{\alpha\in\mathcal{I}^{\mathcal{F}}} \mu_\alpha e_\alpha\|_{JT}$. $\square$

8.2. Forests of tree families in JT

Definition 56. Let $(E_n)_{n=0}^\infty$, $\mathcal{Z}$, $\mathcal{T}_{(\beta,j)}$, $\mathcal{T}$ be as in Definition 43 and for every $(\beta, j) \in \mathcal{Z}$ let $\mathcal{F}_{(\beta,j)} = (x_{(\alpha,j)}, \mathcal{I}_{(\alpha,j)}, \lambda_{(\alpha,j)})_{(\alpha,j)\in\mathcal{T}_{(\beta,j)}}$ be a tree family in *JT* (by identifying $(\beta^\frown\alpha, j)$ with α, for all $\alpha \in \mathcal{D}$).

The family $\mathcal{F} = (\mathcal{F}_{(\beta,j)})_{(\beta,j)\in\mathcal{Z}} = (x_{(\alpha,j)}, \mathcal{I}_{(\alpha,j)}, \lambda_{(\alpha,j)})_{(\alpha,j)\in\mathcal{T}}$ will be called a forest of tree families in *JT* (determined by the sequence $(E_n)_n$), if the following conditions are satisfied.

(1) For every $(\alpha_1, j_1) \neq (\alpha_2, j_2)$ in $\mathcal{T}$ with $|\alpha_1| = |\alpha_2|$, $\mathcal{I}_{(\alpha_1,j_1)} \perp \mathcal{I}_{(\alpha_2,j_2)}$.
(2) For every $n \geqslant 1$, every $(\beta, k) \in \mathcal{Z}$ with $|\beta| = n$ and every $(\alpha, j) \in \mathcal{T}$ with $|\alpha| \leqslant n-1$, either $\mathcal{I}_{(\alpha,j)} \perp \mathcal{I}_{(\beta,k)}$ or $\max \mathcal{I}_{(\alpha,j)} \sqsubset \min \mathcal{I}_{(\beta,k)}$.
(3) For every $\alpha \in \mathcal{D}$, $\lambda_{(\alpha,1)} = 1$.
(4) For every $n \geqslant 1$ and every $\beta \in \mathcal{D}$ with $|\beta| = n$, $\sum_{j\in E_n} |\lambda_{(\beta,j)}|^2 \leqslant 1/2^{2n}$.

Notation 4. Let $\mathcal{F} = (\mathcal{F}_{(\beta,j)})_{(\beta,j)\in\mathcal{Z}} = (x_{(\alpha,j)}, \mathcal{I}_{(\alpha,j)}, \lambda_{(\alpha,j)})_{(\alpha,j)\in\mathcal{T}}$ be a forest of tree families in *JT*.

(1) On the set $\mathcal{T}$ we define the following partial ordering induced by the forest $\mathcal{F}$: $(\alpha_1, j_1) \prec (\alpha_2, j_2)$ if $\max \mathcal{I}_{(\alpha_1,j_1)} \sqsubset \min \mathcal{I}_{(\alpha_2,j_2)}$ and $(\alpha_1, j_1) \preccurlyeq (\alpha_2, j_2)$ if either $(\alpha_1, j_1) = (\alpha_2, j_2)$ or $(\alpha_1, j_1) \prec (\alpha_2, j_2)$. It is easily shown that properties (T1)–(T3) of Subsection 6.2 hold also here (where in place of (T2) we now have that $(\alpha_1, j_1), (\alpha_2, j_2) \in \mathcal{T}$ are incomparable if and only if $\mathcal{I}_{(\alpha_1,j_1)} \perp \mathcal{I}_{(\alpha_2,j_2)}$), and so $(\mathcal{T}, \preccurlyeq)$ is a tree.

(2) For a segment $\mathcal{I}$ of $\mathcal{D}$ let $\mathcal{I}^{\mathcal{F}} = \{(\alpha, j) \in \mathcal{T}: \mathcal{I} \cap \mathcal{I}_{(\alpha,j)} \neq \varnothing\}$. Also for every $(\beta, j) \in \mathcal{Z}$, let $\mathcal{I}^{\mathcal{F}_{(\beta,j)}} = \{(\alpha, j) \in \mathcal{T}_{(\beta,j)}: \mathcal{I} \cap \mathcal{I}_{(\alpha,j)} \neq \varnothing\}$. Notice that $\mathcal{I}^{\mathcal{F}}, \mathcal{I}^{\mathcal{F}_{(\beta,j)}}, (\beta, j) \in \mathcal{Z}$, are segments of $\mathcal{T}$. Moreover setting $\mathcal{I}^{\mathcal{Z}} = \{(\beta, j) \in \mathcal{Z}: \mathcal{I}^{\mathcal{F}_{(\beta,j)}} \neq \varnothing\}$ then $\mathcal{I}^{\mathcal{F}} = \bigcup_{(\beta,j)\in\mathcal{I}^{\mathcal{Z}}} \mathcal{I}^{\mathcal{F}_{(\beta,j)}}$.

(3) If $\mathcal{I}$ is a finite segment and $\mathcal{I}^{\mathcal{F}} \neq \varnothing$ then $\mathcal{I}^{\mathcal{Z}}$ takes the form of an $\prec$-increasing sequence $\mathcal{I}^{\mathcal{Z}} = \{(\beta_1, j_1) \prec \cdots \prec (\beta_l, j_l)\}$, and so we may write $\mathcal{I}^{\mathcal{F}} = \bigcup_{k=1}^{l} \mathcal{I}_k^{\mathcal{F}}$, where $\mathcal{I}_k^{\mathcal{F}} = \mathcal{I}^{\mathcal{F}_{(\beta_k,j_k)}}$ for all $1 \leqslant k \leqslant l$. We will call $\{\mathcal{I}_k^{\mathcal{F}}: 1 \leqslant k \leqslant l\}$ the *analysis* of $\mathcal{I}^{\mathcal{F}}$ in $\mathcal{T}$. It is easy to see that an analogue of Fact 44 holds for $\mathcal{I}^{\mathcal{F}}$ as well.

(4) For every $(\beta, j) \in \mathcal{Z}$ we also set $T_{\mathcal{F}_{(\beta,j)}} = \bigcup_{(\alpha,j)\in\mathcal{T}_{(\beta,j)}} \mathcal{J}_{(\alpha,j)}$, where the segments $\mathcal{J}_{(\alpha,j)} \supseteq \mathcal{I}_{(\alpha,j)}$ are defined for the tree family $\mathcal{F}_{(\beta,j)}$ as in Notation 3. Notice that for $(\beta_1, j_1) \neq (\beta_2, j_2)$, $T_{\mathcal{F}_{(\beta_1,j_1)}} \cap T_{\mathcal{F}_{(\beta_2,j_2)}} = \varnothing$.

(5) The *diagonal family of vectors corresponding to* $\mathcal{F}$ is the family $(y_\alpha)_{\alpha\in\mathcal{D}}$ defined by $y_\alpha = \sum_{j\in\bigcup_{n=0}^{|\alpha|} E_n} \lambda_{(\alpha,j)} x_{(\alpha,j)}$ for all $\alpha \in \mathcal{D}$.

Lemma 57. *Let $\mathcal{F}$ be a forest of tree families in JT and $(y_\alpha)_{\alpha\in\mathcal{D}}$ be the diagonal family of vectors corresponding to $\mathcal{F}$. Let $\mathcal{I}$ be a finite segment of $\mathcal{D}$ with $\mathcal{I}^{\mathcal{Z}} = \{(\beta_1, j_1) \prec \cdots \prec (\beta_l, j_l)\}$, and let $\{\mathcal{I}_k^{\mathcal{F}}: 1 \leqslant k \leqslant l\}$ be the analysis of $\mathcal{I}^{\mathcal{F}}$ in $\mathcal{T}$. Let $(\mu_\alpha)_{\alpha\in\mathcal{D}} \in c_{00}(\mathcal{D})$ and $y = \sum_{\alpha\in\mathcal{D}} \mu_\alpha y_\alpha$. Then the following are satisfied.*

(1) $\mathcal{I}^*(y) = \sum_{k=1}^{l} \mathcal{I}^*(y_{(\beta_k,j_k)})$ *where* $y_{(\beta_k,j_k)} = \sum_{\{\alpha\in\mathcal{D}:\ \beta_k\sqsubseteq\alpha\}} \mu_\alpha \lambda_{(\alpha,j_k)} x_{(\alpha,j_k)}$, *for all* $1 \leqslant k \leqslant l$.
(2) $(\mathcal{I}^*(y))^2 \leqslant 2(\mathcal{I}^*(y_{(\beta_1,j_1)}))^2 + 128 \sum_{k=2}^{l} |\lambda_{(\beta_k,j_k)}|^2 \|\sum_{\alpha\in\mathcal{D}} \mu_\alpha e_\alpha\|_{JT}^2$.

Proof. The proof of (1) is easy. We will only show (2). By (1) we have

$$\bigl(\mathcal{I}^*(y)\bigr)^2 \leqslant 2\bigl(\mathcal{I}^*(y_{(\beta_1,j_1)})\bigr)^2 + 2\left(\sum_{k=2}^{l} \mathcal{I}^*(y_{(\beta_k,j_k)})\right)^2. \tag{54}$$

Notice that $|\mathcal{I}^*(y_{(\beta_k,j_k)})| = |\mathcal{I}^*(\sum_{\alpha\in\mathrm{pr}_{\mathcal{D}}(\mathcal{I}_k^{\mathcal{F}})} \mu_\alpha x_\alpha)|$ and so by Lemma 55 we get that $|\mathcal{I}^*(y_{(\beta_k,j_k)})| \leqslant 8|\lambda_{(\beta_k,j_k)}| \|\sum_{\alpha\in\mathrm{pr}_{\mathcal{D}}(\mathcal{I}_k^{\mathcal{F}})} \mu_\alpha e_\alpha\|_{JT}$, for all $2 \leqslant k \leqslant l$. Therefore

$$\left(\sum_{k=2}^{l}\mathcal{I}^*(y_{(\beta_k,j_k)})\right)^2 \leqslant 64\sum_{k=2}^{l}|\lambda_{(\beta_k,j_k)}|^2\sum_{k=2}^{l}\left\|\sum_{\alpha\in\mathrm{pr}_{\mathcal{D}}(\mathcal{I}_k^{\mathcal{F}})}\mu_\alpha e_\alpha\right\|_{JT}^2$$
$$\leqslant 64\sum_{k=2}^{l}|\lambda_{(\beta_k,j_k)}|^2\left\|\sum_{\alpha\in\mathcal{D}}\mu_\alpha e_\alpha\right\|_{JT}^2 \tag{55}$$

where in the last inequality we used the fact that the segments $\mathrm{pr}_{\mathcal{D}}(\mathcal{I}_k^{\mathcal{F}})$, $2\leqslant k\leqslant l$, are pairwise disjoint. Substituting in (54) the result follows. □

Proposition 58. *Let $\mathcal{F}$ be a forest of tree families in JT determined by a sequence $(E_n)_{n=0}^{\infty}$ of successive intervals of $\mathbb{N}$, $(y_\alpha)_{\alpha\in\mathcal{D}}$ be the diagonal family of vectors corresponding to $\mathcal{F}$ and $\Lambda_0=\sum_{j\in E_0}|\lambda_{(\varnothing,j)}|^2$.*

(1) *For every $n\geqslant 0$ and every sequence of scalars $(\mu_\alpha)_{|\alpha|\leqslant n}$, we have that*

$$\left\|\sum_{|\alpha|\leqslant n}\mu_\alpha e_\alpha\right\|_{JT}\leqslant\left\|\sum_{|\alpha|\leqslant n}\mu_\alpha y_\alpha\right\|_{JT}\leqslant 32(\Lambda_0+1)^{1/2}\left\|\sum_{|\alpha|\leqslant n}\mu_\alpha e_\alpha\right\|_{JT}. \tag{56}$$

(2) *There exists a bounded projection $Q:JT\to\overline{\langle y_\alpha\colon \alpha\in\mathcal{D}\rangle}^{\|\cdot\|}$ with $\|Q\|\leqslant 96(\Lambda_0+1)^{1/2}$.*

Proof. (1) By Proposition 54 we have $\|\sum_{\alpha\in\mathcal{D}}\mu_\alpha e_\alpha\|_{JT}\leqslant\|\sum_{\alpha\in\mathcal{D}}\mu_\alpha x_{(\alpha,1)}\|_{JT}$. Since for all $\alpha\in\mathcal{D}$ and all $(\beta,j)\neq(\varnothing,1)$, $\mathrm{supp}\,x_{(\alpha,1)}\subseteq\mathcal{I}_{(\alpha,1)}\subseteq T_{\mathcal{F}_{(\varnothing,1)}}$, $T_{\mathcal{F}_{(\varnothing,1)}}\cap T_{\mathcal{F}_{(\beta,j)}}=\varnothing$ and $|\lambda_{(\alpha,1)}|=1$, we get that $\|\sum_{\alpha\in\mathcal{D}}\mu_\alpha x_{(\alpha,1)}\|_{JT}\leqslant\|\sum_{\alpha\in\mathcal{D}}\mu_\alpha y_\alpha\|_{JT}$ and the lower estimate follows.

To show the upper estimate let $\mathcal{I}_i$, $1\leqslant i\leqslant m$, be disjoint segments of $\mathcal{D}$. Without loss of generality we may suppose that $\mathcal{I}_i^{\mathcal{F}}\neq\varnothing$, for every $1\leqslant i\leqslant m$. Let $\mathcal{I}_i^{\mathcal{Z}}=\{(\beta_1^i,j_1^i)\prec\cdots\prec(\beta_{l_i}^i,j_{l_i}^i)\}$ and $\{\mathcal{I}_{i,k}^{\mathcal{F}}\colon 1\leqslant k\leqslant l_i\}$ be the analysis of $\mathcal{I}_i^{\mathcal{F}}$.

For every $(\beta,j)\in\mathcal{Z}$, set $[m]_{(\beta,j)}^1=\{i\in\{1,\dots,m\}\colon (\beta_1^i,j_1^i)=(\beta,j)\}$ and $y_{(\beta,j)}=\sum_{\{\alpha\in\mathcal{D}\colon \beta\sqsubseteq\alpha\}}\mu_\alpha\lambda_{(\alpha,j)}x_{(\alpha,j)}$. By Lemma 57 we have that

$$\sum_{i=1}^{m}\big(\mathcal{I}_i^*(y)\big)^2\leqslant 2\sum_{(\beta,j)\in\mathcal{Z}}\sum_{i\in[m]_{(\beta,j)}^1}\big(\mathcal{I}_i^*(y_{(\beta,j)})\big)^2$$
$$+128\sum_{i=1}^{m}\sum_{k=2}^{l_i}|\lambda_{(\beta_k^i,j_k^i)}|^2\left\|\sum_{\alpha\in\mathcal{D}}\mu_\alpha e_\alpha\right\|_{JT}^2. \tag{57}$$

Let ${}_*\mathcal{I}_i^{\mathcal{Z}}=\mathcal{I}_i^{\mathcal{Z}}\setminus\{(\beta_1^i,j_1^i)\}$. Since $\mathcal{I}_i$, $1\leqslant i\leqslant m$ are pairwise disjoint, it is easy to see that ${}_*\mathcal{I}_i^{\mathcal{Z}}\cap{}_*\mathcal{I}_j^{\mathcal{Z}}=\varnothing$ for $i\neq j$ and so by (4) of Definition 56,

$$\sum_{i=1}^{m}\sum_{k=2}^{l_i}|\lambda_{(\beta_k^i,j_k^i)}|^2\leqslant 1. \tag{58}$$

Moreover as the segments $\mathcal{I}_i$, $1 \leqslant i \leqslant m$, are pairwise disjoint we get that

$$\sum_{(\beta,j)\in\mathcal{Z}} \sum_{i\in[m]^1_{(\beta,j)}} \big(\mathcal{I}_i^*(y_{(\beta,j)})\big)^2 \leqslant \sum_{(\beta,j)\in\mathcal{Z}} \|y_{(\beta,j)}\|_{JT}^2.$$

By Proposition 54, we have

$$\|y_{(\beta,j)}\|_{JT} = \Big\|\sum_{\beta\sqsubseteq\alpha} \mu_\alpha \lambda_{(\alpha,j)} x_{(\alpha,j)}\Big\|_{JT} \leqslant 3\Big\|\sum_{\beta\sqsubseteq\alpha} \mu_\alpha \lambda_{(\alpha,j)} e_\alpha\Big\|_{JT}$$

and using (53),

$$\Big\|\sum_{\beta\sqsubseteq\alpha} \mu_\alpha \lambda_{(\alpha,j)} e_\alpha\Big\|_{JT} \leqslant 4|\lambda_{(\beta,j)}|\Big\|\sum_{\beta\sqsubseteq\alpha} \mu_\alpha e_\alpha\Big\|_{JT}.$$

Hence

$$\|y_{(\beta,j)}\|_{JT} \leqslant 12|\lambda_{(\beta,j)}|\Big\|\sum_{\beta\sqsubseteq\alpha} \mu_\alpha e_\alpha\Big\|_{JT} \leqslant 12|\lambda_{(\beta,j)}|\Big\|\sum_{\alpha\in\mathcal{D}} \mu_\alpha e_\alpha\Big\|_{JT}.$$

By the above we conclude that

$$\sum_{(\beta,j)\in\mathcal{Z}} \sum_{i\in[m]^1_{(\beta,j)}} \big(\mathcal{I}_i^*(y_{(\beta,j)})\big)^2 \leqslant 144 \sum_{(\beta,j)\in\mathcal{Z}} |\lambda_{(\beta,j)}|^2 \Big\|\sum_{\alpha\in\mathcal{D}} \mu_\alpha e_\alpha\Big\|_{JT}^2. \tag{59}$$

Substituting (58) and (59) in (57) the result follows.

(2) Let us observe that by (56) and (51) the correspondence $x_{(\alpha,1)} \to y_\alpha$ extends to an isomorphism $T : \overline{\langle x_{(\alpha,1)} \colon \alpha \in \mathcal{D}\rangle}^{\|\cdot\|} \to \overline{\langle y_\alpha \colon \alpha \in \mathcal{D}\rangle}^{\|\cdot\|}$. Moreover by part (2) of Proposition 54 there exists a projection $P : JT \to \overline{\langle x_{(\alpha,1)} \colon \alpha \in \mathcal{D}\rangle}^{\|\cdot\|}$. It follows readily that for each $\alpha \in \mathcal{D}$, $P(y_\alpha) = x_{(\alpha,1)}$. Setting $Q = T \circ P$ we easily check that Q is the desired projection. □

As is well known the space JT^{**} is isomorphic to $JT \oplus \ell_2(\{0,1\}^{\mathbb{N}})$. More precisely for every $\sigma \in \{0,1\}^{\mathbb{N}}$ the sequence $(e_{\sigma|n})_{n\in\mathbb{N}}$ is non-trivial weak Cauchy, $e_{\sigma|n} \xrightarrow{w^*} e_\sigma^{**}$ and $\ell_2(\{0,1\}^{\mathbb{N}})$ is isomorphic to $\overline{\langle e_\sigma^{**} \colon \sigma \in \{0,1\}^{\mathbb{N}}\rangle}^{\|\cdot\|}$. The next proposition is the counterpart of Proposition 23 for subspaces of JT.

Proposition 59. *Let X be a subspace of JT with non-separable dual. Then $X^{**} \cap \ell_2(\{0,1\}^{\mathbb{N}})$ is isomorphic to $\ell_2(\mathfrak{c})$.*

The proof goes along the lines of the proof of Proposition 23 and it is left to the reader.

Let X be a subspace of JT with non-separable dual. As in the case of subspaces of V_2^0 we may assume that there exists an uncountable $\mathcal{H}_0 \subseteq X^{**} \cap \ell_2(\{0,1\}^{\mathbb{N}})$ consisting of elements with infinite and pairwise disjoint support and for every $H \in \mathcal{H}_0$ there exists $\sigma \in \{0,1\}^{\mathbb{N}}$ with $H(\sigma) = 1$. Also the analogue of Lemma 39 for subspaces of JT remains valid.

Proposition 60. *Let X be a subspace of JT with non-separable dual. Then for every $\delta > 0$ there exist a forest $\mathcal{F} = (x_{(\alpha,j)}, \mathcal{I}_{(\alpha,j)}, \lambda_{(\alpha,j)})_{(\alpha,j)\in\mathcal{T}}$ of tree families in JT and a family $\{h_\alpha\}_{\alpha\in\mathcal{D}}$ of elements of X such that $\sum_{\alpha\in\mathcal{D}} \|y_\alpha - h_\alpha\|_{JT} \leqslant \delta$, where $(y_\alpha)_{\alpha\in\mathcal{D}}$ is the diagonal family of vectors corresponding to $\mathcal{F}$.*

The proof of Proposition 60 follows the lines of Lemma 51. Notice that the construction of y_α and h_α does not require monotone tree families which in turn permits us to handle the inductive construction by using the standard concept of condensation points of subsets of $\{0,1\}^{\mathbb{N}}$.

Theorem 61. *Let X be a subspace of JT with non-separable dual. Then X contains a subspace Y complemented in JT and isomorphic to JT.*

Proof. By Proposition 60 for every $\delta > 0$ there exist a forest $\mathcal{F}$ and a family $(h_\alpha)_{\alpha\in\mathcal{D}}$ in X such that $\sum_{\alpha\in\mathcal{D}} \|y_\alpha - h_\alpha\|_{JT} \leqslant \delta$ where $(y_\alpha)_{\alpha\in\mathcal{D}}$ is the diagonal family of vectors corresponding to $\mathcal{F}$. By Proposition 58 the subspace $\overline{\langle y_\alpha \colon \alpha \in \mathcal{D}\rangle}^{\|\cdot\|}$ is complemented in JT and isomorphic to JT. Hence for a small enough $\delta > 0$, the same holds also for the subspace $\overline{\langle h_\alpha \colon \alpha \in \mathcal{D}\rangle}^{\|\cdot\|}$ of X. $\square$

To state some consequences of the above theorem we will need the following.

Lemma 62. *Let X be a subspace of JT. Suppose that JT contains a complemented copy of X and that X contains a complemented copy of JT. Then X is isomorphic to JT.*

Proof. Notice that $JT \approx J \oplus (JT \oplus JT \oplus \cdots)_2 \approx J \oplus (JT \oplus JT \oplus \cdots)_2 \oplus (JT \oplus JT \oplus \cdots)_2 \approx JT \oplus (JT \oplus JT \oplus \cdots)_2 \approx (JT \oplus JT \oplus \cdots)_2$, where J is the James space. The result now follows by applying the Pelczynski decomposition method [11]. $\square$

Corollary 63. *Let X be a complemented subspace of JT with non-separable dual. Then X is isomorphic to JT.*

Proof. Since X is a subspace of JT with non-separable dual, by Theorem 61 we have that X contains a complemented copy of JT. Since X is complemented in JT, by Lemma 62 we have that X is isomorphic to JT. $\square$

Corollary 64. *The space JT is primary.*

Proof. Let $JT = X \oplus Y$. Then either X or Y has non-separable dual and so, by Corollary 63, either X or Y is isomorphic to JT. $\square$

The above result has been proved by A.D. Andrew [1] with a different method. The analogue of Theorem 40 for the space JT remains also valid. In particular the following holds.

Theorem 65. *Let $T : JT \to JT$ be a bounded linear operator such that the dual operator $T^* : JT^* \to JT^*$ has non-separable range. Then there exists a subspace X of JT isomorphic to JT such that the restriction of T on X is an isomorphism and $T[X]$ is complemented in JT.*

The main steps of the proof of Theorem 65 go as follows. First, arguing as in the proof of Theorem 40 we find a subspace Y of *JT* isomorphic to *JT* and such that $T|_Y$ is an isomorphism. Since $T[Y]$ has non-separable dual, by Theorem 61 there exists a subspace X' of $T[Y]$ complemented in *JT* and isomorphic to *JT*. Setting $X = T^{-1}(X') \cap Y$ the result follows.

Remark 5. As we have already mentioned, Theorem 50 yields that every subspace of *TF* with non-separable dual contains a subspace isomorphic to *TF*. Actually it can be shown that the space *TF* satisfies the stronger properties of *JT* stated in the above results. Namely every subspace X of *TF* with non-separable dual contains a subspace Y complemented in *TF* and isomorphic to *TF*. In particular every complemented subspace of *TF* with non-separable dual is isomorphic to *TF* and thus *TF* is primary. Moreover the analogue of Theorem 65 remains valid for the space *TF*.

It is open if *TF* is a complemented subspace of *JF* or more generally if every subspace of *JF* with non-separable dual contains a subspace Y which is isomorphic to *TF* and complemented in *JF*. Also it is unknown if *JF* is primary.

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Kato's square root problem in Banach spaces

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Abstract

Let L be an elliptic differential operator with bounded measurable coefficients, acting in Bochner spaces $L^p(\mathbf{R}^n; X)$ of X-valued functions on $\mathbf{R}^n$. We characterize Kato's square root estimates $\|\sqrt{L}u\|_p \eqsim \|\nabla u\|_p$ and the H^∞-functional calculus of L in terms of R-boundedness properties of the resolvent of L, when X is a Banach function lattice with the UMD property, or a noncommutative L^p space. To do so, we develop various vector-valued analogues of classical objects in Harmonic Analysis, including a maximal function for Bochner spaces. In the special case $X = \mathbf{C}$, we get a new approach to the L^p theory of square roots of elliptic operators, as well as an L^p version of Carleson's inequality.

Keywords: Kato's square root problem; Elliptic operators with bounded measurable coefficients; H^∞-functional calculus; Vector-valued harmonic analysis; UMD spaces; Maximal function; Carleson's inequality

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1. Introduction

The development of a theory of singular integrals for vector functions, which take their values in an infinite-dimensional Banach space, may be viewed as an accelerated replay—with new actors, insight, and considerable improvisation—of the original development in the scalar-valued setting. During the 1980s, this theory advanced from D.L. Burkholder's [13] extension of M. Riesz' classical theorem on the Hilbert transform boundedness, via J. Bourgain's [12], T.R. McConnell's [32] and F. Zimmermann's [42] results on Calderón–Zygmund principal value convolutions and Marcinkiewicz–Mihlin multipliers, to T. Figiel's [19] vector-valued generalization of the $T(1)$ theorem of G. David and J.-L. Journé. More recently, there has been a new boom of activity in developing the vector-valued estimates to match the needs of a wide variety of applications especially in the field of Partial Differential Equations. An important opening move into this direction was made by L. Weis [41]; further developments and references are recorded in [16,30].

The aim of the present paper is to continue the vector-valued program so as to catch up with some of the latest achievements in scalar-valued Harmonic Analysis. More precisely, we are going to develop a Banach space theory for the square roots of elliptic operators appearing in the famous problem of T. Kato, which was recently solved by P. Auscher, S. Hofmann, M. Lacey, A. McIntosh and Ph. Tchamitchian [7], and more generally for the perturbed Dirac operators treated in a subsequent work by A. Axelsson, S. Keith and A. McIntosh [9]. These objects are no longer Calderón–Zygmund operators, and may even fail to have a pointwise defined kernel.

For this reason, their study is considered a move beyond Calderón–Zygmund theory. In the scalar-valued L^p case, this has recently attracted much attention. An extrapolation technique developed by S. Blunck and P. Kunstmann [10] allows to extend L^2 results to the L^p setting for p in an open interval (p_-, p_+), which may be strictly smaller than the whole reflexive range $(1, \infty)$ admissible for classical operators. P. Auscher's memoir [3] presents the large range of applications of this method and demonstrates that the L^p behavior of objects associated with an elliptic operator L (its functional calculus, Riesz transforms, square functions, etc.) is ruled by four critical numbers: $p_-(L)$, $p_+(L)$ (the limits of the range of p's for which the semigroup $(e^{-tL})_{t>0}$ is L^p-bounded), and $q_-(L)$, $q_+(L)$ (the limits of the range of p's for which the family $(\sqrt{t}\nabla e^{-tL})_{t>0}$ is L^p-bounded). In a recent series of papers by P. Auscher and J.M. Martell [4], these results are extended to a more general setting, allowing weighted estimates on spaces of homogeneous type. We also refer to their papers for the history of these developments.

Our work takes a different approach. Since we are aiming at a Banach space-valued theory, where no easier L^2 case is available as a starting point, we cannot rely on an extrapolation

method, but need to work directly in the spaces $L^p(\mathbf{R}^n; X)$. It is interesting, even in the scalar case $X = \mathbf{C}$, to see that the methods from [7] and [9] can in fact be extended to an L^p situation. This requires a set of new techniques. We develop, in particular, a Banach space-valued analogue of the "reduction to the principal part" method used to solve Kato's problem (Theorem 6.2). This is based on adequate off-diagonal estimates (Proposition 6.4), and on the fact that resolvents of an unperturbed Hodge–Dirac operator are, in some sense, equivalent to conditional expectations with respect to the dyadic filtration of $\mathbf{R}^n$ (Corollary 5.6). This result, which is handled in the classical case by a $T(1)$ theorem for Carleson measures (see [5]), is obtained in our context by extending ideas from [9]. To do so, we develop Banach space-valued analogues of classical estimates such as Poincaré's inequality, and Schur's lemma.

Finally we establish an analogue of Carleson's inequality (Theorem 8.2) to handle the principal part. This is a crucial step and requires the L^p boundedness of an appropriate (Rademacher) maximal function which we introduce and study in Section 7. We prove its boundedness in $L^p(\mathbf{R}^n; X)$ when $1 < p < \infty$ provided that X is either a UMD function lattice, or a non-commutative L^q space for some $1 < q < \infty$, or a space with Rademacher type 2. We thus obtain a satisfying result in most of the concrete spaces of interest, but the boundedness of the Rademacher maximal function (and hence the Kato estimates) in general UMD-valued Bochner spaces remains open.

The paper is organized as follows. In Section 2 we provide the reader with a concise introduction to the concepts and results from the theory of Banach spaces and Banach space-valued Harmonic Analysis used in this paper. Section 3 contains the statements of the main results, and their reduction to the main estimate which is then dealt with in the rest of the paper. We develop vector-valued analogues of various classical results, which came to use in the proof of the scalar Kato problem, in Section 4. Section 5 deals with the Banach space-valued analogues of classical inequalities associated with an unperturbed Hodge–Dirac operator, and in particular the relationship with the dyadic conditional expectations. In Section 6 we reduce the main estimate to its principal part. Our Rademacher maximal function is studied in Section 7 and applied in Section 8 to prove an analogue of Carleson's inequality. This is used to reduce the principal part estimate to an analogue of a Carleson measure condition, which is finally verified in Section 9 by essentially the same stopping time argument as in [7] and [9].

Additional results are presented in three appendices. In Appendix A we show how the assumptions of the main theorem can in some cases be checked under appropriate ellipticity conditions. In Appendix B we relate our Carleson inequality to the boundedness of vector-valued paraproducts, and finally Appendix C contains a counterexample related to the Rademacher maximal function.

2. Preliminaries

This work is concerned with resolvent bounds, H^∞-functional calculus, and quadratic estimates for certain partial differential operators acting in L^p spaces of Banach space-valued functions. In order to streamline the actual discussion, we start by recalling the relevant notions and a number of results which will be repeatedly used in the sequel.

To express the typical inequalities "up to a constant" we use the notation $a \lesssim b$ to mean that there exists $C < \infty$ such that $a \leqslant Cb$, and the notation $a \eqsim b$ to mean that $a \lesssim b \lesssim a$. The implicit constants are meant to be independent of other relevant quantities. If we want to mention that the constant C depends on a parameter p, we write $a \lesssim_p b$.

Definition 2.1. Let A be a closed operator acting in a Banach space Y. It is called *bisectorial* with angle θ if its spectrum $\sigma(A)$ is included in a bisector:

$$\sigma(A) \subseteq S_\theta := \Sigma_\theta \cup \{0\} \cup (-\Sigma_\theta), \quad \text{where}$$
$$\Sigma_\theta := \{z \in \mathbf{C} \setminus \{0\};\ |\arg(z)| < \theta\},$$

and outside the bisector it verifies the following resolvent bounds:

$$\forall \theta' \in \left(\theta, \frac{\pi}{2}\right) \exists C > 0\ \forall \lambda \in \mathbf{C} \setminus S_{\theta'} \quad \left\|\lambda(\lambda I - A)^{-1}\right\|_{\mathcal{L}(Y)} \leqslant C. \tag{1}$$

We often omit the angle, and say that A is bisectorial if it is bisectorial with *some* angle $\theta \in [0, \frac{\pi}{2})$. One sees that A is bisectorial if and only if it satisfies the resolvent bound in (1) on the imaginary axis, i.e.,

$$\left\|(I + itA)^{-1}\right\| \leqslant C, \quad t \in \mathbf{R}.$$

For $0 < \nu < \pi/2$, let $H^\infty(S_\nu)$ be the space of bounded functions on S_ν, which are holomorphic in $S_\nu \setminus \{0\}$, and consider the following subspace of functions with decay at zero and infinity:

$$H_0^\infty(S_\nu) := \left\{\phi \in H^\infty(S_\nu)\colon \exists \alpha, C \in (0, \infty)\ \forall z \in S_\nu\ \left|\phi(z)\right| \leqslant C\left|\frac{z}{1+z^2}\right|^\alpha\right\}.$$

For a bisectorial operator A with angle $\theta < \omega < \nu < \pi/2$, and $\psi \in H_0^\infty(S_\nu)$, we define

$$\psi(A)u := \frac{1}{2i\pi} \int\limits_{\partial S_\omega} \psi(\lambda)(\lambda - A)^{-1}u \,\mathrm{d}\lambda,$$

where ∂S_ω is parameterized by arclength and directed anti-clockwise around S_ω.

Definition 2.2. Let A be a bisectorial operator with angle θ, and $\nu \in (\theta, \frac{\pi}{2})$. A is said to admit a *bounded H^∞-functional calculus with angle* ν if $\exists C < \infty\ \forall \psi \in H_0^\infty(S_\nu)$ $\|\psi(A)y\|_Y \leqslant C\|\psi\|_\infty \|y\|_Y$.

On the closure $\overline{\mathsf{R}(A)}$ of the range space $\mathsf{R}(A)$, we then define a bounded operator $f(A)$, for every $f \in H^\infty(S_\nu)$, by $f(A)u = \lim_{n\to\infty} \psi_n(A)u$, where $\psi_n \in H_0^\infty(S_\omega)$ are uniformly bounded and tend to f locally uniformly on $S_\omega \setminus \{0\}$. In a reflexive Banach space, there holds $X = \mathsf{N}(A) \oplus \overline{\mathsf{R}(A)}$ (cf. [21, Proposition 2.1.1], for the sectorial case which is readily adapted to the present context), so that denoting by $\mathbb{P}^0$ the associated projection onto the null space $\mathsf{N}(A)$, we can finally define the bounded operator $f(A)$ by

$$f(A)u = f(0)\mathbb{P}^0 u + \lim_{n\to\infty} \psi_n(A)u.$$

We also often omit the angle and just say that A has an H^∞-functional calculus. The detailed construction of this calculus, and much more information, can be found in [15,21,30].

A crucial aspect of the functional calculus is its harmonic analytic characterization. If Y is a Hilbert space, it is shown in [34] that A has an H^∞-functional calculus with angle ν if and only if the following *quadratic estimate* holds:

$$\left(\int_0^\infty \|\psi(tA)y\|_Y^2 \frac{\mathrm{d}t}{t}\right)^{1/2} \eqsim \|y\|_Y$$

for some non-zero function $\psi \in H_0^\infty(S_\nu)$. In the space $L^p(\mathbf{R}^n;\mathbf{C})$ $(1 < p < \infty)$, it has been shown in [15] that the above norms need to be replaced by

$$\left\|\left(\int_0^\infty |\psi(tA)y|^2 \frac{\mathrm{d}t}{t}\right)^{1/2}\right\|_p$$

as in the Littlewood–Paley theory. In a general Banach space, the correct characterization involves randomized sums of the form

$$\mathbb{E}\left\|\sum_{k\in\mathbf{Z}} \varepsilon_k \psi\big(2^k A\big) y\right\|_Y,$$

where $(\varepsilon_k)_{k\in\mathbf{Z}}$ are independent Rademacher variables on some probability space Ω (i.e., they take each of the two values $+1$ and -1 with probability $1/2$), and $\mathbb{E}$ is the mathematical expectation. These randomized norms provide the right analogue of the quadratic norms used in L^p and for this reason, somewhat loosely speaking, we will occasionally also refer to inequalities for the randomized norms as "quadratic estimates."

Proposition 2.3 *(Khintchine–Kahane inequalities). Let Y be a Banach space, and $(y_k)_{k\in\mathbf{Z}} \subset Y$. Then for each $1 < p < \infty$, there exists $C_p > 0$ such that*

$$\mathbb{E}\left\|\sum_k \varepsilon_k y_k\right\|_Y \leqslant \left(\mathbb{E}\left\|\sum_k \varepsilon_k y_k\right\|_Y^p\right)^{1/p} \leqslant C_p \mathbb{E}\left\|\sum_k \varepsilon_k y_k\right\|_Y.$$

Moreover, if $Y = L^q$ for some $1 < q < \infty$ (or more generally a Banach lattice with finite cotype), then

$$\mathbb{E}\left\|\sum_k \varepsilon_k y_k\right\|_Y \eqsim \left\|\left(\sum_k |y_k|^2\right)^{1/2}\right\|_Y.$$

When using such randomized sums, it is often convenient to introduce the space $\mathrm{Rad}(Y)$ of sequences $(y_k)_{k\in\mathbf{Z}} \subset Y$ such that $\sum_{|k|<n} \varepsilon_k y_k$ converges in $L^1(\Omega;Y)$, with the norm defined by

$$\|(y_k)_{k\in\mathbf{Z}}\|_{\mathrm{Rad}(Y)} = \mathbb{E}\left\|\sum_k \varepsilon_k y_k\right\|_Y.$$

These norms involve discrete rather than continuous sums, but this technical difference is unimportant. In fact, we could avoid discretization by using Banach space-valued stochastic integrals as in [23], but this would only add an unnecessary level of complexity. An important problem, however, is the fact that the quadratic norms are not, outside the Hilbertian setting, independent of the choice of $\phi \in H_0^\infty(S_\theta)$. To ensure such an independence, one has to assume (see [30]) that the family $\{\lambda(\lambda I - A)^{-1};\ \lambda \notin S_\theta\}$ is not only bounded (bisectoriality) but R-bounded (R-bisectoriality) in the following sense.

Definition 2.4. Let X be a Banach space. A family of bounded linear operators $\Psi \subset \mathcal{L}(X)$ is called *R-bounded* if there exists a constant C such that for all $N \in \mathbf{N}$, $T_1, \dots, T_N \in \Psi$, and $x_1, \dots, x_N \in X$, there holds

$$\mathbb{E}\left\|\sum_{j=1}^{N} \varepsilon_j T_j x_j\right\| \leqslant C\mathbb{E}\left\|\sum_{j=1}^{N} \varepsilon_j x_j\right\|.$$

The smallest constant C in the above inequality is called the R-bound of Ψ, and is denoted by $\mathcal{R}(\Psi)$.

A uniformly bounded family of operators is not necessarily R-bounded, as can be seen by considering translations on L^p, $p \neq 2$. In fact, the property that every uniformly bounded family is R-bounded characterizes Hilbert spaces up to isomorphism. This is in contrast to the scalar multiplication where Kahane's principle holds:

Proposition 2.5 *(Contraction principle). Let X be a Banach space, and $\lambda = (\lambda_k)_{k \in \mathbf{Z}} \in \ell^\infty$. Then $\forall N \in \mathbf{N}$, $\forall x_1, \dots, x_N \in X$,*

$$\mathbb{E}\left\|\sum_{j=1}^{N} \varepsilon_j \lambda_j x_j\right\| \leqslant 2\|\lambda\|_\infty \mathbb{E}\left\|\sum_{j=1}^{N} \varepsilon_j x_j\right\|.$$

An immediate but useful consequence of Propositions 2.5 and 2.3 is the following (see e.g. [30]).

Proposition 2.6. *Let X be a Banach space, and $(f_k)_{k \in \mathbf{Z}} \subset L^\infty(\mathbf{R}^n)$ be a bounded sequence of functions. Then the family of multiplication operators defined by $T_k u = f_k u$ is R-bounded on $L^p(\mathbf{R}^n; X)$ for all $1 < p < \infty$.*

The concept of R-boundedness is crucial in Banach space-valued Harmonic Analysis. It is described in detail in [30], where the following characterization can also be found (see [30, Section 12]):

Theorem 2.7 *(Kalton, Kunstmann, Weis). Let Y be a UMD Banach space, and A be an R-bisectorial operator acting on Y. Then A has an H^∞-functional calculus if and only if*

$$\begin{cases} \sup_{1\leqslant|t|\leqslant 2} \mathbb{E}\Big\| \sum_{k\in\mathbf{Z}} \varepsilon_k 2^k tA\big(I+\big(2^k tA\big)^2\big)^{-1} y\Big\|_Y \lesssim \|y\|_Y, & \forall y\in Y,\\ \sup_{1\leqslant|t|\leqslant 2} \mathbb{E}\Big\| \sum_{k\in\mathbf{Z}} \varepsilon_k 2^k tA^*\big(I+\big(2^k tA^*\big)^2\big)^{-1} y^*\Big\|_{Y^*} \lesssim \|y^*\|_{Y^*}, & \forall y^*\in Y^*. \end{cases}$$

The main body of this paper is concerned with proving this kind of estimates when $Y = L^p(\mathbf{R}^n; X^N)$ is the Bochner space of functions with values in the Cartesian product X^N of N copies of a Banach space X, and A is a *perturbed Hodge–Dirac* operator, as defined in the next section. Let us only mention at this point that our operators will be the "simplest" extensions of the classical Hodge–Dirac operators to the Banach space-valued setting, namely tensor products $T \otimes I_X$ of an operator T acting in $L^p(\mathbf{R}^n; \mathbf{C}^N)$ with the identity I_X. The study of such operators is by no means trivial. Already in the case when T is the possibly simplest singular integral operator, the Hilbert transform, the boundedness of $T \otimes I_X$ in $L^p(\mathbf{R}; X)$ is equivalent to X being a so-called *UMD space*, which means the *u*nconditional convergence of *m*artingale *d*ifference sequences in $L^p(\Omega; X)$ for $1 < p < \infty$ and Ω any probability space.

This class of spaces is the most important one for vector-valued Harmonic Analysis. All UMD spaces are reflexive (and even super-reflexive; cf. [11]). The principal examples include the reflexive Lebesgue, Lorentz, Sobolev, and Orlicz spaces, as well as the reflexive *noncommutative L^p spaces*. A recent survey paper on UMD spaces is [14]. The above-mentioned equivalence with the Hilbert transform boundedness, due in one direction to Burkholder [13] and in the other to Bourgain [11], lies at the heart of the theory, and is characteristic of the interaction between probabilistic and analytic methods. It is, for instance, needed in the proof of the following multiplier theorem, which we often resort to in the sequel. The original statement of this kind was obtained by Bourgain [12] and McConnell [32], but the somewhat more general formulation given here is due to Zimmermann [42].

Theorem 2.8 *(Bourgain, McConnell, Zimmermann). Let $n \geqslant 1$. If (and only if) X is a UMD space and $1 < p < \infty$, then every symbol $m : \mathbf{R}^n \setminus \{0\} \to \mathbf{C}$ such that*

$$\sup\big\{|\xi|^{|\alpha|} D^\alpha m(\xi)\colon \alpha \in \{0,1\}^n,\ \xi \in \mathbf{R}^n \setminus \{0\}\big\} < \infty$$

gives rise to a bounded Fourier multiplier $T_m \in \mathcal{L}(L^p(\mathbf{R}^n, X))$ defined by $\mathcal{F}(T_m u)(\xi) = m(\xi)\mathcal{F}(u)(\xi)$, where $\mathcal{F}$ denotes the Fourier transform.

With somewhat stronger conditions on the symbol, we also have stronger conclusions. Let us say that a symbol $m : \mathbf{R}^n \to \mathbf{C}$ has *bounded variation* if for some $C < \infty$ and all $\alpha \in \{0, 1\}^n$, there holds

$$\int_{\mathbf{R}} \dots \int_{\mathbf{R}} \big|D^\alpha m(\xi)\big|\, \mathrm{d}\xi^\alpha \leqslant C < \infty,$$

where the integration is with respect to all the variables ξ_i such that $\alpha_i = 1$, and the estimate is required uniformly in the remaining variables ξ_j. (The case $\alpha = 0$ is understood as the boundedness of $m(\xi)$ by C.) We say that a collection of symbols $\mathcal{M}$ has uniformly bounded variation if

the symbols $m \in \mathcal{M}$ satisfy this condition with the same C. See [30] for the proof of the following useful result.

Proposition 2.9. *Let $n \geqslant 1$, X be a UMD, and $1 < p < \infty$. Let $\mathcal{M}$ be a collection of symbols of uniformly bounded variation. Then the collection of Fourier multipliers T_m, $m \in \mathcal{M}$, is an R-bounded subset of $\mathcal{L}(L^p(\mathbf{R}^n; X))$.*

Another important estimate in UMD spaces, analogous to the previous one, is the following R-boundedness of conditional expectations. It is an extension of a classical quadratic estimate due to Stein [40], which was found in the vector-valued situation by Bourgain [12]. See also [20] for a proof.

Proposition 2.10 *(Stein's inequality). Let X be a UMD Banach space, (Ω, Σ, μ) a measure space, and $1 < p < \infty$. Then any increasing sequence of conditional expectations on $L^p(\Omega; X)$ is R-bounded.*

We will mostly be concerned with the conditional expectations related to the dyadic filtration of $\mathbf{R}^n$. This is defined by the system of *dyadic cubes*

$$\triangle = \bigcup_{k \in \mathbf{Z}} \triangle_{2^k}, \quad \triangle_{2^k} := \big\{ 2^k\big([0,1)^n + m\big) \colon m \in \mathbf{Z}^n \big\}.$$

The corresponding conditional expectation projections are denoted by

$$A_{2^k} u(x) := \langle u \rangle_Q := \fint_Q u(y)\,\mathrm{d}y := \frac{1}{|Q|} \int_Q u(y)\,\mathrm{d}y, \quad x \in Q \in \triangle_{2^k}.$$

The integral average notation above will also be used with other measurable sets from time to time.

Other important Banach space properties are the following.

Definition 2.11. Let X be a Banach space, and $1 \leqslant t \leqslant 2 \leqslant s \leqslant \infty$. Then X is said to have (*Rademacher*) *type t* if

$$\mathbb{E} \Big\| \sum_{k \in \mathbf{Z}} \varepsilon_k x_k \Big\|_X \lesssim \Big(\sum_{k \in \mathbf{Z}} \|x_k\|_X^t \Big)^{1/t}$$

for all $x_k \in X$, and (*Rademacher*) *cotype s* if

$$\Big(\sum_{k \in \mathbf{Z}} \|x_k\|_X^s \Big)^{1/s} \lesssim \mathbb{E} \Big\| \sum_{k \in \mathbf{Z}} \varepsilon_k x_k \Big\|_X$$

for all $x_k \in X$, where the usual modification is understood if $s = \infty$. The space is said to have *non-trivial type* if it has some type $t > 1$, and *non-trivial*, or *finite*, *cotype* if it has some cotype $s < \infty$.

These conditions become stronger with increasing t and decreasing s, and only Hilbert spaces (up to isomorphism) enjoy both the optimal type and cotype $t = s = 2$. For the present purposes, the most important thing is to know that every UMD space has both non-trivial type and cotype. The property of finite cotype is also characterized (see [17, 12.27]) by the comparability of Rademacher and Gaussian random sums,

$$\mathbb{E}\Big\|\sum_{k\in\mathbf{Z}}\varepsilon_k x_k\Big\|_X \eqsim \mathbb{E}\Big\|\sum_{k\in\mathbf{Z}}\gamma_k x_k\Big\|_X \quad\Leftrightarrow\quad X \text{ has finite cotype,} \tag{2}$$

where the γ_k are independent random variables with the standard normal distribution.

These notions, as well as the Khintchine–Kahane inequalities 2.3, are central in a circle of ideas which can be roughly referred to as "averaging in Banach spaces," and which forms the core of vector-valued harmonic analysis. A gentle introduction to this topic can be found in [1].

In addition to the above conditions, which are well known in the theory of Banach spaces, we need to introduce a new class of spaces, the defining property of which is the boundedness of the following *Rademacher maximal function*:

$$M_R u(x) := \sup\left\{\mathbb{E}\Big\|\sum_{k\in\mathbf{Z}}\varepsilon_k\lambda_k A_{2^k}u(x)\Big\|_X:\ \lambda = (\lambda_k)_{k\in\mathbf{Z}} \text{ finitely non-zero with } \|\lambda\|_{\ell^2(\mathbf{Z})} \leqslant 1\right\}.$$

Note that, under the identification $X \eqsim \mathcal{L}(\mathbf{C}, X)$, this is the R-bound of the set

$$\big\{A_{2^k}u(x):\ k\in\mathbf{Z}\big\} = \big\{\langle u\rangle_Q:\ Q \ni x\big\}.$$

In particular, if X is a Hilbert space, we recover the usual dyadic maximal function.

Definition 2.12. We say that the Banach space X has the RMF property, if M_R is bounded from $L^2(\mathbf{R}^n; X)$ to $L^2(\mathbf{R}^n)$.

We do not yet completely understand how this new class of spaces relates to the other Banach space notions discussed above, which forces us to adopt this property as an additional assumption. It would be particularly useful to know if every UMD space has RMF, since this would allow us to state our main theorem in the generality of all UMD spaces, but the question remains open. However, in Section 7 we show that the RMF property does hold in most of the concrete situations of interest. The classes of Banach spaces appearing in the statement are also defined in Section 7.

Proposition 2.13. *A Banach space which is a UMD function lattice, or a noncommutative L^p space for $1 < p < \infty$, or which has Rademacher type 2, has RMF.*

3. Statement of the results

The square root problem originally posed by T. Kato was an operator-theoretic question in an abstract Hilbert space, but it was observed in [31] and [33] that the desired estimate was invalid in this generality (see [7] for references and more historical information). This shifted the attention towards more concrete differentiation and multiplication operators in $L^2(\mathbf{R}^n; \mathbf{C}^N)$, ones of interest in the actual applications that Kato had in mind when formulating his problem. Our

Banach space framework is obtained by modifying the concrete Kato problem, so as the replace $\mathbf{C}^N$ by X^N, and L^2 by L^p. The various differentiation and multiplication operators are simply replaced by their natural tensor extensions acting on X-valued functions. The set-up, which we now present in detail, is closely related to that of [9, Section 3].

Let X be a Banach space, $1 < p < \infty$, and $n, n_1, n_2, N \in \mathbf{Z}_+$ with $N = n_1 + n_2$. Let D be a homogeneous first order partial differential operator with constant $\mathcal{L}(\mathbf{C}^{n_1}, \mathbf{C}^{n_2})$-coefficients, and D^* be its adjoint. We assume that

$$DD^*D = -\Delta D. \tag{3}$$

The principal case of interest is

$$\{n_1, n_2, D, D^*\} = \{1, n, \nabla, -\operatorname{div}\},$$

but it is convenient to consider the abstract formulation, because it makes the assumptions symmetric in D and D^*. (Note that (3) is equivalent to the similar equation with D and D^* reversed by taking adjoints of both sides.) For $i = 1, 2$, let $A_i \in L^\infty(\mathbf{R}^n; \mathcal{L}(\mathbf{C}^{n_i}))$ be bounded matrix-valued functions, which we identify with multiplication operators on $L^p(\mathbf{R}^n; X^{n_i})$ in the natural way. We assume the estimate

$$\|A_i\|_{L^\infty(\mathbf{R}^n;\mathcal{L}(\mathbf{C}^{n_i}))} + \|A_i^{-1}\|_{L^\infty(\mathbf{R}^n;\mathcal{L}(\mathbf{C}^{n_i}))} \leqslant C, \quad i = 1, 2.$$

In the space

$$L^p(\mathbf{R}^n; X^N) \equiv L^p(\mathbf{R}^n; X^{n_1}) \oplus L^p(\mathbf{R}^n; X^{n_2})$$

we consider the operators

$$\Gamma = \begin{pmatrix} 0 & 0 \\ D & 0 \end{pmatrix}, \qquad \Gamma^* = \begin{pmatrix} 0 & D^* \\ 0 & 0 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 & 0 \\ 0 & A_2 \end{pmatrix}.$$

The first two are closed and nilpotent (i.e., the range $\mathsf{R}(\Gamma) \subseteq \mathsf{N}(\Gamma)$, the null space; and the same with Γ^*) operators with their natural dense domains $\mathsf{D}(\Gamma)$ and $\mathsf{D}(\Gamma^*)$, while the latter two are everywhere defined and bounded.

The sum

$$\Pi = \Gamma + \Gamma^* = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix}$$

is called the *Hodge–Dirac* operator. Modified sums of the form

$$\Pi_B = \Gamma + \Gamma_B^* = \Gamma + B_1 \Gamma^* B_2,$$
$$\Pi_{B^*} = \Gamma^* + \Gamma_{B^*} = \Gamma^* + B_2 \Gamma B_1$$

are then called *perturbed Hodge–Dirac* operators. It follows from general Operator Theory, using only the closedness or boundedness of the appropriate operators and the form of the matrices, that Π_B and Π_{B^*} are also closed and densely defined.

In the Hilbert space setting of [9], appropriate ellipticity conditions on B_1 and B_2 further imply, still by abstract operator theoretic methods, the defining resolvent estimates for the (R-)bisectoriality of Π_B and Π_{B^*}. In the present situation, this is no longer the case; in fact, already when $X = \mathbf{C}$ but $p \neq 2$, there exist elliptic second order differential operators which are not sectorial in $L^p(\mathbf{R}^n; \mathbf{C})$ for some values of p (see [6]). Thus we need to redefine the problem slightly, so as to adopt the analogues of some of the operator-theoretic conclusions in [9] as the assumptions for our Harmonic Analysis. In particular, we *assume* the existence of the following resolvents of Π_B for all $t \in \mathbf{R}$:

$$\begin{aligned} R_t^B &:= (I + it\Pi_B)^{-1}, \\ P_t^B &:= \big(I + t^2\Pi_B^2\big)^{-1} = \frac{1}{2}\big(R_t^B + R_{-t}^B\big) = R_t^B R_{-t}^B, \\ Q_t^B &:= t\Pi_B P_t^B = t\Pi_B\big(I + t^2\Pi_B^2\big)^{-1} = \frac{i}{2}\big(R_t^B - R_{-t}^B\big). \end{aligned} \tag{4}$$

We can now state our main result.

Theorem 3.1. *Let X be a UMD Banach space such that both X and X^* have RMF. Let Π_B and Π_{B^*} be perturbed Hodge–Dirac operators defined in $L^p(\mathbf{R}^n; X^N)$ for all $p \in (p_-, p_+) \subseteq (1, \infty)$. Then the following are equivalent*:

$$\Pi_B,\ \Pi_{B^*} \text{ are R-bisectorial in } L^p\big(\mathbf{R}^n; X^N\big) \text{ for all } p \in (p_-, p_+), \tag{5}$$

$$\Pi_B,\ \Pi_{B^*} \text{ have } H^\infty\text{-calculus in } L^p\big(\mathbf{R}^n; X^N\big) \text{ for all } p \in (p_-, p_+). \tag{6}$$

The reason why we are forced to formulate this theorem for L^p estimates valid on open intervals of exponents, instead of an individual p, comes from the limitations in one particular step of the proof (our L^p version of Carleson's inequality); this will be discussed in somewhat more detail in Section 9. Note that we do not require that $2 \in (p_-, p_+)$ here, whereas this is often the case in the scalar-valued results which are based on extrapolation of the L^2 estimates.

The next corollary makes the relation to the square roots of second-order differential operators more explicit.

Corollary 3.2. *Let X be a UMD Banach space such that both X and X^* have RMF. Let A and A^{-1} be multiplications by $L^\infty(\mathbf{R}^n; \mathcal{L}(\mathbf{C}^n))$ functions, and $L = -\operatorname{div} A\nabla$ be a sectorial operator in $L^p(\mathbf{R}^n; X)$ for all $p \in (p_-, p_+) \subseteq (1, \infty)$. Then the following are equivalent*:

$$\begin{cases} \text{The sets } \big\{\big(I + t^2L\big)^{-1}\big\}_{t>0},\ \big\{t\sqrt{-\Delta}\big(I + t^2L\big)^{-1}\big\}_{t>0}, \\ \big\{\big(I + t^2L\big)^{-1}t\sqrt{-\Delta}\,\big\}_{t>0} \text{ and } \big\{t\sqrt{-\Delta}\big(I + t^2L\big)^{-1}t\sqrt{-\Delta}\,\big\}_{t>0} \\ \text{are R-bounded on } L^p\big(\mathbf{R}^n; X\big) \text{ for all } p \in (p_-, p_+), \end{cases} \tag{7}$$

$$\begin{cases} L \text{ has an } H^\infty\text{-functional calculus in } L^p\big(\mathbf{R}^n; X\big) \text{ and} \\ \|\sqrt{L}u\|_p \eqsim \|\nabla u\|_p \text{ for all } p \in (p_-, p_+). \end{cases} \tag{8}$$

Remark 3.3. It is interesting to note the connection between the above condition (7) and (a variant of) L. Weis' characterization of so-called *maximal regularity* [41]: the R-boundedness

of the set $\{(I+t^2L)^{-1}\}_{t>0}$ in $\mathcal{L}(L^p(\mathbf{R}^n;X))$ is equivalent to the existence of a unique solution in $L^q(\mathbf{R};\mathsf{D}(L))\cap W^{2,q}(\mathbf{R};L^p(\mathbf{R}^n;X))$ of the problem $-u''+Lu=f$ for each $f\in L^q(\mathbf{R};L^p(\mathbf{R}^n;X))$, where $1<q<\infty$.

We start with the proof of the corollary.

Proof. Let us first remark that the functional calculus in (8) implies the R-boundedness of $\{(I+t^2L)^{-1}\}_{t>0}$ and $\{t\sqrt{L}(I+t^2L)^{-1}\}_{t>0}$ by [28, Theorem 5.3]. Using also the Kato estimates, we have that (8) $\Rightarrow$ (7). Now consider a perturbed Hodge–Dirac operator Π_B with $A_1=I$, $A_2=A$. Its resolvent can be computed as

$$(I-it\Pi_B)^{-1}=\begin{pmatrix}(I+t^2L)^{-1} & -it(I+t^2L)^{-1}\operatorname{div}A\\ it\nabla(I+t^2L)^{-1} & I+t^2\nabla(I+t^2L)^{-1}\operatorname{div}A\end{pmatrix}.$$

By Theorem 2.8, $\nabla/\sqrt{-\Delta}$ is bounded from $L^p(\mathbf{R}^n;X)$ to $L^p(\mathbf{R}^n;X^n)$, and $\operatorname{div}/\sqrt{-\Delta}$ is bounded from $L^p(\mathbf{R}^n;X^n)$ to $L^p(\mathbf{R}^n;X)$. Using the boundedness of A on $L^p(\mathbf{R}^n;X^n)$, the R-bisectoriality of Π_B thus follows from (7). By Theorem 3.1 the operator Π_B hence has an H^∞-functional calculus. The functional calculus of L follows from the functional calculus of Π_B applied to functions of Π_B^2. The Kato estimates follow from the functional calculus of Π_B applied to the sign function $z\mapsto z/\sqrt{z^2}$, as in [9, Corollary 2.11]. $\square$

Theorem 3.1 is a consequence of the following square function estimate, which is a vector-valued analogue of [9, Proposition 4.8].

Proposition 3.4. *Let X be a UMD Banach space such that both X and X^* have RMF. Consider perturbed Hodge–Dirac operators Π_B and Π_{B^*} in $L^p(\mathbf{R}^n;X^N)$ for p in an open interval $(p_-,p_+)\subseteq(1,\infty)$. Assume that Π_B and Π_{B^*} are R-bisectorial in $L^p(\mathbf{R}^n;X^N)$ for all $p\in(p_-,p_+)$. Then we have*

$$\sup_{1\leqslant|t|\leqslant 2}\mathbb{E}\Big\|\sum_{k\in\mathbf{Z}}\varepsilon_k Q^B_{2^kt}u\Big\|_{L^p(\mathbf{R}^n;X^N)}\lesssim\|u\|_{L^p(\mathbf{R}^n;X^N)},\quad\forall u\in\mathsf{R}(\Gamma).\tag{9}$$

Moreover, the same estimates holds in $L^p(\mathbf{R}^n;X^N)$ if the triple $\{\Gamma,B_1,B_2\}$ is replaced by $\{\Gamma^*,B_2,B_1\}$, and in $L^{p'}(\mathbf{R}^n;(X^*)^N)$ if it is replaced by $\{\Gamma,B_1^*,B_2^*\}$ or $\{\Gamma^*,B_2^*,B_1^*\}$.

This is proven in the rest of the paper. In fact, it suffices to prove the assertion with the triple $\{\Gamma,B_1,B_2\}$, as written out in (9), since the assumptions remain invariant when replacing this triple by any one of the three other possibilities. To simplify notation we will, moreover, only consider $Q^B_{2^k}$ instead of $Q^B_{2^kt}$, since the proofs remain the same in this generality.

We start our journey towards the proof of the main estimate (9) in the next section; in the rest of this section we show how to deduce Theorem 3.1 from Proposition 3.4. We begin with the following:

Lemma 3.5 *(Hodge decomposition). Let X be a reflexive Banach space, $1 < p < \infty$, and Π_B be a perturbed Hodge–Dirac operator which is bisectorial in $L^p(\mathbf{R}^n; X^N)$. Then the space decomposes as the following topological direct sum*:

$$L^p(\mathbf{R}^n; X^N) = \mathsf{N}(\Pi_B) \oplus \overline{\mathsf{R}(\Gamma)} \oplus \overline{\mathsf{R}(\Gamma_B^*)}.$$

Proof. On the abstract level, i.e., without making use of the structure of the Hodge–Dirac operators, the assumptions that X (and then also $L^p(\mathbf{R}^n; X^N)$) is reflexive and Π_B is bisectorial imply the decomposition

$$L^p(\mathbf{R}^n; X^N) = \mathsf{N}(\Pi_B) \oplus \overline{\mathsf{R}(\Pi_B)}.$$

Moreover, the projection on $\overline{\mathsf{R}(\Pi_B)}$ is given by

$$Pu = \lim_{t\to\infty} t^2\Pi_B^2\big(I + t^2\Pi_B^2\big)^{-1}u.$$

In our specific situation, we further have the explicit formula

$$\begin{aligned} &t^2\Pi_B^2\big(I + t^2\Pi_B^2\big)^{-1} \\ &\quad= \begin{pmatrix} t^2A_1D^*A_2D(I + t^2A_1D^*A_2D)^{-1} & 0 \\ 0 & t^2DA_1D^*A_2(I + t^2DA_1D^*A_2)^{-1} \end{pmatrix}. \end{aligned}$$

The projection P thus splits as $P_1 + P_2$, where P_i acts invariantly on $L^p(\mathbf{R}^n; X^{n_i})$ and annihilates $L^p(\mathbf{R}^n; X^{n_j})$ for $j \neq i$. Since $\overline{\mathsf{R}(\Gamma)} \subseteq \mathsf{R}(P) \cap L^p(\mathbf{R}^n; X^{n_2}) = \mathsf{R}(P_2) \subseteq \overline{\mathsf{R}(\Gamma)}$, and $\overline{\mathsf{R}(\Gamma_B^*)} \subseteq \mathsf{R}(P) \cap L^p(\mathbf{R}^{n_1}; X) = \mathsf{R}(P_1) \subseteq \overline{\mathsf{R}(\Gamma_B^*)}$, this gives the Hodge decomposition. $\square$

Proof of Theorem 3.1. The fact that (6) $\Rightarrow$ (5) is essentially contained in [28, Theorem 5.3], where it is stated for sectorial (rather than bisectorial) operators. Likewise, the equivalence between the square function estimates

$$\begin{cases} \sup\limits_{1\leqslant|t|\leqslant 2} \mathbb{E}\Big\| \sum\limits_{k\in\mathbf{Z}} \varepsilon_k Q^B_{2^k t} u\Big\|_p \lesssim \|u\|_p, & \forall u \in L^p(\mathbf{R}^n; X^N), \\ \sup\limits_{1\leqslant|t|\leqslant 2} \mathbb{E}\Big\| \sum\limits_{k\in\mathbf{Z}} \varepsilon_k \big(Q^B_{2^k t}\big)^* u\Big\|_{p'} \lesssim \|u\|_{p'}, & \forall u \in L^{p'}(\mathbf{R}^n; (X^*)^N), \end{cases} \tag{10}$$

and the functional calculus of Π_B is proven in [30, Theorem 12.17] for sectorial operators but the proof carries over to the bisectorial situation.

We thus have to show that (5) implies (10). By Lemma 3.5, it suffices to do this separately for u in each of the three components of the Hodge decomposition. Now $Q^B_{2^k} u = 0$ for all $u \in \mathsf{N}(\Pi_B)$, and Proposition 3.4 gives the first estimate in (10) for $u \in \mathsf{R}(\Gamma)$. On $\mathsf{R}(\Gamma_B^*)$, we then apply Proposition 3.4 with the triple (Γ, B_1, B_2) replaced by (Γ^*, B_2, B_1).

This gives

$$\sup_{1\leqslant|t|\leqslant 2} \mathbb{E}\Big\| \sum_{k\in\mathbf{Z}} \varepsilon_k 2^k t B_2 \Gamma B_1 \big(I + \big(2^k t(\Gamma^* + B_2\Gamma B_1)\big)^2\big)^{-1} u\Big\|_p \lesssim \|u\|_p,$$

for all $u \in R(\Gamma^*)$; by simple manipulation, this is equivalent to

$$\sup_{1\leqslant|t|\leqslant 2} \mathbb{E}\left\| \sum_{k\in\mathbf{Z}} \varepsilon_k 2^k t \Gamma\big(I + \big(2^k t \Pi_B\big)^2\big)^{-1} B_1 u \right\|_p \lesssim \|u\|_p \quad \forall u \in R(\Gamma^*),$$

and then in turn to

$$\sup_{1\leqslant|t|\leqslant 2} \mathbb{E}\left\| \sum_{k\in\mathbf{Z}} \varepsilon_k Q^B_{2^k t} u \right\|_p \lesssim \|u\|_p \quad \forall u \in \mathsf{R}\big(\Gamma_B^*\big),$$

since $\mathsf{R}(\Gamma_B^*) = B_1\mathsf{R}(\Gamma^*)$.

To obtain the dual estimates, one remarks that the above reasoning can be applied to $\Pi_B^* = \Gamma^* + B_2^*\Gamma B_1^*$ and $\Pi_{B^*}^* = \Gamma + B_1^*\Gamma^* B_2^*$. Indeed, these operators are R-bisectorial on $L^{p'}(\mathbf{R}^n; (X^*)^N)$ by the duality of R-bounds ([28, Lemma 3.1]; here one needs the fact that UMD spaces have non-trivial type). $\square$

Remark 3.6. The reader familiar with Hodge–Dirac operators will have noticed the special form of our operators Γ and Γ^*, and, in particular, the fact that we are not working at the level of generality of [9]. However, the proof of Proposition 3.4 carries over to the following situation.

Proposition 3.7. *Let X be a UMD Banach space such that both X and X^* have RMF. Let Γ be a nilpotent first order differential operator with constant coefficients in $\mathcal{L}(\mathbf{C}^N)$ satisfying $\Pi^3 = -\Delta\Pi$, where $\Pi = \Gamma + \Gamma^*$. Let $B_1, B_2 \in L^\infty(\mathbf{R}^n; \mathcal{L}(\mathbf{C}^N))$ be such that $\Gamma^* B_2 B_1 \Gamma^* = 0 = \Gamma B_1 B_2 \Gamma$. Assume that $\Pi_B = \Gamma + B_1\Gamma^* B_2$ is R-bisectorial on $L^q(\mathbf{R}^n; X^N)$ for all $q \in (p-\varepsilon, p+\varepsilon)$, where $\varepsilon > 0$. Then we have*

$$\sup_{1\leqslant|t|\leqslant 2} \mathbb{E}\left\| \sum_{k\in\mathbf{Z}} \varepsilon_k 2^k t \Pi_B\big(I + \big(2^k t \Pi_B\big)^2\big)^{-1} u \right\|_{L^p(\mathbf{R}^n;X^N)} \lesssim \|u\|_{L^p(\mathbf{R}^n;X^N)}, \quad \forall u \in \mathsf{R}(\Gamma).$$

This holds, in particular, in the case where Γ is an exterior derivative. However, the L^p Hodge decomposition of Lemma 3.5 is no more automatic in this situation. To deduce a version of Theorem 3.1 in this more general setting one would thus need to have the existence of the Hodge decomposition as an assumption. Since our main focus is the original square root problem, we chose not to work in this generality in order to keep the paper more readable. The L^p theory of more general Hodge–Dirac operators will be considered elsewhere.

4. Miscellaneous propositions

This section is a smörgåsbord of vector-valued analogues of a number of classical estimates of Analysis, which we need in the subsequent developments. We start with a vector-valued version of the Poincaré inequality. Below, $u \cdot v$ denotes the dot product of $u, v \in \mathbf{R}^n$, τ_h stands for the translation operator defined by $\tau_h f(x) = f(x+h)$, and 1_Q denotes the characteristic function of the set Q.

Proposition 4.1 *(Poincaré inequality). Let X be a Banach space, and $1 \leqslant p < \infty$. For $u \in W^{1,p}(\mathbf{R}^n; X)$, and $m \in \mathbf{Z}^n$ we have*

$$\mathbb{E}\Big\|\sum_{k\in\mathbf{Z}}\varepsilon_k\sum_{Q\in\triangle_{2^k}}1_Q\big(u_k-\langle u_k\rangle_{Q+2^k m}\big)\Big\|_{L^p(\mathbf{R}^n;X)}$$
$$\lesssim \int\limits_{[-1,1]^n}\int\limits_0^1\mathbb{E}\Big\|\sum_{k\in\mathbf{Z}}\varepsilon_k 2^k(m+z)\cdot\nabla\tau_{t2^k(m+z)}u_k\Big\|_{L^p(\mathbf{R}^n;X)}\,\mathrm{d}t\,\mathrm{d}z.$$

Proof. For $x \in Q \in \triangle_{2^k}$, we observe that $Q \subset x + 2^k[-1,1]^n$. Hence

$$u_k(x)-\langle u_k\rangle_{Q+2^k m}$$
$$=\int\limits_{[-1,1]^n}\big[u_k(x)-u_k\big(x+2^k(m+z)\big)\big]1_Q\big(x+2^k z\big)\,\mathrm{d}z$$
$$=\int\limits_{[-1,1]^n}\int\limits_0^1 -2^k(m+z)\cdot\nabla u_k\big(x+t2^k(m+z)\big)\,\mathrm{d}t\,1_Q\big(x+2^k z\big)\,\mathrm{d}z.$$

The assertion follows after bringing the integrals outside the norm and discarding the indicators $1_Q(x+2^k z)$ by the contraction principle 2.5. □

Here is a useful Banach space version of another classical inequality.

Proposition 4.2 *(Schur's estimate). Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be Banach spaces, the last two with finite cotype. For $i, j \in \mathbf{Z}$, let $\alpha(i,j)$ be positive numbers satisfying*

$$\sup_i\sum_j\alpha(i,j)\lesssim 1,\qquad \sup_j\sum_i\alpha(i,j)\lesssim 1,$$

and let $T_{i,j} \in \mathcal{L}(\mathcal{Y},\mathcal{Z})$, $D_i \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ be operators satisfying

$$\mathcal{R}\bigg(\frac{1}{\alpha(i,j)}T_{i,j}\colon i,j\in\mathbf{Z}\bigg)\lesssim 1,\qquad \mathbb{E}\Big\|\sum_i\varepsilon_i D_i x\Big\|_{\mathcal{Y}}\lesssim\|x\|_{\mathcal{X}}$$

for all $x \in \mathcal{X}$. Then there holds

$$\mathbb{E}\Big\|\sum_{i,j}\varepsilon_j T_{i,j}D_i x\Big\|_{\mathcal{Z}}\lesssim\|x\|_{\mathcal{X}}.$$

Proof. Under the assumption that $\mathcal{Y}$ and $\mathcal{Z}$ have finite cotype, we may replace the Rademacher-variables ε_j in the assumptions and the claim by independent standard Gaussian random variables γ_j by (2). We write the left-hand side of the modified assertion as

$$\mathbb{E}\Big\|\sum_j \gamma_j \sum_i \alpha(i,j)^{1/2}\frac{1}{\alpha(i,j)}T_{i,j}\alpha(i,j)^{1/2}D_i x\Big\|_{\mathcal{Z}}.$$

Then, as in [24, Proposition 2.1], let

$$x_{i,j} := \frac{1}{\alpha(i,j)}T_{i,j}\alpha(i,j)^{1/2}D_i x, \qquad y_j := \sum_i \alpha(i,j)^{1/2}x_{i,j}.$$

For $x^* \in \mathcal{X}^*$, we have

$$\sum_j \big|\langle y_j, x^*\rangle\big|^2 \leqslant \sup_j\Big(\sum_i \alpha(i,j)\Big)\sum_{i,j}\big|\langle x_{i,j}, x^*\rangle\big|^2.$$

Now Proposition 3.7 in [37] states that

$$\sum_j \big|\langle y_j, x^*\rangle\big|^2 \leqslant C^2\sum_{i,j}\big|\langle x_{i,j}, x^*\rangle\big|^2 \quad \forall x^* \in X^*$$
$$\Rightarrow \quad \mathbb{E}\Big\|\sum_j \gamma_j y_j\Big\| \leqslant C\mathbb{E}\Big\|\sum_{i,j}\gamma_{i,j}x_{i,j}\Big\|,$$

where $(\gamma_{i,j})_{i,j\in\mathbf{Z}}$ is a double-indexed sequence of independent standard Gaussian variables. Therefore, using our R-boundedness assumption $\mathscr{R}(\frac{1}{\alpha(i,j)}T_{i,j}\colon i,j\in\mathbf{Z}) \lesssim 1$, we have

$$\begin{aligned}
&\mathbb{E}\Big\|\sum_j \gamma_j \sum_i \alpha(i,j)^{1/2}\frac{1}{\alpha(i,j)}T_{i,j}\alpha(i,j)^{1/2}D_i x\Big\|_{\mathcal{Z}} \\
&\leqslant \sup_j\Big(\sum_i \alpha(i,j)\Big)^{1/2}\mathbb{E}\Big\|\sum_{i,j}\gamma_{i,j}\frac{1}{\alpha(i,j)}T_{i,j}\alpha(i,j)^{1/2}D_i x\Big\|_{\mathcal{Z}} \\
&\lesssim \mathbb{E}\Big\|\sum_{i,j}\gamma_{i,j}\alpha(i,j)^{1/2}D_i x\Big\|_{\mathcal{Y}}.
\end{aligned}$$

By reorganization, the last expression is equal to

$$\mathbb{E}\Big\|\sum_i\Big(\sum_j \alpha(i,j)^{1/2}\gamma_{i,j}\Big)D_i x\Big\|_{\mathcal{Y}} =: \mathbb{E}\Big\|\sum_i \tilde{\gamma}_i D_i x\Big\|_{\mathcal{Y}}.$$

By basic properties of Gaussian sums, the random variables $\tilde{\gamma}_i$ are again independent Gaussian, with variance

$$\mathbb{E}\tilde{\gamma}_i^2 = \sum_j \alpha(i,j) \lesssim 1.$$

By the contraction principle 2.5, the random sum with $\tilde{\gamma}_i$'s is then dominated by a random sum with standard Gaussian variables, and using the assumption on the operators D_i we complete the argument. $\square$

In the rest of this section, we make use of the Haar system of functions. Recall that in $\mathbf{R}^n$ there are $2^n - 1$ Haar functions h_Q^η, $\eta \in \{0,1\}^n \setminus \{0\}$, associated with every dyadic cube $Q \in \triangle$. For our purposes, it is most convenient to normalize them in $L^\infty(\mathbf{R}^n)$ so that $|h_Q^\eta| = 1_Q$. We often need only one (say, the "first") of the h_Q^η for each Q, and so we adopt the notation $h_Q := h_Q^{(1,0,\dots,0)} := 1_{Q_+} - 1_{Q_-}$, where Q_+ and Q_- are two halves of Q.

Lemma 4.3 *(Sign-invariance). Let X be any Banach space, $1 \leqslant p < \infty$, and $u_Q \in L^p(\mathbf{R}^n; X)$ for all $Q \in \triangle$. Then*

$$\mathbb{E}\Big\|\sum_k \varepsilon_k \sum_{Q\in\triangle_{2^k}} 1_Q u_Q\Big\|_{L^p(\mathbf{R}^n;X)} \eqsim \mathbb{E}\Big\|\sum_k \varepsilon_k \sum_{Q\in\triangle_{2^k}} h_Q u_Q\Big\|_{L^p(\mathbf{R}^n;X)}$$
$$\eqsim \mathbb{E}\Big\|\sum_{Q\in\triangle} \varepsilon_Q 1_Q u_Q\Big\|_{L^p(\mathbf{R}^n;X)}.$$

Proof. Using Kahane's inequality 2.3, and Fubini's theorem, we have

$$\mathbb{E}\Big\|\sum_k \varepsilon_k \sum_{Q\in\triangle_{2^k}} 1_Q u_Q\Big\|_{L^p(\mathbf{R}^n;X)}$$
$$\eqsim \bigg(\int_{\mathbf{R}^n} \mathbb{E}\Big\|\sum_k \varepsilon_k \sum_{Q\in\triangle_{2^k}} 1_Q(y) u_Q(y)\Big\|_X^p \,\mathrm{d}y\bigg)^{1/p}.$$

For a fixed $y \in \mathbf{R}^n$, and a scale $k \in \mathbf{Z}$, there exists a unique dyadic cube $Q_{k,y} \in \triangle_{2^k}$ containing y. Therefore, by the contraction principle 2.5

$$\mathbb{E}\Big\|\sum_k \varepsilon_k \sum_{Q\in\triangle_{2^k}} 1_Q(y) u_Q(y)\Big\|_X \simeq \mathbb{E}\Big\|\sum_k \varepsilon_k \sum_{Q\in\triangle_{2^k}} h_Q(y) u_Q(y)\Big\|_X.$$

This gives the first equivalence. A similar argument applies to the second. $\square$

We next recall a result of Figiel from [18]. Our need for it is no surprise, since it is also a fundamental ingredient in Figiel's vector-valued $T(1)$ theorem [19].

Proposition 4.4 *(Figiel). Let X be a UMD Banach space, and $1 < p < \infty$. Then for all $m \in \mathbf{Z}^n$ and $x_Q^\eta \in X$*

$$\Big\|\sum_{Q\in\triangle}\sum_\eta x_Q^\eta h_{Q+\ell(Q)m}^\eta\Big\|_{L^p(\mathbf{R}^n;X)} \lesssim \log\big(2+|m|\big)\Big\|\sum_{Q\in\triangle}\sum_\eta x_Q^\eta h_Q^\eta\Big\|_{L^p(\mathbf{R}^n;X)}.$$

Corollary 4.5. *Let X be a UMD Banach space, and $1 < p < \infty$. For $(u_k)_{k\in\mathbf{Z}} \subset L^p(\mathbf{R}^n; X)$, we have*

$$\mathbb{E}\Big\|\sum_k \varepsilon_k \sum_{Q\in\triangle_{2^k}} 1_{Q+2^k m}\langle u_k\rangle_Q\Big\|_{L^p(\mathbf{R}^n;X)} \lesssim \log\big(2+|m|\big)\Big\|\sum_k \varepsilon_k u_k\Big\|_{L^p(\mathbf{R}^n;X)}.$$

Proof. By sign-invariance, and unconditionality of the Haar system, $\log(2+|m|)^{-1}$ times the left-hand side is equivalent to

$$\begin{aligned}
&\frac{1}{\log(2+|m|)}\mathbb{E}\Big\|\sum_k \varepsilon_k \sum_{Q\in\triangle_{2^k}} h_{Q+2^k m}\langle u_k\rangle_Q\Big\| \\
&\quad\lesssim \mathbb{E}\Big\|\sum_k \varepsilon_k \sum_{Q\in\triangle_{2^k}} 1_Q\langle u_k\rangle_Q\Big\| = \mathbb{E}\Big\|\sum_k \varepsilon_k A_{2^k} u_k\Big\| \lesssim \mathbb{E}\Big\|\sum_k u_k\Big\|,
\end{aligned}$$

where Stein's inequality 2.10 was used in the last step. □

The following lemma, too, is closely related to Proposition 4.4, but unlike in the easy corollary above, we now have to employ the techniques of Figiel's proof [18] rather than just his result. Similar martingale arguments inspired by [18] were also recently used in [22].

Lemma 4.6. *Let X be a UMD space, and $1 < p < \infty$. Let further $k \in \mathbf{Z}_+$, $\ell \in \{0, \dots, k\}$, and $x_Q \in X$ for all $Q \in \triangle$. For each $Q \in \triangle$, let $E(Q), F(Q) \subset Q$ be two disjoint subsets such that both $E(Q)$ and $F(Q)$ are unions of some dyadic cubes $R \in \triangle_{2^{-k}\ell(Q)}$, and $|F(Q)| \leqslant |E(Q)|$.*

Then

$$\begin{aligned}
&\bigg(\mathbb{E}\Big\|\sum_{j\equiv\ell} \varepsilon_j \sum_{Q\in\triangle_{2^j}} 1_{F(Q)} x_Q\Big\|^p_{L^p(\mathbf{R}^n;X)}\bigg)^{1/p} \\
&\quad\lesssim_{p,X} \bigg(\mathbb{E}\Big\|\sum_{j\equiv\ell} \varepsilon_j \sum_{Q\in\triangle_{2^j}} 1_{E(Q)} x_Q\Big\|^p_{L^p(\mathbf{R}^n;X)}\bigg)^{1/p},
\end{aligned}$$

where $j \equiv \ell$ is shorthand for $j \equiv \ell \bmod (k+1)$.

Proof. Let

$$E(Q) = \bigcup_{i=1}^{I(Q)} R_i(Q), \qquad F(Q) = \bigcup_{i=1}^{J(Q)} S_i(Q),$$

where $R_i(Q), S_i(Q) \in \triangle_{2^{-k}\ell(Q)}$, the unions are disjoint, and therefore $J(Q) \leqslant I(Q) \leqslant 2^{kn}$ by assumption. Writing $1_{F(Q)} = \sum_i 1_{S_i(Q)}$, $1_{E(Q)} = \sum_i 1_{R_i(Q)}$, and using sign-invariance, the claim is seen to be equivalent to

$$\mathbb{E}\left\|\sum_{j\equiv\ell}\varepsilon_j\sum_{Q\in\triangle_{2^j}}\sum_{i=1}^{J(Q)}h_{S_i(Q)}x_Q\right\|_{L^p(\mathbf{R}^n;X)}^p \lesssim \mathbb{E}\left\|\sum_{j\equiv\ell}\varepsilon_j\sum_{Q\in\triangle_{2^j}}\sum_{i=1}^{I(Q)}h_{R_i(Q)}x_Q\right\|_{L^p(\mathbf{R}^n;X)}^p. \tag{11}$$

We may consider the point in our probability space being fixed for a while, so that the ε_j are just some given signs. For each $j\equiv\ell$ and $Q\in\triangle_{2^j}$, we introduce auxiliary functions as follows:

$$\begin{aligned} d_{Q,i}^{\pm1} &:= \varepsilon_j\frac{1}{2}(h_{R_i(Q)}\pm h_{S_i(Q)})x_Q, && 1\leqslant i\leqslant J(Q),\\ d_{Q,i}^{0} &:= \varepsilon_j h_{R_i(Q)}x_Q, && J(Q)<i\leqslant I(Q), \end{aligned}$$

and finally

$$d_j^{\pm1} := \sum_{Q\in\triangle_{2^j}}\sum_{i=1}^{J(Q)}d_{Q,i}^{\pm1},\qquad d_j^0 := \sum_{Q\in\triangle_{2^j}}\sum_{i=J(Q)+1}^{I(Q)}d_{Q,i}^0.$$

Let us make a key observation. If Q, $Q'\in\triangle$ appear in the claimed estimate (11) and $\ell(Q)>\ell(Q')$, then $\ell(Q)\geqslant 2^{k+1}\ell(Q')$. The functions $d_{Q,i}^{\theta}$ are constant on halves of dyadic cubes of side-length $2^{-k}\ell(Q)$, and hence they are constants on Q'.

We now define the following σ-algebras:

$$\begin{aligned} \mathcal{F}_j^0 &:= \sigma(\triangle_{2^{j-1}}),\\ \mathcal{F}_j^1 &:= \sigma\big(\mathcal{F}_j^0, \big\{d_{Q,i}^{+1}\colon\ Q\in\triangle_{2^j}, 1\leqslant i\leqslant J(Q)\big\}\big),\\ \mathcal{F}_j^2 &:= \sigma\big(\mathcal{F}_j^1, \big\{d_{Q,i}^{-1}\colon\ Q\in\triangle_{2^j}, 1\leqslant i\leqslant J(Q)\big\}\big), \end{aligned}$$

where $\sigma(S)$ denotes the sigma algebra generated by the elements of S, and $\{d_{Q,i}^{\pm1}\colon\ Q\in\triangle_{2^j}, 1\leqslant i\leqslant J(Q)\}$ denotes the sets, indexed by $Q\in\triangle_{2^j}$ and i, of sets $(d_{Q,i}^{\pm1})^{-1}(B)$ where $B\subset\mathbf{R}$ is a Borelian set. Then

$$\cdots\subseteq\mathcal{F}_{\ell+\nu(k+1)}^0\subseteq\mathcal{F}_{\ell+\nu(k+1)}^1\subseteq\mathcal{F}_{\ell+\nu(k+1)}^2\subseteq\mathcal{F}_{\ell+(\nu-1)(k+1)}^0\subseteq\cdots$$

is a filtration of $\mathbf{R}^n$ which generates the Borel σ-algebra, and

$$\ldots,\quad d_{\ell+(\nu+1)(k+1)}^0,\quad d_{\ell+\nu(k+1)}^{+1},\quad d_{\ell+\nu(k+1)}^{-1},\quad d_{\ell+\nu(k+1)}^{0},\quad\ldots$$

is a martingale difference sequence, with respect to this filtration.

By the very definition of UMD spaces, there holds

$$\left\|\sum_{j\equiv\ell}\sum_{\theta\in\{0,\pm1\}}\theta d_j^{\theta}\right\|_{L^p(\mathbf{R}^n;X)}\lesssim\left\|\sum_{j\equiv\ell}\sum_{\theta\in\{0,\pm1\}}d_j^{\theta}\right\|_{L^p(\mathbf{R}^n;X)}.$$

But it is immediate to see that this estimate, after taking the expectation with respect to the ε_j on both sides, is precisely the desired inequality (11). □

Remark 4.7. In the above lemma, the disjointness assumption for $E(Q)$ and $F(Q)$ can be dropped. Writing $1_{F(Q)}$ as $1_{F(Q)}.1_{E(Q)} + 1_{F(Q)\setminus E(Q)}$, one can apply the above proof with $F(Q)$ replaced by $F(Q) \setminus E(Q)$, and handle the other term using sign-invariance and the contraction principle.

5. Vector-valued inequalities for the unperturbed operator

For the unperturbed operator Π, we define R_t, P_t and Q_t by simply dropping the B's from the formulae (4). We also set

$$\mathcal{P}_t = \left(I - t^2\Delta\right)^{-1}, \qquad \mathcal{Q}_t = t\nabla\mathcal{P}_t, \qquad \mathcal{Q}_t^* = -t\mathcal{P}_t \operatorname{div};$$

as it turns out, the assumption (3) often helps to reduce the more complicated Hodge–Dirac resolvents to this canonical family of operators. Note that

$$\mathcal{Q}_t^*\mathcal{Q}_t = -t^2\Delta\left(I - t^2\Delta\right)^{-2}.$$

An important component of our work is the analogue between the harmonic and the dyadic worlds, and in particular the idea that $\mathcal{P}_t$ and A_t are roughly the same. This heuristic will be quantified and proved later on.

Proposition 5.1. *Let X be a UMD Banach space and $1 < p < \infty$. Then the Hodge–Dirac operator Π has an $H^\infty(S_\theta)$-functional calculus on $L^p(\mathbf{R}^n; X^N)$ for every $\theta > 0$.*

Proof. With the help of the Fourier transform and the elementary functional calculus of self-adjoint matrices, the functional calculus of Π may be computed explicitly. In fact, it follows from the assumption (3) that the symbol $\hat{\Pi}(\xi)$ of the differential operator Π satisfies $\hat{\Pi}(\xi)^3 = |\xi|^2\hat{\Pi}(\xi)$, which implies that the only possible eigenvalues of the matrix $\hat{\Pi}(\xi)$ are 0 and $\pm|\xi|$. Functions of such matrices are readily computed, and transforming back we find that

$$f(\Pi) = f_o(\sqrt{-\Delta})\frac{\Pi}{\sqrt{-\Delta}} + \left[f_e(\sqrt{-\Delta}) - f(0)\right]\frac{\Pi^2}{-\Delta} + f(0)I, \tag{12}$$

where $f_o(z) := \frac{1}{2}(f(z) - f(-z))$ and $f_e(z) := \frac{1}{2}(f(z) + f(-z))$ are the odd and even parts of f, respectively. All the operators above are Fourier multipliers, whose boundedness on $L^p(\mathbf{R}^n; X^N)$ follows from the multiplier theorem 2.8. □

Note that (12) and $\Pi^3 = -\Delta\Pi$ imply in particular that

$$g\left(\Pi^2\right)\Pi = g(-\Delta)\Pi, \tag{13}$$

i.e., on $\mathsf{R}(\Pi)$ the functional calculus of Π^2 is just the functional calculus of $-\Delta$.

Lemma 5.2. *Let X be a UMD space, $1 < p < \infty$, and $2M > n + 1$. For $z \in \mathbf{R}^n$, $t \in [0, 1]$, and $u \in L^p(\mathbf{R}^n; X)$, there holds*

$$\mathbb{E}\left\| \sum_k \varepsilon_k 2^k z \cdot \nabla \tau_{t2^k z} \mathcal{P}_{2^k}^M u \right\|_{L^p(\mathbf{R}^n; X)} \lesssim \big(1 + |z|\big)^{n+1} \|u\|_{L^p(\mathbf{R}^n; X)}.$$

Proof. The function inside the norm on the left is a Fourier multiplier transformation of u with the symbol

$$\sigma(\xi) = \sum_k \varepsilon_k 2^k z \cdot i\xi\, e^{it2^k z\cdot\xi} \cdot \big(1 + 2^{2k}|\xi|^2\big)^{-M}.$$

For every $\alpha \in \{0, 1\}^n$, a straightforward computation shows, given the assumption $2M > n + 1$, that

$$|\xi|^{|\alpha|} \big|D^\alpha \sigma(\xi)\big| \lesssim \big(1 + |z|\big)^{1+|\alpha|} \lesssim \big(1 + |z|\big)^{1+n}.$$

The assertion hence follows from the multiplier theorem 2.8. □

Lemma 5.3. *Let X be a UMD Banach space, $1 < p < \infty$, and $M \in \mathbf{Z}_+$. For $u \in L^p(\mathbf{R}^n; X)$ we have*

$$\mathbb{E}\left\| \sum_{k\in\mathbf{Z}} \varepsilon_k \big(\mathcal{P}_{2^k} - \mathcal{P}_{2^k}^M\big) u \right\|_{L^p(\mathbf{R}^n; X^N)} \lesssim \|u\|_{L^p(\mathbf{R}^n; X^N)}.$$

Proof. This is a Fourier multiplier estimate again. One may either directly study the multiplier on the left like in Lemma 5.2, or argue in a slightly more step-by-step fashion as follows. Observe first that $\mathcal{P}_t^{j-1} - \mathcal{P}_t^j = -t^2 \Delta \mathcal{P}_t^j = \mathcal{P}_t^{j-2} \mathcal{Q}_t^* \mathcal{Q}_t$ for all $j = 2, \ldots, N$. The symbols of $\mathcal{P}_t$ have uniformly bounded variation, so the operators are R-bounded by Proposition 2.9, and thus

$$\begin{aligned}
\mathbb{E}\left\| \sum_{k\in\mathbf{Z}} \varepsilon_k \big(\mathcal{P}_{2^k} - \mathcal{P}_{2^k}^M\big) u \right\| &\leqslant \sum_{j=2}^N \mathbb{E}\left\| \sum_{k\in\mathbf{Z}} \varepsilon_k \big(\mathcal{P}_{2^k}^{j-1} - \mathcal{P}_{2^k}^j\big) u \right\| \\
&\lesssim \sum_{j=2}^N \mathbb{E}\left\| \sum_{k\in\mathbf{Z}} \varepsilon_k \mathcal{Q}_{2^k}^* \mathcal{Q}_{2^k} u \right\| \lesssim \|u\|,
\end{aligned}$$

where the final quadratic estimate again follows from the Multiplier Theorem 2.8. □

We have now accumulated enough knowledge to prove the following estimate showing that $\mathcal{P}_t$ is almost like its average $A_t \mathcal{P}_t$, in the precise sense of the quadratic estimate. In the rest of this section we are going to show the "dual" property that also A_t is almost like $A_t \mathcal{P}_t$, thus justifying our heuristic of the "equivalence" of A_t and P_t.

Proposition 5.4. *Let X be a UMD space, and $1 < p < \infty$. Then for all $u \in L^p(\mathbf{R}^n; X)$, there holds*

$$\mathbb{E}\Big\|\sum_k \varepsilon_k (A_{2^k} - I)\mathcal{P}_{2^k} u\Big\|_{L^p(\mathbf{R}^n;X)} \lesssim \|u\|_{L^p(\mathbf{R}^n;X)}.$$

Proof. Since the operators $A_{2^k} - I$ are R-bounded, and the differences $\mathcal{P}_{2^k} - \mathcal{P}_{2^k}^M$ satisfy the quadratic estimate of Lemma 5.3 (taking $2M > n+1$), it suffices to prove the claim with $\mathcal{P}_{2^k}$ replaced by $\mathcal{P}_{2^k}^M$. The left-hand side of the modified claim is

$$\begin{aligned}
&\mathbb{E}\Big\|\sum_k \varepsilon_k \sum_{Q\in\triangle_{2^k}} 1_Q\big(\mathcal{P}_{2^k}^M u - \langle\mathcal{P}_{2^k}^M u\rangle_Q\big)\Big\|_{L^p(\mathbf{R}^n;X)} \\
&\quad\lesssim \int_{[-1,1]^n}\int_0^1 \mathbb{E}\Big\|\sum_k \varepsilon_k 2^k z\cdot\nabla\tau_{t2^k z}\mathcal{P}_{2^k}^M u\Big\|_{L^p(\mathbf{R}^n;X)}\,\mathrm{d}t\,\mathrm{d}z \\
&\quad\lesssim \int_{[-1,1]^n}\int_0^1 \big(1+|z|\big)^{n+1}\|u\|_{L^p(\mathbf{R}^n;X)}\,\mathrm{d}t\,\mathrm{d}z \lesssim \|u\|_{L^p(\mathbf{R}^n;X)},
\end{aligned}$$

by Proposition 4.1 and Lemma 5.2. □

Our next proposition is a vector-valued analogue of [9, Proposition 5.7].

Proposition 5.5. *Let X be a UMD space, and $1 < p < \infty$. For $u \in L^p(\mathbf{R}^n; X)$, we have*

$$\mathbb{E}\Big\|\sum_j \varepsilon_j A_{2^j}(\mathcal{P}_{2^j} - I)u\Big\|_{L^p(\mathbf{R}^n;X)} \lesssim \|u\|_{L^p(\mathbf{R}^n;X)}.$$

Proof. As a preparation, observe that $\sum_{i\in\mathbf{Z}} \mathcal{Q}_{2^i}^*\mathcal{Q}_{2^i}$ is represented by the Fourier multiplier $\sum_{i\in\mathbf{Z}}(2^i|\xi|)^2(1+(2^i|\xi|)^2)^{-2}$ which, as well as its reciprocal, satisfies the conditions of the multiplier theorem 2.8. This implies the two-sided estimate

$$\Big\|\sum_i \mathcal{Q}_{2^i}^*\mathcal{Q}_{2^i} u\Big\|_{L^p(\mathbf{R}^n;X)} \eqsim \|u\|_{L^p(\mathbf{R}^n;X)}.$$

Thus, it suffices to prove

$$\mathbb{E}\Big\|\sum_{i,j} \varepsilon_j A_{2^j}(\mathcal{P}_{2^j} - I)\mathcal{Q}_{2^i}^*\mathcal{Q}_{2^i} u\Big\|_{L^p(\mathbf{R}^n;X)} \lesssim \|u\|_{L^p(\mathbf{R}^n;X)}. \tag{14}$$

Since also

$$\mathbb{E}\Big\|\sum_i \varepsilon_i \mathcal{Q}_{2^i} u\Big\|_{L^p(\mathbf{R}^n;X)} \lesssim \|u\|_{L^p(\mathbf{R}^n;X^n)},$$

again by Theorem 2.8 (say), (14) will follow from Schur's estimate 4.2 (with $\mathcal{X} = \mathcal{Z} = L^p(\mathbf{R}^n; X)$, and $\mathcal{Y} = L^p(\mathbf{R}^n; X^n)$), once we show that

$$\mathcal{R}\big(2^{\delta|j-i|} A_{2^j}(\mathcal{P}_{2^j} - I)\mathcal{Q}^*_{2^i}\colon i, j \in \mathbf{Z}\big) \lesssim 1 \tag{15}$$

for some $\delta > 0$. Since $(I - \mathcal{P}_t)\mathcal{Q}^*_s = \frac{t}{s}(I - \mathcal{P}_s)\mathcal{Q}^*_t$ and $\mathcal{P}_t\mathcal{Q}^*_s = \frac{s}{t}\mathcal{P}_s\mathcal{Q}^*_t$ for all $s, t > 0$, and all the families A_{2^j}, $\mathcal{P}_{2^j}$ and $\mathcal{Q}^*_{2^j}$, $j \in \mathbf{Z}$, are R-bounded on the relevant spaces, it is immediate that

$$\mathcal{R}\big(2^{i-j} A_{2^j}(\mathcal{P}_{2^j} - I)\mathcal{Q}^*_{2^i}\colon i \geqslant j\big) = \mathcal{R}\big(A_{2^j}\mathcal{Q}^*_{2^j}(\mathcal{P}_{2^i} - I)\colon i \geqslant j\big) \lesssim 1,$$
$$\mathcal{R}\big(2^{j-i} A_{2^j}\mathcal{P}_{2^j}\mathcal{Q}^*_{2^i}\colon i < j\big) = \mathcal{R}\big(A_{2^j}\mathcal{P}_{2^i}\mathcal{Q}^*_{2^j}\colon i < j\big) \lesssim 1.$$

It remains to estimate $A_{2^j}\mathcal{Q}^*_{2^i}$ for $i < j$. We divide this task into the countable number of cases where $k = j - i \in \mathbf{Z}_+$ is fixed, aiming to establish sufficiently good R-bounds to be able to sum them up. We start the estimation by writing

$$\begin{aligned}
&\bigg(\mathbb{E}\bigg\|\sum_j \varepsilon_j A_{2^j}\mathcal{Q}^*_{2^{j-k}}u_j\bigg\|^p_{L^p(\mathbf{R}^n;X)}\bigg)^{1/p} \\
&= \bigg(\mathbb{E}\bigg\|\sum_j \varepsilon_j \sum_{Q\in\triangle_{2^j}} 1_Q \fint_Q \mathcal{Q}^*_{2^{j-k}}u_j\bigg\|^p_{L^p(\mathbf{R}^n;X)}\bigg)^{1/p} \\
&\leqslant \sum_{\ell=0}^k \bigg(\mathbb{E}\bigg\|\sum_{j\equiv\ell \bmod (k+1)} \varepsilon_j \sum_{Q\in\triangle_{2^j}} 1_Q \fint_Q \mathcal{Q}^*_{2^{j-k}}u_j\bigg\|^p_{L^p(\mathbf{R}^n;X)}\bigg)^{1/p}.
\end{aligned} \tag{16}$$

We next decompose each of the cubes $Q \in \triangle$ into 2^{k-1} parts inductively as follows. Denoting

$$\partial_\delta E := \big\{x \in E\colon d\big(x, E^c\big) \leqslant \delta\big\},$$

we set

$$Q^1 := \partial_{2^{-k}\ell(Q)}Q, \qquad Q^m := \partial_{2^{-k}\ell(Q)}\Bigg[Q \Big\backslash \bigcup_{\nu=1}^{m-1} Q^\nu\Bigg], \quad m = 2, \ldots, 2^{k-2}.$$

Then Q^m is a union (up to boundaries) of certain dyadic cubes $R \in \triangle_{2^{-k}\ell(Q)}$, and $|Q^m| \geqslant |Q^{m+1}|$ for all $m < 2^{k-1}$. This is preparation for the application of Lemma 4.6 later on.

The right-hand side of (16) may now be rewritten as

$$\sum_{\ell=0}^k \Bigg(\mathbb{E}_{\varepsilon'}\mathbb{E}_\varepsilon\Bigg\|\sum_{m=1}^{2^{k-1}} \varepsilon'_m \sum_{j\equiv\ell} \varepsilon_j \sum_{Q\in\triangle_{2^j}} 1_{Q^m} \fint_Q \mathcal{Q}^*_{2^{j-k}}u_j\Bigg\|^p_{L^p(\mathbf{R}^n;X)}\Bigg)^{1/p},$$

where the randomized j sum, as in Lemma 4.3, does not "see" the introduction of the additional random factors ε'_m. The UMD space X, and then also the Bochner space of functions with values in this space, has some non-trivial Rademacher-type $t > 1$, which gives the estimate

$$\lesssim \sum_{\ell=0}^{k} \left\{ \sum_{m=1}^{2^{k-1}} \left(\mathbb{E}_\varepsilon \left\| \sum_{j\equiv\ell} \varepsilon_j \sum_{Q\in\triangle_{2^j}} 1_{Q^m} \fint_Q \mathcal{Q}^*_{2^{j-k}} u_j \right\|^p_{L^p(\mathbf{R}^n;X)} \right)^{t/p} \right\}^{1/t}.$$

We are now in a position to apply Lemma 4.6. For all $m = 2, \dots, 2^{k-1}$, the sets $E(Q) = Q^1$ and $F(Q) = Q^m$ satisfy the assumptions of that lemma, which means that the summand with $m = 1$ above dominates any one of the other summands with $m = 2, \dots, 2^{k-1}$. Hence, recalling that $Q^1 = \partial_{2^{-k}\ell(Q)} Q$, we may continue with

$$\lesssim \sum_{\ell=0}^{k} 2^{k/t} \left(\mathbb{E} \left\| \sum_{j\equiv\ell} \varepsilon_j \sum_{Q\in\triangle_{2^j}} 1_{\partial_{2^{j-k}} Q} \fint_Q \mathcal{Q}^*_{2^{j-k}} u_j \right\|^p_{L^p(\mathbf{R}^n;X)} \right)^{1/p}. \tag{17}$$

Finally, we start making use of the properties of the operators $\mathcal{Q}^*_t$. For each $Q \in \triangle_{2^j}$, let $\eta_Q \in C_0^\infty(Q)$ be a function with $\eta_Q = 1$ in $Q \setminus \partial_{2^{j-k}} Q$ and $|\nabla \eta_Q| \lesssim 2^{k-j}$. We have

$$\begin{aligned} \int_Q \mathcal{Q}^*_{2^{j-k}} u_j &= \int_Q \eta_Q 2^{j-k} (-\operatorname{div}) \mathcal{P}_{2^{j-k}} u_j + \int_Q (1-\eta_Q) \mathcal{Q}^*_{2^{j-k}} u_j \\ &= 2^{j-k} \int_Q \big[\eta_Q, (-\operatorname{div})\big] \mathcal{P}_{2^{j-k}} u_j + \int_Q (1-\eta_Q) \mathcal{Q}^*_{2^{j-k}} u_j, \end{aligned}$$

where we used the fact that the integral of the divergence of $\eta_Q \mathcal{P}_{2^{j-k}} u_j$ vanishes. We may further observe that $[\eta_Q, (-\operatorname{div})]v = \nabla \eta_Q \cdot v$, and both $\nabla \eta_Q$ and $1_Q - \eta_Q$ are supported on $\partial_{2^{j-k}} Q$, so that both integrals above may be reduced to this smaller set. Thus

$$\begin{aligned} &\left(\mathbb{E} \left\| \sum_{j\equiv\ell} \varepsilon_j \sum_{Q\in\triangle_{2^j}} 1_{\partial_{2^{j-k}} Q} \fint_Q \mathcal{Q}^*_{2^{j-k}} u_j \right\|^p_{L^p(\mathbf{R}^n;X)} \right)^{1/p} \\ &\quad = \Bigg(\mathbb{E} \Bigg\| \sum_{j\equiv\ell} \varepsilon_j \sum_{Q\in\triangle_{2^j}} \frac{|\partial_{2^{j-k}} Q|}{|Q|} 1_{\partial_{2^{j-k}} Q} \\ &\qquad \times \fint_{\partial_{2^{j-k}} Q} \big(2^{j-k} \nabla \eta_Q \cdot \mathcal{P}_{2^{j-k}} u_j + (1-\eta_Q) \mathcal{Q}^*_{2^{j-k}} u_j \big) \Bigg\|^p_{L^p(\mathbf{R}^n;X)} \Bigg)^{1/p}. \end{aligned} \tag{18}$$

The factors $|\partial_{2^{j-k}} Q|/|Q|$ are equal to $1 - (1 - 2^{1-k})^n \lesssim 2^{-k}$ and may be extracted outside the summation and the norm. Then we are left with an expression involving the conditional expectation projections related to the filtration

$$\big(\sigma \big(\partial_{2^{j-k}} Q, Q \setminus \partial_{2^{j-k}} Q \colon Q \in \triangle_{2^j} \big) \big)_{j\equiv\ell \bmod k+1}.$$

These are R-bounded under the UMD assumption, and hence the quantity in (18) is majorized by

$$\lesssim 2^{-k}\biggl(\mathbb{E}\biggl\|\sum_{j\equiv\ell}\varepsilon_j\sum_{Q\in\Delta_{2^j}}\bigl(2^{j-k}\nabla\eta_Q\cdot\mathcal{P}_{2^{j-k}}u_j+(1_Q-\eta_Q)\mathcal{Q}^*_{2^{j-k}}u_j\bigr)\biggr\|^p_{L^p(\mathbf{R}^n;X)}\biggr)^{1/p}$$

$$\lesssim 2^{-k}\biggl(\mathbb{E}\biggl\|\sum_{j\equiv\ell}\varepsilon_j\mathcal{P}_{2^{j-k}}u_j\biggr\|^p_{L^p(\mathbf{R}^n;X^n)}\biggr)^{1/p}+2^{-k}\biggl(\mathbb{E}\biggl\|\sum_{j\equiv\ell}\varepsilon_j\mathcal{Q}^*_{2^{j-k}}u_j\biggr\|^p_{L^p(\mathbf{R}^n;X)}\biggr)^{1/p},$$

where the last estimate used the contraction principle 2.5 and $2^{j-k}|\nabla\eta_Q|\lesssim 1$. Using the R-boundedness of $\mathcal{P}_{2^{j-k}}$ and $\mathcal{Q}^*_{2^{j-k}}$, and substituting back to (17), we have shown that

$$\begin{aligned}\biggl(\mathbb{E}\biggl\|\sum_j\varepsilon_jA_{2^j}\mathcal{Q}^*_{2^{j-k}}u_j\biggr\|^p_{L^p(\mathbf{R}^n;X)}\biggr)^{1/p}&\\ \lesssim\sum_{\ell=0}^{k}2^{k/t}2^{-k}\biggl(\mathbb{E}\biggl\|\sum_j\varepsilon_ju_j\biggr\|^p_{L^p(\mathbf{R}^n;X^n)}\biggr)^{1/p}&\\ =(k+1)2^{-k/t'}\biggl(\mathbb{E}\biggl\|\sum_j\varepsilon_ju_j\biggr\|^p_{L^p(\mathbf{R}^n;X^n)}\biggr)^{1/p}&.\end{aligned}$$

This says that $\mathcal{R}(A_{2^j}\mathcal{Q}^*_{2^{j-k}}\colon\ j\in\mathbf{Z})\lesssim(k+1)2^{-k/t'}$, and allows us to estimate:

$$\begin{aligned}\mathcal{R}\bigl(2^{|i-j|/2t'}A_{2^j}\mathcal{Q}^*_{2^i}\colon\ i,j\in\mathbf{Z},\ i<j\bigr)&\leqslant\sum_{k=1}^{\infty}\mathcal{R}\bigl(2^{k/2t'}A_{2^j}\mathcal{Q}^*_{2^{j-k}}\colon\ j\in\mathbf{Z}\bigr)\\ &\lesssim\sum_{k=1}^{\infty}(k+1)2^{-k/2t'}\lesssim 1.\end{aligned}$$

We have proved the required R-boundedness (15) with $\delta=1/2t'=\frac{1}{2}(1-1/t)>0$, where $t>1$ is a Rademacher-type for $L^p(\mathbf{R}^n;X)$. $\square$

We conclude this section with the following result, which combines most of the estimates achieved so far. Although we will not make direct use of this inequality, but rather the various individual results above, Corollary 5.6 appears worth recording for the potential further applications of the transference between the dyadic and the harmonic estimates, which it provides.

Corollary 5.6. *Let X be a UMD space, and $1<p<\infty$. For $u\in L^p(\mathbf{R}^n;X)$, we have*

$$\mathbb{E}\biggl\|\sum_k\varepsilon_k(A_{2^k}-\mathcal{P}_{2^k})u\biggr\|_{L^p(\mathbf{R}^n;X)}\lesssim\|u\|_{L^p(\mathbf{R}^n;X)}.$$

For $u\in\mathsf{R}(\Pi)$, the same is true with P_{2^k} in place of $\mathcal{P}_{2^k}$.

Proof. The first claim is immediate from Propositions 5.4 and 5.5, and the second follows from (13). □

6. A quadratic $T(1)$ theorem

In this section we show that the proof of certain quadratic estimates can be reduced to similar inequalities for the "principal part" of the operators involved. This will then be applied to our particular operators $Q^B_{2^k}$, and is an analogue of [9, Sections 5.1 and 5.2]. However, we start with the description of a more general situation.

Let $\mathcal{T} = (T_{2^k})_{k\in\mathbf{Z}}$ be an R-bounded sequence of linear operators on $L^p(\mathbf{R}^n; Y)$, where $1 < p < \infty$ and Y is a Banach space, and let $\mathcal{Z} \subseteq L^p(\mathbf{R}^n; Y)$ be a subspace. We say that $\mathcal{T}$ satisfies a *high-frequency estimate* on $\mathcal{Z}$ if

$$\mathbb{E}\left\|\sum_k \varepsilon_k T_{2^k}(I - \mathcal{P}_{2^k})u\right\|_{L^p(\mathbf{R}^n;Y)} \lesssim \|u\|_{L^p(\mathbf{R}^n;Y)} \tag{19}$$

for all $u \in \mathcal{Z}$. Concerning the name, note that the symbol of $I - \mathcal{P}_{2^k}$ is $(2^k|\xi|)^2(1 + (2^k|\xi|)^2)^{-1}$, which can be thought of as a smooth approximation of the characteristic function of $\{\xi \in \mathbf{R}^n\colon |\xi| > 2^{-k}\}$.

We say that $\mathcal{T}$ satisfies *off-diagonal R-bounds* if the following inequality holds for every $M \in \mathbf{N}$, with the implied constant only depending on M: whenever $E_k, F_k \subset \mathbf{R}^n$ are Borel subsets, $u_k \in L^p(\mathbf{R}^n; Y)$, and $(t_k)_{k\in\mathbf{Z}} \subseteq \{2^k\}_{k\in\mathbf{Z}}$ are numbers so that $\operatorname{dist}(E_k, F_k)/t_k > \varrho$ for some $\varrho > 0$ and all $k \in \mathbf{Z}$, there holds

$$\mathbb{E}\left\|\sum_k \varepsilon_k 1_{E_k} T_{t_k} 1_{F_k} u_k\right\|_{L^p(\mathbf{R}^n;Y)} \lesssim (1+\varrho)^{-M}\,\mathbb{E}\left\|\sum_k \varepsilon_k 1_{F_k} u_k\right\|_{L^p(\mathbf{R}^n;Y)}. \tag{20}$$

Note that the case $M = 0$ follows automatically from the assumed R-boundedness of the T_{2^k} and the contraction principle 2.5.

Finally, the *principal part* of the operator T_{2^k} is the operator-valued function $\gamma_{2^k} : \mathbf{R}^n \to \mathcal{L}(Y)$ defined by (intuitively, "$\gamma_{2^k} := T_{2^k}(1)$")

$$\gamma_{2^k}(x)w := T_{2^k}(w)(x) := \sum_{Q\in\Delta_{2^k}} T_{2^k}(w1_Q)(x), \quad x \in \mathbf{R}^n,\ w \in Y. \tag{21}$$

Note that (20) implies that the right-hand side of (21) converges absolutely in $L^p_{\mathrm{loc}}(\mathbf{R}^n; Y)$, and this series defines the action of T_{2^k} on the constant function w, which lies outside its original domain of definition, namely $L^p(\mathbf{R}^n; Y)$.

We are going to prove the following "quadratic $T(1)$ theorem."

Theorem 6.1. *Let Y be a UMD space, and $1 < p < \infty$. Let the R-bounded operator-sequence $\mathcal{T} = (T_{2^k})_{k\in\mathbf{Z}}$ in $\mathcal{L}(L^p(\mathbf{R}^n; Y))$ satisfy the high-frequency estimate* (19) *on a subspace $\mathcal{Z} \subseteq L^p(\mathbf{R}^n; Y)$, and the off-diagonal R-bounds* (20)*. Then there holds*

$$\mathbb{E}\left\|\sum_k \varepsilon_k (T_{2^k} - \gamma_{2^k} A_{2^k})u\right\|_{L^p(\mathbf{R}^n;Y)} \lesssim \|u\|_{L^p(\mathbf{R}^n;Y)}, \quad u \in \mathcal{Z}.$$

Thus $\mathcal{T}$ satisfies a quadratic estimate on $\mathcal{Z}$ if and only if its principal part does.

Before going into the proof, let us indicate the consequences for our primary case of interest, which is the vector-valued analogue of [9, Proposition 5.5].

Theorem 6.2. *Let X be a UMD Banach space, $1 < p < \infty$, and Π_B be an R-bisectorial perturbed Hodge–Dirac operator on $L^p(\mathbf{R}^n; X^N)$. Let γ_{2^k} denote the principal part of $Q^B_{2^k}$. Then there holds*:

$$\mathbb{E}\Bigl\|\sum_{k\in\mathbf{Z}} \varepsilon_k \bigl(Q^B_{2^k} - \gamma_{2^k} A_{2^k}\bigr)u\Bigr\|_{L^p(\mathbf{R}^n;X^N)} \lesssim \|u\|_{L^p(\mathbf{R}^n,X^N)}, \quad \forall u \in \mathsf{R}(\Gamma),$$

and the operators $\gamma_{2^k}(x)$ are multiplications by complex $N \times N$-matrices.

The quadratic estimate is obviously implied by Theorem 6.1 as soon as we check that $(Q^B_{2^k})_{k\in\mathbf{Z}}$ satisfies the high-frequency estimate on $\mathsf{R}(\Gamma)$ and the off-diagonal R-bounds. This is the content of the next two results below. The form of the principal part follows readily from the definition (21) and the fact that the operators $Q^B_{2^k}$ on $L^p(\mathbf{R}^n; X^N)$ are tensor extensions of operators on $L^p(\mathbf{R}^n; \mathbf{C}^N)$.

Lemma 6.3. *The family $(Q^B_{2^k})_{k\in\mathbf{Z}}$ satisfies the high-frequency estimate* (19) *on* $\mathsf{R}(\Gamma) \subset L^p(\mathbf{R}^n; X^N)$.

Proof. It follows from (13) that $\mathcal{P}_{2^k}u = P_{2^k}u$ for $u \in \mathsf{R}(\Gamma)$, so it suffices to prove the modified claim with P_{2^k} in place of $\mathcal{P}_{2^k}$.

Let $\mathbb{P}^1$ denote the projection of

$$L^p\bigl(\mathbf{R}^n; X^N\bigr) = L^p\bigl(\mathbf{R}^n; X^{n_1}\bigr) \oplus L^p\bigl(\mathbf{R}^n; X^{n_2}\bigr)$$

onto $L^p(\mathbf{R}^n; X^{n_1})$. Since $u \in \mathsf{R}(\Gamma)$, a straightforward manipulation using the structure of the operators shows that

$$Q^B_t(I - P_t)u = Q^B_t t\Gamma Q_t u = \bigl(I - P^B_t\bigr)\mathbb{P}^1 Q_t u.$$

Since $\{(I - P^B_t)\mathbb{P}^1;\, t \geqslant 0\}$ is R-bounded, this gives

$$\begin{aligned}\mathbb{E}\Bigl\|\sum_k \varepsilon_k Q^B_{2^k}(I - P_{2^k})u\Bigr\|_{L^p(\mathbf{R}^n;X^N)} &\lesssim \mathbb{E}\Bigl\|\sum_k \varepsilon_k Q_{2^k}u\Bigr\|_{L^p(\mathbf{R}^n;X^N)} \\ &\lesssim \|u\|_{L^p(\mathbf{R}^n;X^N)},\end{aligned}$$

where the last inequality follows from Proposition 5.1. $\square$

The following proposition is the vector-valued analogue of [9, Proposition 5.2].

Proposition 6.4. *The family $(Q^B_{2^k})_{k\in\mathbf{Z}}$ satisfies the off-diagonal R-bounds* (20).

Proof. It is sufficient to prove this result for $R^B_{t_k}$ instead of $Q^B_{t_k}$ since $Q^B_{t_k} = \frac{i}{2}(R^B_{t_k} - R^B_{-t_k})$. We proceed by induction on M. The case $M = 0$ follows from Kahane's contraction principle 2.5 and the R-bisectoriality of Π_B. Now assume it is true for some $M \geqslant 0$, and consider

$$\tilde{E}_k = \left\{x \in \mathbf{R}^n;\ \operatorname{dist}(x, E_k) < \frac{1}{2}\operatorname{dist}(x, F_k)\right\}$$

and η_k a cutoff function supported in $\tilde{E}_k$ with $(\eta_k)_{|E_k} = 1$ and $\|\nabla\eta_k\|_\infty \leqslant 4/\operatorname{dist}(E_k, F_k)$. Denoting by $[T, S] = TS - ST$ the commutator of two operators we have

$$\left[\eta_k I, R^B_{t_k}\right] = i t_k R^B_{t_k}\left([\Gamma, \eta_k I] + B_1\left[\Gamma^*, \eta_k I\right]B_2\right)R^B_{t_k}.$$

Using R-bisectoriality, and the fact that $[\Gamma, \eta_k I] + B_1[\Gamma^*, \eta_k I]B_2$ is a multiplication by an L^∞ function bounded by $\|\nabla\eta_k\|_\infty$, we thus have

$$\begin{aligned}
&\mathbb{E}\left\|\sum_k \varepsilon_k 1_{E_k} R^B_{t_k} 1_{F_k} u_k\right\| \\
&\quad\lesssim \mathbb{E}\left\|\sum_k \varepsilon_k\left[\eta_k I, R^B_{t_k}\right]1_{F_k} u_k\right\| \\
&\quad\lesssim \mathbb{E}\left\|\sum_k \varepsilon_k i t_k R^B_{t_k}\left([\Gamma, \eta_k I] + B_1\left[\Gamma^*, \eta_k I\right]B_2\right)1_{\tilde{E}_k} R^B_{t_k} 1_{F_k} u_k\right\| \\
&\quad\lesssim \sup_{j\in\mathbf{Z}} |t_j|\|\nabla\eta_j\|_\infty \mathbb{E}\left\|\sum_k \varepsilon_k 1_{\tilde{E}_k} R^B_{t_k} 1_{F_k} u_k\right\| \\
&\quad\lesssim \frac{1}{\rho}\mathbb{E}\left\|\sum_k \varepsilon_k 1_{\tilde{E}_k} R^B_{t_k} 1_{F_k} u_k\right\|,
\end{aligned}$$

and we may apply the induction assumption to the remaining quantity. □

This completes the proof that Theorem 6.2 is a consequence of Theorem 6.1. We now return to the quadratic $T(1)$ theorem 6.1. In proving this result, we decompose

$$T_t - \gamma_t A_t = T_t(I - \mathcal{P}_t) + (T_t - \gamma_t A_t)\mathcal{P}_t + \gamma_t A_t(\mathcal{P}_t - I),$$

where the different summands on the right-hand side will be analyzed separately. The first one, of course, is immediately handled by the assumed high-frequency estimate.

Lemma 6.5. *Under the assumptions of Theorem* 6.1*, the principal part operators* $(\gamma_{2^k} A_{2^k})_{k\in\mathbf{Z}}$ *are R-bounded on* $L^p(\mathbf{R}^n; X^N)$.

Proof. For $(u_k)_{k\in\mathbf{Z}} \subset L^p(\mathbf{R}^n; X^N)$ we have

$$
\begin{aligned}
&\mathbb{E}\Big\|\sum_{k\in\mathbf{Z}}\varepsilon_k\gamma_{2^k}A_{2^k}u_k\Big\| \\
&\quad=\mathbb{E}\Big\|\sum_{k\in\mathbf{Z}}\varepsilon_k\sum_{Q\in\triangle_{2^k}}1_Q T_{2^k}\langle u_k\rangle_Q\Big\| \\
&\quad\leqslant\sum_{m\in\mathbf{Z}^n}\mathbb{E}\Big\|\sum_{k\in\mathbf{Z}}\varepsilon_k\sum_{Q\in\triangle_{2^k}}1_Q T_{2^k}\big(1_{Q+2^k m}\langle u_k\rangle_Q\big)\Big\| \\
&\quad\lesssim\sum_{m\in\mathbf{Z}^n}\big(1+|m|\big)^{-M}\mathbb{E}\Big\|\sum_{k\in\mathbf{Z}}\varepsilon_k\sum_{Q\in\triangle_{2^k}}1_{Q+2^k m}\langle u_k\rangle_Q\Big\| \\
&\quad\lesssim\sum_{m\in\mathbf{Z}^n}\big(1+|m|\big)^{-M}\log\big(2+|m|\big)\mathbb{E}\Big\|\sum_{k\in\mathbf{Z}}\varepsilon_k u_k\Big\|,
\end{aligned}
$$

where the last two estimates where applications of the off-diagonal estimates (and sign-invariance), and Corollary 4.5, respectively. The series is summable for $M > n$. □

The next lemma is the vector-valued analogue of [9, Proposition 5.5].

Lemma 6.6. *Under the assumptions of Theorem* 6.1*, for all $u \in L^p(\mathbf{R}^n; Y)$ there holds*

$$
\mathbb{E}\Big\|\sum_{k\in\mathbf{Z}}\varepsilon_k(T_{2^k}-\gamma_{2^k}A_{2^k})\mathcal{P}_{2^k}u\Big\|_{L^p(\mathbf{R}^n,Y)}\lesssim\|u\|_{L^p(\mathbf{R}^n,Y)}.
$$

Proof. We first observe that it suffices to prove a modified assertion with $\mathcal{P}_{2^k}$ replaced by $\mathcal{P}^M_{2^k}$. Indeed, this follows at once from the R-boundedness of T_{2^k} and $\gamma_{2^k}A_{2^k}$ combined with Lemma 5.3.

As for the new claim, denote $v_k := \mathcal{P}^M_{2^k}u$. Then

$$
\begin{aligned}
&\mathbb{E}\Big\|\sum_{k}\varepsilon_k(T_{2^k}-\gamma_{2^k}A_{2^k})v_k\Big\| \\
&\quad=\mathbb{E}\Big\|\sum_{k}\varepsilon_k\sum_{Q\in\triangle_{2^k}}1_Q T_{2^k}\big(v_k-\langle v_k\rangle_Q\big)\Big\| \\
&\quad\leqslant\sum_{m\in\mathbf{Z}^n}\mathbb{E}\Big\|\sum_{k}\varepsilon_k\sum_{Q\in\triangle_{2^k}}1_Q T_{2^k}\big(1_{Q-2^k m}\big(v_k-\langle v_k\rangle_Q\big)\big)\Big\| \\
&\quad\lesssim\sum_{m\in\mathbf{Z}^n}\big(1+|m|\big)^{-M}\mathbb{E}\Big\|\sum_{k}\varepsilon_k\sum_{Q\in\triangle_{2^k}}1_Q\big(v_k-\langle v_k\rangle_{Q+2^k m}\big)\Big\| \qquad (22)
\end{aligned}
$$

where we used the off-diagonal estimates. By the Poincaré inequality (Proposition 4.1) and Lemma 5.2, the last factor is majorized by

$$\int_{[-1,1]^n}\int_0^1 \mathbb{E}\Big\|\sum_k \varepsilon_k 2^k(m+z)\cdot\nabla\,\tau_{t2^k(m+z)}\mathcal{P}_{2^k}^M u\Big\|\,\mathrm{d}t\,\mathrm{d}z \lesssim \big(1+|m|\big)^{n+1}\|u\|.$$

Substituting this back to (22), we find that the series sums up to $\lesssim \|u\|$ provided that we choose $M > 2n+1$. $\square$

Proof of Theorem 6.1. We have

$$\begin{aligned}
&\mathbb{E}\Big\|\sum_k \varepsilon_k(T_{2^k}-\gamma_{2^k}A_{2^k})u\Big\| \\
&\quad\lesssim \mathbb{E}\Big\|\sum_k \varepsilon_k T_{2^k}(I-\mathcal{P}_{2^k})u\Big\| + \mathbb{E}\Big\|\sum_k \varepsilon_k(T_{2^k}-\gamma_{2^k}A_{2^k})\mathcal{P}_{2^k}u\Big\| \\
&\qquad + \mathbb{E}\Big\|\sum_k \varepsilon_k\gamma_{2^k}A_{2^k}(\mathcal{P}_{2^k}-I)u\Big\|.
\end{aligned}$$

For $u \in Z \subset L^p(\mathbf{R}^n;Y)$, the upper bound $\|u\|$ for the first term follows from the assumed high-frequency estimate, for the second term from Lemma 6.6, and for the third one from Lemma 6.5 and Proposition 5.5 together with the observation that $A_{2^k} = A_{2^k}A_{2^k}$. $\square$

In order to estimate the principal term

$$\mathbb{E}\Big\|\sum_{k\in\mathbf{Z}} \varepsilon_k\gamma_{2^k}A_{2^k}u\Big\|_{L^p(\mathbf{R}^n;X^N)}, \qquad u \in \mathsf{R}(\Gamma), \tag{23}$$

we need a version of Carleson's inequality. This is achieved in Section 8 by using the Rademacher maximal function, which we next study.

7. The Rademacher maximal function

We recall the definition of the Rademacher maximal function, here stated in an equivalent but slightly different way from Section 2:

$$\begin{aligned}
M_{\mathrm{R}}u(x) := \sup\bigg\{&\mathbb{E}\Big\|\sum_{Q\ni x}\varepsilon_Q\lambda_Q\langle u\rangle_Q\Big\|_X: \\
&(\lambda_Q)_{Q\in\Delta} \text{ finitely non-zero with } \sum_{Q\in\Delta}|\lambda_Q|^2 \leqslant 1\bigg\}.
\end{aligned}$$

We will also find it convenient to consider the following linearized version:

$$\mathcal{M}_{\mathrm{R}}u(x) : \ell^2(\Delta) \to \mathrm{Rad}(X), \quad (\lambda_Q)_{Q\in\Delta} \mapsto \sum_{Q\ni x}\varepsilon_Q\lambda_Q\langle u\rangle_Q,$$

which satisfies $M_{\mathrm{R}}u(x) = \|\mathcal{M}_{\mathrm{R}}u(x)\|_{\mathcal{L}(\ell^2,\mathrm{Rad}(X))}$.

The RMF property of a Banach space X was defined in terms of the L^2-boundedness of M_{R}, but the next result shows that the exponent 2 is not relevant.

Proposition 7.1. *Let X be a Banach space, and consider the assertion*

$$M_{\mathrm{R}} : L^p(\mathbf{R}^n; X) \to L^p(\mathbf{R}^n) \text{ is bounded.} \tag{24}$$

If (24) *is true for one $p \in (1,\infty)$, then it is true for all $p \in (1,\infty)$.*

Proof. It suffices to prove the same for the equivalent statement

$$\mathcal{M}_{\mathrm{R}} : L^p(\mathbf{R}^n, X) \to L^p(\mathbf{R}^n, \mathcal{L}(\ell^2, \mathrm{Rad}(X))) \text{ is bounded.} \tag{25}$$

Suppose that (25) is true for some $p \in (1,\infty)$. Let a be an atom of the dyadic $H^1(\mathbf{R}^n, X)$ space, i.e., $\operatorname{supp} a \subseteq Q$, a dyadic cube, $\|a\|_\infty \leqslant |Q|^{-1}$ and $\int a(x)\,\mathrm{d}x = 0$. Then $\langle a\rangle_{Q'} \neq 0$ only if $Q' \subset Q$. Hence

$$\begin{aligned}
&\|\mathcal{M}_{\mathrm{R}}u\|_{L^1(\mathbf{R}^n,\mathcal{L}(\ell^2,\mathrm{Rad}(X)))} \\
&\quad = \|\mathcal{M}_{\mathrm{R}}u\|_{L^1(Q,\mathcal{L}(\ell^2,\mathrm{Rad}(X)))} \\
&\quad \leqslant |Q|^{1/p'}\|\mathcal{M}_{\mathrm{R}}u\|_{L^p(\mathbf{R}^n,\mathcal{L}(\ell^2,\mathrm{Rad}(X)))} \\
&\quad \lesssim |Q|^{1/p'}\|u\|_{L^p(\mathbf{R}^n,X)} \leqslant |Q|^{1/p'}|Q|^{1/p}\|u\|_\infty \leqslant 1.
\end{aligned}$$

It follows that $\mathcal{M}_{\mathrm{R}} : H^1(\mathbf{R}^n, X) \to L^1(\mathbf{R}^n, \mathcal{L}(\ell^2, \mathrm{Rad}(X)))$ boundedly.

Let then $u \in L^\infty(\mathbf{R}^n, X)$ and let Q be a dyadic cube. It is easy to see that

$$1_Q[\mathcal{M}_{\mathrm{R}}u - \langle \mathcal{M}_{\mathrm{R}}u\rangle_Q] = \mathcal{M}_{\mathrm{R}}(1_Q[u - \langle u\rangle_Q]).$$

Denoting by *BMO* the dyadic BMO space, it follows that

$$\begin{aligned}
&\|\mathcal{M}_{\mathrm{R}}u\|_{BMO(\mathbf{R}^n,\mathcal{L}(\ell^2,\mathrm{Rad}(X)))} \\
&\quad = \sup_{Q\in\triangle} \frac{1}{|Q|}\big\|\mathcal{M}_{\mathrm{R}}u - \langle\mathcal{M}_{\mathrm{R}}u\rangle_Q\big\|_{L^1(Q,\mathcal{L}(\ell^2,\mathrm{Rad}(X)))} \\
&\quad = \sup_{Q\in\triangle} \big\|\mathcal{M}_{\mathrm{R}}(|Q|^{-1}1_Q[u - \langle u\rangle_Q])\big\|_{L^1(\mathbf{R}^n,\mathcal{L}(\ell^2,\mathrm{Rad}(X)))}.
\end{aligned}$$

But $|Q|^{-1}1_Q[u - \langle u\rangle_Q]$ is $2\|u\|_\infty$ times an atom of $H^1(\mathbf{R}^n, X)$. Hence, by what we already showed, we also find that $\mathcal{M}_{\mathrm{R}} : L^\infty(\mathbf{R}^n, X) \to BMO(\mathbf{R}^n, \mathcal{L}(\ell^2, \mathrm{Rad}(X)))$ boundedly. Now interpolation gives the assertion. □

Remark 7.2. Given a dyadic cube $Q \in \triangle$, it also makes sense to consider M_{R} as an operator acting in $L^p(Q; X)$. In this case one may restrict the summation in the definition to

$$\sum_{R:\, x\in R\subseteq Q} \varepsilon_R \lambda_R \langle u\rangle_R.$$

An obvious restriction argument now shows that $M_{\mathrm{R}} : L^p(Q; X) \to L^p(Q)$, with the norm independent of Q, if X has RMF.

We do not yet fully understand how the RMF property relates to established Banach space notions. Since we need to assume this kind of inequality to be able to carry out the estimates in the subsequent sections, we next provide some sufficient conditions, which imply this property. In Appendix C we also give a counterexample to show that RMF is indeed a non-trivial property not shared by every Banach space; more precisely, it fails in the sequence space ℓ^1. Our first sufficient condition, Rademacher type 2, is the easiest one, but not very useful for our applications, since this condition is not self-dual and the condition that both X and X^* have type 2 is very restrictive, indeed, equivalent to X being isomorphic to a Hilbert space. On the other hand, the other two classes of spaces with RMF—UMD function lattices and reflexive noncommutative L^p spaces—are both self-dual, and they cover the most important concrete examples of UMD spaces.

Spaces of type 2. If X has type 2, then $M_{\mathrm{R}}u(x) \lesssim Mu(x)$, where M is the usual dyadic maximal function. In fact,

$$\mathbb{E}\left\|\sum_k \varepsilon_k \lambda_k A_{2^k} u(x)\right\|_X \lesssim \left(\sum_k |\lambda_k|^2 \left\| A_{2^k} u(x)\right\|_X^2\right)^{1/2} \tag{26}$$

in this case, and the supremum over $\|\lambda\|_{\ell^2(\mathbf{Z})} \leqslant 1$ of the right-hand side is $\sup_k |A_{2^k}u(x)|_X = Mu(x)$.

Remark 7.3. Since R-bounds imply uniform bounds, the reverse estimate $Mu(x) \leqslant M_{\mathrm{R}}u(x)$ holds in any Banach space. Thus there is in fact an equivalence $M_{\mathrm{R}}u(x) \eqsim Mu(x)$ if X has type 2.

Remark 7.4. In [26], James constructed a non-reflexive Banach space with type 2 (and thus with the RMF property). This means, in particular, that RMF does not imply UMD.

UMD function lattices. Suppose now that X is a Banach lattice of (equivalence classes of) measurable functions on some σ-finite measure space (S, Σ, μ). This means that X is a Banach space of such functions and, in addition,

- it contains the pointwise real and imaginary parts of any two functions $\xi, \eta \in X$, and the pointwise maximum and minimum of any two real function $\xi, \eta \in X$;
- if the pointwise absolute values satisfy $|\xi| \leqslant |\eta|$, then $\|\xi\|_X \leqslant \|\eta\|_X$.

Obvious examples are the $L^p(\mu)$ and spaces of continuous functions; also any Banach space with an unconditional basis may be viewed as a Banach lattice of functions defined on $\mathbf{Z}_+$. One can also give an abstract definition of a Banach lattice without a postulated function space structure (see e.g. [2]), but we restrict ourselves to the concrete situation, which is the context where Banach lattices with the UMD property have been studied by Rubio de Francia [39]. In this situation, the harmonic analysis in $L^p(\mathbf{R}^n; X)$ is much closer to the scalar-valued case than on a general UMD space, since one can use square functions similar to their $L^p(\mathbf{R}^n; \mathbf{C})$ counterparts, and there is also the following natural notion of a maximal function.

The (dyadic) lattice maximal function M_{lattice} is defined by

$$M_{\text{lattice}}u(x) := \sup_{Q \ni x} \left| \langle u \rangle_Q \right|,$$

which is again an X-valued function. Suppose X is UMD (and thus has finite cotype), then

$$\begin{aligned} \mathbb{E}\left\| \sum_{Q \ni x} \varepsilon_Q \lambda_Q \langle u \rangle_Q \right\|_X &\lesssim \left\| \left(\sum_{Q \ni x} |\lambda_Q|^2 \left| \langle u \rangle_Q \right|^2 \right)^{1/2} \right\|_X \\ &\leqslant \left\| \left(\sum_{Q \ni x} |\lambda_Q|^2 \right)^{1/2} \sup_{Q \ni x} \left| \langle u \rangle_Q \right| \right\|_X, \end{aligned}$$

so that we have the domination $M_{\mathrm{R}}u(x) \lesssim \|M_{\text{lattice}}u(x)\|_X$. By a result of Rubio de Francia [39], we know that $\|M_{\text{lattice}}u\|_{L^p(\mu,X)} \lesssim \|u\|_{L^p(\mu,X)}$, and hence $\|M_{\mathrm{R}}u\|_{L^p(\mu)} \lesssim \|u\|_{L^p(\mu,X)}$ for all $1 < p < \infty$.

Noncommutative L^p spaces. We now turn to the case where X is a noncommutative L^p space $L^p(N,\tau)$ on a von Neumann algebra N with a normal semifinite faithful trace τ. In this setting, analogues of many important results from Banach space theory and harmonic analysis have recently been found. See [38] for the definition, more information and references. We here presuppose a modest knowledge of these notions, and only mention that the $L^p(N,\tau)$ are spaces of (bounded linear) operators (acting on some Hilbert space), which generalize the "commutative" $L^p(\mu)$ spaces, the trace playing the rôle of an integral. The simplest examples, besides L^p, are the Schatten ideals S^p of bounded linear operators A such that $\mathrm{tr}((A^*A)^{p/2})$ is finite, where tr denotes the usual trace. The reader who is not interested in the applications of our results in the noncommutative context, may very well jump to the beginning of the next section.

The following "noncommutative Doob's maximal inequality" was established by M. Junge in [27].

Theorem 7.5 *(Junge). Let $1 < p \leqslant \infty$ and $u \in L^p(N,\tau)$. Let (N_i) be an increasing sequence of von Neumann subalgebras of N, with associated conditional expectations E_i. Then there exist $a, b \in L^{2p}(N,\tau)$ and contractions $y_i \in N$ such that*

$$E_i u = a y_i b, \qquad \|a\|_{2p}\|b\|_{2p} \lesssim_p \|u\|_p.$$

In particular (cf. [27, Remark 5.5]), Theorem 7.5 applies in the case when

$$N = L^\infty(\mathcal{F}) \mathbin{\bar{\otimes}} M,$$

where $L^\infty(\mathcal{F})$ is a usual commutative L^∞ space, and $N_i = L^\infty(\mathcal{F}_i) \mathbin{\bar{\otimes}} M$ for some sub-σ-algebras $\mathcal{F}_i \subset \mathcal{F}$. Then $L^p(N) \eqsim L^p(\mathcal{F}, L^p(M))$ is the Bochner space of L^p functions with values in the noncommutative space $L^p(M)$, and E_i are the (tensor extensions of) usual conditional expectation operators. In our case $E_i = A_{2^i}$, but the argument is valid for general sequences of conditional expectations.

Corollary 7.6. *Let $1 < p, q < \infty$, let $X = L^q(M)$ and $u \in L^p(\mathcal{F}, X)$. Then*

$$\|M_{\mathrm{R}} u\|_{L^p(\mathcal{F})} \lesssim_{p,q} \|u\|_{L^p(\mathcal{F},X)}.$$

Proof. By Proposition 7.1, it suffices to prove the case $p = q$. Then $L^p(\mathcal{F}, L^p(M)) = L^p(N)$, with $N = L^\infty(\mathcal{F}) \bar{\otimes} M$, is itself a noncommutative L^p space. By Theorem 7.5, there exist $a, b \in L^{2p}(N) = L^{2p}(\mathcal{F}, L^{2p}(M))$ and contractions $y_j \in N$ such that

$$E_j u(x) = a(x) y_j(x) b(x), \qquad \|a\|_{L^{2p}(N)} \|b\|_{L^{2p}(N)} \lesssim_p \|u\|_{L^p(N)}. \tag{27}$$

Then we have, by the noncommutative Hölder inequality,

$$\begin{aligned}
\mathbb{E}\Big\| \sum_j \varepsilon_j \lambda_j E_j u(x) \Big\|_{L^p(M)} &= \mathbb{E}\Big\| a(x) \sum_j \varepsilon_j \lambda_j y_j(x) b(x) \Big\|_{L^p(M)} \\
&\leqslant \mathbb{E}\big\| a(x) \big\|_{L^{2p}(M)} \Big\| \sum_j \varepsilon_j \lambda_j y_j(x) b(x) \Big\|_{L^{2p}(M)}.
\end{aligned}$$

Now $2p > 2$, so that the space $L^{2p}(M)$ has type 2. Hence

$$\begin{aligned}
&\mathbb{E}\Big\| \sum_j \varepsilon_j \lambda_j y_j(x) b(x) \Big\|_{L^{2p}(M)} \\
&\quad \lesssim_p \Big(\sum_j \big\| \lambda_j y_j(x) b(x) \big\|_{L^{2p}(M)}^2 \Big)^{1/2} \\
&\quad \leqslant \Big(\sum_j |\lambda_j|^2 \Big)^{1/2} \big\| b(x) \big\|_{L^{2p}(M)} \leqslant \big\| b(x) \big\|_{L^{2p}(M)}.
\end{aligned}$$

Combining the previous estimates, we have shown that

$$M_{\mathrm{R}} u(x) \lesssim_p \big\| a(x) \big\|_{L^{2p}(M)} \big\| b(x) \big\|_{L^{2p}(M)},$$

and hence, by Hölder's inequality and (27),

$$\|M_{\mathrm{R}} u\|_{L^p(\mathcal{F})} \lesssim_p \|a\|_{L^{2p}(\mathcal{F}; L^{2p}(M))} \|b\|_{L^{2p}(\mathcal{F}; L^{2p}(M))} \lesssim_p \|u\|_{L^p(\mathcal{F}; L^p(M))},$$

which completes the proof. $\square$

The results of this section constitute a proof of Proposition 2.13.

8. An L^p version of Carleson's inequality

We next establish a vector-valued L^p version of Carleson's inequality for Carleson measures. For $p \neq 2$, it appears to be new even in the scalar-valued case. We wish to mention that the proof of this inequality is significantly inspired by the work of N.H. Katz and M.C. Pereyra [29,36], although none of their specific results is explicitly needed.

Let $b = (b_R)_{R\in\triangle}$ be a finitely non-zero sequence of measurable scalar-valued functions, such that $\operatorname{supp} b_Q \subseteq Q$. For each $Q \in \triangle$ we denote

$$\|b\|_{\mathrm{Car}^p(Q)} := \sup_{S\in\triangle,\, S\subseteq Q} \left(\frac{1}{|S|}\int_S \mathbb{E}\left|\sum_{R\subset S} \varepsilon_R b_R(x)\right|^p \mathrm{d}x\right)^{1/p}$$

$$\eqsim \sup_{S\in\triangle,\, S\subseteq Q} \left(\frac{1}{|S|}\int_S \left[\sum_{R\subset S} \left|b_R(x)\right|^2\right]^{p/2} \mathrm{d}x\right)^{1/p}.$$

Let us write $\|b\|_{\mathrm{Car}^p(\mathbf{R}^n)} := \sup_{Q\in\triangle} \|b\|_{\mathrm{Car}^p(Q)}$. For $p = 2$, this is just (the square-root of) the Carleson constant of the measure

$$\mathrm{d}\mu(x,t) = \sum_{Q\in\triangle} b_Q(x) 1_{]\ell(Q)/2,\ell(Q)]}(t)\,\mathrm{d}x\,\frac{\mathrm{d}t}{t}.$$

For the moment, fix a cube $Q \in \triangle$, and denote by μ the normalized Lebesgue measure, $\mu(E) := |E|/|Q|$, on measurable subsets of Q. We recall the definition of Lorentz spaces $L^{p,q}(\mu, X)$. A measurable function $u : Q \to X$ belongs to $L^{p,q}(\mu, X)$ if

$$\|u\|_{L^{p,q}(\mu,X)} := \left(\int_0^\infty \left[t\mu\left(\|u(\cdot)\|_X > t\right)^{1/p}\right]^q \frac{\mathrm{d}t}{t}\right)^{1/q}$$

is finite. We are now ready to state:

Lemma 8.1. *Let X be a Banach space with type $t \geqslant 1$, and let $1 \leqslant p < \infty$. Then*

$$\left(\frac{1}{|Q|}\int_Q \mathbb{E}\left\|\sum_{R\in\triangle,\ R\subseteq Q} \varepsilon_R b_R(x)\langle u\rangle_R\right\|_X^p\right)^{1/p}$$

$$\lesssim \|b\|_{\mathrm{Car}^p(Q)} \cdot \begin{cases} \|M_{\mathrm{R}}u\|_{L^p(\mu)} & \text{if } 1 \leqslant p \leqslant t, \\ \|M_{\mathrm{R}}u\|_{L^{p,t}(\mu)} & \text{if } t < p < \infty. \end{cases}$$

Proof. Let us fix some $A > 0$ and denote

$$\mathcal{G}_k := \left\{S \subseteq Q\colon \sup_{\|\lambda\|_{\ell^2}\leqslant 1} \mathbb{E}\left\|\sum_{R\colon S\subseteq R\subseteq Q} \varepsilon_R \lambda_R \langle u\rangle_R\right\|_X \leqslant A\cdot 2^k\right\}.$$

Let us also denote by $\mathcal{F}_k$ the set of maximal dyadic cubes $S \subseteq Q$ such that $S \notin \mathcal{G}_k$.

Then every $R \notin \mathcal{G}_k$ satisfies $R \subseteq S$ for a unique $S \in \mathcal{F}_k$. Moreover, $\mathcal{G}_k \subseteq \mathcal{G}_{k+1}$, and every $S \subseteq Q$ belongs to $\mathcal{G}_k$ for a sufficiently large k. We write $\mathcal{Q}_0 := \mathcal{G}_0$ and $\mathcal{Q}_k := \mathcal{G}_k \setminus \mathcal{G}_{k-1}$ for $k = 1, 2, \ldots$. Then

$$\sum_{R \subseteq Q} \varepsilon_R b_R(x) \langle u \rangle_R = \sum_{k=0}^{\infty} \sum_{R \in \mathcal{Q}_k} \varepsilon_R b_R(x) \langle u \rangle_R,$$

and, by sign-invariance,

$$\mathbb{E} \left\| \sum_{k=0}^{\infty} \sum_{R \in \mathcal{Q}_k} \varepsilon_R b_R(x) \langle u \rangle_R \right\| \eqsim \mathbb{E}\mathbb{E}' \left\| \sum_{k=0}^{\infty} \varepsilon'_k \sum_{R \in \mathcal{Q}_k} \varepsilon_R b_R(x) \langle u \rangle_R \right\|,$$

where ε'_k are an independent sequence of Rademacher variables. Let us denote $q := \min\{p, t\}$, so that X has type q.

Then, by the definition of type,

$$\mathbb{E}\mathbb{E}' \left\| \sum_{k=0}^{\infty} \varepsilon'_k \sum_{R \in \mathcal{Q}_k} \varepsilon_R b_R(x) \langle u \rangle_R \right\|_X^p \lesssim \left(\sum_{k=0}^{\infty} \mathbb{E} \left\| \sum_{R \in \mathcal{Q}_k} \varepsilon_R b_R(x) \langle u \rangle_R \right\|_X^q \right)^{p/q}.$$

Now consider a fixed $x \in Q$. Suppose first that there is a smallest dyadic cube S such that $x \in S \in \mathcal{Q}_k$. Then

$$\begin{aligned}
&\mathbb{E} \left\| \sum_{R \in \mathcal{Q}_k} \varepsilon_R b_R(x) \langle u \rangle_R \right\|_X^q \\
&\quad = \mathbb{E} \left\| \sum_{S \subseteq R \subseteq Q} \varepsilon_R b_R(x) 1_{\mathcal{Q}_k}(R) \langle u \rangle_R \right\|_X^q \\
&\quad \lesssim \left(A2^k\right)^q \mathbb{E} \left| \sum_{S \subseteq R \subseteq Q} \varepsilon_R b_R(x) 1_{\mathcal{Q}_k}(R) \right|^q \\
&\quad = \left(A2^k\right)^q \mathbb{E} \left| \sum_{R \in \mathcal{Q}_k} \varepsilon_R b_R(x) \right|^q,
\end{aligned} \tag{28}$$

where the estimate employed the fact that $S \in \mathcal{Q}_k \subseteq \mathcal{G}_k$, the defining property of $\mathcal{G}_k$ with $\lambda_R = b_R(x) 1_{\mathcal{Q}_k}(R)$, and the equivalence of the ℓ^2 norm and the randomized norm for scalar sequences.

If there is no smallest S, then (28) remains true with "$\lim_{S \downarrow \{x\}}$" in front of the two intermediate expressions, where S runs through the decreasing sequence of dyadic cubes containing x. In either case, the final estimate between the left-hand and the right-hand side is the same.

Substituting back and using the triangle inequality in $L^{p/q}(\mu)$, we have

$$\left(\frac{1}{|Q|} \int_Q \mathbb{E} \left\| \sum_{R \subseteq Q} \varepsilon_R b_R(x) \langle u \rangle_R \right\|_X^p \mathrm{d}x \right)^{q/p}$$

$$\lesssim \sum_{k=0}^{\infty}\biggl(\frac{1}{|Q|}\int_Q (A2^k)^p\mathbb{E}\biggl|\sum_{R\in\mathcal{Q}_k}\varepsilon_R b_R(x)\biggr|^p\,\mathrm{d}x\biggr)^{q/p}.$$

For $k=0$, it is clear that

$$\frac{1}{|Q|}\int_Q \mathbb{E}\biggl|\sum_{R\in\mathcal{Q}_0}\varepsilon_R b_R(x)\biggr|^p\,\mathrm{d}x \leqslant \|b\|^p_{\mathrm{Car}^p(Q)}.$$

For $k\geqslant 1$, we have, using the definition and disjointness of the cubes $S\in\mathcal{F}_{k-1}$,

$$\begin{aligned}\frac{1}{|Q|}\int_Q \mathbb{E}\biggl|\sum_{R\in\mathcal{Q}_k}\varepsilon_R b_R(x)\biggr|^p\,\mathrm{d}x &\leqslant \sum_{S\in\mathcal{F}_{k-1}}\frac{1}{|Q|}\int_S \mathbb{E}\biggl|\sum_{R\subseteq S}\varepsilon_R b_R(x)\biggr|^p\,\mathrm{d}x\\ &\leqslant \frac{|\bigcup_{S\in\mathcal{F}_{k-1}} S|}{|Q|}\|b\|^p_{\mathrm{Car}^p(Q)}.\end{aligned}$$

Since $\bigcup_{S\in\mathcal{F}_{k-1}} S\subseteq\{M_{\mathrm{R}}u>A\cdot 2^{k-1}\}$, it follows that

$$\begin{aligned}&\biggl(\frac{1}{|Q|}\int_Q \biggl\|\sum_{R\subseteq Q}\varepsilon_R b_R(x)\langle u\rangle_R\biggr\|_X^p\biggr)^{1/p}\\ &\quad\lesssim A\|b\|_{\mathrm{Car}^p(Q)}\biggl[1+\sum_{k=1}^{\infty}2^{kq}\biggl(\frac{|\{M_{\mathrm{R}}u>A\cdot 2^{k-1}\}|}{|Q|}\biggr)^{q/p}\biggr]^{1/q}\\ &\quad\lesssim A\|b\|_{\mathrm{Car}^p(Q)}\biggl[1+\int_0^{\infty}t^q\mu\biggl(\frac{M_{\mathrm{R}}u}{A}>t\biggr)^{q/p}\frac{\mathrm{d}t}{t}\biggr]^{1/q},\end{aligned}$$

and the choice $A=\|M_{\mathrm{R}}u\|_{L^{p,q}(\mu)}$ yields the asserted bound (using the fact that $L^{p,p}(\mu)=L^p(\mu)$). $\square$

Theorem 8.2. *Let X be an RMF space, $1<p<\infty$, and $\epsilon>0$. Then*

$$\biggl(\int_{\mathbf{R}^n}\mathbb{E}\biggl\|\sum_{R\in\Delta}\varepsilon_R b_R(x)\langle u\rangle_R\biggr\|_X^p\biggr)^{1/p}\lesssim \|b\|_{\mathrm{Car}^{p+\epsilon}(\mathbf{R}^n)}\|u\|_{L^p(\mathbf{R}^n,X)},$$

for all $u\in L^p(\mathbf{R}^n;X)$. We may take $\epsilon=0$ if X has type p.

Proof. By standard considerations, it is easy to see that it suffices to prove the estimate with a fixed dyadic cube Q in place of $\mathbf{R}^n$ and $R\in\Delta$ replaced by $R\subseteq Q$. After dividing this modified claim by $|Q|^{1/p}$, the left-hand side becomes identical with that in Lemma 8.1, while the right-hand side is $\|b\|_{\mathrm{Car}^{p+\epsilon}(Q)}\|u\|_{L^p(\mu)}$. If X has type p, the result with $\epsilon=0$ thus follows from Lemma 8.1. We now turn to the case where X has type $t<p$.

By the real method of interpolation, after linearizing $M_{\mathrm{R}}u$ in a standard manner, we have that $\|M_{\mathrm{R}}u\|_{L^{p,q}(\mu)} \lesssim \|u\|_{L^{p,q}(\mu,X)}$ for the same p and $1 \leqslant q \leqslant \infty$. Thus Lemma 8.1 shows that the bilinear map

$$(b,u) \mapsto \sum_{R\subseteq Q} \varepsilon_R b_R(\cdot)\langle u\rangle_R \tag{29}$$

is bounded

$$\mathrm{Car}^p(Q) \times L^{p,t}(\mu,X) \to L^p\big(\mu,\mathrm{Rad}(X)\big) \tag{30}$$

if X has type $t \leqslant p$.

If X does not have type p, it nevertheless has type 1. For a small number $\epsilon > 0$, we already know the following boundedness properties of the Carleson map (29):

$$\begin{aligned} &\mathrm{Car}^{p+\epsilon}(Q) \times L^{p+\epsilon,1}(\mu,X) \to L^{p+\epsilon}\big(\mu,\mathrm{Rad}(X)\big), \\ &\mathrm{Car}^{p+\epsilon}(Q) \times L^{p-\epsilon,1}(\mu,X) \to L^{p-\epsilon}\big(\mu,\mathrm{Rad}(X)\big). \end{aligned} \tag{31}$$

The second line uses the embedding $\mathrm{Car}^{p+\epsilon}(Q) \subseteq \mathrm{Car}^{p-\epsilon}(Q)$. For a fixed $b \in \mathrm{Car}^{p+\epsilon}(Q)$, the lines (31) express the boundedness of the linear operator $u \mapsto \sum_{R\subseteq Q} \varepsilon_R b_R(\cdot)\langle u\rangle_R$ between certain function spaces. Using the real interpolation results

$$\begin{gathered} \big(L^{p+\epsilon,1}(\mu,X), L^{p-\epsilon,1}(\mu,X)\big)_{\theta,p} = L^p(\mu,X), \\ \big(L^{p+\epsilon}\big(\mu,\mathrm{Rad}(X)\big), L^{p-\epsilon}\big(\mu,\mathrm{Rad}(X)\big)\big)_{\theta,p} = L^p\big(\mu,\mathrm{Rad}(X)\big) \end{gathered}$$

for appropriate $\theta \in (0,1)$, we deduce the assertion. □

9. Carleson measure estimate

In Section 6, we reduced the asserted inequality of Proposition 3.4 to the estimation of the principal part (23). We have finally developed the required tools for dealing with this part in this final section.

Let us first see how to make use of the fact that we only need to consider $u \in \mathsf{R}(\Gamma)$. Since Γ is a first-order constant-coefficient partial differential operator in $L^p(\mathbf{R}^n;\mathbf{C}^N)$, it has the form $\Gamma = \Gamma_0\nabla$, where $\Gamma_0 \in \mathcal{L}(\mathbf{C}^n;\mathbf{C}^N)$. Let us write $W_\Gamma := \mathsf{R}(\Gamma_0) \subseteq \mathbf{C}^N$, and let P_Γ be the orthogonal projection of $\mathbf{C}^N$ onto this subspace. As before, we use the same symbol for its tensor extension to X^N. Now, for $u \in \mathsf{R}(\Gamma)$, we have

$$\gamma_{2^k}(x)A_{2^k}u(x) = \gamma_{2^k}(x)P_\Gamma A_{2^k}u(x) = \frac{\gamma_{2^k}(x)P_\Gamma}{\|\gamma_{2^k}(x)P_\Gamma\|}\big\|\gamma_{2^k}(x)P_\Gamma\big\| A_{2^k}u(x),$$

where we denote by $\|\gamma_{2^k}(x)P_\Gamma\|$ the operator norm of $\gamma_{2^k}(x)P_\Gamma$ in $\mathcal{L}(\mathbf{C}^N)$ (and let $0/0 := 0$). Since the tensor extensions of the operators $M \in \mathcal{L}(\mathbf{C}^N)$ with $\|M\| \leqslant 1$ are R-bounded on X^N (by writing out the matrix multiplications and using the contraction principle), it follows from Theorem 8.2

$$
\begin{aligned}
&\mathbb{E}\Big\|\sum_k \varepsilon_k \gamma_{2^k} A_{2^k} u\Big\|_{L^p(\mathbf{R}^n;X^N)} \\
&\quad\lesssim \mathbb{E}\Big\|\sum_k \varepsilon_k \|\gamma_{2^k} P_\Gamma\| A_{2^k} u\Big\|_{L^p(\mathbf{R}^n;X^N)} \\
&\quad= \mathbb{E}\Big\|\sum_{Q\in\triangle} \varepsilon_Q 1_Q \|\gamma_{\ell(Q)} P_\Gamma\| \langle u\rangle_Q\Big\|_{L^p(\mathbf{R}^n;X^N)} \\
&\quad\lesssim \Big\|\big(1_Q \|\gamma_{\ell(Q)} P_\Gamma\|\big)_{Q\in\triangle}\Big\|_{\mathrm{Car}^{p+\epsilon}(\mathbf{R}^n)} \|u\|_{L^p(\mathbf{R}^n;X^N)}.
\end{aligned} \tag{32}
$$

Hence proving the asserted quadratic estimate in $L^p(\mathbf{R}^n; X^N)$ is finally reduced to showing the finiteness of the $\mathrm{Car}^{p+\epsilon}(\mathbf{R}^n)$-norm above.

There are two peculiarities worth pointing out here. First, the space X has completely disappeared from this remaining estimate. Hence, the rest of the proof will be merely an L^p version, no longer Banach space valued, of the L^2 estimates in [9].

Second, to get our desired L^p inequality, we are now required to prove an $L^{p+\epsilon}$-type estimate. This (and only this) is the reason why we formulated the main results—Theorem 3.1, Corollary 3.2, and Proposition 3.4—for p in an open interval (p_-, p_+), instead of just a single exponent p. At this point it could seem that we only need openness at the upper end of the interval, but we also have to be able to repeat the reasoning in the dual case with the interval (p'_+, p'_-).

The reader may also recall that the ϵ could be avoided in (32) if X has type p. But to make the dual argument, we would also require that X^* has type p', and the only exponent for which this can be the case is $p = 2$. Moreover, if both X and X^* have type 2, then X is isomorphic to a Hilbert space, and so we are back to the classical situation. Thus we are able to recover the original L^2 result in Hilbert spaces, but this is also the only situation, where we can work in a fixed L^p space.

Now that we have assumed this extra ϵ, it is clear that completing the proof will only require the following. (Note also that R-bisectoriality of an operator $T \otimes I_X$ in $L^p(\mathbf{R}^n; X^N)$, where X is an arbitrary Banach space, implies R-bisectoriality of T in $L^p(\mathbf{R}^n; \mathbf{C}^N)$ by restricting to a subspace.)

Proposition 9.1. *Let $1 < p < \infty$, and let Π_B and Π_{B^*} be perturbed Hodge–Dirac operators, which are R-bisectorial in $L^p(\mathbf{R}^n; \mathbf{C}^N)$. Then*

$$
\big\|\big(1_Q \|\gamma_{\ell(Q)} P_\Gamma\|_{\mathcal{L}(\mathbf{C}^N)}\big)_{Q\in\triangle}\big\|_{\mathrm{Car}^p(\mathbf{R}^n)} \lesssim 1.
$$

The proof follows closely the Carleson measure estimate in [9, Section 5], and hence we will skip some detail by simply asking the reader to repeat the relevant steps in [9].

Denoting $R_Q := (0, \ell(Q)] \times Q$, a reformulation of the claim is

$$
\mathbb{E}\Big\|\sum_{k\in\mathbf{Z}} \varepsilon_k 1_{R_Q}\big(2^k, \cdot\big) \gamma_{2^k} P_\Gamma\Big\|_{L^p(\mathbf{R}^n;\mathcal{L}(\mathbf{C}^N))}
$$

$$\eqsim \left\| \left(\sum_{k \in \mathbf{Z}} \left\| 1_{R_Q}\left(2^k, \cdot\right) \gamma_{2^k} P_\Gamma \right\|_{\mathcal{L}(\mathbf{C}^N)}^2 \right)^{1/2} \right\|_{L^p(\mathbf{R}^n)} \lesssim |Q|^{1/p}.$$

The equivalence of the first and second form may be justified by Kahane's inequality and using the equivalent Hilbert–Schmidt norm on the finite-dimensional operator space $\mathcal{L}(\mathbf{C}^N)$.

Let us introduce the following subspace of $\mathcal{L}(\mathbf{C}^N)$, which contains our operators of interest $\gamma_{2^k}(x) P_\Gamma$:

$$\mathcal{O}_\Gamma := \left\{ \nu \in \mathcal{L}\left(\mathbf{C}^N\right):\ W_\Gamma^\perp \subseteq \mathsf{N}(\nu) \right\} = \left\{ \nu \in \mathcal{L}\left(\mathbf{C}^N\right):\ \nu = \nu P_\Gamma \right\}.$$

We set $\sigma > 0$ to be chosen later, and consider the cones

$$K_\nu = \left\{ \nu' \in \mathcal{O}_\Gamma \setminus \{0\}:\ \left\| \frac{\nu'}{\|\nu'\|} - \nu \right\| \leqslant \sigma \right\},$$

where ν belongs to a finite set Λ such that $\bigcup_{\nu \in \Lambda} K_\nu = \mathcal{O}_\Gamma \setminus \{0\}$. Writing

$$C_\nu := \left\{ (t, x) \in (0, \infty) \times \mathbf{R}^n:\ \gamma_t(x) P_\Gamma \in K_\nu \right\},$$

we need to show that

$$\mathbb{E} \left\| k \sum_{k \in \mathbf{Z}} \varepsilon_k 1_{R_Q \cap C_\nu}\left(2^k, \cdot\right) \gamma_{2^k} P_\Gamma \right\|_p \lesssim |Q|^{1/p}$$

for each $\nu \in \Lambda$. This in turns reduces to proving the following proposition.

Proposition 9.2. *There exist $\beta \in (0, 1)$ and $C > 0$ which satisfy the following. For all $Q \in \triangle$ and all $\nu \in \mathcal{L}(\mathbf{C}^n)$ with $\|\nu\| = 1$, there is a collection $(Q_j)_{j \in J}$ of disjoint dyadic subcubes of Q such that*: *denoting*

$$E_{Q,\nu} := Q \setminus \bigcup_{j \in J} Q_j, \qquad E_{Q,\nu}^* := R_Q \setminus \bigcup_{j \in J} R_{Q_j}, \tag{33}$$

there holds $|E_{Q,\nu}| > \beta |Q|$ and

$$\left(\mathbb{E} \left\| \sum_{k \in \mathbf{Z}} \varepsilon_k 1_{E_{Q,\nu}^* \cap C_\nu}\left(2^k, \cdot\right) \gamma_{2^k} P_\Gamma \right\|_p^p \right)^{1/p} \leqslant C |Q|^{1/p}.$$

Indeed, assuming this is proven, we have for a fixed $Q \in \triangle$

$$\mathbb{E} \left\| \sum_{k \in \mathbf{Z}} \varepsilon_k 1_{R_Q \cap C_\nu}\left(2^k, \cdot\right) \gamma_{2^k} P_\Gamma \right\|_p^p \leqslant C^p |Q| + \sum_{j \in J} \mathbb{E} \left\| \sum_{k \in \mathbf{Z}} \varepsilon_k 1_{R_{Q_j} \cap C_\nu}\left(2^k, \cdot\right) \gamma_{2^k} P_\Gamma \right\|_p^p.$$

Now, applying Proposition 9.2 for each of the Q_j, and denoting by $(Q_{j,j'})_{j' \in J'}$ the corresponding sequence of subcubes of Q_j, we have

$$
\begin{aligned}
&\mathbb{E}\Big\|\sum_{k\in\mathbf{Z}}\varepsilon_k 1_{R_Q\cap C_\nu}\big(2^k,\cdot\big)\gamma_{2^k}P_\Gamma\Big\|_p^p\\
&\quad\leqslant C^p|Q| + C^p\sum_{j\in J}|Q_j| + \sum_{j\in J}\sum_{j'\in J'}\mathbb{E}\Big\|\sum_{k\in\mathbf{Z}}\varepsilon_k 1_{R_{Q_{j,j'}}\cap C_\nu}\big(2^k,\cdot\big)\gamma_{2^k}P_\Gamma\Big\|_p^p\\
&\quad\leqslant C^p|Q|\big(1+(1-\beta)\big) + \sum_{j\in J}\sum_{j'\in J'}\mathbb{E}\Big\|\sum_{k\in\mathbf{Z}}\varepsilon_k 1_{R_{Q_{j,j'}}\cap C_\nu}\big(2^k,\cdot\big)\gamma_{2^k}P_\Gamma\Big\|_p^p.
\end{aligned}
$$

Reiterating this procedure leads to

$$
\mathbb{E}\Big\|\sum_{k\in\mathbf{Z}}\varepsilon_k 1_{R_Q\cap C_\nu}\big(2^k,\cdot\big)\gamma_{2^k}P_\Gamma\Big\|_p^p \leqslant C^p|Q|\sum_{i=0}^{\infty}(1-\beta)^i = C^p|Q|\beta^{-1}.
$$

We now turn to the proof of Proposition 9.2. Let us fix $\nu\in\mathcal{O}_\Gamma\subseteq\mathcal{L}(\mathbf{C}^N)$ of norm 1, and let $w,\hat{w}\in\mathbf{C}^N$ also be of norm 1, and such that $w=\nu^*(\hat{w})=P_\Gamma\nu^*(\hat{w})$. Hence $w\in W_\Gamma$. We can now construct (as in [8, Lemma 4.10]) the following kind of auxiliary functions for each $Q\in\triangle$:

$$
w_Q\in\mathsf{R}(\Gamma),\qquad \operatorname{supp}w_Q\subseteq 3Q,\qquad w_Q(x)\equiv w\quad\forall x\in 2Q,\qquad \|w_Q\|_\infty\lesssim 1.
$$

To do so, we take an affine function u_Q such that $\Gamma u_Q\equiv w$ and $\|1_Q u_Q\|_\infty\lesssim\ell(Q)$, and a smooth cutoff η_Q supported in $3Q$ and equal to 1 on $2Q$, with $\|\nabla\eta_Q\|_\infty\lesssim\ell(Q)^{-1}$. Then we define $w_Q=\Gamma(\eta_Q u_Q)$.

We now set $f_Q^w := P^B_{\varepsilon\ell(Q)}w_Q$. This satisfies

$$
\big\|f_Q^w\big\|_p\lesssim\|w_Q\|_p\lesssim|Q|^{1/p}, \tag{34}
$$

and, using the identity $Q_sP_t = s/t\cdot Q_tP_s$, also

$$
\begin{aligned}
\mathbb{E}\Big\|\sum_{k\in\mathbf{Z}}\varepsilon_k 1_{R_Q}\big(2^k,\cdot\big)Q^B_{2^k}f_Q^w\Big\|_p &\leqslant \sum_{k:\,2^k\leqslant\ell(Q)}\frac{2^k}{\varepsilon\ell(Q)}\big\|Q^B_{\varepsilon\ell(Q)}P^B_{2^k}w_Q\big\|_p\\
&\lesssim\frac{|Q|^{1/p}}{\varepsilon}.
\end{aligned}\tag{35}
$$

Estimates (34) and (35) are our L^p versions of the first two assertions of [9, Lemma 5.10], and the remaining part of that lemma is dealt with as follows. Note that we write simply $|\cdot|$ for the norm in $\mathbf{C}^N$.

Lemma 9.3. *For some c depending only on p as well as P_t^B, Q_t^B, and Γ, there holds*

$$
\Big|\fint_Q f_Q^w\,\mathrm{d}x - w\Big|\leqslant c\varepsilon^{1/p'}.
$$

Proof. Writing out the definitions,

$$\fint_Q f_Q^w \,\mathrm{d}x - w = \fint_Q \big(P^B_{\varepsilon\ell(Q)} - I\big) w_Q \,\mathrm{d}x$$
$$= \fint_Q -\varepsilon^2 \ell(Q)^2 \Gamma \Pi_B P^B_{\varepsilon\ell(Q)} w_Q \,\mathrm{d}x, \tag{36}$$

where the last equality used the facts that $w_Q \in \mathsf{R}(\Gamma)$ and $\Pi_B^2 = \Gamma\Pi_B$ on $\mathsf{R}(\Gamma)$. We next make use of the following estimate, which depends on the fact that Γ is a first-order differential operator with constant coefficients:

$$\left|\fint_Q \Gamma u \,\mathrm{d}x\right|^p \lesssim \ell(Q)^{1-p} \bigg(\fint_Q |u|^p \,\mathrm{d}x\bigg)^{1/p'} \bigg(\fint_Q |\Gamma u|^p \,\mathrm{d}x\bigg)^{1/p}. \tag{37}$$

This is the L^p version of [9, Lemma 5.6], and is proved by a simple modification of the $p = 2$ case given there.

Using (37) in (36), we obtain

$$\left|\fint_Q f_Q^w \,\mathrm{d}x - w\right|^p \tag{38}$$
$$\lesssim \ell(Q)^{1-p} \bigg(\fint \big|\varepsilon\ell(Q) Q^B_{\varepsilon\ell(Q)} w_Q\big|^p \,\mathrm{d}x\bigg)^{1/p'} \bigg(\fint \big|\big(P^B_{\varepsilon\ell(Q)} - I\big) w_Q\big|^p \,\mathrm{d}x\bigg)^{1/p} \tag{39}$$
$$\lesssim \ell(Q)^{1-p} \big(\varepsilon\ell(Q)\big)^{p/p'} \bigg(|Q|^{-1} \int |w_Q|^p \,\mathrm{d}x\bigg)^{1/p'+1/p} \lesssim \varepsilon^{p-1} \tag{40}$$

by the uniform L^p-boundedness of P_t^B and Q_t^B, together with (34), and this completes the proof. □

Lemma 9.4. *With $\varepsilon = (2c)^{-p'}$, where c is as in Lemma* 9.3*, there exist $\beta, c_1, c_2 > 0$ and for each $Q \in \triangle$ a collection $(Q_j)_{j\in J}$ of disjoint dyadic subcubes such that, with the definitions* (33)*, there holds $|E_{Q,\nu}| > \beta|Q|$ and*

$$\Re\big(w, A_{2^k} f_Q^w(x)\big) \geqslant c_1, \qquad A_{2^k}\big|f_Q^w\big|(x) \leqslant c_2, \quad \text{if } \big(2^k, x\big) \in E^*_{Q,\nu}.$$

Proof. With the given choice of ε, Lemma 9.3 implies that

$$\Re\bigg(w, \fint_Q f_Q^w\bigg) \geqslant \frac{1}{2}.$$

The assertion follows from this together with (34), by a stopping time argument exactly as the corresponding result, in [9, Lemma 5.11]. □

Lemma 9.5. *With* $\sigma := \frac{c_1}{2c_2}$, *there holds*

$$\left|\gamma_{2^k}(x)\big(A_t f_Q^w(x)\big)\right| \geqslant \frac{c_1}{2}\left\|\gamma_{2^k}(x)P_\Gamma\right\|, \qquad \big(2^k, x\big) \in E^*_{Q,\nu} \cap C_\nu.$$

Proof. This is almost like [9, Lemma 5.12]. By Lemma 9.4,

$$\left|\nu\big(A_{2^k} f_Q^w(x)\big)\right| \geqslant \Re\big(\hat{w}, \nu\big(A_{2^k} f_Q^w(x)\big)\big) = \Re\big(w, A_{2^k} f_Q^w(x)\big) \geqslant c_1,$$

and then

$$\begin{aligned}
&\left|\frac{\gamma_{2^k}(x)P_\Gamma}{\|\gamma_{2^k}(x)P_\Gamma\|}\big(A_{2^k} f_Q^w(x)\big)\right| \\
&\quad\geqslant \left|\nu\big(A_{2^k} f_Q^w(x)\big)\right| - \left\|\frac{\gamma_{2^k}(x)P_\Gamma}{\|\gamma_{2^k}(x)P_\Gamma\|} - \nu\right\| \left|A_{2^k} f_Q^w(x)\right| \\
&\quad\geqslant c_1 - \sigma c_2 = c_1/2.
\end{aligned}$$

Finally, recall that $P_\Gamma(A_{2^k} f_Q^w(x)\big) = A_{2^k} f_Q^w(x)$, since $f_Q^w \in \mathsf{R}(\Gamma)$, to complete the proof. □

Proof of Propositions 9.2 and 9.1. We make use of the Khintchine–Kahane inequalities (Proposition 2.3) and Lemma 9.5 to the result:

$$\begin{aligned}
&\left(\mathbb{E}\left\|\sum_{k\in\mathbf{Z}} \varepsilon_k 1_{R_Q \cap E^*_{Q,\nu}}\big(2^k, \cdot\big)\gamma_{2^k} P_\Gamma\right\|^p_{L^p(\mathbf{R}^n;\mathcal{L}(\mathbf{C}^N))}\right)^{1/p} \\
&\quad\eqsim \left\|\left(\sum_{k\in\mathbf{Z}} 1_{R_Q \cap E^*_{Q,\nu}}\big(2^k, \cdot\big)\|\gamma_{2^k} P_\Gamma\|^2\right)^{1/2}\right\|_{L^p(\mathbf{R}^n)} \\
&\quad\lesssim \mathbb{E}\left\|\sum_{k\in\mathbf{Z}} \varepsilon_k 1_{R_Q}\big(2^k, \cdot\big)\gamma_{2^k} A_{2^k} f_Q^w\right\|_{L^p(\mathbf{R}^n;\mathbf{C}^N)} \\
&\quad\leqslant \mathbb{E}\left\|\sum_{k\in\mathbf{Z}} \varepsilon_k \big(Q^B_{2^k} - \gamma_{2^k} A_{2^k}\big) f_Q^w\right\|_{L^p(\mathbf{R}^n;\mathbf{C}^N)} \\
&\qquad + \mathbb{E}\left\|\sum_{k\in\mathbf{Z}} \varepsilon_k 1_{R_Q}\big(2^k, \cdot\big) Q^B_{2^k} f_Q^w\right\|_{L^p(\mathbf{R}^n;\mathbf{C}^N)}.
\end{aligned}$$

Recalling again that $f_Q^w \in \mathsf{R}(\Gamma)$, we may apply the reduction-to-principal part Theorem 6.2, which shows that the first term on the right is dominated by $\|f_Q^w\|_p \lesssim |Q|^{1/p}$. The second term is almost like the quadratic norm in Proposition 3.4 which we started from but with the arbitrary X^N-valued function $u \in \mathsf{R}(\Gamma)$ replaced by the deliberately constructed $\mathbf{C}^N$-valued test function f_Q^w. And indeed the estimate for this test function, which we recorded in (35), is precisely what we need to complete the proof. □

Proof of Proposition 3.4 and Theorem 3.1. By Proposition 9.1 and our analogue of Carleson's inequality (Theorem 8.2) we have:

$$\mathbb{E}\Big\|\sum_{k\in\mathbf{Z}}\varepsilon_k\gamma_{2^k}A_{2^k}u\Big\|_{L^p(\mathbf{R}^n;X^N)}\lesssim\|u\|_{L^p(\mathbf{R}^n;X^N)},\quad\forall u\in\mathsf{R}(\Gamma).$$

Together with our quadratic $T(1)$ theorem 6.2, this completes the proof of Proposition 3.4, and, as pointed out in Section 3, of Theorem 3.1. □

Remark 9.6. Looking back at the structure of the entire proof, it may be interesting to note the difference in the two applications of Theorem 6.2. In Section 6, it was used to replace $Q^B_{2^k}$ in the desired estimate by its principal part $\gamma_{2^k}A_{2^k}$, whereas right above we performed the reverse action. But of course other reductions took place at the same time: the first replacement allowed the application of Carleson's inequality, which reduced the original X^N-valued estimate to an $\mathcal{L}(\mathbf{C}^N)$-valued one, while the second replacement made the further reduction to a $\mathbf{C}^N$-valued inequality for a test function. This strategy was already used in the case when $X=\mathbf{C}$ in [9]; thus the key point was not the reduction of X^N to $\mathbf{C}^N$, but the reduction of u to f^w_Q.

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Appendix A. R-bisectoriality of uniformly elliptic operators

In this section we explain how the R-bisectoriality conditions in Theorem 3.1 can, in some cases, be checked by a simple perturbation argument. Consider the differential operator $L=-\operatorname{div}A\nabla$, where the $\mathcal{L}(\mathbf{C}^n)$-valued function $A(x)$ satisfies the uniform ellipticity (or accretivity) condition

$$\lambda|\xi|^2\leqslant\Re\langle A(x)\xi,\xi\rangle,\qquad\big|\langle A(x)\xi,\eta\rangle\big|\leqslant\Lambda|\xi||\eta|\tag{A.1}$$

for all $x\in\mathbf{R}^n$ and $\xi,\eta\in\mathbf{C}^n$. This implies in particular that $x\mapsto A(x)$ and $x\mapsto A(x)^{-1}$ are in $L^\infty(\mathbf{R}^n;\mathcal{L}(\mathbf{C}^n))$ with norms at most Λ and λ^{-1}, respectively, as required to apply Corollary 3.2. But the ellipticity (A.1) says more: as shown in [35], there exist constants $M,\delta>0$, depending only on λ and Λ, such that $\|MI-A(x)\|\leqslant M-\delta$ for all $x\in\mathbf{R}^n$. Then $A=M(I+M^{-1}[A-MI])=:M(I+K)$, where the norm of K in $L^\infty(\mathbf{R}^n;\mathcal{L}(\mathbf{C}^n))$ is strictly smaller than 1. This obviously implies the same norm bound in $\mathcal{L}(L^p(\mathbf{R}^n;\mathbf{C}^n))$. To be able to make this conclusion even in $\mathcal{L}(L^p(\mathbf{R}^n;X^n))$, we need to use a special norm in the product space X^n. This is given by

$$\big\|(x_i)_{i=1}^n\big\|_{X^n}:=\bigg(\mathbb{E}\bigg|\sum_{i=1}^n\gamma_ix_i\bigg|_X^2\bigg)^{1/2},\tag{A.2}$$

where the γ_i are independent standard Gaussian random variables. This is, of course, equivalent to any of the usual norms that one would use on X^n, and the equivalence constants may be chosen to depend on n only. The crucial property of this norm is the following.

Lemma A.1. *Let $T \in \mathcal{L}(\mathbf{C}^n)$ induce an operator in $\mathcal{L}(X^n)$ in the natural way. If X^n is equipped with the norm* (A.2)*, then*

$$\|T\|_{\mathcal{L}(X^n)} = \|T\|_{\mathcal{L}(\mathbf{C}^n)}.$$

Proof. The inequality $\geqslant$ is clear. The estimate $\leqslant$ follows from [37, Proposition 3.7], once we observe that

$$\begin{aligned}\sum_{i=1}^n \left|\left\langle \sum_{j=1}^n t_{ij} x_j, x^* \right\rangle\right|^2 &= \left|T\left(\langle x_j, x^*\rangle\right)_{j=1}^n\right|_{\mathbf{C}^n}^2 \leqslant \|T\|_{\mathcal{L}(\mathbf{C}^n)}^2 \left|\left(\langle x_j, x^*\rangle\right)_{j=1}^n\right|_{\mathbf{C}^n}^2 \\ &= \|T\|_{\mathcal{L}(\mathbf{C}^n)}^2 \sum_{j=1}^n \left|\langle x_j, x^*\rangle\right|^2\end{aligned}$$

for all $x^* \in X^*$. $\square$

We will now make use of the above observations but applied to A^{-1} in place of A. Note that A^{-1} also satisfies the ellipticity condition (A.1), possibly with different constants, as soon as A does. Since the differential operators L and ML have the same mapping properties, we may assume without loss of generality that $M = 1$. Thus the matrix-multiplication operator A as in (A.1) may be assumed to have an inverse, which is a perturbation of the identity:

$$A^{-1} = I + K, \qquad \|K\|_{\mathcal{L}(L^p(\mathbf{R}^n; X^n))} \leqslant \|K\|_{L^\infty(\mathbf{R}^n; \mathcal{L}(\mathbf{C}^n))} < 1. \tag{A.3}$$

Hence, keeping the notation of Theorem 3.1 and Corollary 3.2, with $A_1 = I$ and $A_2 = A$,

$$\Pi_B = \begin{pmatrix} 0 & -\operatorname{div} A \\ \nabla & 0 \end{pmatrix}, \qquad \Pi_{B^*} = \begin{pmatrix} 0 & -\operatorname{div} \\ A\nabla & 0 \end{pmatrix} \tag{A.4}$$

and then

$$\begin{aligned}(I + it\Pi_B)\begin{pmatrix} I & 0 \\ 0 & A^{-1} \end{pmatrix} &= \begin{pmatrix} I & 0 \\ 0 & A^{-1} \end{pmatrix}(I + it\Pi_{B^*}) = \begin{pmatrix} I & -it\operatorname{div} \\ it\nabla & A^{-1} \end{pmatrix} \\ &= \begin{pmatrix} I & -it\operatorname{div} \\ it\nabla & I \end{pmatrix}\left[I + \begin{pmatrix} I & -it\operatorname{div} \\ it\nabla & I \end{pmatrix}^{-1}\begin{pmatrix} 0 & 0 \\ 0 & K \end{pmatrix}\right] \\ &= (I + it\Pi)\begin{pmatrix} I & it\operatorname{div}(I - t^2\nabla\operatorname{div})^{-1}K \\ 0 & I + (I - t^2\nabla\operatorname{div})^{-1}K \end{pmatrix}.\end{aligned}$$

It follows that

$$\begin{aligned}(I+it\Pi_B)\text{ is invertible}\quad&\Leftrightarrow\quad (I+it\Pi_{B^*})\text{ is invertible}\\&\Leftrightarrow\quad \big(I+\big(I-t^2\nabla\operatorname{div}\big)^{-1}K\big)\text{ is invertible,}\end{aligned}\tag{A.5}$$

and if this is the case, then

$$\begin{aligned}\begin{pmatrix} I & 0\\ 0 & A\end{pmatrix}R_t^B &= R_t^{B^*}\begin{pmatrix} I & 0\\ 0 & A\end{pmatrix}\\ &= \begin{pmatrix} I & it\operatorname{div}(I-t^2\nabla\operatorname{div})^{-1}K\\ 0 & I\end{pmatrix}\begin{pmatrix} I & 0\\ 0 & [I+(I-t^2\nabla\operatorname{div})^{-1}K]^{-1}\end{pmatrix}R_t\end{aligned}\tag{A.6}$$

where, we recall, $R_t^B=(I+it\Pi_B)^{-1}$, $R_t=(I+it\Pi)^{-1}$.

We can now conclude the following.

Proposition A.2. *Let X be a UMD space, $1<p<\infty$, and $A\in L^\infty(\mathbf{R}^n;\mathcal{L}(\mathbf{C}^n))$ satisfy* (A.3). *Then the operators Π_B and Π_{B^*} in* (A.4) *are R-bisectorial in the space $L^p(\mathbf{R}^n;X^{n+1})$ provided that $I+(I-t^2\nabla\operatorname{div})^{-1}K$ is invertible in $L^p(\mathbf{R}^n;X^n)$ for all $t>0$, and*

$$\big\{\big[I+\big(I-t^2\nabla\operatorname{div}\big)^{-1}K\big]^{-1}\big\}_{t>0}\quad \textit{is R-bounded in } L^p\big(\mathbf{R}^n;X^n\big).$$

Hence, if the above condition is valid in an interval $(p-\varepsilon,p+\varepsilon)$, then Π_B and Π_{B^*} have an H^∞-functional calculus in $L^p(\mathbf{R}^n;X^{n+1})$, L has an H^∞-calculus in $L^p(\mathbf{R}^n;X)$, and L satisfies Kato's square root estimates $\|\sqrt{L}u\|_p\eqsim\|\nabla u\|_p$ for all $u\in L^p(\mathbf{R}^n;X)$.

Proof. We have already seen that the inevitability condition is both necessary and sufficient for the existence of the resolvents appearing in the definition of bisectoriality. If X is a UMD space, then the unperturbed operator Π is R-bisectorial, and moreover the family of operators

$$\big\{it\operatorname{div}\big(I-t^2\nabla\operatorname{div}\big)^{-1}\big\}_{t>0}=\big\{it\big(I-t^2\Delta\big)^{-1}\operatorname{div}\big\}_{t>0}$$

is R-bounded from $L^p(\mathbf{R}^n;X)$ to $L^p(\mathbf{R}^n;X^n)$ (by Proposition 2.9, since these are Fourier multiplier operators whose symbols have uniformly bounded variation). From (A.6), and the fact that products of R-bounded sets remain R-bounded, we conclude the first assertion. The second is a consequence of Theorem 3.1 and Corollary 3.2. □

Remark A.3. If $n=1$, then the equivalent inevitability conditions in (A.5) are always satisfied in $L^p(\mathbf{R};X^2)$ respectively $L^p(\mathbf{R};X)$, for all Banach spaces X and all $p\in[1,\infty]$. In fact, in this case $(I-t^2\nabla\operatorname{div})^{-1}=(I-t^2\Delta)^{-1}=\mathcal{P}_t$ is the convolution operator with kernel $(2t)^{-1}e^{-|x|/t}$. This operator contracts all L^p spaces, and hence $I+\mathcal{P}_tK$ has a bounded inverse represented by the convergent Neumann series

$$(I+\mathcal{P}_tK)^{-1}=\sum_{k=0}^{\infty}(-\mathcal{P}_tK)^k,\tag{A.7}$$

since the operator norm of K satisfies $\|K\|<1$.

Corollary A.4. *Let X be a UMD function lattice. Let $A \in L^\infty(\mathbf{R}^n;\mathbf{C})$ satisfy* (A.3)*. Then the operators Π_B and Π_{B^*} in* (A.4) *are R-bisectorial in $L^p(\mathbf{R}^n;X^2)$ for all $p \in]1,\infty[$, and hence $L = -\mathrm{d}/\mathrm{d}x\, A(x)\,\mathrm{d}/\mathrm{d}x$ has an H^∞-calculus and satisfies the Kato's square root estimates in $L^p(\mathbf{R}^n;X)$, for all $p \in]1,\infty[$.*

Proof. By Remark A.3 and (A.5), we already know that the required resolvents exist. To prove the R-boundedness of $(I + \mathcal{P}_t K)^{-1}$, it suffices to show that the R-bounds of the terms in the Neumann series (A.7) converge. Let us investigate the kth term. Our aim is to show that

$$\mathbb{E}\Big\|\sum_j \varepsilon_j (\mathcal{P}_{t_j} K)^k u_j\Big\|_{L^p(\mathbf{R};X)} \lesssim \|K\|_\infty{}^k \mathbb{E}\Big\|\sum_j \varepsilon_j u_j\Big\|_{L^p(\mathbf{R};X)}, \tag{A.8}$$

since this would allow us to sum up the series in k. Since X is a function lattice with finite cotype, (A.8) is equivalent to the quadratic estimate

$$\bigg\|\Big(\sum_j \big|(\mathcal{P}_{t_j} K)^k u_j\big|^2\Big)^{1/2}\bigg\|_p \lesssim \|K\|_\infty^k \bigg\|\Big(\sum_j |u_j|^2\Big)^{1/2}\bigg\|_p. \tag{A.9}$$

Let us denote the convolution kernel of $\mathcal{P}_t$ by $p_t(x) := (2t)^{-1} e^{-|x|/t}$. The positivity of this function is of essential importance in what follows. Now

$$\begin{aligned}
&\big|(\mathcal{P}_t K)^k u(x)\big| \\
&\quad = \bigg|\int \dots \int p_t(x-y_1)K(y_1)\cdots p_t(y_{k-1}-y_k)K(y_k)u(y_k)\,\mathrm{d}y_1\dots\mathrm{d}y_k\bigg| \\
&\quad \leqslant \int \dots \int p_t(x-y_1)\big|K(y_1)\big|\cdots p_t(y_{k-1}-y_k)\big|K(y_k)\big|\big|u(y_k)\big|\,\mathrm{d}y_1\dots\mathrm{d}y_k \\
&\quad \leqslant \|K\|_\infty^k \int \dots \int p_t(x-y_1)\cdots p_t(y_{k-1}-y_k)\big|u(y_k)\big|\mathrm{d}y_1\dots\mathrm{d}y_k \\
&\quad = \|K\|_\infty^k \mathcal{P}_t^k |u|(x).
\end{aligned}$$

Hence we have

$$\bigg\|\Big(\sum_j \big|(\mathcal{P}_{t_j} K)^k u_j\big|^2\Big)^{1/2}\bigg\|_p \leqslant \|K\|_\infty^k \bigg\|\Big(\sum_j \big|(\mathcal{P}_{t_j})^k |u_j|\big|^2\Big)^{1/2}\bigg\|_p.$$

The right-hand side above is dominated by the right-hand side of (A.9), with the implied constant independent of k, since the two-parameter family of operators $\{\mathcal{P}_t^k\colon t>0, k\in\mathbf{Z}_+\}$ is R-bounded in $L^p(\mathbf{R};X)$. In fact, these are Fourier multiplier operators with symbols $(1+t^2|\xi|^2)^{-k}$, and one readily checks that they all have uniformly bounded variation, so that we may apply Proposition 2.9.

This completes the proof of the R-bisectoriality. The final claim concerning the functional calculus and the Kato estimates is just an application of Theorem 3.1 and Corollary 3.2. $\square$

Note that the R-boundedness of $\{\mathcal{P}_t^k\colon\ t > 0, k \in \mathbf{Z}_+\}$, which played a rôle above, is still true in arbitrary UMD spaces; however, without the possibility of replacing the randomized norms by quadratic ones, there does not seem to be a way of extracting the K's out of the operator product $(\mathcal{P}_t K)^k$. In the noncommutative L^p spaces, there are also versions of square functions available, but the proof above does not apply, since the modulus $|\cdot|$ does not satisfy the triangle inequality.

In general, the Neumann series argument shows that Π_{B^*} and Π_B are bisectorial provided the set

$$\left\{\left(I - t^2 \nabla \operatorname{div}\right)^{-1} K;\ t \in \mathbf{R}\right\}$$

is R-bounded with constant $c < 1$. If X is a Hilbert space, and $p = 2$, the R-bounds are just uniform bounds and thus $c \leqslant \|K\|_{\mathcal{L}(L^p(\mathbf{R}^n;X))} < 1$. This gives back the solution of the Kato problem from [7]. Still in the Hilbertian situation, this also implies that, given a perturbation, there exists an open interval $(p_-^A, p_+^A) \subset (1, \infty)$ containing 2 such that (5) holds. This coincides with results from [3]. Computing the precise values of p_-^A and p_+^A seems, unfortunately, to be difficult.

Appendix B. Carleson's inequality and paraproducts

Let us point out some consequences of Theorem 8.2 concerning vector-valued paraproducts

$$P(f, u) := \sum_{Q \in \triangle} \sum_{\eta} \frac{\langle f, h_Q^\eta \rangle \langle u \rangle_Q}{|Q|} h_Q^\eta.$$

These operators play the important rôle of principal parts of Calderón–Zygmund operators in the $T(1)$ and $T(b)$ theorems. Versions of these theorems in UMD spaces have been proved in [19,22,25].

The basic mapping property in the scalar case $X = \mathbf{C}$ is

$$\left\|P(f, u)\right\|_{L^p(\mathbf{R}^n)} \lesssim \|f\|_{BMO(\mathbf{R}^n)} \|u\|_{L^p(\mathbf{R}^n)}, \quad 1 < p < \infty. \tag{B.1}$$

This reduces to the classical Carleson inequality for $p = 2$, and may be extrapolated to the whole range $1 < p < \infty$ by standard Calderón–Zygmund techniques. Alternatively, one may establish the L^2 estimate in all weighted spaces $L^2(\mathbf{R}^n, w(x)\,\mathrm{d}x)$ for w in the Muckenhoupt A_2-class, with uniform dependence on the A_2-constant, and invoke the weighted extrapolation theorem of Rubio de Francia to deduce the corresponding L^p-estimates (cf. [29] for this approach). Figiel [19] has shown (based on an intermediate estimate [20], which he attributes to Bourgain) that one may replace $L^p(\mathbf{R}^n)$ by $L^p(\mathbf{R}^n; X)$ in (B.1) provided that X is a UMD space. His proof employs interpolation between (H^1, L^1) and (L^∞, BMO) type estimates. Thus in all these arguments, the L^p-inequalities in (B.1) when $p \neq 2$ are reached somewhat indirectly.

We next provide an alternative approach to the Bourgain–Figiel result based on Theorem 8.2 (and hence under the additional assumption of the RMF property). This also gives an apparently new "L^p proof" of the classical estimate (B.1). While the proof of Theorem 8.2 was not completely interpolation-free, either, one should note that getting the L^p estimate for a given p only involved interpolation between spaces "in the proximity" of L^p, in contrast to the "far away" end-point spaces in the classical arguments. The proof below will show that the problem of the extra ϵ disappears in this specific situation, thanks to the John–Nirenberg inequality.

Corollary B.1. *Let X be a UMD space with RMF, and $1 < p < \infty$. Then*

$$\big\|P(f,u)\big\|_{L^p(\mathbf{R}^n;X)} \lesssim \|f\|_{BMO(\mathbf{R}^n)}\|u\|_{L^p(\mathbf{R}^n;X)}.$$

Proof. We have the following chain of estimates, where we write simply $\|\cdot\|_p$ for the norm of $L^p(\mathbf{R}^n;X)$:

$$\begin{aligned}
&\big\|P(f,u)\big\|_p \\
&\lesssim \bigg(\int_{\mathbf{R}^n} \mathbb{E}\bigg\|\sum_{Q,\eta} \varepsilon_Q^\eta \frac{\langle f, h_Q^\eta\rangle h_Q^\eta(x)}{|Q|}\langle u\rangle_Q\bigg\|_X^p \,\mathrm{d}x\bigg)^{1/p} \\
&\lesssim \sum_\eta \sup_{S\in\triangle}\bigg(\frac{1}{|S|}\int_S \mathbb{E}\bigg|\sum_{Q\subseteq S} \varepsilon_Q \frac{\langle f, h_Q^\eta\rangle h_Q^\eta(x)}{|Q|}\bigg|^{p+\epsilon} \,\mathrm{d}x\bigg)^{1/(p+\epsilon)} \|u\|_p \\
&\lesssim \sup_{S\in\triangle}\bigg(\frac{1}{|S|}\int_S \bigg|\sum_{Q\subseteq S}\sum_\eta \frac{\langle f, h_Q^\eta\rangle h_Q^\eta(x)}{|Q|}\bigg|^{p+\epsilon} \,\mathrm{d}x\bigg)^{1/(p+\epsilon)} \|u\|_p \\
&= \sup_{S\in\triangle}\bigg(\frac{1}{|S|}\int_S \big|f(x)-\langle f\rangle_S\big|^{p+\epsilon}\,\mathrm{d}x\bigg)^{1/(p+\epsilon)} \|u\|_p \\
&\lesssim \|f\|_{BMO}\|u\|_p.
\end{aligned}$$

The first estimate employed the UMD property of X, the second used Theorem 8.2, the third the UMD property of $\mathbf{C}$, and the final one the John–Nirenberg inequality. □

It is also possible to reverse the rôles of scalar and vector-valued functions in Theorem 8.2 and then in Corollary B.1. We leave the straightforward verification of the details to the reader, and only record the result. The RMF property does not enter this time, because the maximal function estimate is now required for a scalar-valued function.

Corollary B.2. *Let X be a UMD space, and $1 < p < \infty$. Then*

$$\big\|P(f,u)\big\|_{L^p(\mathbf{R}^n;X)} \lesssim \|f\|_{BMO(\mathbf{R}^n;X)}\|u\|_{L^p(\mathbf{R}^n)}.$$

Appendix C. The space ℓ^1 does not have RMF

As mentioned in Section 7, we do not yet understand how the RMF property relates to other properties of Banach spaces, and in particular to the UMD property. In this appendix we show that it is, however, a non-trivial property by proving that ℓ^1 does not enjoy RMF.

Let $n \in \mathbf{N}$, and $u(x) = e_k$ for $x \in [(k-1)2^{-n}, k2^{-n})$ for $k = 1, 2, \ldots, 2^n$. Then

$$\|u\|_{L^p(\mathbf{R}^1,\ell^1)} = 1 \quad \text{for all } p \in [1,\infty].$$

For $x \in [0, 2^{-n})$, we have

$$A_{2^{-n+j}} u(x) = \frac{1}{2^j} \sum_{k=1}^{2^j} e_k, \quad j = 0, 1, \ldots, n.$$

For other $x \in [0, 1)$, we have similar results with a permuted basis $e_{\pi(k)}$ in place of e_k.

Let $n = 2^m$, and consider, given a sequence $\alpha = (\alpha_i)_{i \in \mathbf{N}} \subset \mathbf{R}$ to be chosen later, the sequence λ given by $\lambda_{2^i} = \alpha_i$, $i = 1, \ldots, m$, and $\lambda_j = 0$, otherwise. Then for $0 < x < 2^{-n}$,

$$\begin{aligned}
\mathbb{E}\left\| \sum_{j=0}^{n} \varepsilon_j A_{2^{-n+j}} u(x) \lambda_j \right\|_{\ell^1} &= \mathbb{E}\left\| \sum_{i=1}^{m} \varepsilon_i \frac{1}{2^{2^i}} \sum_{k=1}^{2^{2^i}} e_k \alpha_i \right\|_{\ell^1} \\
&\geqslant \mathbb{E}\left\| \sum_{i=1}^{m} \varepsilon_i \frac{1}{2^{2^i}} \sum_{k=2^{2^{i-1}}+1}^{2^{2^i}} e_k \alpha_i \right\|_{\ell^1} - \sum_{i=1}^{m} \frac{1}{2^{2^i}} 2^{2^{i-1}} |\alpha_i| \\
&= \sum_{i=1}^{m} \frac{2^{2^i} - 2^{2^{i-1}}}{2^{2^i}} |\alpha_i| - \sum_{i=1}^{m} 2^{-2^{i-1}} |\alpha_i| \\
&\gtrsim \|\alpha\|_{\ell^1} - \|\alpha\|_{\ell^\infty}.
\end{aligned}$$

Choosing, say, $\alpha_i = (i+1)^{-1}$, we find that

$$M_{\mathrm{R}} u(x) \gtrsim \log m \gtrsim \log \log n$$

for all $x \in [0, 2^{-n})$, and by the permutation symmetry of the standard basis, for all $x \in [0, 1)$. This shows that $\|M_{\mathrm{R}} u\|_{L^p(\mathbf{R}^1)} \gtrsim \log \log n$. Since the same construction can be repeated with arbitrarily large n, we see that no L^p bound can hold for M_{R} in ℓ^1.

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Rate of convergence for ergodic continuous Markov processes: Lyapunov versus Poincaré

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Abstract

We study the relationship between two classical approaches for quantitative ergodic properties: the first one based on Lyapunov type controls and popularized by Meyn and Tweedie, the second one based on functional inequalities (of Poincaré type). We show that they can be linked through new inequalities (Lyapunov–Poincaré inequalities). Explicit examples for diffusion processes are studied, improving some results in the literature. The example of the kinetic Fokker–Planck equation recently studied by Hérau and Nier, Helffer and Nier, and Villani is in particular discussed in the final section.

Keywords: Ergodic processes; Lyapunov functions; Poincaré inequalities; Hypocoercivity

1. Introduction, framework and first results

Rate of convergence to equilibrium is one of the most studied problem in various areas of mathematics and physics. In the present paper we shall consider a dynamics given by a time

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continuous Markov process $(X_t, \mathbb{P}_x)$ admitting an (unique) ergodic invariant measure μ, and we will try to describe the nature and the rate of convergence to μ.

In the sequel we denote by L the infinitesimal generator (and $D(L)$ the extended domain of the generator, see e.g. [6]), by $P_t(x,.)$ the $\mathbb{P}_x$ law of X_t and by P_t (respectively P_t^*) the associated semi-group (respectively the adjoint or dual semi-group), so that in particular for any regular enough density of probability h with respect to μ, $\int P_t(x,.)h(x)\mu(dx) = P_t^* h\, d\mu$.

Extending the famous Doeblin recurrence condition for Markov chains, Meyn and Tweedie developed stability concepts for time continuous processes and furnished tractable methods to verify stability [22,23]. The most popular criterion certainly is the existence of a so called Lyapunov function for the generator [11,23], yielding exponential (or geometric) convergence, via control of excursions of well-chosen functionals of the process. Sub-geometric or polynomial convergence can also be studied (see [12,28] among others for the diffusion case). A very general form of the method is explained in the recent work by Douc, Fort and Guillin [10], and we shall now explain part of their results in more details.

Definition 1.1. Let ϕ be a positive function defined on $[1,+\infty[$. We say that $V \in D(L)$ is a ϕ-Lyapunov function if $V \geqslant 1$ and if there exist a constant b and a closed petite set C such that for all x

$$LV(x) \leqslant -\phi\big(V(x)\big) + b\mathbb{1}_C(x).$$

Recall that C is a petite set if there exists some probability measure $a(dt)$ on $\mathbb{R}^+$ such that for all $x \in C$, $\int_0^{+\infty} P_t(x,.)\,a(dt) \geqslant \nu(.)$ where $\nu(.)$ is a non-trivial positive measure.

When ϕ is linear ($\phi(u) = au$) we shall simply call V a Lyapunov function. Existence of a Lyapunov function furnishes exponential (geometric) decay [11,23], which is a particular case of

Theorem 1.2. *(See [11, Theorem 5.2.c], and [10, Theorems 3.10, 3.12].) Assume that there exists some increasing smooth and concave ϕ-Lyapunov function V such that V is bounded on the petite set C. Assume in addition that the process is irreducible in some sense (see [10,11] for precise statements). Then there exists a positive constant c such that for all x,*

$$\big\| P_t(x,.) - \mu \big\|_{\mathrm{TV}} \leqslant cV(x)\psi(t),$$

where $\psi(t) = 1/(\phi \circ H_\phi^{-1})(t)$ for $H_\phi(t) = \int_1^t (1/\phi(s))\,ds$, and $\|.\|_{\mathrm{TV}}$ is the total variation distance.

In particular if ϕ is linear, $\psi(t) = e^{-\rho t}$ for some positive explicit ρ.

Actually the result stated in Theorem 1.2 can be reinforced by choosing suitable stronger distances (stronger than the total variation distance actually weighted total variation distances) but to the price of slower rates of convergence (see [10,11] for details). In the same spirit some result for some Wasserstein distance is obtained in [16]. An important drawback of this approach is that there is no explicit control (in general) of c. One of the interest of our approach will be to give explicit constants starting from the same drift condition.

The pointwise Theorem 1.2 of course extends to any initial measure m such that $\int V\,dm$ is finite. In particular, choosing $m = h\mu$ for some nice h, convergence reduces to the study of $P_t^* h$ for large t. Long time behavior of Markov semi-groups is known to be linked to functional

inequalities. The most familiar framework certainly is the $\mathbb{L}^2$ framework and the corresponding Poincaré (or weak Poincaré) inequalities, namely

Theorem 1.3. *The following two statements are equivalent for some positive constant C_P.*
(Exponential decay) *For all $f \in \mathbb{L}^2(\mu)$,*

$$\left\| P_t f - \int f\, d\mu \right\|_2^2 \leqslant e^{-t/C_P} \left\| f - \int f\, d\mu \right\|_2^2.$$

(Poincaré inequality) *For all $f \in D_2(L)$ (the domain of the Fredholm extension of L),*

$$\mathrm{Var}_\mu(f) := \left\| f - \int f\, d\mu \right\|_2^2 \leqslant C_P \int -2 f L f\, d\mu.$$

In the sequel we shall define $\Gamma(f) = -2 f L f$.

Thanks to Cauchy–Schwarz inequality and since P_t and P_t^* have the same $\mathbb{L}^2$ norm, a Poincaré inequality implies an exponential rate of convergence in total variation distance, at least for initial laws with a $\mathbb{L}^2$ density with respect to μ.

As for the Meyn–Tweedie approach, one can get sufficient conditions for slower rates of convergence, namely weak Poincaré inequalities introduced by Roeckner and Wang:

Theorem 1.4. *(See [25, Theorem 2.1].) Let N be such that $N(\lambda f) = \lambda^2 N(f)$, $N(P_t f) \leqslant N(f)$ for all t and $N(f) \geqslant \|f\|_2^2$.*

Assume that there exists a non-increasing function β such that for all $s > 0$ and all nice f the following inequality holds:

$$\text{(WPI)} \quad \left\| f - \int f\, d\mu \right\|_2^2 \leqslant \beta(s) \int \Gamma(f)\, d\mu + s N\left(f - \int f\, d\mu \right).$$

Then

$$\left\| P_t f - \int f\, d\mu \right\|_2^2 \leqslant \psi(t) N\left(f - \int f\, d\mu \right),$$

where $\psi(t) = 2 \inf\{s > 0, \beta(s) \log(1/s) \leqslant t\}$.

In the symmetric case one can state a partial converse to Theorem 1.4 (see [25, Theorem 2.3]). Note that this time one has to assume that $N(h)$ is finite in order to get $\mathbb{L}^2$ convergence for $P_t^* h$. In general (WPI) are written with $N = \|.\|_\infty$ (or the oscillation), criteria and explicit form of β are discussed in [3,25]. A particularly interesting fact is that any μ on $\mathbb{R}^d$ which is absolutely continuous with respect to Lebesgue measure, $d\mu = e^{-F} dx$ with a locally bounded F, satisfies some (WPI).

Actually, as for the Meyn–Tweedie approach, one can show slower rates of convergence for less integrable initial densities, as well as some results for an initial Dirac mass. We refer to [8, Sections 4–6] for such a discussion in particular cases, we shall continue in this paper. Actually [8] is primarily concerned with (weak) logarithmic Sobolev inequalities, that is replacing

the $\mathbb{L}^2$ norm by the Orlicz norm associated to $u \to x^2 \log(1+x^2)$, i.e. replacing $\mathbb{L}^2$ initial densities by densities with finite relative entropy (Kullback–Leibler information) with respect to μ. According to Pinsker–Csiszar inequality, relative entropy dominates (up to a factor 2) the square of the variation distance, hence Gross logarithmic Sobolev inequality (or its weak version introduced in [8]) allows to study the decay to equilibrium in total variation distance too.

Generalizations (interpolating between Poincaré and Gross) have been studied by several authors. We refer to [4,5,24,32,33] for related results on super-Poincaré and general F-Sobolev inequalities, as well as their consequences for the decay of the semi-group in appropriate Orlicz norms. We also refer to [1] for an elementary introduction to the standard Poincaré and Gross inequalities.

If the existence of a ϕ-Lyapunov function is a tractable sufficient condition for the Meyn–Tweedie strategy (actually is a necessary and sufficient condition for the exponential case), general tractable sufficient conditions for Poincaré or others functional inequalities are less known (some of them will be recalled later), and in general no criterion is known (with the notable exception of the one dimensional euclidean space). This is one additional reason to understand the relationship between the Meyn–Tweedie approach and the functional inequality approach, i.e. to link Lyapunov and Poincaré. This is the aim of this paper.

Before to describe the contents of the paper, let us indicate another very attractive related problem.

If μ is symmetric and ergodic, it is known that $\int \Gamma(f)\,d\mu = 0$ if and only if f is a constant. In the non-symmetric case this result is no more true, and we shall call fully degenerate (corresponding to the p.d.e. situation) these cases.

Still in the symmetric case (or if L is normal, i.e. $LL^* = L^*L$), it is known that an exponential decay

$$\left\| P_t f - \int f\,d\mu \right\|_2^2 \leqslant e^{-\rho t} N\left(f - \int f\,d\mu\right),$$

for some N as in Theorem 1.4, actually implies a (true) Poincaré inequality (see [25, Theorem 2.3]).

A similar situation is no more true in the fully degenerate case. Indeed in recent works, Hérau and Nier [18] and then Villani [30] have shown that for the kinetic Fokker–Planck equation (which is fully degenerate) the previous decay holds with $N(g) = \|\nabla g\|_2^2$ (μ being here a log concave measure, $N(g)$ is greater than the $\mathbb{L}^2$ squared norm of g up to a constant), and thanks to the hypoelliptic regularization property, it also holds with $N(g) = C\,\mathrm{Var}_\mu(g)$ for some constant $C > 1$ (recall that if $C \leqslant 1$, the Poincaré inequality holds). Of course the Bakry–Emery curvature of this model is equal to $-\infty$, otherwise an exponential decay with $N(g) = C\,\mathrm{Var}_\mu(g)$ would imply a Poincaré inequality, even for $C > 1$, so that this situation is particularly interesting.

It turns out that this model enters the framework of Meyn–Tweedie approach as shown in [35] (also see [10]). Hence relating Lyapunov to some Poincaré in such a case (called hypocoercive by Villani) should help to understand the picture. We shall also study this problem.

Let us briefly describe now our framework. Recall that in all the paper μ is an invariant measure for the process with generator L.

The main additional hypothesis we shall make is the existence of a "carré du champ," that is we assume that there is an algebra which is a core for the generator and such that for f and g in this algebra

$$L(fg) = fLg + gLf + \Gamma(f,g) \tag{1.5}$$

where $\Gamma(f,g)$ is the polarization of $\Gamma(f)$. We shall also assume that Γ comes from a derivation, i.e. for f, h and g as before

$$\Gamma(fg,h) = f\Gamma(g,h) + g\Gamma(f,h). \tag{1.6}$$

The meaning of these assumptions in terms of the underlying stochastic process is explained in the introduction of [6], to which the reader is referred for more details (also see [2] for the corresponding analytic considerations).

Applying Ito's formula, we then get that for all smooth Ψ, and f as before,

$$L\Psi(f) = \frac{\partial \Psi}{\partial x}(f)Lf + 1/2\frac{\partial^2 \Psi}{\partial x^2}(f)\Gamma(f). \tag{1.7}$$

Our plan will be the following. In Section 2 we show how to get controls in variance or in entropy starting from the result of Theorem 1.2 which will be seen to be quite sharp. Section 3 will be devoted to the introduction of (weak) Lyapunov Poincaré inequalities, leading to tractable criteria enabling us to give explicit control of convergence via (ϕ-)Lyapunov condition, illustrated by the examples of Section 4. The next section presents similar results for the entropy, before presenting in the final section an application in the particular fully degenerate case.

2. From Lyapunov to Poincaré and vice versa

We first show that, in the symmetric case, the Meyn–Tweedie method immediately furnishes some Poincaré inequalities.

Indeed let us assume that the hypothesis of Theorem 1.2 are fulfilled, and let f be a bounded function such that $\int f\,d\mu = 0$. Then, if f does not vanish identically, we may define $h = f_+/\int f_+\,d\mu$ which is a bounded density of probability. Thus, if $V \in \mathbb{L}^1(\mu)$, $\int hV\,d\mu < +\infty$. It follows that $\|P_t^* h - 1\|_{\mathbb{L}^1(\mu)}$, which is the total variation distance between μ and the law at time t (starting from $h\mu$), goes to 0 as $t \to +\infty$, with rate $c\psi(t)$ defined in Theorem 1.2.

Hence for $0 < \beta < 1$,

$$\begin{aligned}
\int \left| P_t^* f_+ - \int f_+\,d\mu \right|^2 d\mu &= (f_+ d)^2 \int \left(P_t^* h - 1\right)^2 d\mu \\
&\leqslant \left(\int f_+\,d\mu\right)^2 \int \left(P_t^* h - 1\right)^\beta \left(P_t^* h - 1\right)^{2-\beta} d\mu \\
&\leqslant \left(\int f_+\,d\mu\right)^2 \left(\int \left|P_t^* h - 1\right| d\mu\right)^\beta \left(\int \left|P_t^* h - 1\right|^{\frac{2-\beta}{1-\beta}} d\mu\right)^{1-\beta} \\
&\leqslant c_\beta \psi^\beta(t) \left(\int f_+ V\,d\mu\right)^\beta \left(\int \left| f_+ - \int f_+\,d\mu \right|^{\frac{2-\beta}{1-\beta}} d\mu\right)^{1-\beta}
\end{aligned}$$

$$\leqslant c_\beta \psi^\beta(t)\left(\int f_+ V\,d\mu\right)^\beta \left(\int (2f_+)^{\frac{2-\beta}{1-\beta}}\,d\mu\right)^{1-\beta}$$

where we have used that P_t^* is an operator with norm equal to 1 in all the $\mathbb{L}^p$'s ($p \geqslant 1$), the elementary $|a+b|^p \leqslant 2^{p-1}(|a|^p + |b|^p)$ for $p \geqslant 1$ and Hölder inequality.

Thus

$$\int \left(P_t^* f\right)^2 d\mu \leqslant 2\int \left|P_t^* f_+ - \int f_+\,d\mu\right|^2 d\mu + 2\int \left|P_t^* f_- - \int f_-\,d\mu\right|^2 d\mu$$
$$\leqslant 2^{4-\beta} c_\beta \psi^\beta(t)\left(\int |f| V\,d\mu\right)^\beta \left(\int |f|^{\frac{2-\beta}{1-\beta}}\,d\mu\right)^{1-\beta}.$$

In the symmetric case (or more generally the normal case) we may thus apply Theorem 2.3 in [25] so that we have shown

Theorem 2.1. *Under the hypotheses of Theorem* 1.2*, for any f such that $\int f\,d\mu = 0$ and any $0 < \beta < 1$ it holds*

$$\int \left(P_t^* f\right)^2 d\mu \leqslant C_\beta \psi^\beta(t)\left(\int |f| V\,d\mu\right)^\beta \left(\int |f|^{\frac{2-\beta}{1-\beta}}\,d\mu\right)^{1-\beta}.$$

The result extends to $\beta = 1$ provided f is bounded, and in this case

$$\int \left|P_t^* f\right|^2 d\mu \leqslant C\left(\int V\,d\mu\right)\|f\|_\infty^2 \psi(t).$$

If in addition μ is a symmetric measure for the process, then μ satisfies a weak Poincaré inequality with $N(f) = C(V)\|f\|_\infty^2$ and

$$\beta(s) = s\inf_{u>0} \frac{1}{u}\psi^{-1}\left(ue^{(1-u/s)}\right) \quad \textit{with } \psi^{-1}(a) := \inf\left\{b > 0,\ \psi(b) \leqslant a\right\}.$$

In particular if $\psi(t) = e^{-\rho t}$, μ satisfies a Poincaré inequality.

The fact that a Lyapunov condition furnishes some Poincaré inequality in the symmetric case is already known, see Wu [34,36], but the techniques used by Wu are different and rely mainly on spectral ideas. Note also that a Lyapunov function is always in $\mathbb{L}^1(\mu)$ by integrating the Lyapunov condition, otherwise only $\phi \circ V$ is integrable with respect to μ (as a direct consequence of the Lyapunov inequality). In fact, the simple use, of this theorem enables us to derive very easily the correct rate of convergence to equilibrium and to extend known sharp weak Poincaré inequality in dimension one to higher dimension. The major drawback is that the constants are quite unknown in the general case, however we refer to [9] to results providing explicit constants.

The same idea furnishes without any effort a similar result for the decay of relative entropy. Indeed, if h is a density of probability (with respect to μ), using the concavity of the logarithm, we get for any $0 < \beta < 1$,

$$\begin{aligned}\int P_t^*h\log P_t^*h\,d\mu &= \int \big(P_t^*h-1\big)\log P_t^*h\,d\mu + \int \log P_t^*h\,d\mu\\ &\leqslant \int \big(P_t^*h-1\big)\log P_t^*h\,d\mu + \log\bigg(\int P_t^*h\,d\mu\bigg)\\ &\leqslant \int \big|P_t^*h-1\big|\big|\log P_t^*h\big|\,d\mu\\ &\leqslant \bigg(\int \big|P_t^*h-1\big|\,d\mu\bigg)^{\beta}\bigg(\int \big|P_t^*h-1\big|\big|\log P_t^*h\big|^{\frac{1}{1-\beta}}\,d\mu\bigg)^{1-\beta}.\end{aligned}$$

It is easily seen that the function $u\mapsto |u-1||\log u|^p$ is convex on $]0,+\infty[$ for $p\geqslant 1$, so that

$$t\mapsto \int \big|P_t^*h-1\big|\big|\log P_t^*h\big|^{\frac{1}{1-\beta}}\,d\mu$$

is decaying on $\mathbb{R}^+$. We thus have obtained

Theorem 2.2. *Under the hypotheses of Theorem* 1.2*, for any non-negative h such that $\int h\,d\mu=1$ and any $0<\beta<1$ it holds*

$$\int P_t^*h\log P_t^*h\,d\mu \leqslant C_\beta\psi^\beta(t)\bigg(\int hV\,d\mu\bigg)^{\beta}\bigg(\int |h-1||\log h|^{\frac{1}{1-\beta}}\,d\mu\bigg)^{1-\beta}.$$

The result extends to $\beta=1$ provided h is bounded, and in this case

$$\int P_t^*h\log P_t^*h\,d\mu \leqslant C\bigg(\int V\,d\mu\bigg)\|h\|_\infty\log\big(\|h\|_\infty\big)\psi(t).$$

Note that (in the symmetric case) there is no analogue converse result for relative entropy as for the variance. Indeed recall that if h is a density of probability, $\int h\log h\,d\mu\leqslant \mathrm{Var}_\mu(h)$, hence relative entropy is decaying exponentially fast, controlled by the initial variance of h as soon as a Poincaré inequality holds. But it is known that a Poincaré inequality may hold without log-Sobolev inequality. However, starting from Theorem 2.2 one can prove some (loose) weak log-Sobolev inequality, see [8, Sections 4 and 5].

Of course Theorems 2.1 and 2.2 furnish (in the non-symmetric case as well) controls depending on the integrability of V. For instance if V has all polynomial moments, we may control $\int |f|V\,d\mu$ by some $\int |f|^p\,d\mu$ in Theorem 2.1 and if $\int e^{qV}\,d\mu<+\infty$ for some $q>0$ we may control $\int hV\,d\mu$ by the $u\log_+ u$ Orlicz norm of h in Theorem 2.2. Recall that a Lyapunov function is in $\mathbb{L}^1(\mu)$.

We have seen that, in the symmetric case, the existence of a Lyapunov function implies a Poincaré inequality. Let us briefly discuss some possible converse.

If P_t is μ symmetric for some μ satisfying a Poincaré inequality, then we know that P_t has a spectral gap, say θ. Let f be an eigenfunction associated with the eigenvalue $-\theta$, i.e. $Lf+\theta f=0$. If the semi-group is regularizing (in the ultracontractive case for instance), f has to be bounded. Assume that f is actually bounded and say continuous. Since $\int f\,d\mu=0$, changing f into $-f$ if necessary, we may assume that $\sup f\geqslant -\inf f=-M$. Then define $g=f+1+M$. $Lg=-\theta g+\theta(1+M)$, so that for all $0<\kappa<1$,

$$Lg \leqslant -\kappa\theta g + \theta\kappa(1+M)\mathbb{1}_C$$

with $C = \{f \leqslant (1+M)\kappa/(1-\kappa)\}$ a non-empty (and non-full) closed set.

Of course the previous discussion only covers very few cases, but it indicates that some converse has to be studied.

Another possible way to prove a converse result is the following. Assume that $d\mu(x) = e^{-2V(x)}\,dx$ where V is C^3 and such that

$$|\nabla V|^2(x) - \Delta V(x) \geqslant -C_{\min} > -\infty \tag{2.3}$$

for a non-negative $C_{\min}$ so that the process defined by (recalling that B_t is an usual Brownian motion in $\mathbb{R}^d$)

$$dX_t = dB_t - (\nabla V)(X_t)\,dt, \qquad Law(X_0) = \nu \tag{2.4}$$

has a unique non-explosive strong solution. Assume also that μ satisfies a Poincaré inequality. The difficulty here is that by using Poincaré inequality we inherit a control for all smooth f with finite variance as

$$\mathrm{Var}_\mu(P_t f) \leqslant e^{-\lambda t}\,\mathrm{Var}_\mu(f).$$

But a drift inequality concerns the generator and its behavior towards some chosen function for all x. However it is known, see Down, Meyn, Tweedie [11, Theorems 5.2, 5.3, and the remarks after Theorem 5.3], that the existence of a drift condition is ensured by

$$\|P_t\delta_x - \pi\|_{\mathrm{TV}} \leqslant M(x)\rho^t$$

for some larger than 1 function M and $\rho < 1$. But it is once again a control local in x. In this direction, one can show (see [26, Theorem 3.2.7]) that $\mathrm{Ent}_\mu P_t\delta_x$ is finite for all $t > 0$. But control in entropy is not useful as our assumption is a Poincaré inequality and thus a control in L^2 is needed. Actually the proof of Royer can be used in order to get the following result. Replacing the convex γ therein by $\gamma(y) = y^2$ we obtain

$$\int (P_t\delta_x)^2\,d\mu \leqslant Ze^{2V(x)}\mathbb{E}\big[e^{-2v(B_t)}e^{-\frac{1}{2}\int_0^t[|\nabla V|^2-\Delta V](B_s)\,ds}\big] \leqslant Ze^{2V(x)}e^{\frac{1}{2}C_m t},$$

where $e^{-2v(y)} = (2\pi t)^{-d/2}e^{-|y-x|^2/2t}$. By the Poincaré inequality, we then get that for some λ and t_0

$$\mathrm{Var}_\mu(P_t\delta_x) \leqslant e^{-\lambda(t-t_0)}\,\mathrm{Var}_\mu(P_{t_0}\delta_x) \leqslant Ze^{2V(x)}e^{\frac{1}{2}C_m t_0}e^{-\lambda(t-t_0)}$$

which ends the work as a control in L^2 enables us to control the L^1 distance, and we thus get the existence of a Lyapunov function. However, the Lyapunov function V is not available in close form (see [11,21] for a precise formula).

Finally, let us mention that it is not possible to get a converse result as previously starting from a weak Poincaré inequality as (1) we do not know how to control $\|P_t\delta_x\|_\infty$ (even if it should be controlled in many case) and (2) there is no converse part in the Meyn–Tweedie framework (even

in the discrete time case) for sub-geometric convergence in total variation towards ϕ-Lyapunov condition.

3. From Lyapunov to Poincaré. Continuation

Since we have seen in the previous section that the existence of Lyapunov functions furnishes functional inequalities in the symmetric case, in this section we shall study relationship between some modified Poincaré inequality (still yielding exponential decay) and the existence of a Lyapunov function (with $\phi(u) = \alpha u$), without assuming symmetry.

3.1. Lyapunov–Poincaré inequalities

The key tool is the following elementary lemma.

Lemma 3.1. *For Ψ smooth enough, $W \in D(L)$ and $f \in L^\infty$, define $I_W^\Psi(t) = \int \Psi(P_t f) W \, d\mu$. Then for all $t > 0$,*

$$\frac{d}{dt} I_W^\Psi(t) = -\int 1/2\Psi''(P_t f)\Gamma(P_t f) W \, d\mu + \int L^* W \Psi(P_t f) \, d\mu.$$

In particular for $\Psi(u) = u^2$ we get (denoting simply by I_W the corresponding I_W^Ψ)

$$I_W'(t) = -\int \Gamma(P_t f) W \, d\mu + \int L^* W P_t^2 f \, d\mu.$$

Proof. Recall that $\int L(\Psi(g)W) \, d\mu = 0$. Using (1.5) and (1.7) with $g = P_t f$ we thus get

$$\begin{aligned}\frac{d}{dt} I_W^\Psi(t) &= \int \Psi'(P_t f) L P_t f W \, d\mu \\ &= \int \big(L\big(\Psi(P_t f)\big) - 1/2\Psi''(P_t f)\Gamma(P_t f)\big) W \, d\mu\end{aligned}$$

hence the result. □

This lemma naturally leads to the following definition and proposition.

Definition 3.2. We shall say that μ satisfies a (W)-Lyapunov–Poincaré inequality, if there exists $W \in D(L)$ with $W \geqslant 1$ and a constant C_{LP} such that for all nice f with $\int f \, d\mu = 0$,

$$\int f^2 W \, d\mu \leqslant C_{\mathrm{LP}} \int \big(W\Gamma(f) - f^2 L W\big) \, d\mu.$$

Here and in all the paper, "nice" means that f belongs to the domain of the generator and the set of nice functions is everywhere dense in the domain of the Dirichlet form (for instance smooth compactly supported functions in the usual Euclidean cases).

Proposition 3.3. *The following statements are equivalent*:

- *μ satisfies a (W)-Lyapunov–Poincaré inequality,*
- *$\int (P_t^* f)^2 W\,d\mu \leqslant e^{-(t/C_{\mathrm{LP}})} \int f^2 W\,d\mu$ for all f with $\int f\,d\mu = 0$.*

In particular for all f such that $\int f^2 W\,d\mu < +\infty$, $P_t f$ and $P_t^* f$ go to $\int f\,d\mu$ in $\mathbb{L}^2(\mu)$ with an exponential rate.

Proof. We consider $I_W^*(t)$ replacing P_t by P_t^*. Taking the derivative at time $t = 0$ furnishes as usual the converse part. For the direct one, we only have to use Gronwall's lemma. Indeed the Lyapunov–Poincaré inequality yields $(I_W^*)'(t) \leqslant -(1/C_{\mathrm{LP}}) I_W^*(t)$. Since $I_W^*(t)$ is non-negative, this shows that $I_W^*(t)$ is non-increasing, hence converges to some limit as t tends to infinity, and this limit has to be 0 (otherwise I_W^* would become negative). Since $I_W^*(+\infty) = 0$, the result follows by integrating the differential inequality above. □

Note that a Lyapunov–Poincaré inequality is not a weighted Poincaré inequality (we still assume that $\int f\,d\mu = 0$) and depends on the generator L (not only on the carré du champ). But as we already mentioned, Theorem 2.3 in [25] tells us that, in the symmetric case, if

$$\int P_t^2 f\,d\mu \leqslant c e^{-\delta t} \|f\|_\infty^2$$

for all f such that $\int f\,d\mu = 0$, then μ satisfies the usual Poincaré inequality, with $C_{\mathrm{P}} = 1/\delta$. Hence

Corollary 3.4. *If L is μ symmetric and μ satisfies a (W)-Lyapunov–Poincaré inequality for some $W \in \mathbb{L}^1$, then μ satisfies the ordinary Poincaré inequality, with $C_{\mathrm{P}} = C_{\mathrm{LP}}$.*

Now we turn to sufficient conditions for a Lyapunov–Poincaré inequality to hold.

Recall that we called V a Lyapunov function if $LV \leqslant -\alpha V + b\mathbb{1}_C$. Note that integrating this relation with respect to μ yields $\alpha \int V\,d\mu \leqslant b\mu(C)$, so that, first we have to assume that $\int V\,d\mu < +\infty$, second since $V \geqslant 1$, b and $\mu(C)$ have to be positive.

Before stating the first result of this section we shall introduce some definition.

Definition 3.5. Let U be a subset of the state space E. We shall say that μ satisfies a local Poincaré inequality on U if there exists some constant κ_U such that for all nice f with $\int_E f\,d\mu = 0$,

$$\int_U f^2\,d\mu \leqslant \kappa_U \int_E \Gamma(f)\,d\mu + \big(1/\mu(U)\big)\bigg(\int_U f\,d\mu\bigg)^2.$$

Notice that the energy integral in the right-hand side is taken over the whole space E. We may now state

Theorem 3.6. *Assume that there exists a Lyapunov function V, i.e. $LV \leqslant -2\alpha V + b\mathbb{1}_C$ for some set C (non-necessarily petite).*

Assume that one can find a (large) set U such that μ satisfies a local Poincaré inequality on U.

Assume in addition that:

(1) *either U contains $C' = C \cap \{V \leqslant b/\alpha\}$ and $\alpha\mu(U) > b\mu(U^c)$,*
(2) *or U contains $\{V \leqslant b/\alpha\}$ and $\mu(U) > \mu(U^c)$.*

Then one can find some $\lambda > 0$ such that if $W = V + \lambda$, μ satisfies a (W)-Lyapunov–Poincaré inequality.

More precisely, corresponding to the two previous cases one can choose:

(1) $\lambda = (b\kappa_U - 1)_+$ *and* $1/C_{\mathrm{LP}} = \alpha(1 - \frac{b\mu(U^c)}{\alpha\mu(U)})/(1+\lambda)$,
(2) *or* $\lambda = (b\kappa_U - 1)_+$ *and* $1/C_{\mathrm{LP}} = \alpha(1 - \frac{\mu(U^c)}{\mu(U)})/(1+\lambda)$.

Proof. First remark the following elementary fact: define $C' = C \cap \{V \leqslant b/\alpha\}$. Then $LV \leqslant -\alpha V + b\mathbb{1}_{C'}$, that is we can always assume that C is included into some level set of V. In the sequel $\theta = b/\alpha$. First we assume that U contains $\{V \leqslant b/\alpha\}$, so that it contains C'.

Let $\int f\,d\mu = 0$. Then for all $\lambda > 0$ it holds

$$\int f^2(V+\lambda)\,d\mu \leqslant (1+\lambda)\int f^2 V\,d\mu$$
$$\leqslant (1+\lambda)/\alpha \int f^2\bigl(-L(V+\lambda) + b\mathbb{1}_{C'}\bigr)\,d\mu.$$

But since $\int_U f\,d\mu = -\int_{U^c} f\,d\mu$ it holds

$$\begin{aligned}\int_{C'} f^2\,d\mu \leqslant \int_U f^2\,d\mu &\leqslant \kappa_U \int \Gamma(f)\,d\mu + \bigl(1/\mu(U)\bigr)\biggl(\int_U f\,d\mu\biggr)^2\\
&\leqslant \kappa_U \int \Gamma(f)\,d\mu + \bigl(1/\mu(U)\bigr)\biggl(\int_{U^c} f\,d\mu\biggr)^2\\
&\leqslant \kappa_U \int \Gamma(f)\,d\mu + \bigl(\mu\bigl(U^c\bigr)/\mu(U)\bigr)\biggl(\int_{U^c} f^2\,d\mu\biggr)\\
&\leqslant \kappa_U \int \Gamma(f)\,d\mu + \bigl(\mu\bigl(U^c\bigr)/\theta\mu(U)\bigr)\biggl(\int_{U^c} f^2 V\,d\mu\biggr),\end{aligned}$$

where we used $V/\theta \geqslant 1$ on U^c. So, if we choose $\lambda = (b\kappa_U - 1)_+$ we get

$$b\int_{C'} f^2\,d\mu \leqslant \int \Gamma(f)(V+\lambda)\,d\mu + \bigl(b\mu\bigl(U^c\bigr)/\theta\mu(U)\bigr)\biggl(\int f^2 V\,d\mu\biggr).$$

It yields

$$\int\bigl(W\Gamma(f) - f^2 LW\bigr)\,d\mu \geqslant \alpha\biggl(1 - \frac{b\mu(U^c)}{\theta\alpha\mu(U)}\biggr)\int f^2 V\,d\mu,$$

hence the result with $1/C_{\mathrm{LP}} = \alpha(1 - \frac{b\mu(U^c)}{\theta\alpha\mu(U)})/(1+\lambda)$ since $\theta = b/\alpha$.

If U does not contain the full level set $\{V \leqslant b/\alpha\}$ but only C', the only difference is that we cannot divide by θ, hence the result. $\square$

Remark 3.7. The conditions on U are not really difficult to check in practice. We have included the first situation because it covers cases where a bounded Lyapunov function exists, hence we cannot assume in general that U contains some level set. The second case is the usual one on Euclidean spaces when V goes to infinity at infinity, so that we may always choose U as a regular neighborhood of a level set of V.

One may think that the constant C_{LP} we have just obtained is a disaster. In particular, contrary to the Meyn–Tweedie approach, the exponential rate given by C_{LP} does not only depend on α but also on b, C, V. But recall that in Meyn–Tweedie approach the non-explicit constant in front of the geometric rate depends on all these quantities (while we here have an explicit $\int f^2 W\, d\mu$). In addition to the stronger type convergence ($\mathbb{L}^2$ type), one advantage of Theorem 3.6 is perhaps to furnish explicit (though disastrous) constants.

3.2. A general sufficient condition for a Poincaré inequality

As we previously said, there are some situations for which a tractable criterion for Poincaré's inequality is known.

The most studied case is of course the Euclidean space equipped with an absolutely continuous measure $\mu(dx) = e^{-2F}dx$ and the usual $\Gamma(f) = |\nabla f|^2$. Dimension one is the only one for which exists a general necessary and sufficient condition (Muckenhoupt criterion, see [1, Theorem 6.2.2]). A more tractable sufficient condition can be deduced (see [1, Theorem 6.4.3]) and can be extended to all dimensions using some isometric correspondence between Fokker–Planck and Schrödinger equations, namely $|\nabla F|^2(x) - \Delta F(x) \geqslant b > 0$ for all $|x|$ large enough (for a detailed discussion of the spectral theory of these operators see [17], in particular Proposition 3.1). Actually this condition can be extended to $\mu = e^{-2F}\nu$ if ν satisfies some log-Sobolev inequality, see [15] (as explained in [7, Proposition 4.4]).

We shall see now that these conditions actually are of Lyapunov type, hence can be extended to a very general setting.

Lemma 3.8. *Let F be a nice enough function. Then if $V = e^{aF}$,*

$$LV - \Gamma(F, V) = aV\left(LF + \left(\frac{a}{2} - 1\right)\Gamma(F)\right).$$

The proof is immediate using (1.5), (1.6) and (1.7). We may thus deduce

Theorem 3.9. *Let ν be a (σ-finite positive measure) and L be ν symmetric. Let $F \in D(L)$ be non-negative and such that $\mu = (1/Z_F)e^{-2F}\nu$ is a probability measure for some normalizing constant Z_F. For $0 < a < 2$ define*

$$H_a = LF + \left(\frac{a}{2} - 1\right)\Gamma(F)$$

and for $\alpha > 0$, $C(a, \alpha) = \{H_a \geqslant -(\alpha/a)\}$.

Assume that for some a and some α, H_a is bounded above on $C(a,\alpha)$.

Assume in addition that for $\varepsilon > 0$ small enough one can find a large subset $U \supseteq C(a,\alpha)$ with $\mu(U) \geqslant 1-\varepsilon$ such that F is bounded on U, and μ satisfies the local Poincaré inequality on U.

Then μ satisfies the Poincaré inequality.

Proof. Recall that the operator $L_F f = Lf - \Gamma(F, f)$ is μ symmetric. According to Lemma 3.8, if $V = e^{aF}$, $L_F V \leqslant -\alpha V$ outside $C(a,\alpha)$. But H_a and V being bounded on $C(a,\alpha)$, one can find some b such that V is a Lyapunov function. We may thus apply Theorem 3.6 which tells us that μ satisfies a Lyapunov–Poincaré inequality. Since we are in the symmetric case, we may conclude thanks to Corollary 3.4. $\square$

We defer to Section 4 further results, applications and comments of this theorem.

3.3. Weak Lyapunov–Poincaré inequalities and weak Poincaré inequalities

We shall conclude this section by extending the two previous subsections to the more general weak framework. We start with the following extension of Theorem 3.6.

Theorem 3.10. *Assume that there exists a 2ϕ-Lyapunov function V, i.e. $LV \leqslant -2\phi(V) + b\mathbb{1}_C$ for some set C (non-necessarily petite). Recall that $\phi(u) \geqslant R > 0$.*

Assume that one can find a (large) set U such that μ satisfies a local Poincaré inequality on U.

Assume in addition that:

(1) *either U contains $C' = C \cap \{\phi(V) \leqslant b\}$ and $R\mu(U) > b\mu(U^c)$,*
(2) *or U contains $\{\phi(V) \leqslant b\}$, ϕ is increasing and $\phi(b)\mu(U) > b\mu(U^c)$.*

Then for $\lambda = (b\kappa_U - 1)_+$ and $W = V + \lambda$, μ satisfies a (W)-weak-Lyapunov–Poincaré inequality, i.e. for all f with $\int f\,d\mu = 0$ and all $s > 0$,

$$\int f^2 W\,d\mu \leqslant C_w \beta_W(s)\left(\int \big(W\Gamma(f) - f^2 LW\big)\,d\mu\right) + s\|f\|_\infty^2$$

with $\beta_W(s) = \inf\{u; \int_{V > u\phi(V)} V\,d\mu \leqslant s\}$, and where C_w is given in the two corresponding cases by:

(1) $1/C_w = (1 - \frac{b\mu(U^c)}{R\mu(U)})/(1+\lambda)$,
(2) *or* $1/C_w = (1 - \frac{b\mu(U^c)}{\phi(b)\mu(U)})/(1+\lambda)$.

Proof. Looking at the proof of Theorem 3.6 we immediately see that, if V is a 2ϕ-Lyapunov function (recall Definition 1.1), then we may replace C by $C' = C \cap \{\phi(V) \leqslant b\}$. In the first situation we obtain as in the proof of Theorem 3.6

$$\int_{C'} f^2\,d\mu \leqslant \kappa_U \int \Gamma(f)\,d\mu + \big(\mu(U^c)/R\mu(U)\big)\left(\int f^2\phi(V)\,d\mu\right),$$

so that

$$\int \big(W\Gamma(f) - f^2 LW\big)\,d\mu \geqslant \left(1 - \frac{b\mu(U^c)}{R\mu(U)}\right)\int f^2\phi(V)\,d\mu.$$

In the second case we may replace R by $\phi(b)$. It remains to note that

$$\int f^2 V\,d\mu \leqslant u \int_{V\leqslant u\phi(V)} f^2\phi(V)\,d\mu + \|f\|_\infty^2\Big(\int_{V>u\phi(V)} V\,d\mu\Big)$$

for all $u > 0$. □

Remark 3.11. It is difficult to compare in full generality the previous weak Poincaré inequality with the one obtained in Theorem 2.1. More precisely, the previous result furnishes some decay for the variance (as Theorem 2.1) but the rate explicitly depends on V (while V only appears through the constants in Theorem 2.1). We shall thus make a more accurate comparison on examples later on.

It is however worthwhile noticing that, in the first case, we do not need to impose any condition on ϕ except that ϕ is bounded below by some positive constant.

Also remark that Theorem 3.1 in [25] establishes a weak Poincaré inequality assuming that one can find an exhausting sequence of sets U_n such that μ satisfies a local Poincaré inequality on each U_n. Here we only need one set U (but large enough). Actually in the examples we have in mind the assumption in [25] is satisfied, but we shall see that we can improve upon the function β_W.

We shall now extend Theorem 3.9 to the weak context.

Corollary 3.12. *Let ν be a (σ-finite positive measure) and L be ν symmetric. Let $F \in D(L)$ be non-negative and such that $\mu = (1/Z_F)e^{-2F}\nu$ is a probability measure for some normalizing constant Z_F. We assume in addition that there exists $p < 2$ such that $\int e^{-pF}\,d\nu = c_p < +\infty$.*

Let η be a non-increasing function such that $u\eta(\log(u))$ is bounded from below by a positive constant. For $0 < a < 2$ define $H_a = LF + (\frac{a}{2} - 1)\Gamma(F)$ and $C(a) = \{H_a \geqslant -\eta(F)\}$.

Assume that for some $0 < a < 2 - p$, H_a is bounded above on $C(a)$. Assume in addition that for $\varepsilon > 0$ small enough one can find a large subset $U \supseteq C(a)$ with $\mu(U) \geqslant 1 - \varepsilon$ and such that F is bounded on U, and μ satisfies a local Poincaré inequality on U.

Then μ satisfies a weak Lyapunov–Poincaré, with $W = e^{aF} + \lambda$ (for some positive λ), inequality with

$$\beta_W(s) = \frac{2}{(a\eta(\frac{\log(c_p/s)}{2-a-p}))} \tag{3.13}$$

hence for $\int f\,d\mu = 0$, $\int (P_t^* f)^2\,d\mu \leqslant \int (P_t^* f)^2 W\,d\mu \leqslant \xi(t)\|f\|_\infty^2$ with

$$\xi(t) = 2\inf\big\{r > 0;\ -C_w\beta_W(r)\log(r) \leqslant t\big\}.$$

Finally μ satisfies a weak Poincaré inequality with

$$\beta(s) = s \inf_{u>0} \xi^{-1}\big(ue^{(1-u/s)}\big) \quad \textit{with } \xi^{-1}(a) := \inf\big\{b > 0, \xi(b) \leqslant a\big\}.$$

We easily remark that β and β_W are of the same order and change only through constants.

Proof. With our hypotheses, for $0 < a < 2$, e^{aF} is a 2ϕ-Lyapunov function for $\phi(u) = \frac{1}{2}au\eta(\log(u)/a)$. Recall that we do not need here ϕ to be increasing nor concave. We may thus apply Theorem 3.10 yielding some weak Lyapunov–Poincaré inequality for μ.

We shall describe the function β_W. Recall that

$$\begin{aligned}
\beta_W(s) &= \inf\left\{u; \int\limits_{V>u\phi(V)} V\,d\mu \leqslant s\right\} \\
&= \inf\left\{u; \int\limits_{2>au\eta(F)} e^{(a-2)F}\,d\nu \leqslant s\right\} \\
&= \inf\left\{u; \int\limits_{F>\eta^{-1}(2/au)} e^{(a-2)F}\,d\nu \leqslant s\right\}.
\end{aligned}$$

But if $2 - a = p + m$,

$$\int\limits_{F>\eta^{-1}(2/au)} e^{(a-2)F}\,d\nu \leqslant \int e^{-pF} e^{-m\eta^{-1}(2/au)}\,d\nu$$

from which we deduce that $\beta_W(s) \leqslant 2/(a\eta(\frac{1}{m}\log(c_p/s)))$.

Using Lemma 3.1 we deduce as usual (see e.g. the proof of Theorem 2.1 in [25]) that $\int (P_t^* f)^2 W\,d\mu \leqslant \xi(t)\|f\|_\infty^2$ with

$$\xi(t) = 2\inf\big\{r > 0; -C_w\beta_W(r)\log(r) \leqslant 2t\big\} \tag{3.14}$$

for $\int f\,d\mu = 0$, W and C_w being as in the previous theorem. $\square$

Remark 3.15. In view of Theorem 2.1 it is interesting to replace the $\mathbb{L}^\infty$ norm above by $\mathbb{L}^p$ norms, with $p > 2$. In the case of usual weak Poincaré inequalities it is known that we may replace the $\mathbb{L}^\infty$ norm by a $\mathbb{L}^p$ norm just changing the β into $\beta_p(s) = c\beta(c's^q)$ for some constants c and c', and $1/p + 1/q = 1$ (see e.g. [37, Theorem 29] for a more general result). But the proof in [37] (inspired by [8, Theorem 3.8]) lies on a capacity-measure characterization of these inequalities introduced in [3].

The situation here is more complicated and a direct modification of the weak-Lyapunov–Poincaré inequality seems to be difficult. However, since we are interested in the rate of convergence to the equilibrium, we may mimic the truncation argument in [8]. Namely, let f be such that $\int f\,d\mu = 0$, denote by $f_K = f \wedge K \vee -K$ and $m_K = \int f_K\,d\mu$, then if a weak-Lyapunov–Poincaré inequality holds we get for all $p > 1$,

$$\begin{aligned}
\int \big(P_t^* f\big)^2 \, d\mu &\leqslant 2\bigg(\int \big(P_t^*(f_K - m_K)\big)^2 W \, d\mu + \int \big(P_t^*(f - f_K + m_K)\big)^2 \, d\mu \bigg) \\
&\leqslant 2\bigg(\int \big(P_t^*(f_K - m_K)\big)^2 W \, d\mu + \int (f - f_K + m_K)^2 \, d\mu \bigg) \\
&\leqslant 2\xi(t) K^2 + 4 \int\limits_{|f|>K} \big(|f| - K\big)^2 \, d\mu + 4m_K^2 \\
&\leqslant 2\xi(t) K^2 + 4 \int\limits_{|f|>K} \big(|f| - K\big)^2 W \, d\mu + 4\bigg(\int\limits_{|f|>K} \big||f| - K\big| \, d\mu \bigg)^2 \\
&\leqslant 2\xi(t) K^2 + 8 \int\limits_{|f|>K} \big(|f| - K\big)^2 \, d\mu \\
&\leqslant 2\xi(t) K^2 + 8\bigg(\int |f|^{2p} \, d\mu \bigg) K^{-2p/q}.
\end{aligned}$$

Now optimizing in K furnishes

$$\int \big(P_t^* f\big)^2 \, d\mu \leqslant C\xi^{1/q}(t) \bigg(\int |f|^{2p} \, d\mu \bigg)^{1/p} \tag{3.16}$$

which is quite the result in Theorem 2.1, but with explicit constants.

Remark 3.17. It is perhaps more natural to try to obtain directly a weak Poincaré inequality starting from the existence of a ϕ-Lyapunov function as follows.

For $\int f \, d\mu = 0$, we have

$$\int f^2 \, d\mu \leqslant \int \frac{-LV}{\phi(V)} f^2 \, d\mu + \int f^2 \frac{b}{\phi(V)} \mathbb{1}_C \, d\mu.$$

We know how to manage the second term if a local Poincaré inequality holds, hence we focus on the first term in the right-hand side of the previous inequality.

Assume that L is μ-symmetric. Integrating by parts we get

$$\int \frac{-LV}{\phi(V)} f^2 \, d\mu = \int \bigg(\frac{f\Gamma(f, V)}{\phi(V)} - \frac{f^2 \phi'(V)\Gamma(V)}{2\phi^2(V)} \bigg) \, d\mu$$

but thanks to our hypotheses

$$\frac{f\Gamma(f, V)}{\phi(V)} \leqslant \frac{a}{2} \Gamma(f) + \frac{1}{2a} \frac{f^2 \Gamma(V)}{\phi^2(V)}$$

for all $a > 0$ so that

$$\int \frac{-LV}{\phi(V)} f^2 \, d\mu \leqslant \int \frac{a}{2} \Gamma(f) \, d\mu + \int \bigg(\frac{f^2 \Gamma(V)}{\phi^2(V)} \bigg) \bigg(\frac{1}{2a} - \phi'(V) \bigg) \, d\mu.$$

Unless ϕ is linear, $\liminf \phi' = 0$ at infinity, so that we get an extra term that cannot be controlled. Of course if ϕ is linear, we may choose a for this term to vanish identically, and so obtain another proof of the Poincaré inequality for μ (with more easily calculable constants).

4. Examples

Due to the local Poincaré property, the most natural framework is the Euclidean space $\mathbb{R}^d$. It will be our underlying space in all examples, but in many cases results extend to a Riemannian manifold as well.

4.1. General weighted Poincaré inequalities

Let F be a smooth enough non-negative function such that $\mu = (1/Z_F)e^{-2F}\,dx$ is a probability measure. We may also assume that $F(x) \to +\infty$ as $x \to \infty$, so that the level sets of F are compact. If

- either $\Delta F - |\nabla F|^2$ is bounded from above,
- or $\int |\nabla F|^2\,d\mu < +\infty$,

it is known than one can build a conservative (i.e. non-exploding) μ symmetric diffusion process with generator $L_F = \frac{1}{2}\Delta - \nabla F.\nabla$. We shall assume for simplicity that the first condition holds.

Assume in addition that

$$\liminf_{|x|\to+\infty} \big(|\nabla F|^2 - \Delta F\big) = \alpha > 0.$$

We may thus apply Theorem 3.9, with $L = \frac{1}{2}\Delta$, ν the Lebesgue measure (which is known to satisfy a Poincaré inequality on Euclidean balls of radius R with $C_{\mathrm{P}} = CR^2$, C being universal, and for $\Gamma(f) = |\nabla f|^2$), U a large enough ball, $a = 1$. Indeed, since ν satisfies a (true) Poincaré inequality on U, μ which is a log-bounded perturbation of ν on U also satisfies a Poincaré inequality on U, hence a local one (since the energy on U is smaller than the one on the full E). This yields

Corollary 4.1. *Let F is a C^2 non-negative function such that, $F(x) \to +\infty$ as $x \to \infty$, $\int e^{-2F}\,dx < +\infty$ and*

- *$\Delta F - |\nabla F|^2$ is bounded from above, and,*
- *$\liminf_{|x|\to+\infty}(|\nabla F|^2 - \Delta F) = \alpha > 0$.*

Then the following (weighted) Poincaré inequality holds for all f smooth enough and some C_{P},

$$\int f^2 e^{-2F}\,dx \leqslant C_{\mathrm{P}} \int |\nabla f|^2 e^{-2F}\,dx + \frac{(\int f e^{-2F}\,dx)^2}{(\int e^{-2F}\,dx)}.$$

This corollary immediately extends to uniformly elliptic operators in divergence form. The degenerate case is more intricate. Indeed, according to results by Jerison, Franchi, Lu [13,19,20]

a Poincaré inequality holds on small metric balls for more general operators of locally subelliptic type. Let us describe the framework we are interested in.

Let $X_1, \dots, X_m$ be C^∞ vector fields defined on $\mathbb{R}^d$. We shall assume for simplicity that they are bounded with all bounded derivatives. We shall make the following Hörmander type assumption.

Assumption 4.2. There exist $N \in \mathbb{N}^*$ and $c > 0$ such that for all x and all $\xi \in R^d$,

$$\sum_Y \langle Y(x), \xi\rangle^2 \geqslant c|\xi|^2,$$

where the sum is taken over all Lie brackets $Y = [X_{i_1}, [\dots X_{i_k}]]$ of length less than or equal to N.

This assumption is enough for ensuring that the natural associated sub-Riemannian metric ρ is locally equivalent to the usual one (see e.g. [13, Theorem 2.3]). According e.g. to [20, Theorem C] (a similar result was first obtained by Jerison), the Lebesgue measure $dx = \nu$ satisfies a Poincaré inequality on small metric balls $B_\rho(y, s)$ for s small enough and $\Gamma(f) = \sum_{i=1}^m |X_i f|^2$. But here we want some local Poincaré inequality on some large set.

If we replace the Euclidean space by a connected unimodular Lie group with polynomial volume growth equipped with left invariant vector fields $X_1, \dots, X_m$ generating the Lie algebra of E, then it is known that a Poincaré inequality holds for all metric balls (the result is due to Varopoulos and we refer to [27, p. 275] for explanations). But in the Euclidean case we can show that Lebesgue measure satisfies some local Poincaré inequality on Euclidean balls centered at the origin.

Indeed let $|.|$ stands for the Euclidean norm. Recall that there exist R and r such that

$$\{|x| \leqslant r\} \subset B_\rho(0, s) \subset \{|x| \leqslant R\}.$$

If $\int_{|x|\leqslant N} f\,dx = 0$, then for all a it holds:

$$\begin{aligned}
\int_{|x|\leqslant N} f^2(x)\,dx &= \int_{|x|\leqslant r} f^2(Nx/r)\left(\frac{N}{r}\right)^d dx \\
&\leqslant \int_{|x|\leqslant r} \big(f(Nx/r) - a\big)^2\left(\frac{N}{r}\right)^d dx \\
&\leqslant \int_{B_\rho(0,s)} \big(f(Nx/r) - a\big)^2\left(\frac{N}{r}\right)^d dx,
\end{aligned}$$

so that if we choose $a = (\int_{B_\rho(0,s)} f(Nx/r)\,dx)/|B_\rho(0, s)|$ (where $|U|$ denotes the Lebesgue volume of U) we may use the Poincaré inequality in the metric ball, and obtain (denoting by $g(x) = f(Nx/r)$)

$$\begin{aligned}
\int_{|x|\leqslant N} f^2(x)\,dx &\leqslant C\left(\frac{N}{r}\right)^d \int_{B_\rho(0,s)} \sum_{i=1}^m |X_i g|^2(x)\,dx \\
&\leqslant C\left(\frac{N}{r}\right)^{d+2} \int_{B_\rho(0,s)} \sum_{i=1}^m |X_i f|^2(Nx/r)\,dx \\
&\leqslant C\left(\frac{N}{r}\right)^{d+2} \int_{|x|\leqslant R} \sum_{i=1}^m |X_i f|^2(Nx/r)\,dx \\
&\leqslant C\left(\frac{N}{r}\right)^{2} \int_{|x|\leqslant (RN/r)} \sum_{i=1}^m |X_i f|^2(x)\,dx.
\end{aligned}$$

Now, since F is locally bounded, it is straightforward to show that μ satisfies a local Poincaré inequality on $\{|x|\leqslant N\}$ with $\kappa_N = C(\frac{N}{r})^2 e^{4\sup_{|x|\leqslant (RN/r)} F(x)}$.

If we define a new vector field as $X_0 = \frac{1}{2}\sum_{i=1}^m \operatorname{div} X_i . X_i$, then dx is symmetric for the generator $L = \frac{1}{2}\sum_{i=1}^m X_i^2 + X_0$ and μ is symmetric for the generator $L_F = \frac{1}{2}\sum_{i=1}^m X_i^2 + X_0 - \sum_{i=1}^m X_i F . X_i$ written in Hörmander form. Hence the following generalizes Corollary 4.1.

Corollary 4.3. *Assume that Assumption* 4.2 *is fulfilled, and let* $L = \frac{1}{2}\sum_{i=1}^m X_i^2 + X_0$ *be as above. If* F *is a* C^2 *non-negative function such that,* $F(x)\to+\infty$ *as* $x\to\infty$, $\int e^{-2F}\,dx < +\infty$ *and*

- $LF - 1/2\sum_{i=1}^m |X_i F|^2$ *is bounded from above, and,*
- $\liminf_{|x|\to+\infty}(1/2\sum_{i=1}^m |X_i F|^2 - LF) = \alpha > 0.$

Then the following (*weighted*) *Poincaré inequality holds for all* f *smooth enough and some* C_{P},

$$\int f^2 e^{-2F}\,dx \leqslant C_{\mathrm{P}} \int \sum_{i=1}^m |X_i f|^2 e^{-2F}\,dx + \frac{(\int f e^{-2F}\,dx)^2}{(\int e^{-2F}\,dx)}.$$

Remark 4.4. The choice $V = e^{aF}$ is not necessarily the best possible. Indeed one wants to get the smallest possible Lyapunov function. For example if $F(x) = |x|^2$ (i.e. the Gaussian case) one can choose $V(x) = 1 + a|x|^2$ for $a > 0$. This is related to some sufficient condition for the Gross logarithmic Sobolev inequality to hold (see [7]). In the same way, if $F(x) = |x|^p$ (at least away from 0 for F to be smooth) for some $2 > p \geqslant 1$, it is easy to see that $V(x) = \exp(a|x|^{2-p})$ is a Lyapunov function (at least for a good choice of a), and of course $2 - p < p$ when $p > 1$, so that this choice is better than e^{aF}. These laws of exponent $1 < p < 2$ are the generic examples of laws satisfying interpolating inequalities (called F-Sobolev inequalities see [4], take care that this F is not the potential). It clearly suggests that the best possible choice for the Lyapunov function is connected with the F-Sobolev inequality satisfied by μ.

4.2. General weighted weak Poincaré inequalities

In this subsection we shall compare various weak Poincaré inequalities obtained in [25], [3, Theorem 2.1 and Corollary 3.12] as well as the various rates of convergence to equilibrium. The framework is the same as in the previous subsection.

4.2.1. Sub-exponential laws

We consider here for $0 < p < 1$ the measures $\mu_p(dx) = C_p e^{-2|x|^p} dx$ where C_p is a normalizing constant and $|.|$ denotes the Euclidean norm. It is shown in [3] that if $d = 1$, μ_p satisfies a weak Poincaré inequality with $\beta_p(s) = d_p \log^{(2/p)-2}(2/s)$ this function being sharp. Note that the previous result does not extend to higher dimensions via the tensorization result [3, Theorem 3.1]. In any dimension however, [25] furnishes $\beta_p(s) = d_p \log^{(4(1-p)/p}(2/s)$. Note that for $d = 1$ the result in [3] improves on the one in [25].

These bounds furnish a sub-exponential decay

$$\int \left(P_t^* f\right)^2 d\mu \leqslant c_1 e^{-c_2 t^\delta} \|f\|_\infty^2$$

for any μ stationary semi-group, with $\delta = p/(2-p)$ if $d = 1$ and $\delta = p/(4-3p)$ for any d.

But sub-exponential laws enter the framework of Section 3.3 with $V = e^{a|x|^p}$, $\eta(u) = cu^{2(1-\frac{1}{p})}$ hence $\beta_W(s) = C \log^{(2/p)-2}(c/s)$ for some constants c and C. Note that we recover the right exponent $(2/p) - 2$ for β_W, hence the right sub-exponential decay in any dimension. Up to the constants, we also recover, thanks to Theorem 2.3 in [25], that β behaves like β_W.

Also note that in this case the rate given by Theorem 2.1 is again $\psi(t) = c_1 e^{-c_2 t^{p/(2-p)}}$.

These results extend to any F going to infinity at infinity and satisfying

$$(1 - a/2)|\nabla F|^2 - \Delta F \geqslant c F^{2(1-\frac{1}{p})}$$

at infinity, generalizing to the weak Poincaré framework similar results for super-Poincaré inequalities (see [4,5]).

4.2.2. Heavy tails laws

Let us deal now with measures $\mu_p(dx) = C_p(1 + |x|)^{-(d+p)}$ where $p > 0$, C_p is a normalizing constant, and $|.|$ denotes once again the usual Euclidean norm. The sharp result in dimension 1 has been given in [3] with $\beta_p(s) = d_p s^{-2/p}$, but cannot be extended to higher dimensions. Röckner, Wang [25] furnishes in any dimension $\beta_p(s) = cs^{-\tau}$ where $\tau = \min\{(d + p + 2)/p, (4p + 4 + 2d)/(p^2 - 4 - 2d - 2p)_+\}$. This result is not sharp in dimension one but enables to quantify the polynomial decay of the variance in any dimension.

Once again, we may use the results of Section 3.3 with $V(x) = (1 + |x|)^{a(d+p)/2}$, so that $F(x) = \frac{d+p}{2} \log(1 + |x|)$ and $\eta(u) = C(p, d)e^{-4u/(p+d)}$. Use now (3.13) to get that $\beta_W(s) = C(p', d)s^{\frac{2}{p'}}$ for any $p' < p$ (and $C(p', d) \to \infty$ as $p' \to p$). This result enables us to be nearly optimal in any dimension and thus improves on the result of [25]. Note that once again, results of [10,12] would give, via Theorem 2.1 the same result, but without explicit constants.

4.3. Drift conditions for diffusion processes

Consider a d-dimensional diffusion process

$$dX_t = \sigma(X_t)\,dB_t + \beta(X_t)\,dt. \tag{4.5}$$

We assume that the (matrix) σ has smooth and bounded entries, and is either uniformly elliptic or hypoelliptic in the sense of Assumption 4.2. We also assume the following drift condition

$$\text{there exist } M \text{ and } r > 0 \quad \text{such that for all } |x| \geqslant M, \quad \langle \beta(x), x \rangle \leqslant -r|x|. \tag{4.6}$$

We also assume that the diffusion has an unique invariant probability measure $d\mu = e^F\,dx$. This is automatically satisfied if σ is uniformly elliptic and (4.6) holds (see [10, Proposition 4.1]).

Consider a smooth function V which coincides with $e^{a|x|}$ outside the ball of radius M, $|x|$ denoting the Euclidean distance. Then on this set

$$LV(x) = a\left\langle \beta(x), \frac{x}{|x|} \right\rangle + a^2\eta(x)$$

where $\eta(x) \to 0$ as $|x| \to +\infty$. Hence, according to (4.6) for all a, V is a Lyapunov function (but C and b depend on a).

We may thus apply Theorem 3.6 (thanks to the local Poincaré property discussed in the previous subsection) and get that for any density of probability h,

$$\int \left| P_t^* h - 1 \right|^2 d\mu \leqslant e^{-\delta_a t} \int (h-1)^2 e^{a|x|}\,d\mu.$$

Indeed we know that μ satisfies a $(V + \lambda)$-Lyapunov–Poincaré inequality, hence apply Proposition 3.3 and then replace W by $1 + \lambda$ in the left-hand side, and $(V + \lambda)$ by $(1 + \lambda)V$ in the right-hand side. Hence we get an exponential convergence for initial densities in $\mathbb{L}^2(e^{a|x|}\mu)$ for some $a > 0$.

Remark that if $\sigma = Id$ and $\beta = -\nabla F$, $d\mu = e^{-2F}\,dx$ and (4.6) which reads

$$\text{there exist } M \text{ and } r > 0 \quad \text{such that for all } |x| \geqslant M, \quad \langle \nabla F(x), x \rangle \geqslant r|x|,$$

thus implies the Poincaré inequality.

We may now complete the picture in the sub-exponential case (the polynomial case being handled similarly), namely we assume

$$\text{there exist } 0 < p < 1, M \text{ and } r > 0 \quad \text{such that for all } |x| \geqslant M, \quad \langle \beta(x), x \rangle \leqslant -r|x|^{1-p}. \tag{4.7}$$

One may then show as in [10] that for sufficiently small a, $V(x) = e^{a|x|^{1-p}}$ is a ϕ-Lyapunov function with $\phi(v) = v \log(v)^{-2\frac{p}{1-p}}$ and get via the use of Theorem 3.10 as in the preceding paragraph a weak Lyapunov–Poincaré inequality with $W(x) = V(x) + \lambda$ and $\beta_W(s) = d_p \log(2/s)^{2p/(1+p)}$. It then implies that for any density of probability h,

$$\int \left| P_t^* h - 1 \right|^2 d\mu \leqslant C_{a,p} e^{-\delta_a t^{\frac{1-p}{1+p}}} \int (h-1)^2 e^{a|x|^{1-p}}\,d\mu.$$

Let us remark that for this diffusion case, the use of Lyapunov function was already present in Röckner, Wang [25, Theorems 3.2 and 3.3] to obtain weak Poincaré inequality. They however always propose as Lyapunov function the distance to the origin, combined with local approximations, which is not optimal as seen in the previous subsections. Remark however that as in [10], Röckner, Wang [25] also considers the case of Markov processes with jumps. We leave this for further research.

5. Entropy and weighted entropy

In all the previous sections we studied the behaviour of the variance or some weighted variance. The only exception is Theorem 2.2 where we obtained the rate of convergence for relative entropy. In many significant cases, for physical relevance, $\mathbb{L}^2$ bounds are too demanding, so that it is of some interest to look at less demanding bounds.

Using Lemma 3.1 the following proposition is obtained exactly as Proposition 3.3, after stating the analogue of Definition 3.2.

Definition 5.1. Let Ψ be a non-negative function such that $\Psi(1) = 0$. We shall say that μ satisfies a (W)-Lyapunov-Ψ-Sobolev inequality, if there exist $W \in D(L)$ with $W \geqslant 1$ and a constant C_Ψ such that for all nice non-negative h with $\int h\,d\mu = 1$,

$$\int \Psi(h) W\,d\mu \leqslant C_\Psi \int \left(\frac{1}{2} W \Psi''(h)\Gamma(h) - \Psi(h) LW\right) d\mu.$$

Proposition 5.2. *Let Ψ be a non-negative function such that $\Psi(1) = 0$. The following statements are equivalent*:

- *μ satisfies a (W)-Lyapunov-Ψ-Sobolev inequality,*
- *$\int \Psi(P_t^* h) W\,d\mu \leqslant e^{-(t/C_\Psi)} \int \Psi(h) W\,d\mu$ for all non-negative h with $\int h\,d\mu = 1$.*

Since the goal of this section is to deal with densities of probability h with very few moments (in particular not in $\mathbb{L}^2$), we shall not discuss the analogous weak versions of these inequalities. The interested reader will easily derive the corresponding results.

Note that for Definition 5.1 to be interesting, we do certainly have to assume that $\Psi''(u) > 0$ for all u. This is a big difference with the (homogeneous) F-Sobolev inequalities studied in [4] where F often vanishes on some neighborhood of 0.

Indeed if we want to mimic what we have done in Theorem 3.6, we have to introduce some local version of some new Ψ-Sobolev inequality, replacing the local Poincaré inequality. Instead of looking at such a complete theory, we shall focus on a typical example which will give the flavor of the results one can obtain. The first remark is, see for instance [14], that the Lebesgue measure satisfies a logarithmic Sobolev inequalities on the interval $I = [-R, R]$ with constant $8R^2/\pi^2$ which by tensorization holds also on the tensor product I^d with the same constant so that we obtain the equivalent of the local Poincaré inequality.

Now if $d\mu = e^{-2F} dx$ is a probability measure, the normalized measure $\bar{\mu} = \mu/\mu(I^d)$ also satisfies a log-Sobolev inequality on I^d as soon as F is locally bounded. But $u \mapsto u \log u$ is not everywhere non-negative so that we have to modify it.

First, since $\bar{\mu}$ also satisfies a Poincaré inequality on I^d, we may apply Lemma 17 in [5] and obtain the following G-Sobolev inequality with $G(u) = (\log u - \log 4)_+$ and some universal C (all universal constants will be denoted by C in the sequel)

$$\int_{I^d} f^2 G\left(\frac{f^2}{\int_{I^d} f^2\, d\bar{\mu}}\right) d\bar{\mu} \leqslant C\left(1 + R^2\right) \int_{I^d} |\nabla f|^2\, d\bar{\mu}. \tag{5.3}$$

Now consider Ψ defined on $\mathbb{R}^+$ by

$$\Psi(u) = (u-1)^2 \mathbb{1}_{u \leqslant 2} + \big(1 + (1 - 4\log 2)(u-2) + 4(u \log u - u - 2\log 2 + 2)\big)\mathbb{1}_{u>2}, \tag{5.4}$$

so that

$$\Psi''(u) = 2\mathbb{1}_{u \leqslant 2} + \frac{4}{u}\mathbb{1}_{u>2},$$

is everywhere positive. Ψ is non-negative and $\Psi(u) = 0$ if and only if $u = 1$. It is easy to see that $u \mapsto \Psi(u)/u$ is non-decreasing on $[1, +\infty[$ and of course Ψ behaves like $4G$ at infinity. Thus combining (5.3) and Lemma 21 in [4] we obtain that for any nice g with $\int_{I^d} g^2\, d\bar{\mu} = 1$ it holds

$$\int_{I^d} \Psi\left(g^2\right)\mathbb{1}_{g^2>1}\, d\bar{\mu} \leqslant C\left(1 + R^2\right) \int_{I^d} |\nabla g|^2\, d\bar{\mu}. \tag{5.5}$$

We may thus state

Theorem 5.6. *Let $\mu = e^{-2F}\, dx$ be a probability measure on $\mathbb{R}^d$ (supposed to be L invariant) satisfying a Poincaré inequality (on the whole $\mathbb{R}^d$) with constant C_{P}. Assume that there exists a Lyapunov function V, i.e. $LV \leqslant -2\alpha V + b\mathbb{1}_C$ for some set C (not necessarily petite), such that either C or the level sets of V are compact.*

Then μ satisfies a (W)-Lyapunov-Ψ-Sobolev inequality for $W = V + \lambda$ where λ is a large enough constant and Ψ is defined in (5.4).

Remark 5.7. According to Corollary 3.4 and Theorem 3.6, if L is μ symmetric, the Poincaré inequality automatically holds here.

Proof. Since we assumed that C or the level sets of V are compact, as for the proof of Theorem 3.6 what we have to do is to control $\int_{I^d} \Psi(h)\, d\mu$ for a large enough I^d and a non-negative h such that $\int_{\mathbb{R}^d} h\, d\mu = 1$. In the sequel we write $h = f^2$ (we may first assume that $f \geqslant \varepsilon > 0$ and then go to the limit if necessary).

First, applying Poincaré inequality we get

$$\int_{I^d} \Psi(h)\mathbb{1}_{h \leqslant 2}\, d\mu = \int_{I^d} (h-1)^2 \mathbb{1}_{h \leqslant 2}\, d\mu = \int_{I^d} (h \wedge 2 - 1)^2 \mathbb{1}_{h \leqslant 2}\, d\mu$$

$$\begin{aligned}
&\leqslant \int_{\mathbb{R}^d} (h \wedge 2 - 1)^2 \, d\mu \\
&\leqslant C_{\mathrm{P}} \int_{\mathbb{R}^d} |\nabla h|^2 \mathbb{1}_{h \leqslant 2} \, d\mu + \left(\int_{\mathbb{R}^d} (h \wedge 2 - 1) \, d\mu \right)^2 \\
&\leqslant C_{\mathrm{P}} \int_{\mathbb{R}^d} |\nabla h|^2 \mathbb{1}_{h \leqslant 2} \, d\mu + \left(\int_{\mathbb{R}^d} \big((h-1)\mathbb{1}_{h<2} + \mathbb{1}_{h \geqslant 2} \big) \, d\mu \right)^2 \\
&\leqslant C_{\mathrm{P}} \int_{\mathbb{R}^d} |\nabla h|^2 \mathbb{1}_{h \leqslant 2} \, d\mu + \left(\int_{\mathbb{R}^d} \big((1-h)\mathbb{1}_{h \geqslant 2} + \mathbb{1}_{h \geqslant 2} \big) \, d\mu \right)^2 \\
&\leqslant C_{\mathrm{P}} \int_{\mathbb{R}^d} |\nabla h|^2 \mathbb{1}_{h \leqslant 2} \, d\mu + \left(\int_{\mathbb{R}^d} (2-h)\mathbb{1}_{h \geqslant 2} \, d\mu \right)^2 \\
&\leqslant C_{\mathrm{P}} \int_{\mathbb{R}^d} |\nabla h|^2 \mathbb{1}_{h \leqslant 2} \, d\mu + \left(\int_{\mathbb{R}^d} (h-2)\mathbb{1}_{h \geqslant 2} \, d\mu \right)^2 \\
&\leqslant C_{\mathrm{P}} \int_{\mathbb{R}^d} |\nabla h|^2 \mathbb{1}_{h \leqslant 2} \, d\mu + \left(\int_{\mathbb{R}^d} h\mathbb{1}_{h \geqslant 2} \, d\mu \right)^2 \\
&\leqslant C_{\mathrm{P}} \int_{\mathbb{R}^d} |\nabla h|^2 \mathbb{1}_{h \leqslant 2} \, d\mu + \int_{\mathbb{R}^d} h\mathbb{1}_{h \geqslant 2} \, d\mu,
\end{aligned}$$

since $\int_{\mathbb{R}^d} h\mathbb{1}_{h \geqslant 2} \, d\mu \leqslant 1$. Since μ satisfies a Poincaré inequality, Remark 22 in [4] shows that

$$\int f^2 \mathbb{1}_{f^2 \geqslant 2 \int f^2 \, d\mu} d\mu \leqslant C \int |\nabla f|^2 \, d\mu,$$

so that (recall that $\Psi''(u) = 2\mathbb{1}_{u \leqslant 2} + \frac{4}{u}\mathbb{1}_{u>2}$) we finally obtain for some constant C,

$$\int_{I^d} \Psi(h)\mathbb{1}_{h \leqslant 2} \, d\mu \leqslant C \int_{\mathbb{R}^d} \Psi''(h)|\nabla h|^2 \, d\mu.$$

For the other part we have to be accurate with normalization in order to use (5.5). Indeed the latter applies for normalized functions for the normalized measure on I^d. Let $m = \int_{I^d} (h \vee a) \, d\bar{\mu}$ for some $2 > a > 0$.

If $m \leqslant 1$ then

$$\Psi(h)\mathbb{1}_{h>2} = \Psi(h \vee a)\mathbb{1}_{h>2} \leqslant \Psi(h \vee a/m)\mathbb{1}_{h>2} \leqslant \Psi(h \vee a/m)\mathbb{1}_{(h \vee a/m)>2}$$

so that we may apply (5.5) with $g = (h \vee a/m)^{\frac{1}{2}}$ (we can of course replace $\mathbb{1}_{g>1}$ by $\mathbb{1}_{g>2}$). Of course $|\nabla g|^2$ is up to some constant (the normalization by m disappears) equal to $\mathbb{1}_{h>a}(|\nabla h|^2/h)$

hence up to the constants to $\mathbb{1}_{h>a}\Psi''(h)|\nabla h|^2$. Remark that we need $h > a$ for $1/h$ to be bounded (since $\Psi''(u) = 2$ when $u \leqslant 2$) at least for $h < 2$.

If $m \geqslant 1$ the situation is more delicate. But

$$m \leqslant \int_{I^d} h\,d\bar{\mu} + a \leqslant \big(1/\mu\big(I^d\big)\big) + a$$

so that if we choose R (the length of the edge of I^d) large enough we may assume that $\mu(I^d) \geqslant 3/4$, choose $a = 1/3$ so that $m \leqslant 5/3 < 2$. In other words on $\{h > 2\}$, $h/m \geqslant 6/5$. It follows that $\Psi(h) = \Psi(m\frac{h}{m}) \leqslant c\Psi(\frac{h}{m})$ on $\{h > 2\}$, for some constant c (recall the form of Ψ). Furthermore $\mathbb{1}_{h>2} \leqslant \mathbb{1}_{\frac{h}{m}>\frac{6}{5}}$ so that one more time we may apply (5.5), and conclude as in the case $m \leqslant 1$.

We have thus shown the existence of some C such that

$$\int_{I^d} \Psi(h)\mathbb{1}_{h>2}\,d\mu \leqslant C\int_{\mathbb{R}^d} \Psi''(h)|\nabla h|^2\,d\mu.$$

With the previous result the proof is completed. □

Remark 5.8. Since $\Psi(u)$ behaves like $u\log u$ at infinity, the previous result has the following consequence: if V has some exponential moment, then

$$\int P_t^*h\log P_t^*h\,d\mu \leqslant Ce^{-\eta t}\left(1\vee\int \Psi(h)\log_+\big(\Psi(h)\big)\,d\mu\right) \leqslant C'e^{-\eta t}\left(1\vee\int h\log_+^2(h)\,d\mu\right).$$

This result is (at a qualitative level) a little bit weaker than the one we obtain in this case in Theorem 2.2, since there we can replace the exponent 2 by any exponent greater than 1.

It should also be interesting to extend this kind of result to (strongly) hypoelliptic operators as in Corollary 4.3. The key would be to prove a local log-Sobolev inequality for the corresponding Γ. We strongly suspect that some inequality of this type is true, but we did not find any reference about it.

6. Fully degenerate cases, towards hypocoercivity

Proposition 3.3 and Theorem 5.6 are hypocoercive results in Villani's terminology. The former shows a coercivity property in $\mathbb{L}^2(W\mu)$ norm, which is stronger than the $\mathbb{L}^2(\mu)$ norm, while the latter can be interpreted in terms of semi-distances. We refer to [29] for a nice presentation of hypocoercivity. In studying fully degenerate cases, Villani introduces higher order functional inequalities (reminding the celebrated Γ_2 criterion for logarithmic Sobolev inequality), see [29, Eq. (11)] and more generally [30]. These higher order inequalities enable him to introduce Lie brackets of the diffusion vector fields with the drift vector field, hence are clearly related to some hypoelliptic situation of Hörmander type. A deep study of the spectral theory of hypoelliptic operators is done in [17], and we refer to the references in both [17,30] for more details and contributors. Also notice that the hypocoercivity phenomenon was first studied by Hérau and Nier (see [18]) by using pseudo-differential calculus (also see some recent work by Hérau on his Web page).

Since the existence of a Lyapunov function does not immediately rely on non degeneracy, it is natural to consider fully degenerate cases from this point of view. Note that Theorem 3.6 requires a local Poincaré inequality, hence is not adapted, while the method in Section 2 furnishes some exponential decay for the variance but controlled by some $\mathbb{L}^p$ norm.

In this section we shall recall the results in [30] for the particular example of the kinetic Fokker–Planck equation. Then we shall see that this example enters the framework of Meyn–Tweedie approach, following [35] and [10] who indicated how to build some Lyapunov function.

First we recall what the kinetic Fokker–Planck equation is. Let F be a smooth function on $\mathbb{R}^d$. We consider on $\mathbb{R}^{2d}$ the stochastic differential system (x stands for position and v for velocity)

$$\begin{aligned} dx_t &= v_t\,dt, \\ dv_t &= dB_t - v_t\,dt - \nabla F(x_t)\,dt \end{aligned} \tag{6.1}$$

associated with

$$L = \frac{1}{2}\Delta_v + v\nabla_x - \big(v + \nabla F(x)\big)\nabla_v.$$

Define

$$\mu(dx, dv) = e^{-(|v|^2 + 2F(x))}\,dx\,dv = e^{-H(x,v)}\,dx\,dv \tag{6.2}$$

which is assumed to be a bounded measure (in the sequel we shall denote again by μ the normalized (probability) measure $\mu/\mu(\mathbb{R}^{2d})$).

If F is bounded from below, it is known that (6.1) has a pathwise unique, non-explosive solution starting from any (x, v). Actually the statement in [35, Lemma 1.1] is for a weak solution since Wu is using Girsanov theory. Let us introduce the stopping time $\tau_R = \inf\{s \geqslant 0;\ |v_t| \geqslant R\}$. Since $|x_{t\wedge\tau_R}| \leqslant Rt + |x|$ pathwise uniqueness holds up to each time τ_R and the explosion time is the limit of the τ_R's as R goes to infinity. That this limit is almost surely $+\infty$ is proved by Wu [35, the top of p. 210]. Furthermore μ is in this case the unique invariant measure.

Let us make three additional remarks:

- μ is not symmetric,
- L is fully degenerate, in particular since $\Gamma f = |\nabla_v f|^2$ any function $f(x, v) = g(x)$ with $\int f\,d\mu = 0$ is such that $\Gamma f = 0$ so that the Poincaré inequality (with Γ) is not true for μ,
- the Bakry–Emery curvature of the semi-group (see [1, Definition 5.3.4]) is equal to $-\infty$.

The main results in [30] about convergence to equilibrium for this equation are collected below.

Theorem 6.1. *(See Villani [30, Theorems 29, 31, 32].)*

(1) *Define $H^1(\mu) := \{f \in \mathbb{L}^2(\mu);\ \nabla f \in \mathbb{L}^2(\mu)\}$ equipped with the semi-norm $\|f\|_{H^1(\mu)} = \|\nabla f\|_{\mathbb{L}^2(\mu)}$.*
Assume that $|\nabla^2 F| \leqslant c(1 + |\nabla F|)$ and that the marginal law $\mu_x(dx) = e^{-2F(x)}\,dx$ satisfies the classical Poincaré inequality for all nice g defined on $\mathbb{R}^d$,

$$\mathrm{Var}_{\mu_x}(g) \leqslant C \int_{\mathbb{R}^d} |\nabla g|^2(x)\mu_x(dx).$$

Then there exist C and λ positive such that for all $f \in H^1(\mu)$,

$$\left\| P_t^* f - \int f\,d\mu \right\|_{H^1(\mu)} \leqslant Ce^{-\lambda t} \|f\|_{H^1(\mu)}.$$

(2) *With the same hypotheses, there exists C such that for all $1 \geqslant \varepsilon > 0$ and all $t > \varepsilon$,*

$$\mathrm{Var}_\mu\big(P_t^* f\big) \leqslant C\varepsilon^{-3/2} e^{-\lambda(t-\varepsilon)}\,\mathrm{Var}_\mu(f).$$

(3) *Assume that $|\nabla^j F| \leqslant c_j$ for all $j \geqslant 2$ and that μ_x satisfies a (classical) log-Sobolev inequality*

$$\mathrm{Ent}_{\mu_x}\big(g^2\big) \leqslant C \int_{\mathbb{R}^d} |\nabla g|^2(x)\mu_x(dx).$$

Then for all $h \geqslant 0$ such that $\int h\,d\mu = 1$ and satisfying

$$\forall k \in \mathbb{N}, \quad \int \big(1 + |x| + |v|\big)^k h(x,v)\,d\mu < +\infty,$$

it holds for some $\lambda > 0$,

$$\int P_t^* h \log\big(P_t^* h\big)\,d\mu \leqslant C(h)e^{-\lambda t}$$

where $C(h)$ depends on the above moments.

It is worthwhile noticing that since μ is a product measure of μ_x and a Gaussian measure, μ inherits the classical Poincaré or log-Sobolev inequality as soon as μ_x satisfies one or the other. Part (2) in the previous result is simply an hypoelliptic regularization property, and some hypotheses can be slightly improved (see [30, Theorems 29–32] for the details). However, it has to be noticed that $C > 1$ (otherwise μ would satisfy a Poincaré inequality with Γ) and that the Bakry–Emery curvature has to be $-\infty$ for the same reason.

In [35], Wu gave some sufficient conditions for the existence of a Lyapunov function for this (and actually more general) model (see [35, Theorem 4.1]). We recall and extend this result below. First define

$$\Lambda_{a,b}(x,v) = aH(x,v) + b\big(\langle v, \nabla G(x)\rangle + G(x)\big), \tag{6.4}$$

where G is smooth, a and b being positive parameters.

Theorem 6.5. *Assume that F is bounded from below and that there exists some G satisfying*

(1) $\liminf_{|x|\to+\infty} \langle \nabla G(x), \nabla F(x)\rangle = 2c > 0$,

(2) $\|\nabla^2 G\|_\infty < c/16d$,
(3) *there exists* $\kappa > 0$ *such that for all* x, $|\nabla G(x)|^2 \leqslant \kappa(1 + |\langle \nabla F(x), \nabla G(x)\rangle|)$,
(4) $\Lambda_{a,b}$ *is bounded from below.*

Then for all $0 < \varepsilon$ *one can find a pair* (a, b) *such that* $\max(a, b) \leqslant \varepsilon$ *for which* $V_{a,b}(x, v) = e^{\Lambda_{a,b}(x,v) - \inf_{x,v} \Lambda_{a,b}(x,v)}$ *is a Lyapunov function.*

Hence if there exists $\eta > 0$ *such that* $\int e^{\Lambda_{\eta,\eta}(x,v)}\, d\mu < +\infty$, *for each* $p > 1$ *one can find a Lyapunov function* $V_p \in \mathbb{L}^p(\mu)$, *so that there exists* $\lambda > 0$ *such that for each* $q > 2$ *there exists* C_q *such that*

$$\operatorname{Var}_\mu\big(P_t^* f\big) \leqslant C_q e^{-(\frac{q-2}{q-1})\lambda t} \left\| f - \int f\, d\mu \right\|_q^2.$$

Proof. Elementary computation yields

$$LV_{a,b}/V_{a,b} = -2a|v|^2(1-a) + ad + 2ab\langle v, \nabla G\rangle + \frac{1}{2}b^2|\nabla G|^2 + b\big\langle \nabla^2 G v, v\big\rangle - b\langle \nabla F, \nabla G\rangle.$$

Our aim is to choose G for the right-hand side to be negative outside some compact set. A rough majorization gives

$$\begin{aligned} LV_{a,b}/V_{a,b} \leqslant{}& \big(-2a(1-a) + b\big|\nabla^2 G(x)\big| + 4ab\big)|v|^2 \\ &- b\langle \nabla G, \nabla F\rangle + \left(\frac{b^2}{2} + 4ab\right)|\nabla G|^2 + ad. \end{aligned}$$

We have thanks to (3),

$$\begin{aligned} &-b\langle \nabla G, \nabla F\rangle + \left(\frac{b^2}{2} + 4ab\right)|\nabla G|^2 + ad \\ &\quad\leqslant b\left(-1 + \kappa\left(\frac{b}{2} + 4a\right)\right)\langle \nabla G, \nabla F\rangle + \left(ad + \kappa b\left(\frac{b}{2} + 4a\right)\right), \end{aligned}$$

so that if we choose a and b small enough for $\kappa(\frac{b}{2} + 4a) \leqslant \frac{1}{2}$ the first term is less than $-cb$ for $|x|$ large enough thanks to (1). Hence if we choose $ad + \kappa b(\frac{b}{2} + 4a) < cb/2$ we get $LV_{a,b}/V_{a,b} \leqslant -cb/2$ for $|x|$ large and all v as soon as

$$-2a(1-a) + b\big|\nabla^2 G(x)\big| + 4ab \leqslant 0.$$

We may thus first choose a and b small enough for $\kappa(\frac{b}{2} + 4a) < c/4$, so that it remains to choose $a < cb/4d$.

Now if $|x| \leqslant L$, $(LV_{a,b}/V_{a,b})(x, v) \to -\infty$ as $|v| \to +\infty$ as soon as

$$-2a(1-a) + b\big|\nabla^2 G(x)\big| + 4ab < 0.$$

We may choose $b \leqslant 1/8$ and $a \leqslant 1/2$ so that we only have to check $-a/2 + b|\nabla^2 G(x)| < 0$, i.e. $a/2 > cb/16d$ thanks to (2). This is possible since our unique constraint is $a/2 < cb/8d$.

We have thus obtained the existence of a Lyapunov function for some pair (a, b) with both a and b as small as we want. This Lyapunov function thus belongs to $\mathbb{L}^p$ if a and b are small enough, according to our integrability hypothesis. It remains to apply Theorem 2.1 to conclude (all the other hypotheses in Theorem 1.2 are satisfied here, see [10,35] for the details). □

Example 6.6. Let us describe some examples.

(1) (Wu [35].) Assume the drift condition $\liminf_{|x|\to+\infty}\langle x, \nabla F(x)\rangle/|x| = 2c > 0$. Then we may choose $G(x) = |x|$ for $|x|$ large, and $|\nabla^2 G(x)| \leqslant \varepsilon$ for all x. This is the situation discussed in [35]. Notice that μ_x satisfies a classical Poincaré inequality (see e.g. Section 4) so that the hypotheses of Theorem 6.1 are satisfied.

(2) A little more general situation is for F going to infinity, satisfying

$$\liminf_{|x|\to+\infty} \left|\nabla F(x)\right|^2 = 2c > 0 \quad \text{and} \quad \left|\nabla^2 F\right| \ll |\nabla F| \quad \text{at infinity.}$$

In this case also μ_x satisfies a classical Poincaré inequality as we saw in Section 4 (if $d = 1$ the converse is true). If $|\nabla^2 F(x)| \to 0$ as $|x| \to +\infty$ we may choose a function G such that $|\nabla^2 G(x)| \leqslant \varepsilon$ for all x and $G(x) = F(x)$ for x large. This function will satisfy all (1), (2), (3). For (4) and the integrability condition to be satisfied it is enough to assume in addition that

$$\left|\nabla F(x)\right|^2/F(x) \text{ goes to } 0 \text{ at infinity.}$$

This is the case for $F(x) = |x|^p$ at infinity for $1 \leqslant p < 2$.

(3) If the latter condition is not satisfied we may take $G = F^\alpha$ for some $\alpha \leqslant 1$. But in this situation we can obtain a better Lyapunov function and study convergence in entropy.

Remark 6.7. The $\mathbb{L}^2$ convergence in Theorem 6.1 is optimal, hence we cannot expect to improve it and actually the controls we obtained in Theorem 6.5 are weaker. In addition, in the last version of his work (see [31]) Villani gives some explicit bounds for the constants involved. As we said, such estimates are not yet available in Theorem 1.2.

However, Villani's approach uses the classical Poincaré inequality in an essential way, and only gives exponential decay results. Examples for the existence of ϕ-Lyapunov functions for this kinetic model are given in [10, Section 4.3]. Indeed consider $F(x) \sim |x|^p$ for large $|x|$ with $0 < p < 1$. Attentive calculations show that one can consider smooth G with $\nabla G(x) = |x|^m$ for large $|x|$ with $1 - p < m \leqslant 1$,

$$e^{s\Lambda^\delta_{a,b}(x,v) - \inf_{x,v} s\Lambda^\delta_{a,b}(x,v)}, \quad (m+1)\delta \leqslant p,$$

as a ϕ-Lyapunov function for well chosen s, a, b, with $\phi(t) = t/\ln^{1/\delta-1} t$. Combined with Theorem 2.1 we thus get a sub-exponential decay in a situation where it is known that there is no exponential decay, thanks to an argument by Wu [35]. We refer to [10] for the polynomial decay case. We shall not go further in this direction here, but Theorems 1.2 and 2.1 thus allow to study a larger field of potentials.

As we said before we turn to the study of entropy decay. This time we shall directly use $\Lambda_{a,b} + M = V_{a,b}$ as a Lyapunov function, for M large enough. Indeed

$$LV_{a,b}(x,v) = ad - 2a|v|^2 - b\langle \nabla F(x), \nabla G(x)\rangle + b\langle \nabla^2 G(x)v, v\rangle.$$

Our aim is to find G and $\eta > 0$ such that $LV_{a,b} \leqslant -\eta V_{a,b}$ outside some compact set. We shall choose $G(x) = F^{1-\alpha}(x)$ for large x, for some $0 \leqslant \alpha < 1$, assuming that F is non-negative outside some compact set. Actually we shall assume that F goes to infinity at infinity. With all these choices

$$\Lambda_{a,b}(x,v) \geqslant a|v|^2 + 2aF(x) - b|v|\frac{|\nabla F(x)|}{F^\alpha(x)}$$

is bounded from below as soon as $|\nabla F(x)|^2/F^{1+2\alpha}(x)$ goes to 0 at infinity or if this ratio is bounded and b/a small enough.

Now if $\alpha > 0$,

$$\langle \nabla^2 G(x)v, v\rangle = (1-\alpha)F^{-\alpha}(x)\langle \nabla^2 F(x)v, v\rangle - \alpha(1-\alpha)F^{-(1+\alpha)}(x)\langle \nabla F(x), v\rangle^2,$$

so that for x large,

$$LV_{a,b}(x,v) \leqslant ad - 2a|v|^2 - b(1-\alpha)F^{-\alpha}(x)\left|\nabla F(x)\right|^2 + b(1-\alpha)F^{-\alpha}(x)\langle \nabla^2 F(x)v, v\rangle. \tag{6.8}$$

To show that $V_{a,b}$ is a Lyapunov function, using the same majorization as in the proof of the latter theorem, it is enough to show that we can find some $\eta > 0$ such that for x large,

$$\begin{aligned}&\left((2-\eta)a - 2b\eta - b(1-\alpha)\frac{|\nabla^2 F(x)|}{F^\alpha(x)}\right)|v|^2 + b\frac{|\nabla F(x)|^2}{F^\alpha(x)}\left(1 - \alpha - \frac{2\eta}{F^\alpha(x)}\right)\\ &\quad - \left(M + ad + 2a\eta F(x) + b\eta F^{1-\alpha}(x)\right) \geqslant 0.\end{aligned} \tag{6.9}$$

Note that the same result holds true for $\alpha = 0$.

The situation is now quite simple: first we shall assume that $|\nabla F(x)|^2 \geqslant \kappa F^{1+\alpha}(x)$ for large x, so that for any b we may choose η small enough for the sum of the last two terms to be positive; next we have to assume that $|\nabla^2 F(x)|/F^\alpha(x)$ is bounded, so that we may choose b small enough for the coefficient of $|v|^2$ to be positive. Of course for $|x| \leqslant L$ (6.9) has to be replaced by the correct one involving G, but G being smooth it is enough again to choose b and η small enough.

Choosing a small enough we see that $\int e^{pV_{a,b}}\,d\mu < +\infty$, so that applying Theorem 2.2 and Hölder–Orlicz inequality to bound $\int Vh\,d\mu$ we have obtained

Theorem 6.10. *Assume that* $F(x) \to +\infty$ *as* $|x| \to +\infty$ (*hence is bounded from below*) *and that there exists* $0 \leqslant \alpha < 1$ *such that the following holds*:

(1) *there exist* c *and* C *such that for* $|x|$ *large,*

$$cF^{1+\alpha}(x) \leqslant \left|\nabla F(x)\right|^2 \leqslant CF^{1+2\alpha}(x),$$

(2) $|\nabla^2 F(x)|/F^\alpha(x)$ *is bounded (for* $|x|$ *large).*

Then for all $p > 1$ *one can find a Lyapunov function* V_p *such that* $\int e^{pV}\,d\mu < +\infty$*. Hence there exists* $\lambda > 0$ *such that for any* $1 > \beta > 0$ *there exists* C_β *such that for all density of probability* h*,*

$$\int P_t^* h \log P_t^* h \, d\mu \leqslant C_\beta e^{-\beta\lambda t}\left(1 + \int h \log h \, d\mu\right)^\beta \left(\int |h-1||\log h|^{\frac{1}{1-\beta}}\, d\mu\right)^{1-\beta}.$$

Example 6.11. If $F(x) = |x|^p$ for some $p \geqslant 2$ and large $|x|$, then we may apply the previous theorem with

$$\frac{p-2}{2p} \leqslant \alpha \leqslant \frac{p-2}{p}.$$

Remark 6.12. As it is shown in [7] the condition $|\nabla F(x)|^2 \geqslant \eta F(x) + \Delta F(x)$ for large x implies a classical logarithmic Sobolev inequality for μ. Hence if $|\nabla^2 F| \leqslant C(1 + \nabla|F|)$ our hypothesis (1) in Theorem 6.10 implies a classical logarithmic Sobolev inequality, as it is asked in Theorem 6.1(3).

But case (3) in Theorem 6.1 is (very) roughly the case where $c|x|^2 \leqslant F(x) \leqslant C|x|^2$ for some positive c. Our result covers more "convex at infinity" cases.

Finally, even if we do not have explicit constants, our hypotheses on h seem to be weaker than the moment conditions in Theorem 6.1. For instance if $F(x) = |x|^2/2$ we may choose with $a > 0$

$$h(x,v) = \frac{e^{|x|^2+|v|^2}}{(1 + |x|^{d+1} + |v|^{d+1})^{a+1}}$$

for any $\beta < 1 - 2/(a(d+1))$, while this h does not fulfill the hypotheses of Theorem 6.1(3) (requires all $\beta < 1$!).

Remark 6.13. Of course, since for any density of probability h it holds $\int h \log h \, d\mu := \mathrm{Ent}_\mu(h) \leqslant \mathrm{Var}_\mu(h)$, the relative entropy is decaying at least with the same rate as the variance, hence Theorem 6.5 furnishes some decay. The study of relative entropy in [18] is based on this argument.

Remark 6.14. Remark that the generator L can be written in Hörmander's form $L = \frac{1}{2}X_1^2 + X_0$ where the vector fields $X_i(x,v)$ are given by $X_1(x,v) = \partial_v$ and $X_0(x,v) = v\partial_x - (v + \nabla F(x))\partial_v$. Hence the Lie bracket $[X_1, X_0](x,v) = \partial_x - \partial_v$ is such that X_1 and $[X_1, X_0]$ generate the tangent space at any (x,v). Furthermore $|X_1|^2 + |[X_1, X_0]|^2$ is uniformly bounded from below by a positive constant. Hence Malliavin calculus allows us to show that, for any $t > 0$, the law of (x_t, v_t) starting from any point (x,v) has a C^∞ density p_t with respect to Lebesgue measure, hence a smooth density h_t with respect to μ. Furthermore p_t satisfies some Gaussian upper bound. However we do not know how to show that $h_t \in \mathbb{L}^2(\mu)$. The latter is shown in [18], but

starting with some particular initial absolutely continuous laws. Due to the Gaussian part of μ, exponent 2 is optimal for such a result.

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A Bożejko–Picardello type inequality for finite-dimensional CAT(0) cube complexes

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Abstract

We prove a Bożejko–Picardello type inequality for finite-dimensional CAT(0) cube complexes and as a consequence we obtain that a group acting properly on a finite-dimensional CAT(0) cube complex is weakly amenable with the Cowling–Haagerup constant 1.

Keywords: C^*-algebra; Weakly amenable; CAT(0) cube complex

1. Introduction

In [1], they considered the characteristic functions χ_n of $X_n = \{(x, y) \in X \times X\colon d(x, y) = n\}$ for a tree X and proved that the norms of their Schur multiplier grow at most linearly. In the present paper, we study the geometry of CAT(0) cube complexes following [1] and prove that in a finite-dimensional CAT(0) cube complex case, the corresponding norms grow at most polynomially. This is a generalization of the result in [1] since a tree is exactly a 1-dimensional CAT(0) cube complex. As a consequence, we obtain that a group acting properly on a finite-dimensional CAT(0) cube complex is weakly amenable with the Cowling–Haagerup constant 1.

In operator algebra, it is very important to consider finite-dimensional approximation properties and among other things, amenability has been one of the central topic since the origin of operator algebra. Amenability of a group is characterized by nuclearity of its reduced group C^*-algebra, which is some kind of finite-dimensional approximation property. The first example

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of a non-nuclear C^*-algebra which has MAP appeared in [14] and it was shown in [9] that it has completely bounded approximation property, or CBAP in short. CBAP was thoroughly studied in [8,15], and the notion of weak amenability of a group is introduced which turned out to be equivalent to CBAP of its reduced group C^*-algebras. Nuclearity obviously implies CBAP, so weak amenability is a weak form of amenability in a sense. There is another important finite-dimensional approximation property called exactness. In [4], it was shown that a group acting properly and cocompactly on a finite-dimensional CAT(0) cube complex is exact. It is known that weak amenability (or, more generally, AP) for a group implies exactness [15] and hence our result extends the above result in [4]. The class of weakly amenable groups is fairly large including all amenable groups, free groups, lattices of simple Lie groups of $\mathbb{R}$-rank 1 [8,9,23] and all Coxeter groups [11,16]. On the other hand, Haagerup [23] showed that lattices of simple Lie groups of higher rank are not weakly amenable. See also [10]. It was proved in [17] that Coxeter groups act properly on (locally finite) finite-dimensional CAT(0) cube complexes and hence our result reproves the result of [11,16] (through [17]).

In [13], they also showed weak amenability for a group acting properly on a finite-dimensional CAT(0) cube complex using uniformly bounded representations.

The paper is organized as follows. In Section 2, we provide some background and notation used in the rest of the paper. In Section 3, we prove our main result, a Bożejko–Picardello type inequality for finite-dimensional CAT(0) cube complexes exploiting the fact that the 1-skeleton of a CAT(0) cube complex is a median graph.

2. Preliminaries

Throughout the paper, $\mathbb{Z}_+$, $\mathbb{N}$ denote the set of non-negative integers, positive integers respectively and $\#S$ denotes the cardinality of a set S.

2.1. Schur multiplier

Let A and B be C^*-algebras and $\varphi : A \to B$ be a linear map. We say that φ is completely bounded if $\|\varphi\|_{\mathrm{cb}} := \sup_n \|\varphi \otimes \mathrm{id}_n\| < \infty$ where id_n denotes the identity map on $n \times n$ matrix algebra and we refer to it as the completely bounded norm, or cb-norm of φ. Let X be a set and k be a function on $X \times X$. For any $T \in \mathbb{B}(\ell^2(X))$, we write it as a matrix $T = [T_{x,y}]_{x,y\in X}$ where $T_{x,y} = \langle T\delta_y, \delta_x\rangle$, δ_x denotes the Dirac function of $\{x\}$. Here we denote by $\mathbb{B}(H)$ the set of all bounded linear maps on a Hilbert space H. The Schur multiplier associated with k is a map m_k which sends $[T_{x,y}]$ to $[k(x,y)T_{x,y}]$. (This may not be an everywhere-defined map on $\mathbb{B}(\ell^2(X))$.) If this correspondence does define a map on all of $\mathbb{B}(H)$, we have that m_k is completely bounded with completely bounded norm $\leqslant C$ iff there exist families of vectors $(\xi_x)_{x\in X}$ and $(\eta_y)_{y\in X}$ in a Hilbert space H such that $k(x,y) = \langle \eta_y, \xi_x\rangle$ for every $x, y \in X$ and $\sup_{x,y\in X} \|\xi_x\|\|\eta_y\| \leqslant C$ [20, p. 110]. The completely bounded norm of m_k is denoted by $\|k\|_{\mathrm{cb}}$. If k is the characteristic function χ_F of a subset $F \subseteq X \times X$, then we abbreviate $\|\chi_F\|_{\mathrm{cb}}$ to $\|F\|_{\mathrm{cb}}$. Assume $X = \Gamma$ is a group. For a function φ on Γ, we associate it with the kernel $(s,t) \mapsto \varphi(st^{-1})$ and call the resulting Schur multiplier as the Herz–Schur multiplier. We still denote its cb-norm by $\|\varphi\|_{\mathrm{cb}}$. A function k on $X \times X$ is said to be positive definite if for any n in $\mathbb{N}$, any complex numbers $\alpha_1, \ldots, \alpha_n$ and any points $x_1, \ldots, x_n$ in X, we have $\sum_{i,j} \bar{\alpha}_i \alpha_j k(x_i, x_j) \geqslant 0$ where $\bar{\alpha}$ denotes the complex conjugate of the complex number α. Also, a function φ on a group Γ is said to be positive definite if the associated kernel is positive definite. The Schur multiplier of a positive definite function whose diagonals are 1 gives rise to a unital completely positive map on $\mathbb{B}(\ell^2(X))$ and

the cb-norm of a unital completely positive map is equal to 1. For details of completely boundedness, complete positivity and Schur multipliers, see [20].

2.2. *Weakly amenable group*

In this paper, all groups are assumed to be discrete and countable. A discrete group Γ is said to be *weakly amenable* if there exists a sequence (φ_n) of finitely supported functions on Γ and a constant C such that $\varphi_n \to 1$ pointwise and $\limsup_{n\to\infty} \|\varphi_n\|_{\mathrm{cb}} \leqslant C$. The Cowling–Haagerup constant $\Lambda_{\mathrm{cb}}(\Gamma)$ is the infimum of all such C for which such a sequence (φ_n) exists. For basics of weak amenability, see [3,8,9,15].

2.3. CAT(0) *cube complex*

A cube complex X is a metric polytopal complex in which each cell is isometric to the Euclidean cube $[-\frac{1}{2}, \frac{1}{2}]^n$ for some $n \in \mathbb{Z}_+$ (we follow the convention that $[-\frac{1}{2}, \frac{1}{2}]^0$ means a single point), and the gluing maps are isometries. We call $[-\frac{1}{2}, \frac{1}{2}]^n$ as an n-cube and the dimension of X, denoted by $\dim X$ is the supremum of such n. A cube complex X is equipped with the metric induced by the Euclidean metric on the cubes and it is called CAT(0) if the metric gives X a CAT(0) metric. If $\dim X < \infty$, then the complex carries a complete geodesic metric. (See [2].) Let us notice that a 1-dimensional CAT(0) cube complex is exactly a tree.

We focus mainly on combinatorics of CAT(0) cube complexes and we also give here the combinatorial description of CAT(0) cube complexes which is known to be equivalent to the above definition (at least in a finite-dimensional case). A cube complex is a non-empty set X with a family $\mathcal{C}$ of non-empty subsets of X called cubes satisfying that (1) $\mathcal{C}$ is a cover of X (2) for C_1 and C_2 in $\mathcal{C}$, $C_1 \cap C_2$ is also in $\mathcal{C}$ unless it is empty (3) for any C in $\mathcal{C}$, there is a bijection $\Phi : C \to \{0,1\}^n$ for some $n \in \mathbb{Z}_+$ (we call this cube C as an n-cube) preserving its faces, i.e. for any $C' \subseteq C$, $C' \in \mathcal{C}$ iff $\Phi(C')$ is a face of $\{0,1\}^n$ where $A \subseteq \{0,1\}^n$ is called a face iff A is of the form $A_1 \times \cdots \times A_n$ for $\emptyset \neq A_i \subseteq \{0,1\}$. The dimension of X is still the supremum of n for which an n-cube exists. Notice that any 1 point set is a cube from (1) and (3) which we refer to as a vertex and also we refer to a 1-cube as an edge. For a vertex x, the vertex link $lk(x)$ of x is the set of vertices which are adjacent to x and it comes equipped with the simplicial structure given as follows: $S \subseteq lk(x)$ is a simplex iff there exists a cube including $\{x\}$ and S. A simplicial complex is said to be flag if any complete subgraph of n vertices is actually the 1-skeleton of an $(n-1)$-simplex. A cube complex is called locally CAT(0) (or non-positively curved) if every vertex link is a flag complex. A (combinatorial) cube complex has a geometric realization through the natural identification of $\{0,1\}^n$ with the Euclidean cube $[0,1]^n$ in $\mathbb{R}^n$. For a cube complex with its geometric realization simply-connected, it is known [12] that it is CAT(0) in the above sense, i.e. the metric induced by the Euclidean metric is CAT(0) iff it is locally CAT(0).

A combinatorial hyperplane is an equivalence class of unoriented edges where two edges e and f are called equivalent if there exists a finite sequence of edges $e = e_1, \ldots, e_n = f$ such that e_i and e_{i+1} are opposite sides of some 2-cube in X for all $i = 1, \ldots, n-1$. If an edge e belongs to a hyperplane H, we say e intersects with H. A combinatorial hyperplane admits a natural geometric realization obtained by the first barycentric subdivision which is called a geometric hyperplane [22]. A CAT(0) cube complex considered as a graph has another natural metric called graph metric. We often refer to the associated distance function on $X \times X$ as the

combinatorial distance. In [22], it was shown that for any points $x, y \in X$ of combinatorial distance n, there are exactly n combinatorial hyperplanes which separates x and y and any geodesic path between x and y intersects with every hyperplanes separating x and y just once. In particular, the combinatorial distance of diagonal points of d-cubes is just d. Also, we will utilize [22, Theorem 4.6], so we cite it here.

Theorem 1. *(See [22, Theorem 4.6].) If x and y are two vertices of X, and α and β are two geodesic paths from x to y, then there exists a finite sequence of geodesic paths α_i from x to y with $\alpha_1 = \alpha$ and $\alpha_n = \beta$, such that α_i and α_{i+1} differ by exchanging two consecutive edges for two edges that run on the opposite side of some* 2*-cube.*

Assume X is a connected graph. We consider it as a metric space via the graph metric and often identify X with its vertex set. For any $x, y \in X$, we define the geodesic interval $[x, y]$ as the set of vertices lying on a shortest path (geodesic path) from x to y. A graph is called median if, for each triple of vertices x, y, z, the geodesic intervals $[x, y]$, $[y, z]$, $[z, x]$ have a unique common point which is called the median of x, y, z. For example, if we divide the Euclidean plane into squares, it becomes trivially a CAT(0) cube complex with the vertex set $\mathbb{Z}^2$ and the combinatorial distance is just the ℓ^1-metric. In this case, for any vertices x, y, the geodesic interval $[x, y]$ is the set of vertices on a rectangle having x, y as one of its diagonal points and then an instant's consideration shows that this graph is median. This holds more generally, that is, in [7], Chepoi showed that the 1-skeleton of a CAT(0) cube complex is a median graph [7, Theorem 6.1]. This fact becomes a crucial ingredient for our paper. See also [5,19,21].

We study the geometry of CAT(0) cube complex and prove a Bożejko–Picardello type inequality for finite-dimensional CAT(0) cube complexes, following the proof for trees in [1].

Theorem 2. *Let X be a finite-dimensional* CAT(0) *cube complex and let $X_n = \{(x, y) \in X \times X \colon d(x, y) = n\}$ for $n \in \mathbb{Z}_+$. Then the norms of Schur multipliers of the characteristic function of X_n increase polynomially, i.e. there exists a polynomial p such that $\|X_n\|_{\mathrm{cb}} \leqslant p(n)$.*

Combining Theorem 2 with the previously established fact [18] that the combinatorial distance function on X is conditionally negative definite, we obtain

Theorem 3. *A group Γ which acts cellularly and properly on a finite-dimensional* CAT(0) *cube complex X is weakly amenable with $\Lambda_{\mathrm{cb}}(\Gamma) = 1$.*

Guentner–Higson has obtained the same result [13] using uniformly bounded representations.

3. Main results

Let X be a CAT(0) cube complex. We call ω is an infinite geodesic in X if ω is an isometric map from $\mathbb{Z}_+$ into X where X is equipped with the combinatorial distance. If we fix a point $x \in X$ and add an infinite geodesic ω to X which starts at x, then the resulting cube complex $\tilde{X} = X \cup \{\omega\}$ is CAT(0) [2, p. 347] (or we can easily see this from the definition of CAT(0) cube complex). Note that the embedding of X into $\tilde{X}$ is isometric in the combinatorial distance.

Henceforth, we assume there exists an infinite geodesic ω_0 in X and fix it once and for all. We say that two infinite geodesics ω_1 and ω_2 eventually flow with if there exists $N \in \mathbb{Z}$ such that for any $n \in \mathbb{Z}_+$ with $n \geqslant |N|$, $\omega_1(n + N) = \omega_2(n)$. The following lemma is seen in [24, p. 246].

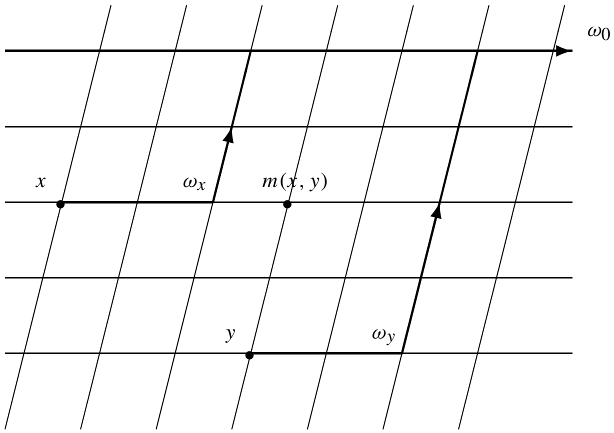

Fig. 1.

Lemma 1. *Let X be a connected graph and ω be an infinite geodesic in X. Then for any $x \in X$ there exists an infinite geodesic ω_x which starts at x and eventually flows with ω.*

Recall that X is a median graph and for any three vertices x, y, z in a median graph, there exists a unique point $m(x, y, z)$ called the median of x, y, z which is on some geodesics connecting each pair of them.

Lemma 2. *For $x, y \in X$, there exists a unique point $m(x, y)$ in X with the following property: For all but finitely many z on ω_0, $m(x, y)$ is the median of x, y, z (Fig. 1).*

Proof. Uniqueness is clear from the uniqueness of the median for three points in X. For the existence, take any infinite geodesics ω_x, ω_y which start at x, y respectively and eventually flow with ω_0, which exist by Lemma 1. Since they eventually flow with, there is a point z which is on both ω_x and ω_y. It is easy to see that if ω is an infinite geodesic and if we take two points x', y' on ω and take another geodesic connecting x' and y' and substitute it for the corresponding geodesic on ω, then the resulting infinite path is a geodesic. With this observation in hand, the existence of $m(x, y)$ follows by considering geodesics between x and z, y and z which pass through $m(x, y, z)$ and substituting it for the corresponding geodesics on ω_x, ω_y, respectively. $\square$

For any $x \in X$, we denote by $A(x, k)$ the set of points of distance k from $x \in X$ which is on some infinite geodesic ω which starts at x and eventually flows with ω_0 (Fig. 2), i.e.

$$A(x,k) = \big\{y \in X\colon \text{ there exists an infinite geodesic } \omega \\ \text{which starts at } x, \omega(k) = y, \text{ and eventually flows with } \omega\big\}.$$

By Lemma 1, these sets are non-empty for all $x \in X$ and for all $k \in \mathbb{Z}_+$.

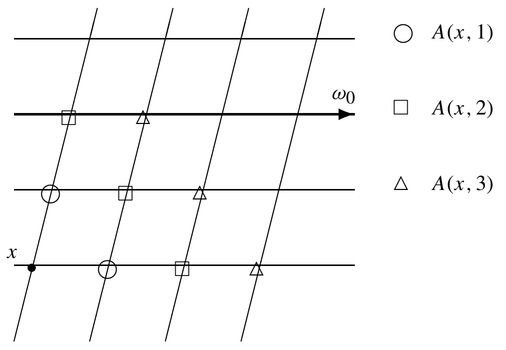

Fig. 2.

Lemma 3. *For $x_1, x_2 \in X$, we write $y = m(x_1, x_2)$, $\ell_1 = d(x_1, y)$, $\ell_2 = d(x_2, y)$. Then for $k_1, k_2 \in \mathbb{Z}_+$, $A(x_1, k_1) \cap A(x_2, k_2)$ is not empty iff $k_1 = \ell_1 + m$ and $k_2 = \ell_2 + m$ for some $m \in \mathbb{Z}_+$, and in this case, we have $A(x_1, k_1) \cap A(x_2, k_2) = A(y, m)$.*

Proof. Assume $A(x_1, k_1) \cap A(x_2, k_2)$ is not empty and take z in $A(x_1, k_1) \cap A(x_2, k_2)$. Then there exist infinite geodesics ω_{x_1}, ω_{x_2} which start at x_1, x_2 respectively and eventually flow with ω_0 and pass through z. Consider the median $m(x_1, x_2, z)$ and take geodesics connecting x_1 to z and x_2 to z which pass through $m(x_1, x_2, z)$, and substitute them for the corresponding geodesics on ω_{x_1}, ω_{x_2}, then we obtain two infinite geodesics $\tilde{\omega}_{x_1}$, $\tilde{\omega}_{x_2}$ which start at x_1, x_2 respectively, eventually flow with ω_0 and pass through $m(x_1, x_2, z)$ and z. By the uniqueness, $m(x_1, x_2, z)$ coincides with y. If we define m to be $d(y, z)$, then $k_1 = \ell_1 + m$ and $k_2 = \ell_2 + m$ and z is in $A(y, m)$. Conversely, if $k_1 = \ell_1 + m$, $k_2 = \ell_2 + m$ for some $m \in \mathbb{Z}_+$, then we take two geodesics ω_{x_1}, ω_{x_2} which start at x_1, x_2 respectively, pass through y and eventually flow with ω_0. We can assume that ω_{x_1} and ω_{x_2} coincide after passing y by arguing as above. Then $\omega_{x_1}(k_1) = \omega_{x_2}(k_2)$ is in $A(x_1, k_1) \cap A(x_2, k_2)$ and it is not empty. Hence we have proved that the first statement and $A(x_1, k_1) \cap A(x_2, k_2) \subseteq A(y, m)$. Assume z is in $A(y, m)$ and take any geodesic ω which starts at y and eventually flows with ω_0 and $\omega(m) = z$. If we take any geodesic ω_{x_1} which starts at x_1 and passes through y and eventually flows with ω_0, there exists a point w which is on ω, ω_{x_1} with $d(y, w) \geqslant m$ since ω and ω_{x_1} eventually flow with ω_0. Then substituting the geodesic connecting y to w passing through z for the corresponding geodesic on ω_{x_1}, we obtain an infinite geodesic $\tilde{\omega}_{x_1}$ which starts at x_1 and passes through y, z and eventually flows with ω_0 and hence z is in $A(x_1, k_1)$. Similarly, we can show z is in $A(x_2, k_2)$ and we complete the proof. □

We consider the following polytopal structure. Its polytopes are obtained by sections of cubes, which are perpendicular to some diagonal lines of cubes: 0-polytopes are just vertices of X and for $d \geqslant 1$, $P \subseteq X$ is a d-polytope if there exists a $(d+1)$-cube C, $z \in C$ and l, $1 \leqslant l \leqslant d$, such that $P =$ the set of vertices of all lth points on geodesics connecting z to $d_C(z)$ where we denote by $d_C(z)$ the point diagonal to z with respect to C. We make $A(x, k)$ into polytopal complexes by the above definition and denote this polytopal complex by $\mathcal{A}(x, k)$ (whose underlying vertex set is $A(x, k)$). Though we could give $\mathcal{A}(x, k)$ a topological structure and argue along that line, we treat it in a purely combinatorial way.

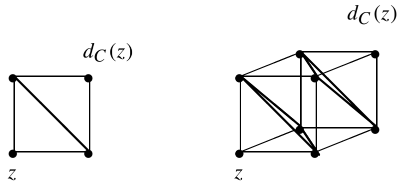

Fig. 3. 1- and 2-dimensional polytopes.

For example, a 1-polytope is just a line segment corresponding to a diagonal line of some 2-cube, a 2-polytope is an equilateral triangle and for $d \geqslant 3$, there are several shapes of d-polytopes (Fig. 3). For any d-polytope P, we write $d = \dim P$.

Lemma 4. *For any d-polytope $P \in \mathcal{A}(x,k)$, the $(d+1)$-cube C is unique in the above definition. Moreover, there exist infinite geodesics ω_p, $p \in P$, which start at x, pass through z, p, $d_C(z)$ (not necessarily in this order) and eventually flow with ω_0.*

Proof. We first prove this for 1- and 2-polytopes. Assume $x_1, x_2 \in A(x,k)$ and a 2-cube C on which x_1, x_2 form a 1-polytope is given. Then the median of $\{x, x_1, x_2\}$, $m(x, x_1, x_2)$ is one vertex of C and another is $m(x_1, x_2)$. With the observation used in Lemma 2 in hand, the statement holds in this case. Assume $P = \{x_1, x_2, x_3\}$ is a 2-polytope. Since P is contained in a 3-cube, the 2-cubes formed by $\{x_1, x_2, m(x, x_1, x_2), m(x_1, x_2)\}$ and $\{x_1, x_3, m(x, x_1, x_3), m(x_1, x_3)\}$ intersects by an edge and hence either $m(x, x_1, x_2) = m(x, x_1, x_3)$ or $m(x_1, x_2) = m(x_1, x_3)$. Similarly, the 2-cube formed by $\{x_2, x_3, m(x, x_2, x_3), m(x_2, x_3)\}$ has common edges with these 2-cubes and hence there is a vertex which is in common for all these 2-cubes, and three points, either $\{m(x, x_1, x_2), m(x, x_1, x_3), m(x, x_2, x_3)\}$ or $\{m(x_1, x_2), m(x_1, x_3), m(x_2, x_3)\}$, each of which is in each 2-cubes form another 2-polytope in either $A(x, k-1)$ or $A(x, k+1)$. Continuing the above process with this new 2-polytope, we obtain the desired property. The general cases are essentially the same. We prove by induction in the dimension of cubes. Assume P is a d-polytope and a cube C on which P forms a d-polytope is given. Fix two points y, z in P and consider two points $m(x, y, z)$ and $m(y, z)$. We identify C with $\{0,1\}^{d+1}$ and we may assume $y = (1, \dots, 1, 0, \dots, 0)$ (the first l coordinates are 1), $m(x, y, z) = (1, \dots, 1, 0, \dots, 0)$ (the first $l-1$ coordinate are 1) and $m(y, z) = (1, \dots, 1, 0, \dots, 0)$ (the first $l+1$ coordinates are 1) and P is the polytope formed by the points whose coordinates have 1 l-times with $l < d$. Considering $(l+1)$-cube formed by the first $l+1$ coordinates, we obtain an infinite geodesic ω which starts at x and passes through $(0, \dots, 0)$, y and eventually flows with ω_0 by inductive hypothesis and hence we have that $(0, \dots, 0)$ is in $A(x, k-l)$. By definition, there exist infinite geodesics ω_w for any $w \in P$ which start at x and pass through w and eventually flow with ω_0. Taking any geodesic between $(0, \dots, 0)$ and w (which is necessarily in C) and tying it with a geodesic between x and $(0, \dots, 0)$, we obtain a geodesic between x and w passing through $(0, \dots, 0)$ and hence we may assume ω_w pass through $(0, \dots, 0)$ by substituting the corresponding geodesics if necessary. Then, considering the medians $m(w, w')$ for $w, w' \in P$, it is easy to see that the set of $(l+1)$th point of geodesics from $(0, \dots, 0)$ to $(1, \dots, 1)$ forms a d-polytope in $A(x, k+1)$ and continuing these process, we obtain an infinite geodesic which starts at x, passes through $(0, \dots, 0)$, $(1, \dots, 1)$ and eventually flows with ω_0 and this implies the statement except for uniqueness.

Uniqueness follows from the construction. For example, the $(l+1)$th points are obtained by the medians $m(w, w')$ and the $(l-1)$th points are obtained by the medians $m(x, w, w')$, $w, w' \in P$, and the $(l+2)$th points are obtained by the medians of two points in the $(l+1)$th points and so on. □

Assume $A(x,1) = \{x_1, \ldots, x_n\}$. We claim that for any i, j $(i \neq j)$ there are 2-cubes $C_{\{i,j\}}$ which is formed by x, x_i, x_j, $m(x_i, x_j)$. To see this, first we see that the distance between x_i and x_j is 2. Indeed, if we take a geodesic α between them and form a loop by tying α and edges $\{x, x_i\}$, $\{x, x_j\}$ together, we can conclude that the length of α is 2 since the length of any loop is even number [6, Lemma 4.5]. Hence $m(x_i, x_j)$ is adjacent to x_i and x_j. Considering two geodesics between x_i and x_j formed by (x_i, x, x_j) and $(x_i, m(x_i, x_j), x_j)$, and using the cited theorem [22, Theorem 4.6], we conclude the existence of 2-cubes. Hence by the link condition in the definition of CAT(0) cube complex, there is an n-cube C which includes x and $A(x,1)$. Assume $k \geqslant 1$ and in this case, we have $A(x,k) = \bigcup_{i=1}^{n} A(x_i, k-1)$.

Proposition 1. *For non-empty $I \subseteq \{1, \ldots, n\}$, we have the following*:

$$\bigcap_{i \in I} A(x_i, k-1) = \begin{cases} \emptyset & (\text{if } k \leqslant \#I - 1), \\ A(d_I(x), k - \#I) & (\text{if } k \geqslant \#I), \end{cases}$$

where $d_I(x)$ is the point diagonal to x with respect to the $\#I$-cube spanned by $\{x_i\}_{i \in I}$.

Proof. We prove by induction in $\#I$. The case for $\#I = 1$ is trivial and we assume $\#I \geqslant 2$ and the statement holds for $J \subseteq \{1, \ldots, n\}$ with $\#J < \#I$. First we treat the case for $k \leqslant \#I - 1$. Take $J \subseteq I$ with $\#J = \#I - 1$. If $k < \#I - 1$, then $\bigcap_{i \in I} A(x_i, k-1) \subseteq \bigcap_{i \in J} A(x_i, k-1) = \emptyset$ by the assumption. If $k = \#I - 1$ and we write $J \cup \{j\} = I$, then $\bigcap_{i \in J} A(x_i, k-1) = A(d_J(x), k - \#J)$ by the assumption and noting that $d(x_j, d_J(x)) = \#I$, we obtain $\bigcap_{i \in I} A(x_i, k-1) = \bigcap_{i \in J} A(x_i, k-1) \cap A(x_j, k-1) = \emptyset$ by Lemma 3. Next we treat the case for $k \geqslant \#I$. Again, if we write I as $J \cup \{j\}$ and by the assumption we have $\bigcap_{i \in J} A(x_i, k-1) = A(d_J(x), k - \#J)$. Noting that $m(x_j, d_J(x)) = d_I(x)$ by Lemma 4, we obtain $\bigcap_{i \in I} A(x_i, k-1) = \bigcap_{i \in J} A(x_i, k-1) \cap A(x_j, k-1) = A(d_I(x), k - \#I)$ by Lemma 3. □

Let $\mathcal{I}(x,k)$ be the set of non-empty subsets I of $\{1, \ldots, n\}$ such that $\bigcap_{i \in I} A(x_i, k-1) = \emptyset$ and let $\mathcal{P}(x,k)$ be the set of polytopes P in $\mathcal{A}(x,k)$ which are not in the union $\bigcup_{i=1}^{n} \mathcal{A}(x_i, k-1)$. By using the above proposition, it is not difficult to check that there is a bijection between $\mathcal{I}(x,k)$ and $\mathcal{P}(x,k)$ given by $I \mapsto P_I :=$ the $(\#I - 1)$-polytope which is on the cube formed by $\{x_i\}_{i \in I}$ and $P \mapsto I_P :=$ the set of indexes which are involved in the cube which P is on.

We denote the Euler characteristic of $\mathcal{A}(x,k)$ by $\chi(\mathcal{A}(x,k))$, which is defined by $\chi(\mathcal{A}(x,k)) = \sum_{d=0}^{N-1} (-1)^d \#\mathcal{A}(x,k)^{(d)}$, where $\mathcal{A}(x,k)^{(d)}$ means the set of d-polytopes in $\mathcal{A}(x,k)$ and N is the dimension of X. For any subset (not necessarily subcomplex) $\mathcal{S} \subseteq \mathcal{A}(x,k)$, we denote by $\chi(\mathcal{S})$ the number $\sum_{d=0}^{N-1} \#\mathcal{S}^{(d)}$ where $S^{(d)}$ denotes the set of d-polytopes contained in $\mathcal{S}$. By the definition, we have $\chi(\mathcal{A}(x,k)) = \chi(\mathcal{S}) + \chi(\mathcal{A}(x,k) \setminus \mathcal{S})$.

Proposition 2. *The Euler characteristic of $\mathcal{A}(x,k)$ is 1 for all $x \in X$ and $k \in \mathbb{Z}_+$.*

Proof. We proceed by induction. If $k = 0$, then $\mathcal{A}(x,0)$ consists of one point and hence the statement holds. Assume that $k \geqslant 1$ and $A(x,1) = \{x_1, \ldots, x_n\}$. Recall that $A(x,k) =$

$\bigcup_{i=1}^{n} A(x_i, k-1)$. If this is a union not only as a vertex set but also as a complex, i.e. if every d-polytope which is in $\mathcal{A}(x,k)$ is also in $\mathcal{A}(x_i, k-1)$ for some i, $i = 1, \dots, n$, then the statement holds as follows: By using the inclusion–exclusion principle, we see that

$$\chi\big(\mathcal{A}(x,k)\big) = \sum_{\emptyset \neq I \subseteq \{1,\dots,n\}} (-1)^{\#I-1} \chi\Big(\bigcap_{i \in I} \mathcal{A}(x_i, k-1)\Big)$$

holds. Then using Proposition 1 and the observation preceding this proposition, we have that the terms $\chi(\bigcap_{i\in I} \mathcal{A}(x_i, k-1))$ are all 1 by the inductive hypothesis and consequently we obtain

$$\chi\big(\mathcal{A}(x,k)\big) = \sum_{d=1}^{n} (-1)^{d-1} \binom{n}{d} = 1$$

by the binomial theorem.

In general, we have the following equations:

$$\begin{aligned}\chi\big(\mathcal{A}(x,k)\big) &= \chi\Bigg(\bigcup_{i=1}^{n} \mathcal{A}(x_i, k-1)\Bigg) + \chi\big(\mathcal{P}(x,k)\big) \\ &= \chi\Bigg(\bigcup_{i=1}^{n} \mathcal{A}(x_i, k-1)\Bigg) + \sum_{P \in \mathcal{P}(x,k)} (-1)^{\dim P}.\end{aligned}$$

On the other hand,

$$\chi\Bigg(\bigcup_{i=1}^{n} \mathcal{A}(x_i, k-1)\Bigg) = \sum_{\emptyset \neq I \subseteq \{1,\dots,n\}} (-1)^{\#I-1} \chi\Big(\bigcap_{i \in I} \mathcal{A}(x_i, k-1)\Big)$$

holds by the inclusion–exclusion principle and thus we have

$$\chi\Bigg(\bigcup_{i=1}^{n} \mathcal{A}(x_i, k-1)\Bigg) = 1 - \sum_{I \in \mathcal{I}(x,k)} (-1)^{\#I-1}$$

by the binomial theorem, the inductive hypothesis and the following paragraph of Proposition 1 since $\chi(\emptyset) = 0$. Since there is a natural one-to-one correspondence between $\mathcal{I}(x,k)$ and $\mathcal{P}(x,k)$ as given at the following paragraph of Proposition 1, we have

$$\sum_{P \in \mathcal{P}(x,k)} (-1)^{\dim P} = \sum_{I \in \mathcal{I}(x,k)} (-1)^{\#I-1},$$

and the statement holds also in this case. $\square$

Here we evaluate the number of vertices in $A(x,k)$. In [6], they showed that it is bounded above by polynomial in k. We prove a similar result and calculate the exact bound. Intuitively, the maximal number is attained if the cubes of the maximal dimension are stuffed and in this

case an easy calculation shows the number is equal to $\binom{k+N-1}{N-1}$ where N is the dimension of X. Indeed, this is the case.

Lemma 5. *There exists a polynomial q such that for any $x \in X$ and for any $k \in \mathbb{Z}_+$, $\#A(x,k) \leqslant q(k)$. More precisely, we can take q to be $q_N(k) = \binom{k+N-1}{N-1}$ where N is the dimension of X.*

Proof. We prove by double induction in the dimension of the cubes which are involved and in k. The cases that $k=0$ with arbitrary dimension, and $\dim X = 1$ with arbitrary k are trivial. Assume $A(x,1) = \{x_1,\dots,x_n\}$ and write $A(x,k)$ as $A(x_1,k-1) \cup (A(x,k) \setminus A(x_1,k-1))$. We consider the hyperplane H separating x and x_1. Then H separates $A(x_1,k-1)$ from $A(x,k) \setminus A(x_1,k-1)$. To see this, take any $y \in A(x_1,k-1)$ and assume H does not separate x from y. Then there are k hyperplanes which separate x from y. Note that these hyperplanes are different from H since it does not separate x from y. Moreover, note that these hyperplanes separating x from y also separate x_1 from y since x and x_1 are separated only by H and so there exist $k+1$ hyperplanes separating x_1 and y, a contradiction. Conversely, take $y \in A(x,k)$ and assume H does not separate x_1 from y. Then there are just $k-1$ hyperplanes separating x_1 from y since H does not separate them, i.e. $d(x_1,y) = k-1$. It is easy to see that there exists an infinite geodesic which starts at x and passes through x_1, y and eventually flows with ω_0, and hence in particular, y is in $A(x_1,k-1)$.

We claim by induction in k that for all y in $A(x,k) \setminus A(x_1,k-1)$ there exists z in $A(x_1,k)$ such that y and z are adjacent by an edge and this edge intersects with H. To prove this, take any y' in $A(x,k-1)$ which is adjacent to y. Then it is necessarily contained in $A(x,k-1) \setminus A(x_1,k-2)$ for if y' was in $A(x_1,k-2)$, then the hyperplane separating y from y' must be H, but H does not separate y from x, there must be $k+1$ hyperplanes which separates x from y', a contradiction. Hence, by induction, there exists z' in $A(x_1,k-1)$ such that y' and z' are adjacent by an edge and this edge intersects with H. If we define z to be $m(y,z')$ which is in $A(x,k+1)$, it is easily seen that y', z', y, z form a 2-cube by arguing as in the preceding paragraph of Proposition 1. Note that the edges connecting y' to z' and y to z are hyperplane equivalent and so the edge connecting y to z intersects with H and hence we have proved the claim. If we consider $A(x,k) \setminus A(x_1,k-1)$ only, then the dimension of cubes that are involved is no more than $N-1$. Hence, by induction, we obtain

$$\begin{aligned}\#A(x,k) &= \#A(x_1,k-1) + \#\bigl(A(x,k) \setminus A(x_1,k-1)\bigr)\\ &\leqslant q_N(k-1) + q_{N-1}(k)\\ &= q_N(k)\end{aligned}$$

and we are done. □

Now, we can prove Theorem 2 using the above results and the construction used in [1]. For the reader's convenience, we sketch the proof of Theorem 2 here for the case of a tree seen in [1]. For any $x \in X$, there exists a unique geodesic ω_x which starts at x and eventually flows with ω_0. We define maps f_k from X into $\ell^2(X)$ by $f_k(x) = \delta_{\omega_x(k)}$. Then it turns out that $\theta_n(x,y) := \sum_{k=0}^{n} \langle f_k(x), f_{n-k}(y)\rangle = \sum_{k=0}^{\lfloor n/2\rfloor} \chi_{n-2k}(x,y)$ for any $x, y \in X$. Since $\|f_k(x)\| = 1$ for all $x \in X$, we have $\|\theta_n\|_{\mathrm{cb}} \leqslant n+1$ and hence $\|\chi_n\|_{\mathrm{cb}} \leqslant 2n$ since $\chi_n = \theta_n - \theta_{n-2}$. The variant for CAT(0) cube complexes is more complicated, but we have prepared enough to prove it.

Proof of Theorem 2. Let χ_n be the characteristic function of X_n. To evaluate the cb-norm of m_{χ_n}, we consider the following Hilbert spaces. For any l, we define $\mathcal{X}^{(l)}$ to be the set of all l-polytopes in X and consider ℓ^2-spaces of them. Then we define maps $f_k^{(l)}$ from X into $\bigoplus_{d=0}^{N-1} \ell^2(\mathcal{X}^{(d)})$ as follows

$$f_k^{(l)}(x) = \sum_{K \in \mathcal{A}(x,k)^{(l)}} \delta_K \in \ell^2\bigl(\mathcal{X}^{(l)}\bigr) \subseteq \bigoplus_{d=0}^{N-1} \ell^2\bigl(\mathcal{X}^{(d)}\bigr)$$

where δ_K denotes the Dirac function of K, N is the dimension of X and $\ell^2(\mathcal{X}^{(l)})$ sits in $\bigoplus_{d=0}^{N-1} \ell^2(\mathcal{X}^{(d)})$ naturally. We define functions θ_n on $X \times X$ by

$$\theta_n(x, y) = \sum_{k=0}^{n} \sum_{l=0}^{N-1} (-1)^l \bigl\langle f_k^{(l)}(x), f_{n-k}^{(l)}(y) \bigr\rangle$$

for any $x, y \in X$. Fix $x, y \in X$. If $d(x, y) \neq n - 2k$ for any k, $k \in \mathbb{Z}_+$, we have $\theta_n(x, y) = 0$ by Lemma 3 and if $d(x, y) = n - 2k$ for some k, $k \in \mathbb{Z}_+$, then $\theta_n(x, y) =$ the Euler characteristic of $\mathcal{A}(m(x, y), \frac{n-d(x,y)}{2})$ by Lemma 3 which equals to 1 by Proposition 2. Hence we conclude that $\theta_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \chi_{n-2k}$. Since $\chi_n = \theta_n - \theta_{n-2}$, to prove that cb-norms of m_{χ_n} increase polynomially, it suffices to show that those of m_{θ_n} do. Note that the square of the norm of $f_k^{(l)}$ is equal to $\#\mathcal{A}(x, k)^{(l)}$. Recall that there are several shapes of l-polytopes and they consist of $\binom{l+1}{1}, \ldots, \binom{l+1}{l/2}$ vertices for l even, and $\binom{l+1}{1}, \ldots, \binom{l+1}{(l+1)/2}$ vertices for l odd respectively and hence $\#\mathcal{A}(x, k)^{(l)}$ is bounded above by $\sum_{m=1}^{\lfloor l/2 \rfloor + 1} \binom{\#A(x,k)}{\binom{l+1}{m}}$ (here we follow the convention that $\binom{n}{k} = 0$ if $n < k$). We define $r(n)$ to be $\sum_{l=0}^{N-1} \sum_{m=1}^{\lfloor l/2 \rfloor + 1} \binom{n}{\binom{l+1}{m}}$. Since $\binom{n}{\binom{l+1}{m}}$ is an increasing function in n for fixed l, m, $\|f_k^{(l)}(x)\|^2 \leqslant r(q(k))$ holds for all $0 \leqslant l \leqslant N-1$, $k \in \mathbb{Z}_+$ and $x \in X$ where $q(k)$ is a polynomial in Lemma 5. Hence the completely bounded norm of θ_n is bounded above by $N \sum_{k=0}^{n} \sqrt{r(q(k))}\sqrt{r(q(n-k))}$ which is bounded by $N \sum_{k=0}^{n} r(q(k)) r(q(n-k))$ since $\sqrt{t} \leqslant t$ for $t \geqslant 1$. $\square$

We know from [18] that the combinatorial distance d on X gives rise to a conditionally negative definite function on $X \times X$ and also that, by Schoenberg's theorem, $(x, y) \mapsto \exp(-\frac{1}{n} d(x, y))$ is a positive definite function on $X \times X$ for all $n \in \mathbb{N}$. If Γ acts on X cellularly (and hence isometrically), then we obtain unital positive definite functions on Γ defined by $s \mapsto \exp(-\frac{1}{n} d(s x_0, x_0))$ where $x_0 \in X$ is a fixed point in X. If, moreover, Γ-action is proper, then these functions are in $c_0(\Gamma)$, where $c_0(\Gamma)$ denotes the set of functions on Γ which vanish at infinity. Recall that a unital positive definite function gives rise to a unital completely positive Herz–Schur multiplier and hence its cb-norm is equal to 1. To obtain finitely supported functions with controlled cb-norms, we truncate the functions by using Theorem 2. This kind of idea has first appeared in [14] and been used by many authors.

Proof of Theorem 3. We fix $x_0 \in X$ as above and consider the functions $\psi_n(s) := \exp(-\frac{1}{n} d(s x_0, x_0))$ and $\chi_n(s) := \chi_n(s x_0, x_0)$. Then $\psi_n(s)\chi_k(s) = \exp(-\frac{k}{n})\chi_k(s)$ holds for all $s \in \Gamma$. We already know that there exists a polynomial p such that $\|X_n\|_{\mathrm{cb}} \leqslant p(n)$. For fixed

$n \in \mathbb{N}$, $\sum_{k=0}^{\infty} \exp(-\frac{k}{n}) p(k)$ converges and hence $\|\sum_{k=K}^{\infty} \psi_n \chi_k\|_{\mathrm{cb}}$ tends to 0, or equivalently, $\|\sum_{k=0}^{K} \psi_n \chi_k\|_{\mathrm{cb}}$ tends to $\|\psi_n\|_{\mathrm{cb}} = 1$ as K tends to ∞. Assume $s_1, \dots, s_n$ and $\varepsilon > 0$ are given. Since ψ_n tends to 1 pointwise as n tends to ∞, we can take N so that $|1 - \psi_N(s_i)| < \varepsilon$ holds for all i, $i = 1, \dots, n$. Also, we can take K so that $\|\sum_{k \geqslant K} \psi_N \chi_k\|_{\mathrm{cb}} < \varepsilon$ holds. Then $\varphi(s) := \sum_{k \leqslant K} \psi_N(s) \chi_k(s)$ satisfies that $|1 - \varphi(s_i)| < 2\varepsilon$ for all i, $i = 1, \dots, n$, and $\|\varphi\|_{\mathrm{cb}} < 1 + \varepsilon$. Hence, if we choose N_n and K_n suitably for all $n \in \mathbb{N}$, $\varphi_n(s) = \sum_{k \leqslant K_n} \psi_{N_n}(s) \chi_k(s)$ defines a sequence of finitely supported functions on Γ which tends to 1 pointwise whose cb-norm tends to 1. $\square$

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Lorentz space estimates for the Ginzburg–Landau energy

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Abstract

In this paper we prove novel lower bounds for the Ginzburg–Landau energy with or without magnetic field. These bounds rely on an improvement of the "vortex-balls construction" estimates by extracting a new positive term in the energy lower bounds. This extra term can be conveniently estimated through a Lorentz space norm, on which it thus provides an upper bound. The Lorentz space $L^{2,\infty}$ we use is critical with respect to the expected vortex profiles and can serve to estimate the total number of vortices and get improved convergence results.

Keywords: Ginzburg–Landau; Vortex balls; Ball construction; Lorentz spaces Best regards; Sylvia Serfaty

1. Introduction

1.1. Motivation

In this paper we consider the Ginzburg–Landau "free energy"

$$F_\varepsilon(u, A) = \frac{1}{2}\int_\Omega |\nabla_A u|^2 + |\operatorname{curl} A|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2}. \tag{1.1}$$

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Here Ω is a bounded regular two-dimensional domain of $\mathbb{R}^2$, u is a complex-valued function, and $A \in \mathbb{R}^2$ is a vector field in Ω. This functional is the free energy of the model of superconductivity developed by Ginzburg and Landau. In the model, A is the vector-potential of the magnetic field, the function $h := \operatorname{curl} A = \partial_1 A_2 - \partial_2 A_1$ is the induced magnetic field, and the complex-valued function u is the "order parameter" indicating the local state of the material (normal or superconducting): $|u|^2$ is the local density of superconducting electrons. The notation ∇_A refers to the covariant gradient, which acts according to $\nabla_A u = (\nabla - iA)u$.

We are interested in the regime of small ε: ε corresponds to a material constant, and small ε implies type-II superconductivity. In this regime, u (because it is complex-valued) can have zeroes with a non-zero topological degree. These defects are called the *vortices* of u and are the crucial objects of interest.

By setting $A \equiv 0$ we are led to studying the simpler Ginzburg–Landau energy "without magnetic field":

$$E_\varepsilon(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2}. \tag{1.2}$$

All our results will thus apply to this energy as well, by setting $A \equiv 0$.

These functionals, and in particular the vortices arising in their minimizers or critical points, have been studied intensively in the mathematics literature. We refer in particular to the books [1] for E_ε and [7] for the functional with magnetic field. The interested reader can find there more information on the physical and mathematical background.

We are interested in proving lower bounds on F_ε, and in particular estimates which relate $F_\varepsilon(u, A)$ and $\|\nabla_A u\|_{L^{2,\infty}}$, the norm of $\nabla_A u$ in the Lorentz space $L^{2,\infty}$. Noticeably, Lorentz spaces were already used in the context of the Ginzburg–Landau energy by Lin and Rivière in [5]. Their goal there was to study energy critical points in 3 dimensions, but what they used was interpolation ideas and the duality between Lorentz spaces $L^{2,1}$ and $L^{2,\infty}$.

The Ginzburg–Landau energy is generally unbounded as $\varepsilon \to 0$; it blows up roughly like $\pi n |\log|\varepsilon$, where n is the number (or total degree) of vortices. Our investigation of estimates for $\|\nabla_A u\|_{L^{2,\infty}}$ is thus part of a quest for intrinsic quantities in $\nabla_A u$ which do not blow up as $\varepsilon \to 0$, but rather remain of the order of n.

1.2. Heuristics for idealized vortices

Let us now try to explain the interest and relevance of the Lorentz space $L^{2,\infty}$ for this problem. The space $L^{2,\infty}$, also known as "weak-L^2," is a functional space which is just "slightly larger" than the Lebesgue space L^2. One simple way of defining the $L^{2,\infty}$ norm is by

$$\|f\|_{L^{2,\infty}} = \sup_{|E|<\infty} |E|^{-\frac{1}{2}} \int_E |f(x)|\, dx, \tag{1.3}$$

where $|E|$ denotes the Lebesgue measure of E. An equivalent way is through the super-level sets of f:

$$\|f\|_{L^{2,\infty}} = \sup_{t>0} t \lambda_f(t)^{\frac{1}{2}}, \tag{1.4}$$

where $\lambda_f(t) = |\{x \in \Omega \mid |f(x)| > t\}|$. For more information on Lorentz spaces we refer for example to [2,8]. A simple application of the Cauchy–Schwarz inequality in (1.3) allows to check that if f is in L^2 then it is in $L^{2,\infty}$ with $\|f\|_{L^{2,\infty}} \leqslant \|f\|_{L^2}$.

Let us now consider vortices of a complex-valued function u in the context of Ginzburg–Landau. In the regime of small ε, u can have zeroes, but because of the strong penalization of the term $\int_\Omega (1-|u|^2)^2$, $|u|$ can be small only in (small) regions of characteristic size ε.

Then around a zero at a point x_0, u has a degree defined as the topological degree of $u/|u| = e^{i\varphi}$ as a map from a circle to $\mathbb{S}^1$, or in other words

$$d = \frac{1}{2\pi} \int_{\partial B(x_0,r)} \frac{\partial \varphi}{\partial \tau} \in \mathbb{Z}, \tag{1.5}$$

where r is sufficiently small. One can describe the situation very roughly as follows: $|u|$ is small in a ball of radius $C\varepsilon$, and $|u| \approx 1$ outside of this ball, say in an annulus $B(x_0,R) \setminus B(x_0,C\varepsilon)$. The size of R is meant to account for possible neighboring zeroes. In this annulus, the model case is that of a radial vortex of degree d, i.e

$$u(r,\theta) = f(r)e^{id\theta}, \tag{1.6}$$

where (r,θ) are the polar coordinates centered at x_0, and f is a real-valued function, close to 1 in $B(x_0,R) \setminus B(x_0,C\varepsilon)$. When computing the L^2 norm of ∇u, we find that $|\nabla u| \approx \frac{|d|}{r}$ in the annulus and thus, using polar coordinates,

$$\begin{aligned} \|\nabla u\|^2_{L^2(B(x_0,R))} &\geqslant \int_{B(0,R)\setminus B(0,C\varepsilon)} \left|\frac{d}{r}\right|^2 = \int_{C\varepsilon}^{R} \frac{2\pi d^2}{r}\,dr \\ &\geqslant 2\pi d^2 \log \frac{R}{C\varepsilon}. \end{aligned} \tag{1.7}$$

This tells us that the (square of the) L^2 norm of ∇u blows up like $2\pi d^2|\log\varepsilon|$ as $\varepsilon \to 0$. This is a crucial fact in the analysis of Ginzburg–Landau, much used since [1]. Jerrard [3] and Sandier [6] showed that this picture is actually accurate even for arbitrary configurations: without assuming that the vortex profile is radial, the inequality (1.7) still holds (the radial profile is actually the one that is minimal for the L^2 norm). Moreover, any configuration with an arbitrary number of vortices can be understood as many such annuli, possibly at very close distance to each other, glued together. Good lower bounds like (1.7) can be added up together by keeping annuli with the same conformal type. This was the basis of the “vortex-balls construction” that they formulated and which was used extensively to understand Ginzburg–Landau minimizers, in particular in [7].

On the other hand, let us calculate (roughly) the $L^{2,\infty}$ norm of ∇u for the above vortex. We recall that $|\nabla u| \approx \frac{|d|}{r}$ in the annulus $B(x_0,R) \setminus B(x_0,C\varepsilon)$. Using the definition (1.4), we have $|\nabla u| > t$ if and only if $r < |d|/t$. Thus

$$\lambda_{|\nabla u|}(t) \approx \pi d^2/t^2,$$

and we find

$$\|\nabla u\|_{L^{2,\infty}(B(x_0,R)\setminus B(x_0,C\varepsilon))} \approx \sqrt{\pi}|d|. \tag{1.8}$$

So in contrast, the $L^{2,\infty}$ norm of ∇u *does not blow up* as $\varepsilon \to 0$. One can see that this space is critical in the sense that $1/|x|$ (barely) fails to be in L^2 or in $L^{2,q}$ for any $q < \infty$ (its norm blows up logarithmically in all cases) but is in $L^{2,\infty}$ and in all L^p for $p < 2$.

Moreover, from this formula (1.8), it is expected that the $L^{2,\infty}$ norm can serve to estimate the total degree $\sum |d_i|$ of all the vortices of a configuration. This is convenient since the total degree $\sum |d_i|$ is generally obtained via a "ball construction" that is non-unique. On the other hand, $\|\nabla u\|_{L^{2,\infty}}$ provides a unique and intrinsic quantity useful to evaluate the number of vortices.

Because of these remarks and because of the paper [5], it could be expected that Lorentz spaces are a suitable functional setting in which to study Ginzburg–Landau vortices. One may point out that there are other spaces that would be critical for the profile $1/|x|$, such as Besov spaces; however, it seems difficult to find an effective way of using them in connection with the Ginzburg–Landau energy.

The main goal of our results is to give a rigorous basis to the above observations. The connection with the Lorentz norm of ∇u is made through the "vortex-balls construction" of Jerrard and Sandier, as formulated in [7]. Our estimates will in fact provide an improvement of these lower bounds by adding an extra positive term in the lower bounds, which is then related to the Lorentz norm. Just as in the ball construction method, one of the interests of the result is that it is valid under very few assumptions: only a very weak upper bound on the energy, even when u has a large number of vortices, unbounded as $\varepsilon \to 0$. This creates serious technical difficulties but is important since such situations occur for energy minimizers when there is a large applied magnetic field, as proved in [7].

1.3. Main results

Let us point out that the estimates we prove are not on the Lorentz norm of ∇u but rather on that of $\nabla_A u$. The reason is that the energy F_ε is *gauge-invariant*: it satisfies $F_\varepsilon(u, A) = F_\varepsilon(ue^{i\Phi}, A + \nabla\Phi)$ for any smooth function Φ. Thus the quantity $|\nabla u|$ is not a gauge-invariant quantity, hence not an intrinsic physical quantity. This is why it is replaced by the gauge-invariant "covariant derivative" $|\nabla_A u|$.

Our method consists in proving the following improvement of the "ball construction" lower bounds (see [7, Chapter 4]):

Theorem 1 *(Improved lower bounds). Let $\alpha \in (0, 1)$. There exists $\varepsilon_0 > 0$ (depending on α) such that for $\varepsilon \leqslant \varepsilon_0$ and u, A both C^1 such that $F_\varepsilon(|u|, \Omega) \leqslant \varepsilon^{\alpha-1}$, the following hold.*

For any $1 > r > C\varepsilon^{\alpha/2}$, where C is a universal constant, there exists a finite, disjoint collection of closed balls, denoted by $\mathcal{B}$, with the following properties.

1. *The sum of the radii of the balls in the collection is r.*
2. *Defining $\Omega_\varepsilon = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$, we have $\{x \in \Omega_\varepsilon \mid ||u(x)| - 1| \geqslant \delta\} \subset V := \Omega_\varepsilon \cap (\bigcup_{B \in \mathcal{B}} B)$, where $\delta = \varepsilon^{\alpha/4}$.*
3. *We have*

$$\frac{1}{2}\int_V |\nabla_A u|^2 + \frac{1}{2\varepsilon^2}\left(1 - |u|^2\right)^2 + r^2(\operatorname{curl} A)^2$$

$$\geqslant \pi D\left(\log\frac{r}{\varepsilon D} - C\right) + \frac{1}{18}\int_V |\nabla_{A+G} u|^2 + \frac{1}{2\varepsilon^2}\left(1 - |u|^2\right)^2, \tag{1.9}$$

where G is some explicitly constructed vector field, d_B denotes $\deg(u, \partial B)$ *if* $B \subset \Omega_\varepsilon$ *and* 0 *otherwise,*

$$D = \sum_{\substack{B \in \mathcal{B} \\ B \subset \Omega_\varepsilon}} |d_B|$$

is assumed to be non-zero, and C is universal.

The improvement with respect to Theorem 4.1 in [7] is the addition of the extra term $\frac{1}{18}\int |\nabla_{A+G} u|^2$. The term G is a vector-field constructed in the course of the ball construction, which essentially compensates for the expected behavior of $\nabla_A u$ in the vortices. One can take it to be $\tau d/r$ in every annulus of the ball construction where u has a constant degree d, τ denotes the unit tangent vector to each circle centered at x_0, the center of the annulus, and $r = |x - x_0|$. By extending G to be zero outside of the union of balls V, we easily deduce:

Corollary 1.1. *Let* (u, A) *be as above, then*

$$\int_\Omega |\nabla_A u - iGu|^2 \leqslant C\left(F_\varepsilon(u, A) - \pi D \log\left(\frac{r}{\varepsilon D} - C\right)\right) \tag{1.10}$$

where G is the explicitly constructed vector field of Theorem 1*, and C a universal constant.*

The right-hand side of this inequality can be considered as the "energy-excess," the difference between the total energy and the expected vortex energy provided by the ball construction lower bounds. Thus we control $\int_\Omega |\nabla_A u - iGu|^2$ by the energy-excess. This fact is used repeatedly in the sequel paper [9] to better understand the behavior of $\nabla_A u$ for minimizers and almost minimizers of the Ginzburg–Landau energy with applied magnetic field.

One can also note that such a control (1.10) has a similar flavor to a result of Jerrard and Spirn [4] where they control the difference (in a weaker norm but with better control) of the Jacobian of u to a measure of the form $\sum d_i \delta_{a_i}$ by the energy-excess.

Once Theorem 1 is proved, we turn to obtaining an $L^{2,\infty}$ estimate from which G has disappeared. In order to do so, we can bound below $\|\nabla_{A+G} u\|_{L^2}$ by $\|\nabla_{A+G} u\|_{L^{2,\infty}}$; the more delicate task is then to control $\|G\|_{L^{2,\infty}}$ in a way that only depends on the final data of the theorem, that is, on the degrees of the final balls constructed above and on the energy. This task is complicated by the possible presence of large numbers of vortices very close to each other, and compensations of vortices of large positive degrees with vortices of large negative degrees. To overcome this, G is not defined exactly as previously said, but in a modified way, and $\|G\|_{L^{2,\infty}}$ is controlled not only through the degrees but also through the total energy.

We then arrive at the following main result:

Theorem 2 *(Lorentz norm bound). Assume the hypotheses and results of Theorem* 1*. Then there exists a universal constant C such that*

$$\begin{aligned} &\frac{1}{2}\int_V |\nabla_A u|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2} + r^2(\operatorname{curl} A)^2 + \pi \sum |d_B|^2 \\ &\quad \geqslant C\|\nabla_A u\|^2_{L^{2,\infty}(V)} + \pi \sum |d_B|\left(\log\frac{r}{\varepsilon \sum |d_B|} - C\right), \end{aligned} \tag{1.11}$$

where the sums are taken over all the balls B in the final collection $\mathcal{B}$ that are included in Ω_ε.

This theorem bounds below the energy contained in the union of balls V in terms of the $L^{2,\infty}$ norm on V. It is a simple matter to extend these estimates to all of Ω, and deduce a control of the $L^{2,\infty}$ norm of $\nabla_A u$ by the energy-excess, plus the term $\sum |d_B|^2$. This is the content of the following corollary.

Corollary 1.2. *Assuming the hypotheses and results of Theorem* 1*, there exists a universal constant C such that*

$$\|\nabla_A u\|^2_{L^{2,\infty}(\Omega)} \leqslant C\left(F_\varepsilon(u,A) - \pi \sum |d_B| \log \frac{r}{\varepsilon \sum |d_B|} + \sum |d_B|^2\right), \tag{1.12}$$

where the sums are taken over all the balls B in the final collection $\mathcal{B}$ that are included in Ω_ε.

These estimates can indeed help to bound from above $\|\nabla_A u\|^2_{L^{2,\infty}(\Omega)}$ by the total number of vortices, provided we can control the energy-excess by that number of vortices. This can in turn serve to obtain stronger convergence results when a weak limit of $\nabla_A u$ is known. For example, if one considers the energy E_ε (which we recall amounts to setting $A \equiv 0$), it is known from Bethuel, Brezis and Hélein [1] that $\pi \sum |d_B||\log \varepsilon| = \pi n |\log \varepsilon|$ is the leading order of the energy (at least for minimizers) and that the next order term is a term of order 1, called the "renormalized energy" W, that accounts for the interaction between the vortices. The upper bound of Corollary 1.2 roughly tells us that

$$\|\nabla u\|^2_{L^{2,\infty}(\Omega)} \leqslant C\left(W + \sum |d_B|^2 + \sum |d_B| \log \sum |d_B|\right).$$

It is expected that the total cost of interaction of the vortices in W is of order of n^2, where $n = \sum |d_B|$ is the total vorticity mass (here n can blow up as $\varepsilon \to 0$). Thus, we obtain a bound of the form

$$\|\nabla u\|^2_{L^{2,\infty}(\Omega)} \leqslant C n^2,$$

which indeed bounds the $L^{2,\infty}$ norm of ∇u by an order of n, the total vorticity mass, as expected in the heuristic calculations of Section 1.2.

In the simplest case where we know that $E_\varepsilon(u_\varepsilon) \leqslant \pi n |\log \varepsilon| + C$, which happens for energy minimizers when n is bounded, as proved in [1], we then deduce that $\|\nabla u\|_{L^{2,\infty}} \leqslant C$. To be more precise, for the minimizers of E_ε found in [1], we have

Proposition 1.3 *(Application to minimizers of E_ε with Dirichlet boundary conditions). Let Ω be starshaped and u_ε minimize E_ε under the constraint $u_\varepsilon = g$ on $\partial\Omega$, where g is a fixed $\mathbb{S}^1$-valued map of degree $d > 0$ on the boundary of Ω, as studied in* [1]*. Then there exists a universal constant C such that*

$$\|\nabla u_\varepsilon\|^2_{L^{2,\infty}(\Omega)} \leqslant C\left(\min_{\Omega^d} W + d(\log d + 1)\right) + o_\varepsilon(1).$$

Moreover, as $\varepsilon \to 0$,

$$\nabla u_\varepsilon \rightharpoonup \nabla u_\star \quad \textit{weakly-}{*} \textit{ in } L^{2,\infty}(\Omega),$$

where $u_\star$ is the S^1-valued "canonical harmonic map" of [1] *to which converges u in C^k_{loc} outside of a set of d vortex points.*

Note that the renormalized energy W depends on g (hence on d), and the $d\log d$ is not optimal here; rather, it should be d. It is more delicate to obtain this kind of improvement to the estimate; this is one of the things done in [9] in the context of the energy with applied magnetic field. Also the convergence of ∇u_ε *cannot be strengthened*, convergence in $L^{2,\infty}$ strong does not hold, as illustrated by the following model case: let V_ε be the vector field $\frac{(x-p_\varepsilon)^\perp}{|x-p_\varepsilon|^2}$ and $V=\frac{(x-p)^\perp}{|x-p|^2}$ with $p_\varepsilon \neq p$ but $p_\varepsilon \to p$ as $\varepsilon \to 0$. Then $2\sqrt{\pi} \leqslant \|V_\varepsilon - V\|_{L^{2,\infty}} \leqslant 4\sqrt{\pi}$, while clearly $V_\varepsilon \rightharpoonup V$ weakly-$*$ in $L^{2,\infty}$.

We have focused on proving upper bounds on $\|\nabla_A u\|_{L^{2,\infty}}$ in terms of its L^2 norm and Ginzburg–Landau energy. It is not difficult to obtain some adapted, though not optimal, lower bounds. For example, we can prove the following:

Proposition 1.4. *Let $f \in L^\infty(\Omega)$ be such that $\|f\|_{L^\infty(\Omega)} \leqslant \frac{C}{\varepsilon}$ for some $\varepsilon < 1$. Then*

$$\|f\|^2_{L^{2,\infty}(\Omega)} \geqslant \frac{1}{2|\log\varepsilon|}\int_\Omega |f|^2 - \frac{C^2|\Omega|}{2|\log\varepsilon|}. \tag{1.13}$$

This proposition is a direct consequence of the definition of the $L^{2,\infty}$ norm. Its short proof is presented in Section 6.1.

For critical points of the Ginzburg–Landau energy, it is known that the gradient bound $\|\nabla_A u\|_{L^\infty(\Omega)} \leqslant \frac{C}{\varepsilon}$ holds. Thus applying Proposition 1.4 to $f = \nabla_A u$, we find

$$\|\nabla_A u\|^2_{L^{2,\infty}(\Omega)} \geqslant \frac{1}{2|\log\varepsilon|}\int_\Omega |\nabla_A u|^2 - o(1).$$

Knowing some lower bounds (provided by the ball construction) of the type $\int_\Omega |\nabla_A u|^2 \geqslant 2\pi n|\log\varepsilon|$, where n is the total degree of the vortices, we find lower bounds of the type $\|\nabla_A u\|^2_{L^{2,\infty}(\Omega)} \geqslant \pi n$, also relating the $L^{2,\infty}$ norm of $\nabla_A u$ to the total number of vortices.

In [9], which is the sequel of this paper, the ideas and main results of this paper are extended to the case of the full Ginzburg–Landau energy with an applied magnetic field, getting better estimates on $\|\nabla_A u\|_{L^{2,\infty}(\Omega)}$ in terms of the number of vortices. These results lead to a somewhat stronger (than previously known results) convergence of $\nabla_A u$ and of the Jacobian determinants of u when certain energy conditions are fulfilled.

1.4. Plan

The paper is organized as follows: in Section 2, for the convenience of the reader, we give a review (with slight modifications) of the crucial definitions and ingredients for the vortex-balls construction following Chapter 4 of [7].

In Section 3 we present the main argument, with the introduction of the function G and the "trick" that allows us to gain an extra term in the lower bounds for the energy on annuli.

In Section 4 we show how this extra term incorporates into the estimates through the growing and merging of balls, and hence through the whole ball construction.

In Section 5 we deduce the proof of the main results.

In Section 6 we estimate the $L^{2,\infty}$ norm of G in order to pass from Theorem 1 to Theorem 2. This is the only section in which $L^{2,\infty}$ comes into play.

In Section 7 we show how the methods of this paper can be adapted to work with the version of the ball construction formulated by Jerrard in [3], at the expense of less control of $\|G\|_{L^{2,\infty}}$.

2. Reminders for the vortex-balls construction

2.1. The ball growth method

In finding lower bounds for the Ginzburg–Landau energy of a configuration (u, A) it is most convenient to work on annuli, the deleted interior discs of which contain the set where u is near 0, and in particular the vortices. On each annulus, a lower bound is found in terms of a topological term (the degree of the vortex) and a conformal factor, which we define to be the logarithm of the ratio of the outer and inner radii of the annulus. Therefore, to find useful lower bounds we must be able to identify the set where u is near 0 and then create a family of annuli with large conformal type outside this set. The first component of the process uses energy methods to find a covering of the set by small, disjoint balls, and is addressed later. The second component is known as the general ball growth method and is presented in this section. Here we follow the construction of Chapter 4 from [7].

As a technical tool we will need the ability to merge two tangent or overlapping balls into a single ball that contains the original balls, and with the property that its radius is equal to the sum of the radii of the original balls. Our first lemma recalls how to do such a merging. We write $r(B)$ for the radius of a ball B.

Lemma 2.1. *Let B_1 and B_2 be closed balls in $\mathbb{R}^n$ such that $B_1 \cap B_2 \neq \emptyset$. Then there is a closed ball B such that $r(B) = r(B_1) + r(B_2)$ and $B_1 \cup B_2 \subset B$.*

Proof. If $B_1 = B(a_1, r_1)$ and $B_2 = B(a_2, r_2)$, then $B = B(\frac{r_1 a_1 + r_2 a_2}{r_1 + r_2}, r_1 + r_2)$ has the desired properties. □

The ball growth lemma now provides an algorithm for growing an initial collection of small balls into a final collection of large balls. Essentially, the balls in a collection are grown concentrically by increasing their radii by the same conformal factor. This is continued until a tangency occurs, at which point the previous lemma is used to merge the tangent balls. The process is then repeated in stages until the collection is of the desired size. The annuli of interest at each stage are formed by deleting the initial collection of balls from the final collection; the construction guarantees that all of the annuli in a stage have the same conformal type.

Given a finite collection of disjoint balls, $\mathcal{B}$, we define the radius of the collection, $r(\mathcal{B})$, to be the sum of the radii of the balls in the collection, i.e.

$$r(\mathcal{B}) = \sum_{B \in \mathcal{B}} r(B).$$

For any $\lambda > 0$ and any ball $B = B(a, r)$, we define $\lambda B = B(a, \lambda r)$. Extending this notation to collections of balls, we write $\lambda\mathcal{B} = \{\lambda B \mid B \in \mathcal{B}\}$. For an annulus $A = B(a, r_1) \setminus B(a, r_0)$, we define the conformal factor by $\tau = \log(r_1/r_0)$. We can now state the ball growth lemma, the proof of which can be found in Theorem 4.2 of [7].

Lemma 2.2 *(Ball growth lemma). Let $\mathcal{B}_0$ be a finite collection of disjoint, closed balls. There exists a family $\{\mathcal{B}(t)\}_{t\in\mathbb{R}_+}$ of collections of disjoint, closed balls such that the following hold.*

1. $\mathcal{B}_0 = \mathcal{B}(0)$.
2. *For $s \geqslant t \geqslant 0$,*

$$\bigcup_{B\in\mathcal{B}(t)} B \subseteq \bigcup_{B\in\mathcal{B}(s)} B.$$

3. *There exists a finite set $T \subset \mathbb{R}^+$ such that if $[t,s] \subset \mathbb{R}^+ \setminus T$, then $\mathcal{B}(s) = e^{s-t}\mathcal{B}(t)$. In particular, if $B(s) \in \mathcal{B}(s)$ and $B(t) \in \mathcal{B}(t)$ are such that $B(t) \subset B(s)$, then $B(s) = e^{s-t}B(t)$ and the conformal factor of the annulus $B(s) \setminus B(t)$ is $\tau = s - t$.*
4. *For every $t \in \mathbb{R}^+$, $r(\mathcal{B}(t)) = e^t r(\mathcal{B}_0)$.*

We now show how to couple lower bounds to the geometric construction. We may think of a function $\mathcal{F} : \mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{R}^+$ as being defined also for collections of balls, $\mathcal{B}$, via the identifications

$$\mathcal{F}\big(B(x,r)\big) = \mathcal{F}(x,r)$$

and

$$\mathcal{F}(\mathcal{B}) = \sum_{B\in\mathcal{B}} \mathcal{F}(B).$$

Here and for the rest of the paper we employ the notation $\bar{B}$ to refer to a specific ball $\bar{B}$ in some collection, and not to refer to the closure of B. We will also abuse notation by writing $\bar{B} \cap \mathcal{B}(t)$ for the collection $\{\bar{B} \cap B \mid B \in \mathcal{B}(t)\}$.

Lemma 2.3. *Let $\mathcal{B}_0$ be a finite collection of disjoint, closed balls, and suppose that $\mathcal{B}(t)$ is the collection of balls obtained from $\mathcal{B}_0$ by growing them according to the ball growth lemma. Fix a time $s > 0$ and suppose that $0 < s_1 < \cdots < s_K \leqslant s$ denote the times at which mergings occur in the ball growth lemma, i.e. let the s_i be an increasing enumeration of the set T defined there. Then*

$$\mathcal{F}\big(\mathcal{B}(s)\big) - \mathcal{F}(\mathcal{B}_0) = \int_0^s \sum_{B(x,r)\in\mathcal{B}(t)} r\frac{\partial\mathcal{F}}{\partial r}(x,r)\,dt + \sum_{k=1}^K \mathcal{F}\big(\mathcal{B}(s_k)\big) - \mathcal{F}\big(\mathcal{B}(s_k)\big)^-, \tag{2.1}$$

where $\mathcal{F}(\mathcal{B}(s_k))^- = \lim_{t\to s_k^-} \mathcal{F}(\mathcal{B}(t))$. Moreover, for any $\bar{B} \in \mathcal{B}(s)$, the following localized version of (2.1) *holds*:

$$\mathcal{F}(\bar{B}) - \mathcal{F}(\bar{B}\cap\mathcal{B}_0) = \int_0^s \sum_{B(x,r)\in\bar{B}\cap\mathcal{B}(t)} r\frac{\partial\mathcal{F}}{\partial r}\,dt + \sum_{k=1}^K \mathcal{F}\big(\bar{B}\cap\mathcal{B}(s_k)\big) - \mathcal{F}\big(\bar{B}\cap\mathcal{B}(s_k)\big)^-. \tag{2.2}$$

Proof. The proof is the same as in Proposition 4.1 of [7], but here we keep the second sum in (2.1) rather than bounding it below by 0. □

Note that in the case that

$$\mathcal{F}(x,r) = \int_{B(x,r)} e(u)$$

for some u-dependent energy density $e(u)$, the first term on the right of (2.1) corresponds to integration in polar coordinates on each annulus, and the second corresponds to the energy contained in the non-annular parts of $\mathcal{B}(s)$.

2.2. The radius of a set

In order to effectively use the ball growth lemma to generate lower bounds, it is necessary to first produce a collection of disjoint balls covering the set where u is near 0. We do this by using the concept of the radius of a set, which is useful in two ways. First, it is defined as an infimum over all coverings of the set by collections of balls, so that by exceeding the infimum we may find a particular covering of the set by balls. Second, it is comparable to the $\mathcal{H}^1$ Hausdorff measure of the boundary, and so it can be used with the co-area formula to produce coverings by balls of the set where $|u|$ is far from unity.

We define the radius of a compact set $\omega \subset \mathbb{R}^2$, written $r(\omega)$, by

$$r(\omega) = \inf\left\{ r(B_1) + \cdots + r(B_k) \;\middle|\; \omega \subset \bigcup_{i=1}^{k} B_i \text{ and } k < \infty \right\}.$$

We make the following remarks.

(1) In the definition we may assume that the balls are disjoint. If they are not, then we merge balls that meet into a single ball with radius equal to the sum of the radii of the merged balls according to Lemma 2.1.
(2) If $A \subseteq B$ then $r(A) \leqslant r(B)$.
(3) The infimum is not necessarily achieved.

It is necessary also to introduce a modification of the radius that measures the radius of the connected components of a compact set ω that lie inside an open set Ω. Indeed, we define

$$r_\Omega(\omega) = \sup\left\{ r(K \cap \omega) \;\middle|\; K \subset \Omega \text{ s.t. } K \text{ is compact and } \partial K \cap \omega = \emptyset \right\}.$$

The following lemmas record the crucial properties of these quantities. The omitted proofs may be found in Section 4.4 of [7].

Lemma 2.4. *Let ω be a compact subset of $\mathbb{R}^2$. Then*

$$2r(\omega) \leqslant \mathcal{H}^1(\partial\omega). \tag{2.3}$$

Lemma 2.5. *Let Ω be open and $\omega \subset \Omega$ be a compact set. Then*

$$2r_\Omega(\omega) \leqslant \mathcal{H}^1(\partial\omega \cap \Omega). \tag{2.4}$$

Lemma 2.6. *Let ω_1, ω_2 be compact subsets of $\mathbb{R}^2$. Then*

$$r(\omega_1 \cup \omega_2) \leqslant r(\omega_1) + r(\omega_2). \tag{2.5}$$

Lemma 2.7. *Let ω_1, ω_2 be compact sets, and let $\Omega \subset \mathbb{R}^2$ be an open set. Then*

$$r_\Omega(\omega_1 \cup \omega_2) \leqslant r_\Omega(\omega_1) + r_\Omega(\omega_2). \tag{2.6}$$

Proof. If $\Omega \subset \omega_1 \cup \omega_2$, then the result is trivial. Suppose otherwise. Let $K \subset \Omega$ be such that K is compact and $\partial K \cap (\omega_1 \cup \omega_2) = \emptyset$. Then $(\partial K \cap \omega_1) \cup (\partial K \cap \omega_2) = \emptyset$, which implies that $\partial K \cap \omega_1 = \emptyset$ and $\partial K \cap \omega_2 = \emptyset$. Hence,

$$\begin{aligned} r\big(K \cap (\omega_1 \cup \omega_2)\big) &= r\big((K \cap \omega_1) \cup (K \cap \omega_2)\big) \\ &\leqslant r(K \cap \omega_1) + r(K \cap \omega_2) \\ &\leqslant r_\Omega(\omega_1) + r_\Omega(\omega_2). \end{aligned} \tag{2.7}$$

Taking the supremum over all such K, we get $r_\Omega(\omega_1 \cup \omega_2) \leqslant r_\Omega(\omega_1) + r_\Omega(\omega_2)$. $\square$

We will now use these concepts to compare the energy of a real-valued function ρ, defined on an open set Ω, to the radius of the set where ρ is far from unity.

Lemma 2.8. *Let $\rho \in C^1(\Omega, \mathbb{R})$ with $\Omega \subset \mathbb{R}^2$ open and bounded. Let*

$$F_\varepsilon(\rho, \Omega) = \frac{1}{2}\int_\Omega |\nabla\rho|^2 + \frac{1}{2\varepsilon^2}\big(1-\rho^2\big)^2. \tag{2.8}$$

Then there is a universal constant C such that

$$r_\Omega\big(\{\rho \leqslant 1/2\} \cup \{\rho \geqslant 3/2\}\big) \leqslant \varepsilon C F_\varepsilon(\rho, \Omega). \tag{2.9}$$

Proof. By the Cauchy–Schwarz inequality and the co-area formula we have that

$$\begin{aligned} F_\varepsilon(\rho, \Omega) &= \frac{1}{2}\int_\Omega |\nabla\rho|^2 + \frac{1}{2\varepsilon^2}\big(1-\rho^2\big)^2 \\ &\geqslant \frac{1}{\varepsilon\sqrt{2}}\int_\Omega |\nabla\rho|\big|1-\rho^2\big| \\ &= \frac{1}{\varepsilon\sqrt{2}}\int_0^\infty \int_{\{\rho=t\}\cap\Omega} \big|1-\rho^2\big|\,d\mathcal{H}^1\,dt \\ &= \frac{1}{\varepsilon\sqrt{2}}\int_0^\infty \big|1-t^2\big|\mathcal{H}^1\big(\{\rho=t\}\cap\Omega\big)\,dt. \end{aligned} \tag{2.10}$$

We break the last integral into two parts and bound

$$\begin{aligned}
&\frac{1}{\varepsilon\sqrt{2}}\int_0^\infty \left|1-t^2\right|\mathcal{H}^1\big(\{\rho=t\}\cap\Omega\big)\,dt\\
&\quad\geqslant \frac{1}{\varepsilon\sqrt{2}}\int_{\frac{1}{2}}^{\frac{3}{4}}\big(1-t^2\big)\mathcal{H}^1\big(\{\rho=t\}\cap\Omega\big)\,dt+\frac{1}{\varepsilon\sqrt{2}}\int_{\frac{5}{4}}^{\frac{3}{2}}\big(t^2-1\big)\mathcal{H}^1\big(\{\rho=t\}\cap\Omega\big)\,dt\\
&\quad=\frac{1}{\varepsilon 4\sqrt{2}}\big(1-t_0^2\big)\mathcal{H}^1\big(\{\rho=t_0\}\cap\Omega\big)+\frac{1}{\varepsilon 4\sqrt{2}}\big(t_1^2-1\big)\mathcal{H}^1\big(\{\rho=t_1\}\cap\Omega\big),
\end{aligned}\tag{2.11}$$

where the last equality follows from the mean value theorem, and $t_0\in(\frac{1}{2},\frac{3}{4})$ and $t_1\in(\frac{5}{4},\frac{3}{2})$. The bounds on t_0 and t_1 imply that

$$\begin{aligned}
\big(1-t_0^2\big)&\geqslant 1-\frac{9}{16}=\frac{7}{16},\quad\text{and}\\
\big(t_1^2-1\big)&\geqslant\frac{25}{16}-1=\frac{9}{16}.
\end{aligned}\tag{2.12}$$

Combining (2.10), (2.11), and (2.12), we get

$$\begin{aligned}
F_\varepsilon(\rho,\Omega)&\geqslant\frac{7}{\varepsilon 64\sqrt{2}}\mathcal{H}^1\big(\{\rho=t_0\}\cap\Omega\big)+\frac{9}{\varepsilon 64\sqrt{2}}\mathcal{H}^1\big(\{\rho=t_1\}\cap\Omega\big)\\
&\geqslant\frac{7}{\varepsilon 64\sqrt{2}}\big(\mathcal{H}^1\big(\{\rho=t_0\}\cap\Omega\big)+\mathcal{H}^1\big(\{\rho=t_1\}\cap\Omega\big)\big).
\end{aligned}\tag{2.13}$$

Write S_{t_0} and S^{t_1} for the $\mathbb{R}^2$-closures of the sets $\{x\in\Omega\mid\rho(x)\leqslant t_0\}$ and $\{x\in\Omega\mid\rho(x)\geqslant t_1\}$ respectively. The bounds $t_0\geqslant\frac{1}{2}$, $t_1\leqslant\frac{3}{2}$ imply the inclusions $\{\rho\leqslant 1/2\}\subset S_{t_0}$ and $\{\rho\geqslant 3/2\}\subset S^{t_1}$. We may then apply Lemmas 2.5 and 2.7 to find the bounds

$$\begin{aligned}
\mathcal{H}^1\big(\{\rho=t_0\}\cap\Omega\big)+\mathcal{H}^1\big(\{\rho=t_1\}\cap\Omega\big)&=\mathcal{H}^1(\partial S_{t_0}\cap\Omega)+\mathcal{H}^1\big(\partial S^{t_1}\cap\Omega\big)\\
&\geqslant 2r_\Omega(S_{t_0})+2r_\Omega\big(S^{t_1}\big)\\
&\geqslant 2r_\Omega\big(\{\rho\leqslant 1/2\}\big)+2r_\Omega\big(\{\rho\geqslant 3/2\}\big)\\
&\geqslant 2r_\Omega\big(\{\rho\leqslant 1/2\}\cup\{\rho\geqslant 3/2\}\big).
\end{aligned}\tag{2.14}$$

Putting (2.14) into (2.13) yields the desired estimate with $C=\frac{32\sqrt{2}}{7}$. □

3. Improved lower bounds on annuli

In this section we will show how to obtain lower bounds for the Ginzburg–Landau energy in terms of the degree. We begin by constructing estimates on circles. The primary difference between our estimates and those constructed previously is that we arrive at our lower bounds by introducing an auxiliary function G and using a completion of the square trick. This allows us to

retain terms involving G and thereby create an energy bound with a novel term. Before properly defining G let us prove the lower bounds on circles.

We first record a simple lemma (see for example Lemma 3.4 in [7]).

Lemma 3.1. *Let $u \in H^1(\Omega, \mathbb{C})$ be written (at least locally) $u = \rho v$, where $\rho = |u|$ and $v = e^{i\varphi}$. Then $|\nabla_A u|^2 = |\nabla \rho|^2 + \rho^2 |\nabla \varphi - A|^2 = |\nabla \rho|^2 + \rho^2 |\nabla_A v|^2$.*

Now we prove the lower bounds on circles.

Lemma 3.2. *Let $B := B(a, r) \subset \mathbb{R}^2$, and suppose that $v : \partial B \to \mathbb{S}^1$ and $A : B \to \mathbb{R}^2$ are both C^1. Let $G : \partial B \to \mathbb{R}^2$ be given by $G = \frac{c\tau}{r}$, where τ is the oriented unit tangent vector field to ∂B and c is a constant. Write $d_B := \deg(v, \partial B)$. Then for any $\lambda > 0$,*

$$\frac{1}{2} \int_{\partial B} |\nabla_A v|^2 + \frac{\lambda}{2} \int_B (\operatorname{curl} A)^2 \geqslant \frac{1}{2} \int_{\partial B} |\nabla_{A+G} v|^2 + \frac{\pi}{r} \left(2 c d_B - c^2\right) - \frac{\pi c^2}{2\lambda}. \tag{3.1}$$

Proof. Define the quantity

$$X := \int_B \operatorname{curl} A = \int_{\partial B} A \cdot \tau. \tag{3.2}$$

We write $v = e^{i\varphi}$ and recall that $2\pi d_B = \int_{\partial B} \nabla \varphi \cdot \tau$. Using Lemma 3.1, we see

$$\begin{aligned} \int_{\partial B} |\nabla_{A+G} v|^2 &= \int_{\partial B} |\nabla \varphi - A - G|^2 \\ &= \int_{\partial B} |G|^2 - 2 \int_{\partial B} G \cdot (\nabla \varphi - A) + \int_{\partial B} |\nabla \varphi - A|^2 \\ &= \frac{2\pi r c^2}{r^2} - \frac{2c}{r} \int_{\partial B} \nabla \varphi \cdot \tau + \frac{2c}{r} \int_{\partial B} A \cdot \tau + \int_{\partial B} |\nabla_A v|^2 \\ &= \frac{2\pi c^2}{r} - \frac{2c}{r} 2\pi d_B + \frac{2c}{r} X + \int_{\partial B} |\nabla_A v|^2 \\ &= \frac{2\pi (c^2 - 2 c d_B)}{r} + \frac{2c}{r} X + \int_{\partial B} |\nabla_A v|^2. \end{aligned} \tag{3.3}$$

An application of Hölder's inequality shows that

$$\int_B (\operatorname{curl} A)^2 \geqslant \frac{1}{\pi r^2} \left(\int_B \operatorname{curl} A \right)^2 = \frac{1}{\pi r^2} X^2. \tag{3.4}$$

Combining (3.3) and (3.4) yields the inequality

$$\frac{1}{2}\int_{\partial B} |\nabla_A v|^2 + \frac{\lambda}{2}\int_B (\operatorname{curl} A)^2 \geqslant \frac{1}{2}\int_{\partial B} |\nabla_{A+G} v|^2 + \frac{\pi(2cd_B - c^2)}{r} - \frac{c}{r}X + \frac{\lambda}{2\pi r^2}X^2. \quad (3.5)$$

As X varies, the minimum value of the right-hand side occurs when $X = \frac{\pi c r}{\lambda}$. Plugging this into (3.5) yields (3.1). □

For this lemma to be useful we must construct a function $G : \Omega \to \mathbb{R}^2$ compatible with the ball growth lemma. That is, since estimates will ultimately be added up over balls B, G must have the property that on each ∂B, $G = \tau_{\partial B} \frac{c}{r}$ with r the distance to the center of B. We will take advantage of the fact that c was an arbitrary constant; many of the following results are thus valid with any choice of constants, and it is only much later that we choose specific values. Observe already, though, that taking $c = d_B$ yields an improvement by the $\int |\nabla_{A+G} v|^2$ term to the bounds constructed in Lemma 4.4 of [7]. Unfortunately, we must choose a more complicated constant c to make the estimates in Sections 5 and 6 work. We now show how to define such a G so that it will be useful analytically.

Let $\Omega \subset \mathbb{R}^2$ be open and let $\{\mathcal{B}(t)\}_{t\in[0,s]}$ be a family of collections of closed, disjoint balls grown via the ball growth lemma from an initial collection $\mathcal{B}_0$ that covers the set on which u is near 0. Let $\mathcal{G}$ denote the subcollection of balls in $\mathcal{B}(s)$ entirely contained in Ω, and let $\mathcal{G}(t)$ denote the balls in $\mathcal{B}(t)$ that are contained in a ball from $\mathcal{G}$, i.e. that remain inside Ω for all t. For each ball $B \in \mathcal{G}(t)$ we define several quantities. Let $\tau_{\partial B} : \partial B \to \mathbb{R}^2$ denote the oriented unit tangent vector field to ∂B, and let a_B denote the center of B. Let $d_B = \deg(u/|u|, \partial B)$; this is well defined since the set on which u vanishes is contained in $\mathcal{B}_0$. Let β_B denote a constant, to be specified later, with the property that if $B_1 \in \mathcal{G}(t_1)$, $B_2 \in \mathcal{G}(t_2)$, and $B_2 = e^{t_2 - t_1} B_1$ (i.e. B_2 is grown from B_1 without any mergings) then $\beta_{B_1} = \beta_{B_2}$. In other words, the β_B are constant over each annulus produced by the ball construction. Let $T \subset [0, s]$ denote the finite set of times from the ball growth lemma at which a merging occurs in the growth of $\mathcal{G}(t)$. We then define the function $G : \Omega \to \mathbb{R}^2$ by

$$G(x) = \begin{cases} \tau_{\partial B}(x) \frac{d_B \beta_B}{|x - a_B|} & \text{if } x \in \partial B \text{ for some } B \in \mathcal{G}(t),\ t \in [0, s] \setminus T, \\ 0 & \text{otherwise.} \end{cases} \quad (3.6)$$

The ball growth lemma guarantees that if $x \in \partial B$ for some $B \in \mathcal{G}(t)$, $t \in [0, s] \setminus T$, then that t is unique, and so $G(x)$ is well defined. By construction, $G = 0$ in $\bigcup_{B\in\mathcal{G}(0)} B$, and so we can use the above definition of G to extend any function previously defined on $\bigcup_{B\in\mathcal{G}(0)} B$. We will frequently do so.

Fig. 1 shows a simple example of balls grown near the boundary of Ω. Four initial balls, colored light gray, are grown into three final balls, labeled B_1, B_2, B_3. The initial balls are first grown with by a conformal factor of $\tau = \log 2$ until a merging in required in the balls that become B_1. The result of this merging is the white ball contained in B_1. The growth is then continued with a conformal factor of $\tau = \log(6/5)$ to produce the final balls. The annuli on which G is defined are colored in dark gray and black. Since B_3 leaves the domain, G is set to zero on the annuli inside it. G also vanishes on the white region contained in B_1.

With G now properly defined we can show how to couple Lemma 3.2 to the ball growth lemma to produce lower bounds on annuli.

Proposition 3.3. *Let $\mathcal{B}_0$ be a finite, disjoint collection of closed balls and let $\Omega \subseteq \mathbb{R}^2$ be open. Let $\omega = \bigcup_{B\in\mathcal{B}_0} B$ and denote the collection of balls obtained from $\mathcal{B}_0$ via the ball growth lemma*

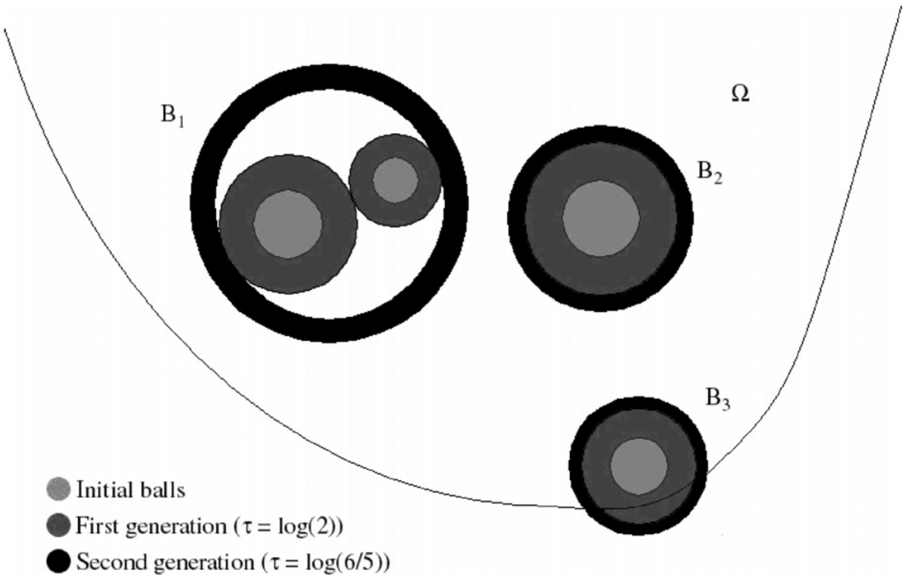

Fig. 1. Balls grown near the boundary of Ω.

by $\{\mathcal{B}(t)\}$, $t \geqslant 0$. *Suppose that* $v : \Omega \setminus \omega \to \mathbb{S}^1$ *and* $A : \Omega \to \mathbb{R}^2$ *are both* C^1, *and let* $G : \Omega \to \mathbb{R}^2$ *be the function defined by* (3.6). *Fix* $s > 0$ *such that* $r(\mathcal{B}(s)) \leqslant 1$. *Then, for any* $\bar{B} \in \mathcal{B}(s)$ *such that* $\bar{B} \subset \Omega$, *and any* $\lambda > 0$, *we have*

$$\frac{1}{2}\int\limits_{\bar{B}\setminus\omega} |\nabla_A v|^2 + \frac{r(\bar{B})\lambda}{2}\int\limits_{\bar{B}} (\operatorname{curl} A)^2 - \sum_{B \in \bar{B}\cap\mathcal{B}_0} \frac{r(B)\lambda}{2}\int\limits_{B} (\operatorname{curl} A)^2$$

$$\geqslant \frac{1}{2}\int\limits_{\bar{B}\setminus\omega} |\nabla_{A+G} v|^2 + \int\limits_0^s \sum_{B \in \bar{B}\cap\mathcal{B}(t)} \pi d_B^2\left(2\beta_B - \beta_B^2 - \frac{\beta_B^2 r(\mathcal{B}(t))}{2\lambda}\right) dt, \tag{3.7}$$

where we have written $d_B = \deg(u/|u|, \partial B)$.

Proof. In order to utilize Lemma 2.3 we define the function

$$\mathcal{F}(x,r) = \frac{1}{2}\int\limits_{B(x,r)} |\nabla_A v|^2 + \frac{r\lambda}{2}\int\limits_{B(x,r)} (\operatorname{curl} A)^2. \tag{3.8}$$

Differentiating and using (3.1) with $c = \beta_B d_B$, we arrive at the bound

$$\frac{\partial \mathcal{F}}{\partial r} \geqslant \frac{1}{2}\int\limits_{\partial B(x,r)} |\nabla_A v|^2 + \frac{\lambda}{2}\int\limits_{B(x,r)} (\operatorname{curl} A)^2$$

$$\geqslant \frac{1}{2}\int\limits_{\partial B(x,r)} |\nabla_{A+G} v|^2 + \frac{\pi d_B^2}{r}\big(2\beta_B - \beta_B^2\big) - \frac{\pi d_B^2 \beta_B^2}{2\lambda}. \tag{3.9}$$

We now recall the notation of Lemma 2.3: $0 < s_1 < \cdots < s_K \leqslant s$ denote the times at which merging occurs in the growth of $\mathcal{B}_0$ to $\mathcal{B}(s)$ via the ball growth lemma, and

$$\mathcal{F}\big(\bar{B} \cap \mathcal{B}(s_k)\big)^- = \lim_{t \to s_k^-} \mathcal{F}\big(\bar{B} \cap \mathcal{B}(t)\big). \tag{3.10}$$

By discarding the terms involving curl A, we see that

$$\begin{aligned} &\sum_{k=1}^{K} \mathcal{F}\big(\bar{B} \cap \mathcal{B}(s_k)\big) - \mathcal{F}\big(\bar{B} \cap \mathcal{B}(s_k)\big)^- \\ &\quad \geqslant \sum_{k=1}^{K} \Bigg(\sum_{B \in \bar{B} \cap \mathcal{B}(s_k)} \frac{1}{2} \int_B |\nabla_A v|^2 - \lim_{t \to s_k^-} \sum_{B \in \bar{B} \cap \mathcal{B}(t)} \frac{1}{2} \int_B |\nabla_A v|^2 \Bigg), \end{aligned} \tag{3.11}$$

which corresponds to the integral of $\frac{1}{2}|\nabla_{A+G} v|^2$ over the non-annular parts of $\bar{B} \setminus \omega$ since $G = 0$ there. Since the ball growth lemma makes

$$\frac{d}{dt} r\big(\mathcal{B}(t)\big) = r\big(\mathcal{B}(t)\big),$$

the expression

$$\int_0^s \sum_{B \in \bar{B} \cap \mathcal{B}(t)} \frac{r(B)}{2} \int_{\partial B} |\nabla_{A+G} v|^2 \, dt$$

corresponds to the integral of $\frac{1}{2}|\nabla_{A+G} v|^2$ over the annular parts of $\bar{B} \setminus \omega$. We now combine this observation, inequalities (3.9) and (3.11), and equality (2.2) to conclude that

$$\begin{aligned} &\mathcal{F}(\bar{B}) - \mathcal{F}(\bar{B} \cap \mathcal{B}_0) \\ &\quad \geqslant \frac{1}{2} \int_{\bar{B} \setminus \omega} |\nabla_{A+G} v|^2 + \int_0^s \sum_{B \in \bar{B} \cap \mathcal{B}(t)} \pi d_B^2 \left(2\beta_B - \beta_B^2 - \frac{\beta_B^2 r(B)}{2\lambda} \right) dt \\ &\quad \geqslant \frac{1}{2} \int_{\bar{B} \setminus \omega} |\nabla_{A+G} v|^2 + \int_0^s \sum_{B \in \bar{B} \cap \mathcal{B}(t)} \pi d_B^2 \left(2\beta_B - \beta_B^2 - \frac{\beta_B^2 r(\mathcal{B}(t))}{2\lambda} \right) dt. \end{aligned} \tag{3.12}$$

This is (3.7). $\square$

The following corollary shows that our method, using G, can be used to recover the same estimates found in Proposition 4.3 of [7].

Corollary 3.4. *Under the same assumptions as in Proposition* 3.3 *we have*

$$\frac{1}{2} \int_{\bar{B} \setminus \omega} |\nabla_A v|^2 + \frac{r(\bar{B})(r_1 - r_0)}{2} \int_{\bar{B}} (\operatorname{curl} A)^2 \geqslant \int_0^s \sum_{B \in \bar{B} \cap \mathcal{B}(t)} \pi d_B^2 \left(1 - \frac{r(\mathcal{B}(t))}{2(r_1 - r_0)} \right) dt, \tag{3.13}$$

and

$$\frac{1}{2}\int_{\bar{B}\setminus\omega}|\nabla_A v|^2+\frac{r(\bar{B})(r_1-r_0)}{2}\int_{\bar{B}}(\operatorname{curl}A)^2\geqslant\pi|d_{\bar{B}}|\left(\log\frac{r_1}{r_0}-\log 2\right),\tag{3.14}$$

where $r_0:=r(\mathcal{B}_0)$ *and* $r_1:=r(\mathcal{B}(s))=e^s r_0$.

Proof. Set $\lambda=r_1-r_0$, each $\beta_B=1$, and disregard the $|\nabla_{A+G}v|$ term and the curl terms on $\mathcal{B}_0$ in (3.7) to get (3.13). If $\log\frac{r_1}{r_0}<\log 2$, then (3.14) follows trivially. On the other hand, if $\log\frac{r_1}{r_0}\geqslant\log 2$, then $r_1\geqslant 2r_0$, which implies

$$1-\frac{r(\mathcal{B}(t))}{2(r_1-r_0)}\geqslant 1-\frac{r_1}{2(r_1-r_0)}=\frac{r_1-2r_0}{2(r_1-r_0)}\geqslant 0.\tag{3.15}$$

Then (3.14) follows by noting that $r_1=e^s r_0$,

$$\frac{d}{dt}r\big(\mathcal{B}(t)\big)=r\big(\mathcal{B}(t)\big),\tag{3.16}$$

and (see Lemma 4.2 in [7])

$$\sum_{B\in\bar{B}\cap\mathcal{B}(t)}d_B^2\geqslant\sum_{B\in\bar{B}\cap\mathcal{B}(t)}|d_B|\geqslant|d_{\bar{B}}|.\qquad\square\tag{3.17}$$

We will need the following modification of the previous corollary later. It is a slight modification of Proposition 4.3 from [7].

Lemma 3.5. *Under the same assumptions as in Proposition* 3.3 *we have*

$$\frac{1}{2}\int_{\bar{B}\setminus\omega}|\nabla_A v|^2+\frac{r(\bar{B})r_1}{2}\int_{\bar{B}}(\operatorname{curl}A)^2\geqslant\frac{2\pi}{3}\int_0^s\sum_{B\in\bar{B}\cap\mathcal{B}(t)}d_B^2\,dt.\tag{3.18}$$

Proof. Lemma 4.4 from [7] provides the lower bound on circles, $\partial B=\partial B(a,r)$:

$$\frac{1}{2}\int_{\partial B}|\nabla_A v|^2+\frac{\lambda}{2}\int_B(\operatorname{curl}A)^2\geqslant\pi\frac{d_B^2}{r}\left(\frac{2\lambda}{2\lambda+r}\right).\tag{3.19}$$

We now set $\lambda=r_1$, bound

$$\frac{2r_1}{2r_1+r}\geqslant\frac{2}{3},$$

and proceed as before to conclude. $\square$

4. Initial and final balls

In this section we record the energy estimates that couple to the ball construction. For technical reasons that will arise in the proof of Theorem 1 we must use the ball growth lemma in two phases, just as in Chapter 4 of [7]. The first phase produces a collection of initial balls that cover the set where $|u|$ is far from unity and on which lower bounds of a type needed in the proof of Theorem 1 are satisfied. This initial collection contains as a subset a collection of balls on which we initially define the function G. The second phase produces a collection of final balls, grown from the initial balls, of a chosen size and on which nice lower bounds hold. In the final section we finally specify the values of the β_B used to define G and show that certain lower bounds hold with this choice of constants.

4.1. The initial balls

Before we can produce the collection of initial balls, we must first produce a collection of balls that covers the set where $|u|$ is far from unity. This is accomplished via the following lemma (Proposition 4.8 from [7]), which shows how the radius of this set is controlled by the energy of $|u|$.

Lemma 4.1. *Let $M, \varepsilon, \delta > 0$ be such that $\varepsilon, \delta < 1$, and let $u \in C^1(\Omega, \mathbb{C})$ satisfy the bound $F_\varepsilon(|u|, \Omega) \leqslant M$. Then*

$$r\left(\left\{x \in \Omega_\varepsilon \;\middle|\; \left|\left|u(x)\right| - 1\right| \geqslant \delta\right\}\right) \leqslant C\frac{\varepsilon M}{\delta^2} \tag{4.1}$$

where C is a universal constant and $\Omega_\varepsilon = \{x \in \Omega \mid d(x, \partial\Omega) > \varepsilon\}$.

The next technical result shows how to bound from below the modified radius of sub- and super-level sets.

Lemma 4.2. *Let $\Omega \subset \mathbb{R}^2$ be open, $\Omega_\varepsilon = \{x \in \Omega \mid d(x, \partial\Omega) > \varepsilon\}$, and suppose $\mathcal{B}$ is a finite collection of disjoint, closed balls that cover the set*

$$\left\{x \in \Omega_\varepsilon \;\middle|\; \left|\left|u(x)\right| - 1\right| \geqslant \delta\right\}.$$

Let $\mathcal{B}_b$ denote the subcollection of balls in $\mathcal{B}$ that intersect $\partial\Omega_\varepsilon$, and let $\mathcal{B}_i$ denote the subcollection of balls in $\mathcal{B}$ contained in the interior of Ω_ε (i.e. $\mathcal{B} = \mathcal{B}_b \cup \mathcal{B}_i$). Define $\tilde{\Omega} = \Omega_\varepsilon \setminus (\bigcup_{B\in\mathcal{B}_b} B)$. For $0 < s \leqslant t$ define the sets $\omega_t = \{x \in \Omega_\varepsilon \mid |u| \leqslant t\}$, $\omega^t = \{x \in \Omega_\varepsilon \mid |u| \geqslant t\}$, and $\omega_s^t = \omega_s \cup \omega^t$. Then

$$\begin{aligned}
&r_{\Omega_\varepsilon}(\omega_t) \geqslant r(\omega_t \cap \tilde{\Omega}) \quad \textit{for } t \in (0, 1-\delta),\\
&r_{\Omega_\varepsilon}\left(\omega^t\right) \geqslant r\left(\omega^t \cap \tilde{\Omega}\right) \quad \textit{for } t \in (1+\delta, \infty), \textit{ and}\\
&r_{\Omega_\varepsilon}\left(\omega_s^t\right) \geqslant r\left(\omega_s^t \cap \tilde{\Omega}\right) \quad \textit{for } s \in (0, 1-\delta),\ t \in (1+\delta, \infty).
\end{aligned} \tag{4.2}$$

Proof. Suppose that $t \in (0, 1-\delta)$ and let $\operatorname{Int}(\cdot)$ denote the interior of a set. Write $V = \bigcup_{B\in\mathcal{B}} B$ and $V_i = \bigcup_{B\in\mathcal{B}_i} B$. Since the inclusions

$$\mathrm{Int}(V) \supseteq \mathrm{Int}\big(\big\{x \in \Omega_\varepsilon \;\big|\; \big|\big|u(x)\big| - 1\big| \geqslant \delta\big\}\big) \supset \omega_t \tag{4.3}$$

hold, we have that $\omega_t \cap \tilde{\Omega} = \omega_t \cap V_i$, and hence $r(\omega_t \cap \tilde{\Omega}) = r(\omega_t \cap V_i)$. When combined with the fact that V_i is a compact subset of Ω_ε and $\partial V_i \cap \omega_t = \emptyset$, this yields the first estimate in (4.2). Similar arguments prove the second and third assertions. □

We now construct the initial balls. The following proposition is the analogue of Proposition 4.7 of [7], but here we have an extra term of the form

$$\int |\nabla_{A+G} v|^2.$$

Note that items 1, 2, and 3 are the same as those found in [7]; item 4 is new.

Proposition 4.3. *Let $\alpha \in (0,1)$. There exists $\varepsilon_0 > 0$ (depending on α) such that for $\varepsilon \leqslant \varepsilon_0$ and $u \in C^1(\Omega, \mathbb{C})$ with $F_\varepsilon(|u|, \Omega) \leqslant \varepsilon^{\alpha-1}$, the following hold.*

There exists a finite, disjoint collection of closed balls, denoted by $\mathcal{B}_0$, with the following properties.

1. *$r(\mathcal{B}_0) = C\varepsilon^{\alpha/2}$, where C is a universal constant.*
2. *$\{x \in \Omega_\varepsilon \mid ||u(x)| - 1| \geqslant \delta\} \subset V_0 := \Omega_\varepsilon \cap (\bigcup_{B \in \mathcal{B}_0} B)$, where $\delta = \varepsilon^{\alpha/4}$.*
3. *Write $v = u/|u|$. For $t \in (0, 1-\delta)$ we have the estimate*

$$\frac{1}{2} \int\limits_{V_0 \setminus \omega_t} |\nabla_A v|^2 + \frac{r(\mathcal{B}_0)^2}{2} \int\limits_{V_0} (\operatorname{curl} A)^2 \geqslant \pi D_0 \left(\log \frac{r(\mathcal{B}_0)}{r_{\Omega_\varepsilon}(\omega_t)} - C \right), \tag{4.4}$$

where

$$D_0 = \sum_{\substack{B \in \mathcal{B}_0 \\ B \subset \Omega_\varepsilon}} |d_B|. \tag{4.5}$$

4. *There exists a family of finite collections of closed, disjoint balls $\{\mathcal{C}(s)\}_{s \in [0,\sigma]}$, all of which are contained in V_0, and that are grown according to the ball growth lemma from an initial collection, $\mathcal{C}(0)$, that covers the set $\omega_{1/2}^{3/2} \cap V_0$. The number σ is such that $r(\mathcal{C}(\sigma)) = \frac{3}{8} r(\mathcal{B}_0)$. Let $G : V_0 \to \mathbb{R}^2$ be the function defined by using Ω_ε and $\{\mathcal{C}(s)\}_{s \in [0,\sigma]}$ in (3.6) and then extended by zero to the rest of V_0. For each $\lambda > 0$ we have the estimate*

$$\frac{1}{2} \int\limits_{V_0 \setminus \omega_{1/2}^{3/2}} |\nabla_A v|^2 + \sum_{B \in \mathcal{B}_0} \frac{r(B)\lambda}{2} \int\limits_{B \cap \Omega} (\operatorname{curl} A)^2$$
$$\geqslant \int_0^\sigma \sum_{\substack{\bar{B} \in \mathcal{C}(\sigma) \\ \bar{B} \subset \Omega_\varepsilon}} \sum_{B \in \bar{B} \cap \mathcal{C}(t)} \pi d_B^2 \left(2\beta_B - \beta_B^2 - \frac{\beta_B^2 r(\mathcal{C}(t))}{2\lambda} \right) dt + \frac{1}{2} \int\limits_{V_0 \setminus \omega_{1/2}^{3/2}} |\nabla_{A+G} v|^2. \tag{4.6}$$

Proof. We break the proof into six steps. The first four consist of finding four collections of balls that are used to create the initial collection $\mathcal{B}_0$. The last two steps prove the estimates of items 3 and 4.

Step 1. Using $M = \varepsilon^{\alpha-1}$ and $\delta = \varepsilon^{\alpha/4}$ in Lemma 4.1 produces a collection of disjoint, closed balls $\mathcal{E}$ that cover the set $\{x \in \Omega_\varepsilon \mid ||u(x)| - 1| \geqslant \delta\}$ such that $R := r(\mathcal{E}) \leqslant C\varepsilon^{\alpha/2}$. We will eventually need to use Lemma 4.2, so we employ its notation by breaking the collection $\mathcal{E}$ into subcollections $\mathcal{E}_i$ and $\mathcal{E}_b$ and defining the set $\tilde{\Omega} = \Omega_\varepsilon \setminus (\bigcup_{B\in\mathcal{E}_b} B)$.

Step 2. By the definition of the radius of a set, for any $t \in (0, 1-\delta)$ we can cover $\omega_t \cap \tilde{\Omega}$ by a collection of disjoint balls, denoted by $\mathcal{B}_t^0$, with total radius less than $2r(\omega_t \cap \tilde{\Omega})$. Since $r(\omega_t \cap \tilde{\Omega}) \leqslant R$, we can use Lemma 2.2 to grow the collection $\mathcal{B}_t^0$ into a collection $\mathcal{B}_t$ such that $r(\mathcal{B}_t) = 2R$. We then utilize Corollary 3.4 on each of the balls in $\mathcal{B}_t$ that is contained in $\tilde{\Omega}$ and sum to get the estimate

$$\frac{1}{2}\int_{V_t\setminus\omega_t} |\nabla_A v|^2 + \frac{4R^2}{2}\int_{V_t} (\operatorname{curl} A)^2 \geqslant \pi D_t\left(\log\frac{2R}{2r(\omega_t\cap\tilde{\Omega})} - \log 2\right), \tag{4.7}$$

where

$$V_t = \tilde{\Omega} \cap \left(\bigcup_{B\in\mathcal{B}_t} B\right), \quad \text{and}$$

$$D_t = \sum_{\substack{B\in\mathcal{B}_t\\ B\subset\tilde{\Omega}}} |d_B|.$$

Choose $\bar{t} \in (0, 1-\delta)$ such that $D_{\bar{t}}$ is minimal.

Step 3. Let m denote the supremum of

$$\mathcal{F}(K) := \frac{1}{2}\int_{(K\cap\tilde{\Omega})\setminus\omega} |\nabla_A v|^2 + \frac{4R^2}{2}\int_{K\cap\tilde{\Omega}} (\operatorname{curl} A)^2$$

over compact $K \subset \Omega$ such that $r(K) < 2R$. Choose K so that $r(K) < 2R$ and $\mathcal{F}(K) \geqslant m - 1$. Cover K by a collection of disjoint, closed balls $\mathcal{K}$ such that $r(\mathcal{K}) = 2R$ (the existence of such a collection is guaranteed by the ball growth lemma).

Step 4. We can cover $\omega_{1/2}^{3/2} \cap \tilde{\Omega}$ by a collection of disjoint balls, denoted by $\mathcal{C}_0$, with radius less than $\frac{3}{2}r(\omega_{1/2}^{3/2} \cap \tilde{\Omega})$. We use the ball growth lemma, applied to $\mathcal{C}_0$, to produce a family of collections $\{\mathcal{C}(s)\}$ with $s \in (0, \sigma)$,

$$\sigma = \log\left(\frac{3R}{r(\mathcal{C}_0)}\right).$$

Let $\mathcal{C} = \mathcal{C}(\sigma)$ and note that by construction $r(\mathcal{C}) = 3R$.

Step 5. Define $\mathcal{B}_0$ to be a collection of disjoint balls that cover the balls in $\mathcal{B}_{\bar{t}}$, $\mathcal{K}$, $\mathcal{C}$, and $\mathcal{E}$. We may choose such a collection so that $r(\mathcal{B}_0) = 8R$. Let $V_0 = \Omega_\varepsilon \cap (\bigcup_{B\in\mathcal{B}_0} B)$. Then

$$I := \frac{1}{2}\int\limits_{V_0\setminus\omega_t} |\nabla_A v|^2 + \frac{r(\mathcal{B}_0)^2}{2}\int\limits_{V_0} (\operatorname{curl} A)^2 \geqslant \mathcal{F}(K) + \frac{1}{2}\int\limits_{\omega\setminus\omega_t} |\nabla_A v|^2, \tag{4.8}$$

and by the construction of K and V_t for any $t \in (0, 1-\delta)$, this implies

$$\begin{aligned} I + 1 &\geqslant \mathcal{F}(V_t) + \frac{1}{2}\int\limits_{\omega\setminus\omega_t} |\nabla_A v|^2 \\ &\geqslant \frac{1}{2}\int\limits_{V_t\setminus\omega_t} |\nabla_A v|^2 + \frac{4R^2}{2}\int\limits_{V_t} (\operatorname{curl} A)^2 \\ &\geqslant \pi D_t\left(\log\frac{2R}{2r(\omega_t\cap\tilde{\Omega})} - \log 2\right) \\ &\geqslant \pi D_t\left(\log\frac{r(\mathcal{B}_0)}{r_{\Omega_\varepsilon}(\omega_t)} - C\right), \end{aligned} \tag{4.9}$$

where the last line follows from (4.2) and the fact that $r(\mathcal{B}_0) = 8R$. By the choice of $\bar{t}$,

$$D_t \geqslant D_{\bar{t}} = \sum_{\substack{B\in\mathcal{B}_{\bar{t}}\\ B\subset\tilde{\Omega}}} |d_B|. \tag{4.10}$$

We break the collection of balls in the last sum in (4.10) into two subcollections:

$$\begin{aligned} I_1 &:= \{B \in \mathcal{B}_{\bar{t}} \mid B \subseteq \tilde{\Omega},\ B \subseteq B' \in \mathcal{B}_0 \text{ so that } B' \cap \partial\Omega_\varepsilon \neq \emptyset\}, \\ I_2 &:= \{B \in \mathcal{B}_{\bar{t}} \mid B \subseteq \tilde{\Omega},\ B \subseteq B' \in \mathcal{B}_0 \text{ so that } B' \subseteq \Omega_\varepsilon\}. \end{aligned}$$

Then

$$\sum_{\substack{B\in\mathcal{B}_{\bar{t}}\\ B\subset\tilde{\Omega}}} |d_B| = \sum_{B\in I_1} |d_B| + \sum_{B\in I_2} |d_B| \geqslant 0 + \sum_{\substack{B\in\mathcal{B}_0\\ B\subset\Omega_\varepsilon}} |d_B| = D_0, \tag{4.11}$$

where the inequality follows from Lemma 4.2 in [7]. Combining (4.9), (4.10), and (4.11) yields (4.4).

Step 6. Let U be the union of the balls in $\mathcal{C}_0$ that are contained in Ω_ε and W be the union of the balls in $\mathcal{C}$ that are contained in Ω_ε. Then applying Proposition 3.3 to each $\bar{B} \in \mathcal{C}$ such that $\bar{B} \subset \Omega_\varepsilon$ and summing, we get the estimate

$$\frac{1}{2}\int_{W\setminus U}|\nabla_A v|^2+\sum_{\substack{\bar{B}\in\mathcal{C}\\ \bar{B}\subset\Omega_\varepsilon}}\frac{r(\bar{B})\lambda}{2}\int_{\bar{B}}(\operatorname{curl}A)^2$$
$$\geqslant\frac{1}{2}\int_{W\setminus U}|\nabla_{A+G}v|^2+\int_0^\sigma\sum_{\substack{\bar{B}\in\mathcal{C}\\ \bar{B}\subset\Omega_\varepsilon}}\sum_{B\in\bar{B}\cap\mathcal{C}(t)}\pi d_B^2\left(2\beta_B-\beta_B^2-\frac{\beta_B^2 r(\mathcal{C}(t))}{2\lambda}\right)dt. \tag{4.12}$$

G vanishes in the regions $V_0\setminus W$ and $U\setminus\omega_{1/2}^{3/2}$, so

$$\frac{1}{2}\int_{(V_0\setminus W)\cup(U\setminus\omega_{1/2}^{3/2})}|\nabla_A v|^2=\frac{1}{2}\int_{(V_0\setminus W)\cup(U\setminus\omega_{1/2}^{3/2})}|\nabla_{A+G}v|^2. \tag{4.13}$$

Adding (4.13) to both sides of (4.12) and noting that

$$\sum_{\substack{\bar{B}\in\mathcal{C}\\ \bar{B}\subset\Omega_\varepsilon}}\frac{r(\bar{B})\lambda}{2}\int_{\bar{B}}(\operatorname{curl}A)^2\leqslant\sum_{B\in\mathcal{B}_0}\frac{r(B)\lambda}{2}\int_{B\cap\Omega}(\operatorname{curl}A)^2 \tag{4.14}$$

yields (4.6). □

4.2. The final balls

The next proposition constructs the final balls from the initial ones constructed in Proposition 4.3. Items 1, 2, and 3 are the same as those of Theorem 4.1 of [7]; item 4 contains the novel estimate with the G-term.

Proposition 4.4. *Let $\alpha\in(0,1)$. There exists $\varepsilon_0>0$ (depending on α) such that for $\varepsilon\leqslant\varepsilon_0$ and $u\in C^1(\Omega,\mathbb{C})$ with $F_\varepsilon(|u|,\Omega)\leqslant\varepsilon^{\alpha-1}$, the following hold.*

For any $1>r>C\varepsilon^{\alpha/2}$, where C is a universal constant, there exists a finite, disjoint collection of closed balls, denoted by $\mathcal{B}$, with the following properties.

1. $r(\mathcal{B})=r$.
2. $\{x\in\Omega_\varepsilon\mid ||u(x)|-1|\geqslant\delta\}\subset V:=\Omega_\varepsilon\cap(\bigcup_{B\in\mathcal{B}}B)$, *where* $\delta=\varepsilon^{\alpha/4}$.
3. *Write $v=u/|u|$. For $t\in(0,1-\delta)$ we have the estimate*

$$\frac{1}{2}\int_{V\setminus\omega_t}|\nabla_A v|^2+\frac{r^2}{2}\int_V(\operatorname{curl}A)^2\geqslant\pi D\left(\log\frac{r}{r_{\Omega_\varepsilon}(\omega_t)}-C\right), \tag{4.15}$$

where

$$D=\sum_{\substack{B\in\mathcal{B}\\ B\subset\Omega_\varepsilon}}|d_B|. \tag{4.16}$$

4. *Let* $G : \Omega \to \mathbb{R}^2$ *be the extension, according to* (3.6), *of the G from item* 4 *in Proposition* 4.3. *Write* $s = \log \frac{r}{r(\mathcal{B}_0)}$. *Then*

$$\frac{1}{2} \int_{V \setminus \omega_{1/2}^{3/2}} |\nabla_A v|^2 + \sum_{\bar{B} \in \mathcal{B}} \frac{r(\bar{B})(r - r(\mathcal{B}_0))}{2} \int_{\bar{B} \cap \Omega} (\operatorname{curl} A)^2$$
$$\geqslant \frac{1}{2} \int_{V \setminus \omega_{1/2}^{3/2}} |\nabla_{A+G} v|^2 + \int_0^s \sum_{\substack{\bar{B} \in \mathcal{B} \\ \bar{B} \subset \Omega_\varepsilon}} \sum_{B \in \bar{B} \cap \mathcal{B}(t)} \pi d_B^2 \left(2\beta_B - \beta_B^2 - \frac{\beta_B^2 r(\mathcal{B}(t))}{2(r - r(\mathcal{B}_0))} \right) dt$$
$$+ \int_0^\sigma \sum_{\substack{\bar{B} \in \mathcal{C}(\sigma) \\ \bar{B} \subset \Omega_\varepsilon}} \sum_{B \in \bar{B} \cap \mathcal{C}(t)} \pi d_B^2 \left(2\beta_B - \beta_B^2 - \frac{\beta_B^2 r(\mathcal{C}(t))}{2(r(\mathcal{C}(\sigma)) - r(\mathcal{C}_0))} \right) dt. \tag{4.17}$$

Proof. Lemma 4.3 provides an initial set of disjoint, closed balls $\mathcal{B}_0$. We grow these according to the ball growth lemma to produce $\{\mathcal{B}(t)\}_{t \in [0,s]}$ with s chosen so that $r(\mathcal{B}(s)) = r$, i.e. $s = \log \frac{r}{r(\mathcal{B}_0)}$. By construction, items 1 and 2 are proved. Let $\mathcal{B} = \mathcal{B}(s)$, and write $V = \Omega_\varepsilon \cap \bigcup_{B \in \mathcal{B}} B$, $V_0 = \Omega_\varepsilon \cap \bigcup_{B \in \mathcal{B}_0} B$. Let $G : V_0 \to \mathbb{R}^2$ be the function defined in item 4 of Proposition 4.3. We then use $\mathcal{B}_0$ and $\mathcal{B}$ to extend $G : \Omega \to \mathbb{R}^2$ according to (3.6).

We analyze the balls in $\mathcal{B}$ according to whether or not they are contained entirely in Ω_ε. For balls $\bar{B} \in \mathcal{B}$ such that $\bar{B} \subset \Omega_\varepsilon$, we use (3.14), and for the other balls we use the trivial non-negative bound. Summing over all balls in $\mathcal{B}$, we get

$$\frac{1}{2} \int_{V \setminus V_0} |\nabla_A v|^2 + \sum_{\bar{B} \in \mathcal{B}} \frac{r(\bar{B})(r - r(\mathcal{B}_0))}{2} \int_{\bar{B} \cap \Omega} (\operatorname{curl} A)^2 \geqslant \pi D \left(\log \frac{r}{r(\mathcal{B}_0)} - \log 2 \right). \tag{4.18}$$

Adding (4.4) to (4.18) and noting that $D_0 \geqslant D$ then yields (4.15).

To prove (4.17) we proceed similarly, using different estimates for the balls in $\mathcal{B}$ according to whether or not they are contained in Ω_ε. For balls $\bar{B} \in \mathcal{B}$ such that $\bar{B} \subset \Omega_\varepsilon$ we use Proposition 3.3 to get the estimate

$$\frac{1}{2} \int_{\bar{B} \setminus V_0} |\nabla_A v|^2 + \frac{r(\bar{B})\lambda}{2} \int_{\bar{B}} (\operatorname{curl} A)^2 - \sum_{B \in \bar{B} \cap \mathcal{B}_0} \frac{r(B)\lambda}{2} \int_B (\operatorname{curl} A)^2$$
$$\geqslant \frac{1}{2} \int_{\bar{B} \setminus V_0} |\nabla_{A+G} v|^2 + \int_0^s \sum_{B \in \bar{B} \cap \mathcal{B}(t)} \pi d_B^2 \left(2\beta_B - \beta_B^2 - \frac{\beta_B^2 r(\mathcal{B}(t))}{2\lambda} \right). \tag{4.19}$$

On the other hand, the construction of G guarantees that it vanishes on all balls $\bar{B} \in \mathcal{B}$ such that $\bar{B} \cap \partial \Omega_\varepsilon \neq \emptyset$, and so for such $\bar{B}$ we trivially have the estimate

$$\frac{1}{2} \int_{(\bar{B} \cap \Omega) \setminus V_0} |\nabla_A v|^2 + \frac{r(\bar{B})\lambda}{2} \int_{\bar{B} \cap \Omega} (\operatorname{curl} A)^2 - \sum_{B \in \bar{B} \cap \mathcal{B}_0} \frac{r(B)\lambda}{2} \int_{B \cap \Omega} (\operatorname{curl} A)^2$$
$$\geqslant \frac{1}{2} \int_{(\bar{B} \cap \Omega) \setminus V_0} |\nabla_{A+G} v|^2. \tag{4.20}$$

Summing (4.19) and (4.20) over all balls in $\mathcal{B}$ then yields the estimate

$$\frac{1}{2}\int_{V\setminus V_0}|\nabla_A v|^2+\sum_{\bar{B}\in\mathcal{B}}\frac{r(\bar{B})\lambda}{2}\int_{\bar{B}\cap\Omega}(\operatorname{curl}A)^2-\sum_{B\in\mathcal{B}_0}\frac{r(B)\lambda}{2}\int_{B\cap\Omega}(\operatorname{curl}A)^2$$
$$\geqslant\frac{1}{2}\int_{V\setminus V_0}|\nabla_{A+G}v|^2+\int_0^s\sum_{\substack{\bar{B}\in\mathcal{B}\\ \bar{B}\subset\Omega_\varepsilon}}\sum_{B\in\bar{B}\cap\mathcal{B}(t)}\pi d_B^2\left(2\beta_B-\beta_B^2-\frac{\beta_B^2 r(\mathcal{B}(t))}{2\lambda}\right)dt. \tag{4.21}$$

We insert $\lambda=r-r(\mathcal{B}_0)$ into (4.21) and $\lambda=r(\mathcal{C}(\sigma))-r(\mathcal{C}_0)=\frac{3r(\mathcal{B}_0)}{8}-r(\mathcal{C}_0)$ into (4.6) and add the estimates together. Noting that

$$\frac{3r(\mathcal{B}_0)}{8}-r(\mathcal{C}_0)-r+r(\mathcal{B}_0)\leqslant C\varepsilon^{\alpha/2}-r\leqslant 0, \tag{4.22}$$

we arrive at the estimate (4.17). □

4.3. *Degree analysis and selection of the β_B values*

We will now select the values of the β_B used to define G. Ultimately, later in Theorem 2, we will get rid of G altogether by bounding its $L^{2,\infty}$ norm by a term of the order D^2. This bound, the proof of which is Proposition 6.4, requires the values of the β_B to be small. However, since they play a role in the lower bounds of Proposition 4.4, we cannot choose the β_B to be too small. We balance these two demands by introducing a parameter η to measure when β_B must be small and when it can assume the natural choice for its value, 1.

The next two results establish that for a ball $\bar{B}\in\mathcal{B}(s)$ there is a transition time (depending on η) in the family $\bar{B}\cap\mathcal{B}(t)$ before which we can take $\beta_B=1$, and after which we must use something more complicated.

Lemma 4.5. *Let $\mathcal{B}_0$ be a finite collection of disjoint, closed balls. Suppose further that the collection $\mathcal{B}_0$ has the degree covering property that for all balls $B\subset\Omega\setminus(\bigcup_{S\in\mathcal{B}_0}S)$, it is the case that $d_B=0$. In other words, the collection $\mathcal{B}_0$ covers all of the vortices. Let $\mathcal{B}(t)$, $t\in[0,s]$, be a t-parameterized family of finite collections of disjoint, closed balls. Suppose that $\mathcal{B}_0=\mathcal{B}(0)$ and that*

$$\bigcup_{B\in\mathcal{B}(t_1)}B\subseteq\bigcup_{B\in\mathcal{B}(t_2)}B\quad\textit{for } t_1\leqslant t_2. \tag{4.23}$$

Fix $\bar{B}\in\mathcal{B}(s)$. Define the negative and positive vorticity masses by

$$N(t):=\sum_{\substack{B\in\bar{B}\cap\mathcal{B}(t)\\ d_B<0}}|d_B|,$$
$$P(t):=\sum_{\substack{B\in\bar{B}\cap\mathcal{B}(t)\\ d_B>0}}d_B. \tag{4.24}$$

Then for any $\eta \in (0,1)$*, the following hold.*

1. *If* $d_{\bar{B}} \geqslant 0$ *and the inequality*

$$N(s_0) \leqslant \eta P(s_0) \tag{4.25}$$

holds for some $s_0 \in [0,s]$*, then* $N(t) \leqslant \eta P(t)$ *for all* $t \in [s_0, s]$.

2. *If* $d_{\bar{B}} < 0$ *and the inequality*

$$P(s_0) \leqslant \eta N(s_0) \tag{4.26}$$

holds for some $s_0 \in [0,s]$*, then* $P(t) \leqslant \eta N(t)$ *for all* $t \in [s_0, s]$.

Proof. Take $d_{\bar{B}} \geqslant 0$; the following proves (4.25), and a similar argument with $d_{\bar{B}} < 0$ proves (4.26). Let $n(t) = \#\mathcal{B}(t)$. Then by the inclusion property (4.23), $n(t)$ is a decreasing $\mathbb{N}$-valued function. Hence there exist finitely many times $0 = t_0 < \cdots < t_K = s$ such that $n(t)$ is constant on (t_i, t_{i+1}). This implies that for $t_i < s < t < t_{i+1}$ and $B \in \mathcal{B}(t)$, there exists exactly one ball $B' \in \mathcal{B}(s)$ such that $B' \subseteq B$, and by the degree covering property, $d_B = d_{B'}$. It follows that $N(t)$ and $P(t)$ are also constant on each (t_i, t_{i+1}). Then it suffices to show that if $N(t_k) \leqslant \eta P(t_k)$, then $N(t_{k+1}) \leqslant \eta P(t_{k+1})$.

Given a ball $C \in \mathcal{B}(t_{k+1})$, the inclusion property guarantees that there is a finite collection $\{B_1, \dots, B_j\} \subseteq \mathcal{B}(t_k)$ such that $B_i \subseteq C$ for $i = 1, \dots, j$. We then get

$$\begin{aligned} |d_C| &= -\sum_{\substack{i \in \{1,\dots,j\} \\ d_{B_i} \geqslant 0}} d_{B_i} + \sum_{\substack{i \in \{1,\dots,j\} \\ d_{B_i} < 0}} |d_{B_i}| \quad \text{if } d_C < 0, \quad \text{and} \\ |d_C| &= \sum_{\substack{i \in \{1,\dots,j\} \\ d_{B_i} \geqslant 0}} d_{B_i} - \sum_{\substack{i \in \{1,\dots,j\} \\ d_{B_i} < 0}} |d_{B_i}| \quad \text{if } d_C \geqslant 0. \end{aligned} \tag{4.27}$$

We must now subdivide the collection $\bar{B} \cap \mathcal{B}(t_k)$ according to the degrees of balls in $\bar{B} \cap \mathcal{B}(t_{k+1})$. Define the collections

$$\begin{aligned} I_{-,-} &= \bigl\{ B \in \bar{B} \cap \mathcal{B}(t_k) \bigm| d_B < 0,\ \exists B' \in \bar{B} \cap \mathcal{B}(t_{k+1}) \text{ s.t. } B \subset B',\ d_{B'} < 0 \bigr\}, \\ I_{-,+} &= \bigl\{ B \in \bar{B} \cap \mathcal{B}(t_k) \bigm| d_B < 0,\ \exists B' \in \bar{B} \cap \mathcal{B}(t_{k+1}) \text{ s.t. } B \subset B',\ d_{B'} \geqslant 0 \bigr\}, \\ I_{+,-} &= \bigl\{ B \in \bar{B} \cap \mathcal{B}(t_k) \bigm| d_B \geqslant 0,\ \exists B' \in \bar{B} \cap \mathcal{B}(t_{k+1}) \text{ s.t. } B \subset B',\ d_{B'} < 0 \bigr\}, \\ I_{+,+} &= \bigl\{ B \in \bar{B} \cap \mathcal{B}(t_k) \bigm| d_B \geqslant 0,\ \exists B' \in \bar{B} \cap \mathcal{B}(t_{k+1}) \text{ s.t. } B \subset B',\ d_{B'} \geqslant 0 \bigr\}. \end{aligned}$$

Now we can estimate

$$\begin{aligned} \eta \sum_{B \in I_{-,+}} |d_B| + \sum_{B \in I_{-,-}} |d_B| &\leqslant \sum_{B \in I_{-,+}} |d_B| + \sum_{B \in I_{-,-}} |d_B| = N(t_k) \\ &\leqslant \eta P(t_k) = \eta \sum_{B \in I_{+,-}} d_B + \eta \sum_{B \in I_{+,+}} d_B \\ &\leqslant \sum_{B \in I_{+,-}} d_B + \eta \sum_{B \in I_{+,+}} d_B. \end{aligned} \tag{4.28}$$

After regrouping terms according to containment and using (4.27) and (4.28) we conclude

$$N(t_{k+1}) = \sum_{B\in I_{-,-}} |d_B| - \sum_{B\in I_{+,-}} d_B \leqslant \eta \sum_{B\in I_{+,+}} d_B - \eta \sum_{B\in I_{-,+}} |d_B| = \eta P(t_{k+1}). \qquad \square \tag{4.29}$$

We use this lemma to define the transition times.

Corollary 4.6. *Assume the hypotheses and notation of Lemma* 4.5. *If $d_{\bar B} \geqslant 0$ then there exists $t_0 \in [0,s]$ such that $\eta P(t) < N(t)$ for $t \in [0,t_0)$ and $N(t) \leqslant \eta P(t)$ for $t \in [t_0,s]$. Similarly, if $d_{\bar B} < 0$ then there exists $t_0 \in [0,s]$ such that $\eta N(t) < P(t)$ for $t \in [0,t_0)$ and $P(t) \leqslant \eta N(t)$ for $t \in [t_0,s]$. We call these times, t_0, the transition times.*

Proof. Assume $d_{\bar B} \geqslant 0$. Since there is only one ball in $\bar B \cap \mathcal{B}(s)$, and the degree in $\bar B$ is non-negative, the inequality $N(s) \leqslant \eta P(s)$ is satisfied trivially. An application of Lemma 4.5 proves the existence of t_0. A similar argument works for the case when $d_{\bar B} < 0$. $\square$

With the transition times defined we can finally set the values of the β_B. Define the collection $\{\mathcal{D}(t)\}_{t\in[0,s+\sigma]}$ by

$$\mathcal{D}(t) = \begin{cases} \mathcal{C}(t), & t \in [0,\sigma), \\ \mathcal{B}(t-\sigma), & t \in [\sigma, s+\sigma]. \end{cases} \tag{4.30}$$

Let $\eta \in (0,1)$. For each $\bar B \in \mathcal{B}$ let $t_{\bar B} \in [0, s+\sigma]$ denote the transition time for the collection $\bar B \cap \mathcal{D}(t)$ obtained from Corollary 4.6 (the times depend on η). We now specify the values of β_B in the definition of G. Note that the construction of G only requires specifying the values of β_B for those balls B such that $B \subset \bar B \in \mathcal{B}$ with $\bar B \subset \Omega_\varepsilon$. Then for $B \in \bar B \cap \mathcal{D}(t)$ for some $\bar B \in \mathcal{B}$, we define

$$\beta_B = \begin{cases} 1, & \text{if } t \in [0, t_{\bar B}), \\ |d_{\bar B}|^{\frac12} (\sum_{B'\in \bar B\cap\mathcal{D}(t)} d_{B'}^2)^{-\frac12}, & \text{if } t \in [t_{\bar B}, s+\sigma]. \end{cases} \tag{4.31}$$

Note that if

$$\sum_{B'\in \bar B\cap\mathcal{D}(t)} d_{B'}^2 = 0,$$

then $d_{\bar B} = 0$ as well, and we take the second case in (4.31) to equal 0. Further, note that in the second case, the β_B are chosen so that for $t \in [t_{\bar B}, s+\sigma]$

$$\sum_{B\in \bar B\cap\mathcal{D}(t)} d_B^2 \beta_B^2 = |d_{\bar B}|. \tag{4.32}$$

The following proposition shows that G is still useful for the lower bounds with these values of β_B.

Proposition 4.7. *With G defined as above, and under the assumptions of Proposition* 4.4, *we have the estimate*

$$\frac{1}{2}\int_{V\setminus\omega_{1/2}^{3/2}} |\nabla_A v|^2 + \frac{r^2}{2}\int_V (\operatorname{curl} A)^2 \geqslant \frac{1}{2}\int_{V\setminus\omega_{1/2}^{3/2}} |\nabla_{A+G} v|^2 + \pi D\left(\log\frac{r}{r_{\Omega_\varepsilon}(\omega_{1/2}^{3/2})} - C\right). \tag{4.33}$$

Proof. To prove (4.33) we must deal with the sums in the integrands in (4.17). We begin by showing that the terms in parentheses are non-negative. Since $r(\mathcal{B}_0) = C\varepsilon^{\alpha/2}$ and $\beta_B \leqslant 1$, we can estimate

$$\begin{aligned}
2\beta_B - \beta_B^2 - \frac{\beta_B^2 r(\mathcal{B}(t))}{2(r - r(\mathcal{B}_0))} &= \beta_B^2\left(\frac{2}{\beta_B} - 1 - \frac{r(\mathcal{B}(t))}{2(r - r(\mathcal{B}_0))}\right) \\
&\geqslant \beta_B^2\left(1 - \frac{r(\mathcal{B}(t))}{2(r - r(\mathcal{B}_0))}\right) \geqslant \beta_B^2\left(\frac{r - 2r(\mathcal{B}_0)}{2(r - r(\mathcal{B}_0))}\right) \geqslant 0.
\end{aligned} \tag{4.34}$$

By construction,

$$r(\mathcal{C}_0) < \frac{3}{2} r\big(\omega_{1/2}^{3/2} \cap \tilde{\Omega}\big) \leqslant \frac{3}{2} R = \frac{1}{2} r\big(\mathcal{C}(\sigma)\big), \tag{4.35}$$

and so we can similarly conclude that

$$2\beta_B - \beta_B^2 - \frac{\beta_B^2 r(\mathcal{B}(t))}{2(r(\mathcal{C}(\sigma)) - r(\mathcal{C}_0))} \geqslant 0. \tag{4.36}$$

A simple change of variables $t \mapsto t + \sigma$ allows us to rewrite

$$\begin{aligned}
&\int_0^s \sum_{\substack{\bar{B}\in\mathcal{B}\\ \bar{B}\subset\Omega_\varepsilon}} \sum_{B\in\bar{B}\cap\mathcal{B}(t)} \pi d_B^2\left(2\beta_B - \beta_B^2 - \frac{\beta_B^2 r(\mathcal{B}(t))}{2(r - r(\mathcal{B}_0))}\right) dt \\
&\quad + \int_0^\sigma \sum_{\substack{\bar{B}\in\mathcal{C}(\sigma)\\ \bar{B}\subset\Omega_\varepsilon}} \sum_{B\in\bar{B}\cap\mathcal{C}(t)} \pi d_B^2\left(2\beta_B - \beta_B^2 - \frac{\beta_B^2 r(\mathcal{C}(t))}{2(r(\mathcal{C}(\sigma)) - r(\mathcal{C}_0))}\right) dt \\
&= \sum_{\substack{\bar{B}\in\mathcal{B}\\ \bar{B}\subset\Omega_\varepsilon}} \int_0^{s+\sigma} \sum_{B\in\bar{B}\cap\mathcal{D}(t)} \pi d_B^2\left(2\beta_B - \beta_B^2 - \frac{\beta_B^2 r(\mathcal{D}(t))}{2\lambda(t)}\right) dt,
\end{aligned} \tag{4.37}$$

where

$$\lambda(t) = \begin{cases} r(\mathcal{C}(\sigma)) - r(\mathcal{C}_0), & t \in [0, \sigma), \\ r - r(\mathcal{B}_0), & t \in [\sigma, s+\sigma]. \end{cases}$$

Fix $\bar{B} \in \mathcal{B}$ such that $\bar{B} \subset \Omega_\varepsilon$. For $t \in [0, t_{\bar{B}})$ we have that $\beta_B = 1$, and hence

$$\sum_{B\in\bar{B}\cap\mathcal{D}(t)} \pi d_B^2\left(2\beta_B-\beta_B^2-\frac{\beta_B^2 r(\mathcal{D}(t))}{2\lambda(t)}\right)\geqslant \pi d_{\bar{B}}\left(1-\frac{r(\mathcal{D}(t))}{2\lambda(t)}\right). \tag{4.38}$$

For $t\in[t_{\bar{B}},s+\sigma]$ we similarly estimate

$$\begin{aligned}\sum_{B\in\bar{B}\cap\mathcal{D}(t)} d_B^2\left(2\beta_B-\beta_B^2-\frac{\beta_B^2 r(\mathcal{D}(t))}{2\lambda(t)}\right) &= 2|d_{\bar{B}}|^{\frac12}\left(\sum_{B\in\bar{B}\cap\mathcal{D}(t)} d_B^2\right)^{\frac12}-|d_{\bar{B}}|\left(1+\frac{r(\mathcal{D}(t))}{2\lambda(t)}\right)\\ &\geqslant 2|d_{\bar{B}}|^{\frac12}|d_{\bar{B}}|^{\frac12}-|d_{\bar{B}}|\left(1+\frac{r(\mathcal{D}(t))}{2\lambda(t)}\right)\\ &= |d_{\bar{B}}|\left(1-\frac{r(\mathcal{D}(t))}{2\lambda(t)}\right).\end{aligned} \tag{4.39}$$

This proves that

$$\begin{aligned}&\sum_{\substack{\bar{B}\in\mathcal{B}\\ \bar{B}\subset\Omega_\varepsilon}}\int_0^{s+\sigma}\sum_{B\in\bar{B}\cap\mathcal{D}(t)}\pi d_B^2\left(2\beta_B-\beta_B^2-\frac{\beta_B^2 r(\mathcal{D}(t))}{2\lambda(t)}\right)dt\\ &\geqslant \pi\sum_{\substack{\bar{B}\in\mathcal{B}\\ \bar{B}\subset\Omega_\varepsilon}}|d_{\bar{B}}|\int_0^{s+\sigma}\left(1-\frac{r(\mathcal{D}(t))}{2\lambda(t)}\right)dt\\ &=\pi D(s+\sigma-1),\end{aligned} \tag{4.40}$$

where the last equality follows since $r(\mathcal{D}(t))'=r(\mathcal{D}(t))$ for $t\in[0,s+\sigma]\setminus\{\sigma\}$ and $\lambda(t)$ is piecewise constant.

An application of Lemma 4.2 and the bound (4.35) show that

$$r(\mathcal{C}_0)\leqslant\frac{3}{2}r_{\Omega_\varepsilon}\big(\omega_{1/2}^{3/2}\big). \tag{4.41}$$

Recall that $r(\mathcal{C}(\sigma))=3r(\mathcal{B}_0)/8$. This and (4.41) provide the bound

$$\begin{aligned}s+\sigma-1&=\left(\log\frac{r}{r(\mathcal{B}_0)}+\log\frac{r(\mathcal{C}(\sigma))}{r(\mathcal{C}_0)}-1\right)\\ &\geqslant\left(\log\frac{r}{r(\mathcal{B}_0)}+\log\frac{r(\mathcal{B}_0)}{4r_{\Omega_\varepsilon}(\omega_{1/2}^{3/2})}-1\right)\\ &=\left(\log\frac{r}{r_{\Omega_\varepsilon}(\omega_{1/2}^{3/2})}-C\right).\end{aligned} \tag{4.42}$$

Plugging (4.40) and (4.42) into (4.17) yields (4.33). $\square$

5. Proof of the main results

With our technical tools sufficiently developed, we may now assemble them for use in proving the main theorems.

We begin with a lemma on the use of the co-area formula in conjunction with sub- and superlevel sets.

Lemma 5.1. *Let $u:\Omega\to\mathbb{C}$ and $A:\Omega\to\mathbb{R}^2$ both be C^1 and write (at least locally) $u=\rho v$ with $\rho=|u|$. Fix $t_0>0$ and $V\subset\Omega$ to be compact. Then*

$$\begin{aligned}\frac{1}{2}\int_{V\cap\{\rho\geqslant t_0\}}\rho^2|\nabla_A v|^2&=\int_{t_0}^{\infty}-t^2\frac{d}{dt}\left(\frac{1}{2}\int_{V\cap\{\rho\geqslant t\}}|\nabla_A v|^2\right)dt\\&=\frac{t_0^2}{2}\int_{V\cap\{\rho\geqslant t_0\}}|\nabla_A v|^2+\int_{t_0}^{\infty}2t\left(\frac{1}{2}\int_{V\cap\{\rho\geqslant t\}}|\nabla_A v|^2\right)dt\end{aligned}\tag{5.1}$$

and

$$\begin{aligned}\frac{1}{2}\int_{V\cap\{\rho\leqslant t_0\}}\rho^2|\nabla_A v|^2&=\int_{0}^{t_0}-t^2\frac{d}{dt}\left(\frac{1}{2}\int_{V\cap\{\rho\geqslant t\}\cap\{\rho\leqslant t_0\}}|\nabla_A v|^2\right)dt\\&=\int_{0}^{t_0}2t\left(\frac{1}{2}\int_{V\cap\{\rho\geqslant t\}\cap\{\rho\leqslant t_0\}}|\nabla_A v|^2\right)dt.\end{aligned}\tag{5.2}$$

Proof. The first equality in (5.1) follows from the co-area formula, and the second follows by integrating by parts. The same argument proves (5.2). □

5.1. Proof of Theorem 1

Theorem 1 is an improvement on Theorem 4.1 of [7] that incorporates the G term into the lower bounds on the vortex balls. The crucial difference between this result and those in the previous section is that this one bounds the energy of the function $u:\Omega\to\mathbb{C}$, whereas the previous results were for the $\mathbb{S}^1$-valued map $v=u/|u|:\Omega\to\mathbb{S}^1\hookrightarrow\mathbb{C}$. The statement made in the introduction of Theorem 1 should be understood with $G:\Omega\to\mathbb{R}^2$ the function defined in item 4 of Proposition 4.4 with β_B values given by (4.31).

Proof of Theorem 1. Proposition 4.4 produces the collection $\mathcal{B}$ and guarantees items 1 and 2. The rest of the proof is devoted to showing that (1.9) holds. By Lemma 3.1 we have, writing $u=\rho v$,

$$\begin{aligned}&\frac{1}{2}\int_V|\nabla_A u|^2+\frac{1}{2\varepsilon^2}\left(1-|u|^2\right)^2+r^2(\operatorname{curl}A)^2\\&\quad=\frac{1}{2}\int_V|\nabla\rho|^2+\frac{1}{2\varepsilon^2}\left(1-\rho^2\right)^2+\rho^2|\nabla_A v|^2+r^2(\operatorname{curl}A)^2.\end{aligned}\tag{5.3}$$

An application of the co-area formula and integration by parts, the same as that used in Lemma 5.1, shows that

$$\frac{1}{2}\int_V \rho^2|\nabla_A v|^2 = \int_0^\infty 2t\left(\frac{1}{2}\int_{V\setminus\omega_t}|\nabla_A v|^2\right)dt. \tag{5.4}$$

Then

$$\begin{aligned}
&\frac{1}{2}\int_V \rho^2|\nabla_A v|^2 + r^2(\operatorname{curl} A)^2 \\
&\geqslant \int_0^\infty 2t\left(\frac{1}{2}\int_{V\setminus\omega_t}|\nabla_A v|^2\right)dt + \int_0^{1-\delta} 2t\left(\frac{r^2}{2}\int_V(\operatorname{curl} A)^2\right)dt \\
&= \int_0^{\frac{1}{2}} 2t\left(\frac{1}{2}\int_{V\setminus\omega_t}|\nabla_A v|^2 + \frac{r^2}{2}\int_V(\operatorname{curl} A)^2\right)dt + \int_{1-\delta}^\infty 2t\left(\frac{1}{2}\int_{V\setminus\omega_t}|\nabla_A v|^2\right)dt \\
&\quad + \int_{\frac{1}{2}}^{1-\delta} 2t\left(\frac{1}{2}\int_{V\setminus\omega_t}|\nabla_A v|^2 + \frac{r^2}{2}\int_V(\operatorname{curl} A)^2\right)dt \\
&:= A_1 + A_2 + A_3.
\end{aligned} \tag{5.5}$$

We further break up the first term on the right-hand side of (5.5):

$$\begin{aligned}
A_1 &= \int_0^{\frac{1}{2}} 2t\left(\frac{1}{2}\int_{V\setminus\omega_t}|\nabla_A v|^2 + \frac{r^2}{2}\int_V(\operatorname{curl} A)^2\right)dt \\
&= \int_0^{\frac{1}{2}} 2t\left(\frac{1}{2}\int_{\omega_{1/2}^{3/2}\setminus\omega_t}|\nabla_A v|^2\right)dt + \int_0^{\frac{1}{2}} 2t\left(\frac{1}{2}\int_{V\setminus\omega_{1/2}^{3/2}}|\nabla_A v|^2 + \frac{r^2}{2}\int_V(\operatorname{curl} A)^2\right)dt.
\end{aligned} \tag{5.6}$$

Then, by writing $\omega_{1/2}^{3/2}\setminus\omega_t = \omega^{3/2}\cup\omega_{1/2}\setminus\omega_t$, noting that $\omega_{1/2}\subset V$, and applying (5.2) with $t_0 = 1/2$, we may conclude that

$$\begin{aligned}
\int_0^{\frac{1}{2}} 2t\left(\frac{1}{2}\int_{\omega_{1/2}^{3/2}\setminus\omega_t}|\nabla_A v|^2\right)dt &= \int_0^{\frac{1}{2}} 2t\left(\frac{1}{2}\int_{\omega^{3/2}}|\nabla_A v|^2 + \frac{1}{2}\int_{\omega_{1/2}\setminus\omega_t}|\nabla_A v|^2\right)dt \\
&= \frac{1}{8}\int_{\omega^{3/2}}|\nabla_A v|^2 + \frac{1}{2}\int_{\omega_{1/2}}\rho^2|\nabla_A v|^2.
\end{aligned} \tag{5.7}$$

Since the integrand does not depend on t, we have

$$\int_0^{\frac{1}{2}} 2t\left(\frac{1}{2}\int_{V\setminus\omega_{1/2}^{3/2}} |\nabla_A v|^2 + \frac{r^2}{2}\int_V (\operatorname{curl} A)^2\right) dt = \frac{1}{4}\left(\frac{1}{2}\int_{V\setminus\omega_{1/2}^{3/2}} |\nabla_A v|^2 + \frac{r^2}{2}\int_V (\operatorname{curl} A)^2\right). \tag{5.8}$$

From (5.1), applied with $t_0 = 1-\delta$, we bound the second term in (5.5)

$$\begin{aligned} A_2 = \int_{1-\delta}^{\infty} 2t\left(\frac{1}{2}\int_{V\setminus\omega_t} |\nabla_A v|^2\right) dt &= \frac{1}{2}\int_{V\setminus\omega_{1-\delta}} \left(\rho^2-(1-\delta)^2\right)|\nabla_A v|^2 \\ &\geqslant \frac{1}{2}\int_{\omega^{3/2}} \left(\rho^2-1\right)|\nabla_A v|^2. \end{aligned} \tag{5.9}$$

When $\rho \geqslant \frac{3}{2}$, the inequality $\rho^2 - \frac{3}{4} \geqslant \frac{2}{3}\rho^2$ holds; hence,

$$\frac{1}{2}\int_{\omega^{3/2}} \left(\rho^2-1\right)|\nabla_A v|^2 + \frac{1}{8}\int_{\omega^{3/2}} |\nabla_A v|^2 \geqslant \frac{1}{3}\int_{\omega^{3/2}} \rho^2|\nabla_A v|^2. \tag{5.10}$$

We now combine (5.5)–(5.10), leaving A_3 as it was, and arrive at the bound

$$\begin{aligned} &\frac{1}{2}\int_V \rho^2|\nabla_A v|^2 + r^2(\operatorname{curl} A)^2 \\ &\quad\geqslant \frac{1}{4}\left(\frac{1}{2}\int_{V\setminus\omega_{1/2}^{3/2}} |\nabla_A v|^2 + \frac{r^2}{2}\int_V (\operatorname{curl} A)^2\right) + \int_{\frac{1}{2}}^{1-\delta} 2t\left(\frac{1}{2}\int_{V\setminus\omega_t} |\nabla_A v|^2 + \frac{r^2}{2}\int_V (\operatorname{curl} A)^2\right) dt \\ &\qquad + \frac{1}{3}\int_{\omega_{1/2}^{3/2}} \rho^2|\nabla_A v|^2. \end{aligned} \tag{5.11}$$

Recalling the notation

$$F_\varepsilon(\rho, V) = \frac{1}{2}\int_V |\nabla\rho|^2 + \frac{1}{2\varepsilon^2}\left(1-\rho^2\right)^2$$

and the decomposition (5.3), we can use (5.11) to see that

$$\begin{aligned} \frac{1}{2}\int_V |\nabla_A u|^2 + \frac{1}{2\varepsilon^2}\left(1-|u|^2\right)^2 + r^2(\operatorname{curl} A)^2 &= F_\varepsilon(\rho, V) + \frac{1}{2}\int_V \rho^2|\nabla_A v|^2 + r^2(\operatorname{curl} A)^2 \\ &\geqslant B_1 + B_2 + B_3, \end{aligned} \tag{5.12}$$

where

$$B_1 := \frac{1}{4}\left(F_\varepsilon(\rho, V) + \frac{1}{2}\int\limits_{V\setminus\omega_{1/2}^{3/2}} |\nabla_A v|^2 + \frac{r^2}{2}\int\limits_V (\operatorname{curl} A)^2\right),$$

$$B_2 := \frac{3\beta}{4}F_\varepsilon(\rho, V) + \int\limits_{\frac{1}{2}}^{1-\delta} 2t\left(\frac{1}{2}\int\limits_{V\setminus\omega_t} |\nabla_A v|^2 + \frac{r^2}{2}\int\limits_V (\operatorname{curl} A)^2\right)dt,$$

$$B_3 := \frac{3(1-\beta)}{4}F_\varepsilon(\rho, V) + \frac{1}{3}\int\limits_{\omega_{1/2}^{3/2}} \rho^2|\nabla_A v|^2,$$

and $\beta \in (0,1)$ is to be chosen later in the proof.

To bound B_1, we employ Proposition 4.7 to see that

$$\begin{aligned}
&\frac{1}{2}\int\limits_{V\setminus\omega_{1/2}^{3/2}} |\nabla_A v|^2 + \frac{r^2}{2}\int\limits_V (\operatorname{curl} A)^2 + \frac{1}{2}\int\limits_V |\nabla\rho|^2 + \frac{1}{2\varepsilon^2}\left(1-\rho^2\right)^2 \\
&\quad \geqslant \frac{1}{2}\int\limits_{V\setminus\omega_{1/2}^{3/2}} |\nabla_{A+G} v|^2 + \pi D\left(\log\frac{r}{r_{\Omega_\varepsilon}(\omega_{1/2}^{3/2})} - C\right) + F_\varepsilon(\rho, V).
\end{aligned} \tag{5.13}$$

Then, an application of Lemma 2.8 shows that

$$\begin{aligned}
\pi D\left(\log\frac{r}{r_{\Omega_\varepsilon}(\omega_{1/2}^{3/2})} - C\right) + F_\varepsilon(\rho, V) &\geqslant \pi D\left(\log\frac{r}{C\varepsilon F_\varepsilon(\rho, V)} - C\right) + F_\varepsilon(\rho, V) \\
&\geqslant \pi D\left(\log\frac{r}{\varepsilon D} - C\right) + F_\varepsilon(\rho, V) - \pi D\log\frac{F_\varepsilon(\rho, V)}{\pi D} \\
&\geqslant \pi D\left(\log\frac{r}{\varepsilon D} - C\right),
\end{aligned} \tag{5.14}$$

where the last line follows from the inequality $x - a\log\frac{x}{a} \geqslant 0$. On the set $V \setminus \omega_{1/2}^{3/2}$ it is the case that $1/2 \leqslant \rho \leqslant 3/2$, and so $1 \geqslant 4\rho^2/9$. Hence, from this bound, (5.13), and (5.14), we may conclude that

$$\begin{aligned}
B_1 &\geqslant \frac{1}{4}\left(\frac{1}{2}\int\limits_{V\setminus\omega_{1/2}^{3/2}} |\nabla_{A+G} v|^2 + \pi D\left(\log\frac{r}{\varepsilon D} - C\right)\right) \\
&\geqslant \frac{1}{18}\int\limits_{V\setminus\omega_{1/2}^{3/2}} \rho^2|\nabla_{A+G} v|^2 + \frac{\pi D}{4}\left(\log\frac{r}{\varepsilon D} - C\right).
\end{aligned} \tag{5.15}$$

To control B_2, we begin by using (2.10) and Lemma 2.5 to find the bound

$$\frac{3\beta}{4}F_\varepsilon(\rho, V) \geqslant \frac{3\sqrt{2}\beta}{8\varepsilon}\int_0^\infty \left|1-t^2\right|\mathcal{H}^1\big(\{\rho = t\}\big)\,dt$$
$$\geqslant \frac{3\sqrt{2}\beta}{4\varepsilon}\int_{\frac{1}{2}}^{1-\delta}\big(1-t^2\big)r_{\Omega_\varepsilon}(\omega_t)\,dt. \tag{5.16}$$

Then (5.16) and (4.15) prove that

$$B_2 \geqslant \int_{\frac{1}{2}}^{1-\delta}\left(2t\pi D\left(\log\frac{r}{r_{\Omega_\varepsilon}(\omega_t)} - C\right) + \frac{3\sqrt{2}\beta}{4\varepsilon}\big(1-t^2\big)r_{\Omega_\varepsilon}(\omega_t)\right)dt. \tag{5.17}$$

As $r_{\Omega_\varepsilon}(\omega_t)$ varies, the integrand on the right-hand side of (5.17) achieves its minimum at

$$r_{\Omega_\varepsilon}(\omega_t) = \frac{8\pi Dt\varepsilon}{3\sqrt{2}\beta(1-t^2)}.$$

Plugging this in, we get the estimate

$$B_2 \geqslant \int_{\frac{1}{2}}^{1-\delta} 2\pi Dt\left(\log\frac{3\sqrt{2}r\beta(1-t^2)}{8\pi Dt\varepsilon} - C + 1\right)dt$$
$$= \int_{\frac{1}{2}}^{1-\delta} 2\pi Dt\left(\log\frac{r}{\varepsilon D} + \log\frac{3\sqrt{2}\beta(1-t^2)}{8\pi t} - C\right)dt$$
$$= \pi D\left(\left((1-\delta)^2 - \frac{1}{4}\right)\log\frac{r}{\varepsilon D} - C\right). \tag{5.18}$$

We now choose $\beta = \frac{23}{27}$ so that $\frac{3(1-\beta)}{8} = \frac{1}{18}$. Then

$$B_1 + B_3 \geqslant \frac{1}{18}\int_V |\nabla_{A+G}u|^2 + \frac{1}{2\varepsilon^2}\big(1-|u|^2\big)^2 + \frac{\pi D}{4}\left(\log\frac{r}{\varepsilon D} - C\right). \tag{5.19}$$

Using (5.18) and (5.19) in (5.12) then shows that

$$\frac{1}{2}\int_V |\nabla_A u|^2 + \frac{1}{2\varepsilon^2}\big(1-|u|^2\big)^2 + r^2(\operatorname{curl} A)^2$$
$$\geqslant \pi D\left((1-\delta)^2\log\frac{r}{\varepsilon D} - C\right) + \frac{1}{18}\int_V |\nabla_{A+G}u|^2 + \frac{1}{2\varepsilon^2}\big(1-|u|^2\big)^2. \tag{5.20}$$

Now, by assumption $r \leqslant 1 \leqslant D$, so $\log \frac{r}{D} \leqslant 0$. Since $\delta = \varepsilon^{\alpha/4}$, we have that for $\varepsilon \leqslant \varepsilon_0 = \varepsilon_0(\alpha)$, the inequalities

$$\begin{aligned} &\delta^2 - \delta \leqslant 0, \\ &\left(2\delta - \delta^2\right) \log \varepsilon \geqslant -1 \end{aligned} \tag{5.21}$$

both hold. Hence, for $\varepsilon \leqslant \varepsilon_0$,

$$\begin{aligned} (1-\delta)^2 \log \frac{r}{\varepsilon D} - C &= \log \frac{r}{\varepsilon D} - C + \left(\delta^2 - 2\delta\right) \log \frac{r}{D} + \left(2\delta - \delta^2\right) \log \varepsilon \\ &\geqslant \log \frac{r}{\varepsilon D} - C - 1. \end{aligned} \tag{5.22}$$

Combining (5.20) with (5.22) gives (1.9). □

5.2. Proof of Theorem 2 and corollaries

Theorem 2 justifies the selection of the function G. It has been chosen so that $\|G\|_{L^{2,\infty}}$ only depends on the final data of Theorem 1, that is on natural quantities. This estimate of $\|G\|_{L^{2,\infty}}$, Proposition 6.4, is quite technical and is thus reserved for the next section. A more thorough discussion of the space $L^{2,\infty}$, also known as weak-L^2, is also reserved for the next section.

Proof of Theorem 2. We begin by noting that $\nabla_A u = \nabla_{A+G} u + iGu$. This and the fact that $\|f\|_{L^{2,\infty}(V)} \leqslant \|g\|_{L^{2,\infty}(V)}$ if $|f| \leqslant |g|$ allow us to estimate

$$\begin{aligned} \frac{1}{2} \|\nabla_A u\|^2_{L^{2,\infty}(V)} &\leqslant \|\nabla_{A+G} u\|^2_{L^{2,\infty}(V)} + \|iGu\|^2_{L^{2,\infty}(V)} \\ &\leqslant \|\nabla_{A+G} u\|^2_{L^2(V)} + \frac{9}{4} \|G\|^2_{L^{2,\infty}(V)}. \end{aligned} \tag{5.23}$$

The second inequality follows since $|u| \leqslant \frac{3}{2}$ on the support of G. Write

$$F_\varepsilon^r(u, A, V) = \frac{1}{2} \int_V |\nabla_A u|^2 + \frac{1}{2\varepsilon^2}\left(1 - |u|^2\right)^2 + r^2 (\operatorname{curl} A)^2.$$

We now employ Theorem 1 to bound

$$\|\nabla_{A+G} u\|^2_{L^2(V)} \leqslant 18\left(F_\varepsilon^r(u, A, V) - \pi D\left(\log \frac{r}{\varepsilon D} - C\right)\right). \tag{5.24}$$

We will show in Proposition 6.4 that

$$\begin{aligned} \frac{9}{4} \|G\|^2_{L^{2,\infty}(V)} \leqslant{}& \frac{216(1+\eta)}{2\eta - 1}\left(F_\varepsilon^r(u, A, V) - \pi D\left(\log \frac{r}{\varepsilon D} - C\right)\right) \\ &+ \pi \frac{9(1+\eta)}{1-\eta} \sum_{\substack{\bar{B} \in \mathcal{B} \\ \bar{B} \subset \Omega_\varepsilon}} d_{\bar{B}}^2. \end{aligned} \tag{5.25}$$

Now choose $\eta = \frac{5+\sqrt{2785}}{60} \approx .962$ so that

$$18 + \frac{216(1+\eta)}{2\eta - 1} = \frac{9(1+\eta)}{1-\eta}.$$

Combining (5.23)–(5.25) yields (1.11) with constant $C = (1-\eta)/(18(1+\eta)) \approx 1/951$. □

The previous theorem dealt with the energy content of the set $V \subset \Omega$. We can deduce a slightly stronger version of Corollary 1.2.

Corollary 5.2. *Assume the hypotheses of Theorem* 1. *Then*

$$C\|\nabla_A u\|^2_{L^{2,\infty}(\Omega)} \leqslant F^r_\varepsilon(u, A, \Omega) - \pi D\left(\log\frac{r}{\varepsilon D} - C\right) + \pi \sum_{\substack{B \in \mathcal{B} \\ B \subset \Omega_\varepsilon}} d_B^2. \tag{5.26}$$

Proof. Add $F^r_\varepsilon(u, A, \Omega \setminus V)$ to both sides of (1.11). We then bound

$$\begin{aligned} &C\|\nabla_A u\|^2_{L^{2,\infty}(V)} + F^r_\varepsilon(u, A, \Omega \setminus V) \\ &\quad\geqslant C\|\nabla_A u\|^2_{L^{2,\infty}(V)} + \|\nabla_A u\|^2_{L^2(\Omega\setminus V)} \\ &\quad\geqslant C\|\nabla_A u\|^2_{L^{2,\infty}(V)} + \|\nabla_A u\|^2_{L^{2,\infty}(\Omega\setminus V)} \\ &\quad\geqslant C\|\nabla_A u\|^2_{L^{2,\infty}(\Omega)}, \end{aligned} \tag{5.27}$$

where the last inequality follows by using the convexity of norms, and C is a different constant. The result follows. □

Proof of Proposition 1.3. It is proved in Theorem 0.5 of [1] that minimizers of E_ε with this constraint have exactly d zeroes of degree 1 which converge to d distinct points $a_1, \dots, a_d$, minimizing W_g. They also prove that their energy is

$$\min E_\varepsilon = \pi d|\log\varepsilon| + \min W_g + d\gamma + o(1), \tag{5.28}$$

where γ is a universal constant. Let us apply the vortex-ball construction to these solutions, choosing for final radius $r = \frac{1}{4}\min_{i,j}(\mathrm{dist}(a_i, \partial\Omega), |a_i - a_j|)$. Since the final balls $B \in \mathcal{B}$ cover all the zeroes of u, and there is exactly one zero b_i^ε with non-zero degree, converging to each a_i, there is one ball B_i in the collection containing b_i^ε. Since $d_i = \deg(u_\varepsilon, \partial B_i) = 1$, and there are no other zeroes of u_ε, we have $D = d$ (with our previous notation) and Corollary 1.2 (taken with $A \equiv 0$) gives us

$$E_\varepsilon(u_\varepsilon) + \pi d \geqslant C\|\nabla u_\varepsilon\|^2_{L^{2,\infty}(\Omega)} + \pi d\big(|\log\varepsilon| - C - \log d\big),$$

where C is a universal constant. In view of (5.28), this implies that

$$C\|\nabla u_\varepsilon\|^2_{L^{2,\infty}(\Omega)} \leqslant \min W_g + d\gamma + Cd + \pi d\log d + o(1),$$

and the first result follows.

Since $L^{2,\infty}$ is a dual Banach space, we deduce from this bound that, as $\varepsilon \to 0$, up to extraction, ∇u_ε converges weakly-$*$ in $L^{2,\infty}$, to its distributional limit. But it is proved in [1] that $\nabla u_\varepsilon \to \nabla u_\star$ uniformly away from $a_1, \dots, a_d$ (in fact in C^k_{loc}), where $u_\star$ is given by

$$u_\star(x) = e^{iH(x)} \prod_{k=1}^{d} \frac{x - a_k}{|x - a_k|}$$

with H a harmonic function. Note in particular that $u_\star \in W^{1,p}(\Omega)$ for $p < 2$.

We claim that $\nabla u_\varepsilon \to \nabla u_\star$ in the sense of distributions on Ω. Indeed, let X be a smooth compactly supported test vector field. Fix $\rho > 0$ and let us write

$$\int_{\Omega} (\nabla u_\varepsilon - \nabla u_\star) \cdot X = \int_{\Omega \setminus \bigcup_i B(a_i,\rho)} (\nabla u_\varepsilon - \nabla u_\star) \cdot X + \sum_i \int_{B(a_i,\rho)} (\nabla u_\varepsilon - \nabla u_\star) \cdot X.$$

The first term in the right-hand side tends to 0 by uniform convergence of ∇u_ε to $\nabla u_\star$ away from the a_i's. The second term is bounded by Hölder's inequality by $C\|X\|_{L^\infty}\|\nabla u_\varepsilon - \nabla u_\star\|_{L^p(\Omega)}\rho^{2/q}$, where $p < 2$ and $1/p + 1/q = 1$. This is bounded by $C\rho^{2/q}\|X\|_{L^\infty}$ since $\nabla u_\star \in L^p(\Omega)$ for all $p < 2$ and ∇u_ε is bounded in $L^p(\Omega)$ for all $p < 2$ ($L^{2,\infty}(\Omega)$ embeds in $L^p(\Omega)$ for all $p < 2$). Letting ρ tend to 0 we conclude that $\int_\Omega (\nabla u_\varepsilon - \nabla u_\star) \cdot X \to 0$ and finally that $\nabla u_\varepsilon \rightharpoonup \nabla u_\star$ weakly-$*$ in $L^{2,\infty}(\Omega)$. $\square$

6. The $L^{2,\infty}$ norm of G

6.1. Definitions and preliminary results

We begin with a discussion of the various quantities needed to define and norm the space $L^{2,\infty}$. For a function $f : \Omega \to \mathbb{R}^k$, $k \geqslant 1$, we define the distribution function of f by

$$\lambda_f(t) = \left|\left\{x \in \Omega \;\middle|\; \left|f(x)\right| > t\right\}\right|. \tag{6.1}$$

This allows us to define the decreasing rearrangement of f as $f^* : \mathbb{R}^+ \to \mathbb{R}^+$, where

$$f^*(t) = \inf\left\{s > 0 \;\middle|\; \lambda_f(s) \leqslant t\right\}. \tag{6.2}$$

We then define the quantity

$$|||f|||_{L^{2,\infty}} = \sqrt{\sup_{t>0} t^2 \lambda_f(t)} = \sup_{t>0} t\lambda_f(t)^{\frac{1}{2}} = \sup_{t>0} t^{\frac{1}{2}} f^*(t), \tag{6.3}$$

and $L^{2,\infty}(\Omega) = \{f \mid |||f|||_{L^{2,\infty}} < \infty\}$. Unfortunately, this does not define a norm, but rather a quasi-norm. That is, $|||\cdot|||_{L^{2,\infty}}$ satisfies

$$\begin{cases} |||\alpha f|||_{L^{2,\infty}} = |\alpha|\,|||f|||_{L^{2,\infty}}, & \\ |||f|||_{L^{2,\infty}} = 0 & \text{if and only if } f = 0 \text{ a.e.,} \\ |||f + g|||_{L^{2,\infty}} \leqslant C(|||f|||_{L^{2,\infty}} + |||g|||_{L^{2,\infty}}) & \text{for some } C \geqslant 1. \end{cases}$$

It can be shown that with $|||\cdot|||_{L^{2,\infty}}$, $L^{2,\infty}$ is a quasi-Banach space, i.e. a linear space in which every quasi-norm Cauchy sequence converges in the quasi-norm. However, as the next lemma shows, the space can, in fact, be normed. We define

$$\begin{aligned}\|f\|_{L^{2,\infty}} &= \sup_{|E|<\infty} |E|^{-1/2} \int_E \left|f(x)\right| dx \\ &= \sup_{t>0} \frac{1}{t^{\frac{1}{2}}} \sup_{|E|=t} \int_E \left|f(x)\right| dx \\ &= \sup_{t>0} \frac{1}{t^{\frac{1}{2}}} \int_0^t f^*(s)\, ds, \end{aligned} \tag{6.4}$$

which is obviously a norm.

Lemma 6.1. *$L^{2,\infty}$ is a Banach space with norm $\|\cdot\|_{L^{2,\infty}}$, and*

$$|||f|||_{L^{2,\infty}} \leqslant \|f\|_{L^{2,\infty}} \leqslant 2|||f|||_{L^{2,\infty}}. \tag{6.5}$$

Proof. Since f^* is decreasing, we see that

$$\|f\|_{L^{2,\infty}} = \sup_{t>0} \frac{1}{t^{\frac{1}{2}}} \int_0^t f^*(s)\, ds \geqslant \sup_{t>0} \frac{1}{t^{\frac{1}{2}}} t f^*(t) = \sup_{t>0} t^{\frac{1}{2}} f^*(t) = |||f|||_{L^{2,\infty}}. \tag{6.6}$$

For the second inequality we note that

$$\frac{1}{t^{\frac{1}{2}}} \int_0^t f^*(s)\, ds = \frac{1}{t^{\frac{1}{2}}} \int_0^t \left(s^{\frac{1}{2}} f^*(s)\right) \frac{ds}{s^{\frac{1}{2}}} \leqslant |||f|||_{L^{2,\infty}} \frac{2t^{\frac{1}{2}}}{t^{\frac{1}{2}}} = 2|||f|||_{L^{2,\infty}}. \tag{6.7}$$

This also shows how to construct a function that makes the inequalities sharp: any f so that $f^*(s) = \frac{c}{\sqrt{s}}$ will do. This is the case for $f(x) = 1/|x|$ in $\mathbb{R}^2$. □

We now present the

Proof of Proposition 1.4. First rewrite the L^2 integral using the distribution function:

$$\int_\Omega |f|^2 = \int_0^\infty 2t\lambda_f(t)\, dt. \tag{6.8}$$

We break this integral into two parts and utilize the boundedness of f and the trivial inequality $\lambda_f(t) \leqslant |\Omega|$ for all $t > 0$. Indeed,

$$\begin{aligned}\int_0^\infty 2t\lambda_f(t)\,dt &= \int_0^C 2t\lambda_f(t)\,dt + \int_C^{\frac{C}{\varepsilon}} 2t\lambda_f(t)\,dt \\ &\leqslant |\Omega| \int_0^C 2t\,dt + 2\sup_{t>0}\bigl(t^2\lambda_f(t)\bigr)\int_C^{\frac{C}{\varepsilon}} \frac{dt}{t} \\ &\leqslant |\Omega| C^2 + 2\|f\|^2_{L^{2,\infty}(\Omega)} \log\frac{C}{C\varepsilon},\end{aligned} \tag{6.9}$$

where we have used Lemma 6.1 in the last inequality. The result follows by dividing both sides by $2|\log\varepsilon|$. □

6.2. The calculation

Before proving the main result we prove some quasi-norm estimates for simplified versions of G. The main result breaks G into various simplified components in order to utilize these estimates.

Lemma 6.2. *Suppose we are given a collection of disjoint annuli $\{A_i\}$, $i = 1, \dots, n$, where*

$$A_i = \bigl\{r_i < |x - c_i| \leqslant s_i\bigr\} \subset \mathbb{R}^2,$$

c_i denotes the center of A_i, and r_i and s_i are the inner and outer radii respectively. Let

$$f(x) = \sum_{i=1}^n \chi_{A_i}(x) v_i(x) \frac{a_i}{|x - c_i|}, \tag{6.10}$$

where $v_i : A_i \to \mathbb{R}^k$ is a vector field so that $|v_i| = 1$ and a_i is a constant for $i = 1, \dots, n$. Write $\tau_i = \log\frac{s_i}{r_i}$ for the conformal factor of A_i. Then for $t > 0$,

$$t^2\lambda_f(t) \leqslant \pi \sum_{i=1}^n a_i^2\bigl(1 - e^{-2\tau_i}\bigr). \tag{6.11}$$

Proof. We begin by noting that on the annulus A_i it is the case that

$$\frac{|a_i|}{s_i} \leqslant |f| < \frac{|a_i|}{r_i}. \tag{6.12}$$

Then for any $t > 0$ and any annulus A_i, the measure of the set in A_i where $f > t$ is simple to calculate. Indeed, if $t \leqslant |a_i|/s_i$, then $f > t$ on the whole annulus, which has measure $\pi(s_i^2 - r_i^2)$. If $t \geqslant |a_i|/r_i$, then $f < t$ everywhere on the annulus, and so the measure is zero. Finally, if $|a_i|/s_i < t < |a_i|/r_i$, then $f > t$ exactly on the sub-annulus $\{r_i < |x - c_i| \leqslant \rho_i\}$, where

$$\rho_i = \frac{|a_i|}{t}, \tag{6.13}$$

which has measure $\pi(a_i^2/t^2 - r_i^2)$.

Combining these, for any $t > 0$ we may then write

$$\lambda_f(t) = \sum_{\{i \mid \frac{|a_i|}{s_i} < t < \frac{|a_i|}{r_i}\}} \pi\left(\frac{a_i^2}{t^2} - r_i^2\right) + \sum_{\{i \mid t \leqslant \frac{|a_i|}{s_i}\}} \pi\left(s_i^2 - r_i^2\right). \tag{6.14}$$

Then

$$\begin{aligned} t^2\lambda_f(t) &= \sum_{\{i \mid \frac{|a_i|}{s_i} < t < \frac{|a_i|}{r_i}\}} \pi\left(a_i^2 - t^2 r_i^2\right) + \sum_{\{i \mid t \leqslant \frac{|a_i|}{s_i}\}} \pi\left(s_i^2 - r_i^2\right)t^2 \\ &\leqslant \sum_{\{i \mid \frac{|a_i|}{s_i} < t < \frac{|a_i|}{r_i}\}} \pi a_i^2\left(1 - \frac{r_i^2}{s_i^2}\right) + \sum_{\{i \mid t \leqslant \frac{|a_i|}{s_i}\}} \pi a_i^2\left(1 - \frac{r_i^2}{s_i^2}\right) \\ &\leqslant \sum_{i=1}^n \pi a_i^2\left(1 - \frac{r_i^2}{s_i^2}\right). \end{aligned} \tag{6.15}$$

Plugging in $\tau_i = \log \frac{s_i}{r_i}$ proves the result. $\square$

The next lemma tells us that a collection of annuli with uniformly bounded degrees and the property that they can be rearranged to fit concentrically inside each other can, for the purposes of estimating the $L^{2,\infty}$ quasi-norm, be regarded as a single annulus.

Lemma 6.3. *Suppose $\{A_i\}$, $i = 1, \ldots, n$, is a collection of disjoint annuli, where*

$$A_i = \{r_i < |x - c_i| \leqslant s_i\} \subset \mathbb{R}^2,$$

c_i denotes the center of A_i, and r_i and s_i are the inner and outer radii respectively. Suppose further that the annuli can be arranged concentrically without overlap. That is, suppose that

$$r_1 < s_1 \leqslant r_2 < s_2 \leqslant r_3 \leqslant \cdots < s_{n-1} \leqslant r_n < s_n.$$

Let

$$f(x) = \sum_{i=1}^n \chi_{A_i}(x) v_i(x) \frac{a_i}{|x - c_i|}, \tag{6.16}$$

where the a_i are constants such that $|a_i| \leqslant |a|$ and $v_i : A_i \to \mathbb{R}^k$ is a vector field so that $|v_i| = 1$ for $i = 1, \ldots, n$. Then

$$t^2\lambda_f(t) = t^2 \sum_{i=1}^n \left|A_i \cap \left\{|f| > t\right\}\right| \leqslant \pi a^2. \tag{6.17}$$

Proof. Since the distribution function is invariant under translations, without loss of generality we may assume that the annuli are concentric with common center c. This reduces f to the form

$$f(x) = \sum_{i=1}^{n} \chi_{A_i}(x) v_i(x) \frac{a_i}{|x-c|}. \tag{6.18}$$

Consider the function

$$g(x) = \frac{ae_1}{|x-c|}, \tag{6.19}$$

where $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^k$. The pointwise bound $|f(x)| \leqslant |g(x)|$ yields the bound $\lambda_f(t) \leqslant \lambda_g(t)$ for all $t > 0$. It is a simple matter to see that

$$\lambda_g(t) = \pi \frac{a^2}{t^2}, \tag{6.20}$$

and hence,

$$t^2 \lambda_f(t) \leqslant t^2 \lambda_g(t) = \pi a^2. \qquad \square \tag{6.21}$$

We are now ready to prove the main result of this section.

Proposition 6.4. *Let $G : \Omega \to \mathbb{R}^2$ be the function defined in Proposition* 4.4 *with $\eta \in (0, 1)$ fixed and the β_B values given by* (4.31). *Write*

$$F_\varepsilon^r(u, A, V) = \frac{1}{2} \int_V |\nabla_A u|^2 + \frac{1}{2\varepsilon^2}\bigl(1 - |u|^2\bigr)^2 + r^2 (\operatorname{curl} A)^2.$$

Then

$$\|G\|^2_{L^{2,\infty}(V)} \leqslant \frac{96(1+\eta)}{2\eta - 1}\left(F_\varepsilon^r(u, A, V) - \pi D\left(\log \frac{r}{\varepsilon D} - C\right)\right) + \pi \frac{4(1+\eta)}{1-\eta} \sum_{\substack{\bar{B} \in \mathcal{B} \\ \bar{B} \subset \Omega_\varepsilon}} d_{\bar{B}}^2. \tag{6.22}$$

Proof. Step 1. To begin we must translate the notation used to define G into different notation that is more cumbersome but that will allow a more exact enumeration of the objects generated by the ball construction. Recall that to define G, the collection $\{\mathcal{D}(t)\}_{t \in [0, s+\sigma]}$ defined by (4.30) is refined to the subcollection $\{\mathcal{G}(t)\}_{t \in [0, s+\sigma]}$ that consists of all balls that stay entirely inside Ω_ε. Let N be the number of balls in $\mathcal{G}(s+\sigma) = \{\bar{B}_1, \dots, \bar{B}_N\}$, i.e. the number of final balls. Let T be the finite set of merging times in the growth of $\mathcal{G}(t)$, where here we count $t = \sigma$, the time when the collection shifts from $\mathcal{C}(\sigma)$ to $\mathcal{B}(0)$, as a merging time. Let $0 = t_0 < t_1 < \cdots < t_{K-1} < t_K = s + \sigma$ be an enumeration of $T \cup \{0, s+\sigma\}$. For $k = 1, \dots, K$ and $t \in [t_{k-1}, t_k)$ we call all balls in $\mathcal{G}(t)$ members of the kth generation. We write $\mathcal{G}(t_k^-)$ for the collection of balls obtained

as $t \to t_k^-$, i.e. the collection of pre-merged balls at time $t = t_k$. Similarly, when we write $\mathcal{G}(t_k)$ we refer to the post-merged balls. For $k = 1, \dots, K$ and $n = 1, \dots, N$ we enumerate

$$\{B_{i,k,n}\}_{i=1}^{M_{k,n}} = \{B \in \mathcal{G}(t_k^-) \mid B \subset \bar{B}_n\}, \quad \text{and}$$
$$\{\tilde{B}_{i,k,n}\}_{i=1}^{M_{k,n}} = \{B \in \mathcal{G}(t_{k-1}) \mid B \subset \bar{B}_n\},$$

in such a way that $\tilde{B}_{i,k,n} \subset B_{i,k,n}$. We define the annuli $A_{i,k,n} = B_{i,k,n} \setminus \tilde{B}_{i,k,n}$ and write $d_{i,k,n} = \deg(u, \partial B_{i,k,n})$ for the degree of u in the annulus $A_{i,k,n}$. For fixed $k = 1, \dots, K$ we say the annuli $\{A_{i,k,n}\}$ are kth generation annuli. Without loss of generality we may assume that the indices are ordered so that $|d_{i,k,n}|$ is a decreasing sequence with respect to i for k and n fixed. Write $D_n = d_{\bar{B}_n}$. We define the conformal growth factor in the kth generation, denoted τ_k, by

$$\tau_k = \log \frac{r(\mathcal{G}(t_k^-))}{r(\mathcal{G}(t_{k-1}))}.$$

Recall that for each $\bar{B}_n$, $n = 1, \dots, N$, Corollary 4.6 provides a transition time $t_{\bar{B}_n}$ (depending on η). In the current setting, the more natural notion is that of transition generation, and in fact, the proof of Lemma 4.5 shows that the transition time actually occurs at one of the t_k for $k = 0, \dots, K-1$. We then define the transition generation k_n as the unique k such that $t_{\bar{B}_n} \in [t_{k-1}, t_k)$. If we define generational versions of the negative and positive vorticity masses $N(t)$ and $P(t)$ from (4.24) by

$$N(k,n) := \sum_{\substack{1 \leqslant i \leqslant M_{k,n} \\ d_{i,k,n} < 0}} |d_{i,k,n}|,$$
$$P(k,n) := \sum_{\substack{1 \leqslant i \leqslant M_{k,n} \\ d_{i,k,n} \geqslant 0}} d_{i,k,n},$$

then the definition of k_n and Corollary 4.6 allow us to conclude

$$D_n \geqslant 0 \quad \Rightarrow \quad \begin{cases} \eta P(k,n) < N(k,n) & \text{for } 1 \leqslant k \leqslant k_n - 1, \\ N(k,n) \leqslant \eta P(k,n) & \text{for } k_n \leqslant k \leqslant K, \end{cases} \tag{6.23}$$

$$D_n < 0 \quad \Rightarrow \quad \begin{cases} \eta N(k,n) < P(k,n) & \text{for } 1 \leqslant k \leqslant k_n - 1, \\ P(k,n) \leqslant \eta N(k,n) & \text{for } k_n \leqslant k \leqslant K. \end{cases} \tag{6.24}$$

Translating the definition of the β_B from (4.31) into the new notation, we see that

$$\beta_{i,k,n} = \begin{cases} 1 & \text{for } 1 \leqslant k < k_n,\ 1 \leqslant i \leqslant M_{k,n}, \\ |D_n|^{1/2} (\sum_{i=1}^{M_{k,n}} d_{i,k,n}^2)^{-1/2} & \text{for } k_n \leqslant k \leqslant K,\ 1 \leqslant i \leqslant M_{k,n}. \end{cases} \tag{6.25}$$

This means that G can be written

$$G(x) = \sum_{n=1}^{N} \sum_{k=1}^{K} \sum_{i=1}^{M_{k,n}} \chi_{A_{i,k,n}}(x) \frac{d_{i,k,n} \beta_{i,k,n}}{|x - c_{i,k,n}|} \tau_{i,k,n}(x), \tag{6.26}$$

where $\tau_{i,k,n}$ is the unit tangent vector field in $A_{i,k,n}$. In order to somewhat ease the notational burden, we define the following sets of indices. The early and later generations are given respectively by

$$S_e = \{(n,k) \mid 1 \leqslant n \leqslant N,\ 1 \leqslant k \leqslant k_n - 1\},$$
$$S_l = \{(n,k) \mid 1 \leqslant n \leqslant N,\ k_n \leqslant k \leqslant K\},$$

and we similarly define the sets of early and later annuli by

$$T_e = \{(n,k,i) \mid (n,k) \in S_e,\ 1 \leqslant i \leqslant M_{k,n}\},$$
$$T_l = \{(n,k,i) \mid (n,k) \in S_l,\ 1 \leqslant i \leqslant M_{k,n}\}.$$

Step 2. In this step we will prove an intermediate bound on $t^2\lambda_G(t)$. We begin by breaking the distribution function for G up into two components determined by the value of k_n. Indeed,

$$\begin{aligned}\lambda_G(t) &= \sum_{n,k,i} \left|A_{i,k,n} \cap \{|G| > t\}\right| \\ &= \sum_{T_e} \left|A_{i,k,n} \cap \{|G| > t\}\right| + \sum_{T_l} \left|A_{i,k,n} \cap \{|G| > t\}\right| \\ &:= A_1 + A_2. \end{aligned} \tag{6.27}$$

Applying Lemma 6.2 to A_1, we see that

$$t^2 A_1 \leqslant \pi \sum_{T_e} d_{i,k,n}^2 \left(1 - e^{-2\tau_k}\right). \tag{6.28}$$

To analyze the A_2 term we must take advantage of all of the notation created in the first step. Particular attention must be paid to the generations after k_n that come about as the result of mergings in which balls of non-zero degree are merged only with balls of zero degree. These generations, which we call zero-merging generations, throw off a counting argument that we will use to bound the number of later generations (after k_n) in terms of the degrees of the balls in the k_nth generation. Generations that are not zero-merging generations we call effective-merging generations. The degrees of the annuli are not changed in a zero-merging generation, and the annuli of such a generation can be rearranged to fit concentrically outside the annuli of the previous generation. Our strategy for dealing with zero-merging generations, then, is to collect successive zero-merging generations, group them with the preceding effective-merging generation, and utilize Lemma 6.3 to regard the group as a single collection of annuli.

To this end, for each n we define the sets

$$Z_n = \big\{k \in \{k_n, \dots, K\} \bigm| \text{each ball in } \mathcal{G}(t_k) \text{ contains at most one ball in } \mathcal{G}(t_k^-) \text{ of non-zero degree}\big\},$$

and

$$I_n = \{k_n, \dots, K\} \setminus Z_n.$$

The generations in Z_n are the zero-merging generations, and those in I_n are the effective-merging generations.

Since $|d_{i,k,n}|$ is a decreasing sequence with respect to i for k, n fixed, there must exist an integer $P_{k,n} \in \{1, \dots, M_{k,n}\}$ so that $d_{i,k,n} \neq 0$ for $i = 1, \dots, P_{k,n}$ and $d_{i,k,n} = 0$ for $i = P_{k,n} + 1, \dots, M_{k,n}$. Since the annuli of a zero-merging generation have the same degrees as the previous generation, we have that $P_{k,n} = P_{k-1,n}$. We may assume, without loss of generality, that the ball ordering is such that $B_{i,k-1,n} \subset B_{i,k,n}$ and $d_{i,k,n} = d_{i,k-1,n}$ for $k \in Z_n$ and $i = 1, \dots, P_{k,n}$. To identify sequences of zero-merging generations that happen one after the other we write $Z_n = Z_n^1 \cup \dots \cup Z_n^{m_n}$, where the Z_n^j are maximal subsets of sequential integers, i.e. the integer connected components of Z_n. All of the generations in Z_n^j will be grouped with the generation preceding Z_n^j and analyzed as a single entity with Lemma 6.3. This preceding effective generation occurs at generation $l_n^j := \min(Z_N^j) - 1$. We group it together with the generations in Z_n^j by forming the collections $\tilde{Z}_n^j = Z_n^j \cup \{l_n^j\}$. Write the modified collection $\tilde{Z}_n = \tilde{Z}_n^1 \cup \dots \cup \tilde{Z}_n^{m_n}$, and $\tilde{I}_n = I_n \setminus \tilde{Z}_n$. Note that $P_{k,n}$ is constant for $k \in \tilde{Z}_n^j$; we call this common value P_n^j.

We now split A_2 again:

$$\begin{aligned}
A_2 &= \sum_{T_l} \left| A_{i,k,n} \cap \{|G| > t\} \right| \\
&= \sum_{n=1}^{N} \sum_{k \in \tilde{I}_n} \sum_{i=1}^{P_{k,n}} \left| A_{i,k,n} \cap \{|G| > t\} \right| + \sum_{n=1}^{N} \sum_{k \in \tilde{Z}_n} \sum_{i=1}^{P_{k,n}} \left| A_{i,k,n} \cap \{|G| > t\} \right| \\
&:= B_1 + B_2. \qquad (6.29)
\end{aligned}$$

Applying Lemma 6.2 to B_1, we get

$$t^2 B_1 \leqslant \pi \sum_{n=1}^{N} \sum_{k \in \tilde{I}_n} \sum_{i=1}^{P_{k,n}} (d_{i,k,n} \beta_{i,k,n})^2 \left(1 - e^{-2\tau_k}\right) \leqslant \pi \sum_{n=1}^{N} \sum_{k \in \tilde{I}_n} \sum_{i=1}^{P_{k,n}} (d_{i,k,n} \beta_{i,k,n})^2. \qquad (6.30)$$

Upon inserting the values of $\beta_{i,k,n}$ from (6.25), we find that

$$t^2 B_1 \leqslant \pi \sum_{n=1}^{N} \sum_{k \in \tilde{I}_n} |D_n| = \pi \sum_{n=1}^{N} \#(\tilde{I}_n) |D_n|, \qquad (6.31)$$

where $\#(\tilde{I}_n)$ denotes the cardinality of $\tilde{I}_n$.

To handle the B_2 term we note that

$$\begin{aligned}
&\left\{(n,k,i) \,\middle|\, 1 \leqslant n \leqslant N,\ k \in \tilde{Z}_n,\ 1 \leqslant i \leqslant P_{k,n}\right\} \\
&\quad = \bigcup_{\substack{1 \leqslant n \leqslant N \\ 1 \leqslant j \leqslant m_n}} \left\{(n,k,i) \,\middle|\, 1 \leqslant i \leqslant P_n^j,\ k \in \tilde{Z}_n^j\right\}, \qquad (6.32)
\end{aligned}$$

and hence

$$B_2 = \sum_{n=1}^{N} \sum_{j=1}^{m_n} \sum_{i=1}^{P_n^j} \sum_{k \in \tilde{Z}_n^j} \left| A_{i,k,n} \cap \left\{ |G| > t \right\} \right|. \tag{6.33}$$

When a zero-merging happens to a ball B of non-zero degree, it is merged with a number of balls of zero degree. The resulting ball has the same degree as B, and its radius is strictly larger than the radius of B. Thus, we see that the radii hypothesis of Lemma 6.3 is satisfied by $\{A_{i,k,n}\}$ for $k \in \tilde{Z}_n^j$, $i = 1, \dots, P_{k,n}$. Moreover, for $k \in \tilde{Z}_n^j$, we have that $d_{i,k,n} = d_{i,l_n^j,n}$ and $\beta_{i,k,n} = \beta_{i,l_n^j,n}$. All hypotheses of Lemma 6.3 are thus satisfied; applying it, for each j, n we may bound

$$t^2 \sum_{k \in \tilde{Z}_n^j} \left| A_{i,k,n} \cap \left\{ |G| > t \right\} \right| \leqslant \pi \left(d_{i,l_n^j,n} \beta_{i,l_n^j,n} \right)^2. \tag{6.34}$$

Plugging in the values of $\beta_{i,k,n}$ from (6.25) then shows that

$$t^2 B_2 \leqslant \sum_{n=1}^{N} \sum_{j=1}^{m_n} \pi |D_n| = \sum_{n=1}^{N} \pi m_n |D_n|. \tag{6.35}$$

Recall that $I_n = \tilde{I}_n \cup \{l_n^1, \dots, l_n^{m_n}\}$. Hence $\#(I_n) = \#(\tilde{I}_n) + m_n$. We then combine (6.29), (6.31), and (6.35) to get the estimate

$$t^2 A_2 \leqslant \pi \sum_{n=1}^{N} \#(I_n) |D_n|. \tag{6.36}$$

Together, (6.27), (6.28), and (6.36) prove that

$$t^2 \lambda_G(t) \leqslant \pi \sum_{T_e} d_{i,k,n}^2 \left(1 - e^{-2\tau_k} \right) + \pi \sum_{n=1}^{N} \#(I_n) |D_n|, \tag{6.37}$$

where $\#(I_n)$ is the cardinality of I_n.

Step 3. In this step we will utilize the η inequalities (6.23) and (6.24) to show that the energy excess, $F_\varepsilon(u, A) - \pi D(\log \frac{r}{\varepsilon D} - C)$, controls the first term on the right-hand side of (6.37). To begin we modify an argument from the beginning of the proof of Theorem 1. Define V to be the union of the balls in $\mathcal{G}(s + \sigma)$. Then, copying (5.5), we can bound

$$\begin{aligned} F_\varepsilon^r(u, A, V) &= \frac{1}{2} \int_V \rho^2 |\nabla_A v|^2 + \frac{1}{2\varepsilon^2} \left(1 - \rho^2 \right)^2 + |\nabla \rho|^2 + r^2 (\operatorname{curl} A)^2 \\ &\geqslant F_\varepsilon(\rho, V) + \int_0^{\frac{1}{2}} 2t \left(\frac{1}{2} \int_{V \setminus \omega_t} |\nabla_A v|^2 \right) dt + \frac{r^2}{8} \int_V (\operatorname{curl} A)^2 \\ &\quad + \int_{\frac{1}{2}}^{1-\delta} 2t \left(\frac{1}{2} \int_{V \setminus \omega_t} |\nabla_A v|^2 + \frac{r^2}{2} \int_V (\operatorname{curl} A)^2 \right) dt. \end{aligned} \tag{6.38}$$

For $t \in [0, 1/2]$ the inclusions

$$V \setminus \omega_t \supseteq V \setminus \omega_{1/2} \supseteq V \setminus \omega_{1/2}^{3/2} \tag{6.39}$$

hold, and hence

$$\begin{aligned}\int_0^{\frac{1}{2}} 2t\left(\frac{1}{2}\int_{V\setminus\omega_t} |\nabla_A v|^2\right) dt &\geqslant \int_0^{\frac{1}{2}} 2t\left(\frac{1}{2}\int_{V\setminus\omega_{1/2}^{3/2}} |\nabla_A v|^2\right) dt \\ &= \frac{1}{8}\int_{V\setminus\omega_{1/2}^{3/2}} |\nabla_A v|^2.\end{aligned} \tag{6.40}$$

We now use (5.18) and (5.22) from Theorem 1 to bound

$$\begin{aligned}&\int_{1/2}^{1-\delta} 2t\left(\frac{1}{2}\int_{V\setminus\omega_t} |\nabla_A v|^2 + \frac{r^2}{2}\int_V (\operatorname{curl} A)^2\right) dt + \frac{3}{4}F_\varepsilon(\rho, V) \\ &\quad \geqslant \pi D\left(\frac{3}{4}\log\frac{r}{\varepsilon D} - C\right).\end{aligned} \tag{6.41}$$

Here we have used $D = \sum_{n=1}^N D_n$. Assembling the bounds (6.38), (6.40), and (6.41) produces the bound

$$\begin{aligned}&F_\varepsilon^r(u, A, V) - \pi D\left(\log\frac{r}{\varepsilon D} - C\right) \\ &\quad\geqslant \frac{1}{4}\left(F_\varepsilon(\rho, V) + \frac{1}{2}\int_{V\setminus\omega_{1/2}^{3/2}} |\nabla_A v|^2 + \frac{r^2}{2}\int_V (\operatorname{curl} A)^2 - \pi D\log\frac{r}{\varepsilon D}\right).\end{aligned} \tag{6.42}$$

The argument in (5.14) shows that

$$F_\varepsilon(\rho, V) - \pi D\left(\log\frac{r}{\varepsilon D} - C\right) \geqslant \pi D\left(\log\frac{r}{r_{\Omega_\varepsilon}(\omega_{1/2}^{3/2})} - C\right). \tag{6.43}$$

In order to use the logarithm terms they must be translated into the new notation. Recalling (4.42) and changing the constant C (larger but still universal), we see that

$$\begin{aligned}\log\frac{r}{r_{\Omega_\varepsilon(\omega_{1/2}^{3/2})}} - C &= \log\frac{r}{r(\mathcal{B}_0)} + \log\frac{3r(\mathcal{B}_0)}{16 r_{\Omega_\varepsilon}(\omega_{1/2}^{3/2})} \\ &\leqslant \log\frac{r}{r(\mathcal{B}_0)} + \log\frac{r(\mathcal{C}(\sigma))}{r(\mathcal{C}_0)} = \sum_{k=1}^K \tau_k.\end{aligned} \tag{6.44}$$

Combining (6.42)–(6.44) and again changing the constant, we arrive at

$$
\begin{aligned}
& F_\varepsilon^r(u, A, V) - \pi D\left(\log \frac{r}{\varepsilon D} - C\right) \\
& \quad \geqslant \frac{1}{4}\left(\frac{1}{2} \int\limits_{V \backslash \omega_{1/2}^{3/2}} |\nabla_A v|^2 + \frac{r^2}{2} \int\limits_V (\operatorname{curl} A)^2 - \pi D \sum_{k=1}^K \tau_k\right).
\end{aligned} \tag{6.45}
$$

We now translate the term on the right-hand side of inequality (6.45) into the new notation and break it into two parts according to whether the generation is before or after generation k_n. Indeed,

$$
\begin{aligned}
& \frac{1}{2} \int\limits_{V \backslash \omega_{1/2}^{3/2}} |\nabla_A v|^2 + \frac{r^2}{2} \int\limits_V (\operatorname{curl} A)^2 - \pi D \sum_{k=1}^K \tau_k \\
& \quad \geqslant \frac{1}{2} \sum_{T_e} \int\limits_{A_{i,k,n}} |\nabla_A v|^2 - \pi \sum_{S_e} |D_n| \tau_k + \frac{1}{2} \sum_{T_l} \int\limits_{A_{i,k,n}} |\nabla_A v|^2 + r^2 (\operatorname{curl} A)^2 \\
& \qquad - \pi \sum_{S_l} |D_n| \tau_k + \sum_{n=1}^N \sum_{B \in \bar{B}_n \cap \mathcal{G}(t_{\bar{B}_n})} \frac{r^2}{2} \int\limits_B (\operatorname{curl} A)^2.
\end{aligned} \tag{6.46}
$$

For each $\bar{B}_n \in \mathcal{G}(s+\sigma)$ we consider $\bar{B}_n$ to have been grown from $\bar{B}_n \cap \mathcal{G}(t_{\bar{B}_n})$ and apply Corollary 3.4; summing over n gives

$$
\frac{1}{2} \sum_{T_l} \int\limits_{A_{i,k,n}} |\nabla_A v|^2 + r^2 (\operatorname{curl} A)^2 \geqslant \pi D\left(\log \frac{r}{r(\mathcal{G}(t_{\bar{B}_n}))} - \log 2\right). \tag{6.47}
$$

Note that if $t_{\bar{B}_n} \geqslant \sigma$, then

$$
\sum_{k=k_n}^K \tau_k = \log \frac{r}{r(\mathcal{G}(t_{\bar{B}_n}))},
$$

whereas if $t_{\bar{B}_n} < \sigma$, then

$$
\sum_{k=k_n}^K \tau_k = \log \frac{r}{r(\mathcal{B}_0)} + \log \frac{r(\mathcal{G}(\sigma))}{r(\mathcal{G}(t_{\bar{B}_n}))} = \log \frac{r}{r(\mathcal{G}(t_{\bar{B}_n}))} + \log \frac{3}{8}
$$

since $r(\mathcal{G}(\sigma)) = r(\mathcal{C}(\sigma)) = 3r(\mathcal{B}_0)/8$ (see item 4 of Proposition 4.3). Then

$$
\frac{1}{2} \sum_{T_l} \int\limits_{A_{i,k,n}} |\nabla_A v|^2 + r^2 (\operatorname{curl} A)^2 - \pi \sum_{S_l} |D_n| \tau_k \geqslant -\pi C D, \tag{6.48}
$$

where C is universal.

It remains to control the term corresponding to the early generations:

$$Q := \frac{1}{2}\sum_{T_e}\int_{A_{i,k,n}} |\nabla_A v|^2 - \pi\sum_{S_e}|D_n|\tau_k + \sum_{n=1}^{N}\sum_{B\in\bar{B}_n\cap\mathcal{G}(t_{\bar{B}_n})}\frac{r^2}{2}\int_B(\operatorname{curl} A)^2.$$

We apply Lemma 3.5 to each $B \in \bar{B}_n \cap \mathcal{G}(t_{\bar{B}_n})$ and sum to get

$$Q \geqslant \pi\sum_{S_e}\tau_k\left(\frac{2}{3}\sum_{i=1}^{M_{k,n}} d_{i,k,n}^2 - |D_n|\right). \tag{6.49}$$

In order to control the difference in (6.49) we must now turn to the η inequalities for generations before k_n. If $D_n \geqslant 0$, $1 \leqslant k < k_n$, the inequality (6.23) allows us to estimate

$$\begin{aligned}\sum_{i=1}^{M_{k,n}} d_{i,k,n}^2 &\geqslant \sum_{i=1}^{M_{k,n}}|d_{i,k,n}| = \sum_{\substack{1\leqslant i\leqslant M_{k,n}\\ d_{i,k,n}\geqslant 0}} d_{i,k,n} + \sum_{\substack{1\leqslant i\leqslant M_{k,n}\\ d_{i,k,n}<0}}|d_{i,k,n}|\\ &> (1+\eta)\sum_{\substack{1\leqslant i\leqslant M_{k,n}\\ d_{i,k,n}\geqslant 0}} d_{i,k,n}\\ &\geqslant (1+\eta)D_n = (1+\eta)|D_n|. \end{aligned}\tag{6.50}$$

If $D_n < 0$, we similarly get

$$\sum_{i=1}^{M_{k,n}} d_{i,k,n}^2 > (1+\eta)|D_n|,$$

and so in either case we arrive at the estimate

$$-|D_n| \geqslant -\frac{1}{1+\eta}\sum_{i=1}^{M_{k,n}} d_{i,k,n}^2. \tag{6.51}$$

Putting (6.51) into (6.49) then shows that

$$\begin{aligned}Q &\geqslant \pi\frac{2\eta-1}{3(1+\eta)}\sum_{T_e}\tau_k d_{i,k,n}^2\\ &\geqslant \pi\frac{2\eta-1}{6(1+\eta)}\sum_{T_e} d_{i,k,n}^2\big(1-e^{-2\tau_k}\big),\end{aligned}\tag{6.52}$$

where in the last inequality we have used the fact that

$$x \geqslant \frac{1}{2}\big(1-e^{-2x}\big) \quad \text{for } x \geqslant 0.$$

Finally, we use (6.45)–(6.48) and (6.52) to conclude

$$F_\varepsilon(u, A, V) - \pi D\left(\log\frac{r}{\varepsilon D} - C\right) \geqslant \pi\frac{2\eta - 1}{24(1+\eta)}\sum_{T_e} d_{i,k,n}^2\left(1 - e^{-2\tau_k}\right). \tag{6.53}$$

Step 4. In this step we use the η inequalities to provide an upper bound for the second term on the right-hand side of (6.37) by bounding $\#(I_n)$ in terms of $|D_n|$ and η. Fix n and suppose that $k_n \leqslant k \leqslant K$. For now take $D_n \geqslant 0$. The inequality (6.23) allows us to bound

$$\sum_{\substack{1\leqslant i\leqslant M_{k,n}\\ d_{i,k,n}\geqslant 0}} d_{i,k,n} = D_n + \sum_{\substack{1\leqslant i\leqslant M_{k,n}\\ d_{i,k,n}<0}} |d_{i,k,n}| \leqslant D_n + \eta\sum_{\substack{1\leqslant i\leqslant M_{k,n}\\ d_{i,k,n}\geqslant 0}} d_{i,k,n},$$

and so we can conclude that

$$\sum_{\substack{1\leqslant i\leqslant M_{k,n}\\ d_{i,k,n}\geqslant 0}} d_{i,k,n} \leqslant \frac{|D_n|}{1-\eta}. \tag{6.54}$$

We can use this estimate to bound $\#(I_n)$. Each generation in I_n is an effective-merging generation. As such, the mergings of that generation include at least one ball of non-zero degree merging with another ball of non-zero degree, resulting in a decrease in the number of balls of non-zero degree. So, the number of effective generations, $\#(I_n)$, is bounded by the number of non-zero degree balls in the k_n generation. This quantity can then be bounded in terms of D_n and η. Indeed,

$$\begin{aligned}
\#(I_n) &\leqslant \#\text{ of non-zero degree balls in generation } k_n \\
&\leqslant \sum_{i=1}^{M_{k_n,n}} |d_{i,k_n,n}| = \sum_{\substack{1\leqslant i\leqslant M_{k_n,n}\\ d_{i,k_n,n}\geqslant 0}} |d_{i,k_n,n}| + \sum_{\substack{1\leqslant i\leqslant M_{k_n,n}\\ d_{i,k_n,n}<0}} |d_{i,k_n,n}| \\
&\leqslant (1+\eta)\sum_{\substack{1\leqslant i\leqslant M_{k_n,n}\\ d_{i,k_n,n}\geqslant 0}} d_{i,k_n,n} \\
&\leqslant \frac{1+\eta}{1-\eta}|D_n|.
\end{aligned} \tag{6.55}$$

If $D_n < 0$ then (6.24) and a similar argument show that (6.55) still holds. Hence

$$\pi\sum_{n=1}^N \#(I_n)|D_n| \leqslant \pi\frac{1+\eta}{1-\eta}\sum_{n=1}^N |D_n|^2. \tag{6.56}$$

Step 5. We now conclude the proof by combining (6.37), (6.53), and (6.56) to get the inequality

$$t^2\lambda_G(t) \leqslant \pi\frac{1+\eta}{1-\eta}\sum_{n=1}^N |D_n|^2 + \frac{24(1+\eta)}{2\eta-1}\left(F_\varepsilon^r(u, A, V) - \pi D\left(\log\frac{r}{\varepsilon D} - C\right)\right). \tag{6.57}$$

Using Lemma 6.1 and switching back to our original notation then proves (6.22). $\square$

7. Jerrard's construction

In the above results we have modified and improved the vortex-ball construction of Sandier, introduced in [6], and presented in an updated form in [7]. The purpose of this section is to show that the methods of this paper can be applied equally well to the other version of the vortex-ball construction, developed by Jerrard in [3]. The two constructions are not at all dissimilar, so it is no surprise that the above methods still work. For completeness, though, we highlight the differences in the two constructions and outline the modifications necessary to make the above ideas work with Jerrard's construction. In the interest of brevity we discuss only the case without magnetic field.

There are three main differences between the ball construction employed above and that of [3]. The Jerrard construction grows finite collections of disjoint balls from an initial small collection to a final large collection, employing mergings when grown balls become tangent. However, a collection of disjoint balls $\{B_i\}$ is not grown uniformly, as we grow them above, but instead according to the parameter

$$s = \min_i \frac{r_i}{|d_i|},$$

where $d_i = \deg(u, \partial B_i)$ and r_i is the radius of B_i. There is no guarantee that this parameter is uniform throughout the collection (hence the minimum in the definition of s), and as a result, only balls for which the minimum s is achieved are grown. Note that as a ball is grown without merging, its degree does not vary, so increasing s amounts to increasing the radius of the ball. Moreover, for the subcollection of balls in $\{B_i\}$ that achieve s, if we write s^{new} for the increased parameter and r_i^{new} for the increased radii, we see that

$$\frac{s^{\mathrm{new}}}{s} = \frac{r_i^{\mathrm{new}}}{d_i}\frac{d_i}{r_i} = \frac{r_i^{\mathrm{new}}}{r_i},$$

and so all of the annuli formed by deleting the old balls from the new ones have the same conformal type. The use of this parameter causes trouble above since $r(\mathcal{B}(t)) \neq e^t r(\mathcal{B}_0)$.

The second major difference in the two methods is in how they pass from lower bounds on circles, which in both methods are most conveniently calculated by estimating $\frac{1}{2}\int_{\partial B(a,r)} |\nabla v|^2$ from below, to lower bounds of $\frac{1}{2}\int |\nabla u|^2$ on annuli and balls. Above we employ the co-area formula in Lemma 5.1 and in (5.5) of Theorem 1 to accomplish this. The Jerrard method writes $u = \rho v$, with $\rho = |u|$, and expands the energy as

$$\frac{1}{2}\int_{\partial B(a,r)} |\nabla u|^2 + \frac{1}{2\varepsilon^2}\left(1-|u|^2\right)^2 = \frac{1}{2}\int_{\partial B(a,r)} |\nabla \rho|^2 + \frac{1}{2\varepsilon^2}\left(1-\rho^2\right)^2 + \frac{1}{2}\int_{\partial B(a,r)} \rho^2|\nabla v|^2.$$

Lemmas 2.4 and 2.5 of [3] then show that

$$\frac{1}{2}\int_{\partial B(a,r)} \rho^2|\nabla v|^2 \geqslant \pi\frac{m^2 d^2}{r},$$

and

$$\frac{1}{2}\int_{\partial B(a,r)} |\nabla\rho|^2 + \frac{1}{2\varepsilon^2}\left(1-\rho^2\right)^2 \geqslant \frac{1}{c\varepsilon}(1-m)^2,$$

where c is a universal constant and $m = \min\{1, \inf_{\partial B(a,r)} \rho\}$. These two bounds are combined with the energy expansion to find

$$\frac{1}{2}\int_{\partial B(a,r)} |\nabla u|^2 + \frac{1}{2\varepsilon^2}\left(1-|u|^2\right)^2 \geqslant \inf_{m\in[0,1]}\left(\pi\frac{m^2d^2}{r} + \frac{1}{c\varepsilon}(1-m)^2\right) =: \lambda_\varepsilon(r,d).$$

One readily verifies that $\lambda_\varepsilon(r,d) \geqslant \lambda_\varepsilon(r/|d|, 1)$ and that

$$\lambda_\varepsilon(r,1) = \frac{\pi}{r + c\varepsilon\pi}. \tag{7.1}$$

The function $\Lambda_\varepsilon(s) = \int_0^s \lambda_\varepsilon(r,1)\,dr = \pi\log(1 + \frac{s}{c\varepsilon\pi})$ is then introduced, and lower bounds on annuli are calculated by integrating on circles:

$$\begin{aligned}\frac{1}{2}\int_{B(a,r_1)\setminus B(a,r_0)} |\nabla u|^2 + \frac{1}{2\varepsilon^2}\left(1-|u|^2\right)^2 &\geqslant \int_{r_0}^{r_1} \lambda_\varepsilon(r,d)\,dr \geqslant |d|\int_{r_0/|d|}^{r_1/|d|} \lambda_\varepsilon(r,1)\,dr \\ &= |d|\left(\Lambda_\varepsilon\left(r_1/|d|\right) - \Lambda_\varepsilon\left(r_0/|d|\right)\right).\end{aligned}$$

Note that this bound justifies the use of $s = r/d$ as the growth parameter.

The third major difference is in the nature of the lower bounds. The method above produces lower bounds on the total collection of balls but cannot say much about the energy content of any given ball in the collection. Because of its use of the Λ_ε function, which only depends on the parameter s, the Jerrard construction can localize the lower bounds to each ball in the collection. In particular, Proposition 4.1 of [3], the analogue of our Theorem 1, shows that there exists a σ_0 such that for any $0 \leqslant \sigma \leqslant \sigma_0$ there exists a collection of disjoint balls $\{B_i\}$ with radii r_i and degrees d_i such that

$$\frac{1}{2}\int_{B_i\cap\Omega} |\nabla u|^2 + \frac{1}{2\varepsilon^2}\left(1-|u|^2\right)^2 \geqslant \frac{r_i}{s}\Lambda_\varepsilon(s),$$

where $s = \min_i(r_i/|d_i|) \in [\sigma/2, \sigma]$. In particular this implies that

$$\frac{1}{2}\int_{B_i\cap\Omega} |\nabla u|^2 + \frac{1}{2\varepsilon^2}\left(1-|u|^2\right)^2 \geqslant \pi|d_i|\log\left(1 + \frac{\sigma}{2c\pi\varepsilon}\right).$$

The proof of this result follows from a line of reasoning similar to what led to Theorem 1. An initial collection of balls $\{B_i\}$ with radii $r_i \geqslant \varepsilon$ is found (Proposition 3.3 of [3]) that covers $\{|u| \leqslant 1/2\}$ and on which

$$\frac{1}{2}\int_{B_i\cap\Omega} |\nabla u|^2 + \frac{1}{2\varepsilon^2}\left(1-|u|^2\right)^2 \geqslant c_0\frac{r_i}{\varepsilon} \geqslant \frac{r_i}{s}\Lambda_\varepsilon(s), \tag{7.2}$$

where c_0 is a universal constant. These balls are then grown into the final balls according to the ball growth lemma, but used with the parameter s as the growth parameter. It is then shown that growth and merging preserves the form of the lower bound (7.2), i.e. that if the bound holds with one value of s, it also holds with the value of s obtained after growing the balls.

In order to utilize our completion of the square trick to extract the new term we must only present a modification of Lemma 3.2 designed to work with the minimization of m trick. The rest of the argument follows from simple modifications of the arguments in [3] that we will only sketch.

Lemma 7.1. *Let $B = B(a, r)$ and suppose that $u : \partial B \to \mathbb{C}$ is C^1 and that $|u| > c \geqslant 0$ on ∂B. Write $u = \rho v$ with $\rho = |u|$, and define the function*

$$G = \frac{dm^2\beta}{\rho^2 r}\tau, \tag{7.3}$$

where $d = \deg(u, \partial B)$, $m = \min\{1, \inf_{\partial B(a,r)} \rho\}$, τ is the oriented unit tangent vector field to ∂B, and $\beta \in [0, 1]$ is a constant. Then

$$\frac{1}{2}\int_{\partial B} \rho^2|\nabla v|^2 \geqslant \frac{1}{2}\int_{\partial B} \rho^2|\nabla v - G|^2 + \pi\frac{d^2m^2\beta}{r}. \tag{7.4}$$

Proof. Arguing as in Lemma 3.2, we find that

$$\frac{1}{2}\int_{\partial B} \rho^2|\nabla v|^2 = \frac{1}{2}\int_{\partial B} \rho^2|\nabla v - G|^2 + 2\pi d\frac{dm^2\beta}{r} - \frac{d^2m^4\beta^2}{2r^2}\int_{\partial B} \frac{1}{\rho^2}. \tag{7.5}$$

Then the definition of m implies that

$$2\pi d\frac{dm^2\beta}{r} - \frac{d^2m^4\beta^2}{2r^2}\int_{\partial B} \frac{1}{\rho^2} \geqslant \pi\frac{d^2m^2}{r}\left(2\beta - \beta^2\right) \geqslant \pi\frac{d^2m^2\beta}{r}, \tag{7.6}$$

where the last inequality follows from the fact that $0 \leqslant \beta \leqslant 1$. This proves the result. $\square$

This result may be used in conjunction with Lemma 2.5 of [3], borrowing half of that energy to absorb into the novel term, to arrive at the lower bound

$$\frac{1}{2}\int_{\partial B} |\nabla u|^2 + \frac{1}{2\varepsilon^2}\left(1 - |u|^2\right)^2 \geqslant \frac{1}{4}\int_{\partial B} |\nabla u - iuG|^2 + \inf_{m\in[0,1]}\left(\pi\frac{m^2d^2\beta}{r} + \frac{1}{c\varepsilon}(1 - m)^2\right). \tag{7.7}$$

In order to gain the ability to localize the estimates in each ball, we must have that $\lambda_\varepsilon(r, d)$ is independent of β and that the homogeneity inequality $\lambda_\varepsilon(r, d) \geqslant \lambda_\varepsilon(r/|d|, 1)$ holds. The first of these requires us to set $\beta = 1$ in the above, which precludes the special choice of β needed to make Proposition 6.4 work. The second requires us to throw away the d^2 terms in favor of $|d|$. So, there is a tradeoff: the price we pay for localizing the estimates is a loss of control of the $L^{2,\infty}$ norm of the auxiliary function G. This choice leads to the lower bound on circles

$$\frac{1}{2}\int_{\partial B}|\nabla u|^2+\frac{1}{2\varepsilon^2}\big(1-|u|^2\big)^2\geqslant\frac{1}{4}\int_{\partial B}|\nabla u-iuG|^2+\lambda_\varepsilon\big(r/|d|,1\big),\tag{7.8}$$

where λ_ε is as defined in (7.1), but with the universal constant doubled, and $G=\frac{dm^2}{\rho^2 r}\tau$. The bound on circles leads to bounds on annuli by integrating; indeed,

$$\begin{aligned}\frac{1}{2}\int_{B(a,r_1)\setminus B(a,r_0)}|\nabla u|^2+\frac{1}{2\varepsilon^2}\big(1-|u|^2\big)^2\geqslant{}&\frac{1}{4}\int_{B(a,r_1)\setminus B(a,r_0)}|\nabla u-iuG|^2\\&+|d|\big(\Lambda_\varepsilon\big(r_1/|d|\big)-\Lambda_\varepsilon\big(r_0/|d|\big)\big),\end{aligned}\tag{7.9}$$

where now we take $G(x)=\frac{dm^2}{\rho(x)^2|x-a|}\tau(x)$.

Now, to achieve a bound of the form (7.2) but with the L^2 difference with iuG included, we use Lemma 7.1 in the Jerrard construction. As above, we define the function G to vanish in the initial collection of balls obtained in Proposition 3.3 of [3]. Then we trivially modify (7.2) to read (since $G=0$ there)

$$\begin{aligned}\frac{1}{2}\int_{B_i\cap\Omega}|\nabla u|^2+\frac{1}{2\varepsilon^2}\big(1-|u|^2\big)^2&\geqslant\frac{c_0r_i}{2\varepsilon}+\frac{1}{4}\int_{B_i\cap\Omega}|\nabla u|^2\\&\geqslant\frac{r_i}{s}\Lambda_\varepsilon(s)+\frac{1}{4}\int_{B_i\cap\Omega}|\nabla u-iuG|^2.\end{aligned}\tag{7.10}$$

We then take G to vanish in all of the non-annular regions of the balls constructed in Proposition 4.1 of [3]. The estimates in these balls, like the original Sandier estimates, discard the energy of the non-annular regions. We retain it and rewrite it as a $\int|\nabla u-iuG|^2$ term, which is possible since $G=0$ there. Then, adding in the extra G term in the annular regions, we arrive at the modification.

Proposition 7.2. *There exists a σ_0 such that for any $0\leqslant\sigma\leqslant\sigma_0$ there exists a collection of disjoint balls $\{B_i\}$ with radii r_i and degrees d_i such that*

$$\frac{1}{2}\int_{B_i\cap\Omega}|\nabla u|^2+\frac{1}{2\varepsilon^2}\big(1-|u|^2\big)^2\geqslant\frac{1}{4}\int_{B_i\cap\Omega}|\nabla u-iuG|^2+\frac{r_i}{s}\Lambda_\varepsilon(s),$$

where $s=\min_i(r_i/|d_i|)\in[\sigma/2,\sigma]$. In particular this implies that

$$\frac{1}{2}\int_{B_i\cap\Omega}|\nabla u|^2+\frac{1}{2\varepsilon^2}\big(1-|u|^2\big)^2\geqslant\frac{1}{4}\int_{B_i\cap\Omega}|\nabla u-iuG|^2+\pi|d_i|\log\bigg(1+\frac{\sigma}{2c\pi\varepsilon}\bigg).$$

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Transference between Laguerre and Hermite settings ☆

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Abstract

In this paper we present a new method in order to transfer boundedness results for operators associated with Hermite functions to boundedness results for operators associated with Laguerre functions. The technique relies on an exact point-wise identity relating the heat kernels of both systems. The method that we present here has the novelty that can be used backwards, that is, boundedness results for Laguerre systems can be also transfered to boundedness results for Hermite systems. We apply our method in order to get new properties of some operators in the Laguerre setting. Among others, we mention the description of Riesz transforms as principal value operators. As an application of the reversibility of the method we characterize the class of Banach spaces B for which the Riesz transforms (in the Laguerre setting) are bounded from L^p_B into itself. It is shown that this class coincides with the UMD class.

Keywords: Laguerre expansions; Hermite expansions; Riesz transforms UMD spaces

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1. Introduction

We consider the Laguerre differential operator

$$\mathbf{L}_\alpha = \frac{1}{2}\left\{-\frac{d^2}{dy^2} + y^2 + \frac{1}{y^2}\left(\alpha^2 - \frac{1}{4}\right)\right\}, \quad y \in (0,\infty),\ \alpha > -1.$$

The operator $\mathbf{L}_\alpha$ is selfadjoint with respect to the Lebesgue measure on $(0,\infty)$, its eigenfunctions are the complete family of Laguerre orthonormal functions, $\{\varphi_n^\alpha\}_{n=0}^\infty$, defined as

$$\varphi_n^\alpha(y) = \left(\frac{\Gamma(n+1)}{\Gamma(n+1+\alpha)}\right)^{1/2} e^{-y^2/2} y^\alpha L_n^\alpha\big(y^2\big)(2y)^{1/2},$$

where $\{L_n^\alpha\}_{n=0}^\infty$ are the Laguerre polynomials of type α, see [27, p. 100] and [28, p. 7]. In fact,

$$\mathbf{L}_\alpha\big(\varphi_n^\alpha\big) = (2n+\alpha+1)\varphi_n^\alpha.$$

The operator $\mathbf{L}_\alpha$ can be "factorized" as

$$\mathbf{L}_\alpha = \frac{1}{2}\mathbf{D}_\alpha^*\mathbf{D}_\alpha + \alpha + 1,$$

where $\mathbf{D}_\alpha f = (-\frac{\alpha+1/2}{x} + x + \frac{d}{dx})f = x^{\alpha+1/2}\frac{d}{dx}(x^{-(\alpha+1/2)}f) + xf$, and $\mathbf{D}_\alpha^*$ represents the formal adjoint operator of $\mathbf{D}_\alpha$ into $L^2((0,\infty), dx)$.

In these circumstances, following Stein [24], an important part of the main objects in harmonic analysis (heat and Poisson semigroups, "Riesz transforms" ...) associated with the "Laplacian" $\mathbf{L}_\alpha$ can be defined and studied.

The heat semigroup $\{e^{-t\mathbf{L}_\alpha}\}_{t>0}$ associated with $\mathbf{L}_\alpha$, was studied in [25] for $\alpha \geqslant -1/2$ and in [6] for $\alpha > -1$. In [25] it was shown that the maximal operator of the heat and Poisson semigroups are bounded from L^p into itself, $1 < p < \infty$, and also from L^1 into weak-L^1. For $-1 < \alpha < -1/2$, a particular boundedness result has appeared, namely there exists an optimal interval of p's depending on α for which the heat semigroup is L^p bounded, see [6,13,14].

The Riesz transforms

$$\mathbf{R}_\alpha = \mathbf{D}_\alpha(\mathbf{L}_\alpha)^{-1/2} \tag{1.1}$$

were studied in [22] for $\alpha \geqslant -1/2$ and in [3] for $\alpha > -1$. Their boundedness from L^p, $1 < p < \infty$, into itself (for certain classes of weights) were proved in [22] for the case $\alpha \geqslant -1/2$. While for $-1 < \alpha < -1/2$, see [3], the particular boundedness result appears (also in a weighted version) parallel to the case of the maximal operator of the heat semigroup.

This topic of describing operators associated with a Laplacian was initiated in the 1960s with a series of papers authored by Muckenhoupt and Stein, see among them [15–17,19]. In these papers the case of Hermite, Laguerre and ultraspherical polynomials were treated. In the last fifteen years and in the particular case of Laguerre function systems a lot of effort has been employed in this line of thought. Even more, the following Laguerre function systems have also been considered:

$$\mathcal{L}_n^\alpha(x) = (2\sqrt{x})^{-1/2}\varphi_n^\alpha(\sqrt{x}), \quad x \in (0,\infty), \tag{1.2}$$

and

$$\ell_n^\alpha(x) = x^{-\alpha/2}\mathcal{L}_n^\alpha(x), \quad x \in (0, \infty). \tag{1.3}$$

The boundedness in L^p and the $(1, 1)$ weak type of the maximal heat semigroups were proved in [25] for $\alpha \geqslant 0$ in the case of the system $\{\mathcal{L}_n^\alpha\}_{n=0}^\infty$ and for $\alpha > -1$ in the case of the system $\{\ell_n^\alpha\}_{n=0}^\infty$. The proof uses some ideas, previously introduced in [16], of breaking the kernel of the operators into a "local part" and a "global part." For general $\alpha > -1$ and for the system $\{\mathcal{L}_n^\alpha\}_{n=0}^\infty$, (strong and weak-)$L^p$ boundedness of these maximal operators with power weights has been proved in [6,13,14]. Again the technique is to analyze the local and the global parts. At this point we want to be more precise. The best result, contained in [6], it is achieved by proving that the local part is a "local Calderón–Zygmund operator" (concept introduced in [21]), see Definition 3.5, and the "global part" is controlled by some Hardy operators. Again a particular boundedness result appears for $-1 < \alpha < 0$ in the case $\{\mathcal{L}_n^\alpha\}_{n=0}^\infty$.

Riesz transforms for the system $\{\mathcal{L}_n^\alpha\}_{n=0}^\infty$ and $\alpha > -1$ were studied in [10]. Also the optimal interval of p's depending on α, for which the operators are L^p bounded, was found. In the paper [10], the technique was to use a kind of transference from Hermite function systems (in dimensions d) to Laguerre function systems of particular index $\alpha = \frac{d}{2} - 1$. This transference relays on some classic formulae relating Hermite and Laguerre polynomials and was used previously in [8]. It can be shown that the Riesz transforms are given by a kernel $\mathbf{R}_\alpha(x, y)$ in the following sense. For any function $f \in L^2((0, \infty), dx)$ with compact support

$$\mathbf{R}_\alpha(f)(x) = \int_0^\infty \mathbf{R}_\alpha(x, y) f(y)\, dy, \quad x \notin \operatorname{supp}(f), \tag{1.4}$$

where the kernel $\mathbf{R}_\alpha(x, y)$ is given by the formula

$$\mathbf{R}_\alpha(x, y) = \int_0^\infty \mathbf{D}_\alpha W_t^\alpha(x, y) \frac{dt}{\sqrt{t}},$$

where $W_t^\alpha(x, y)$, $t, x, y \in (0, \infty)$, represents the heat kernel for the system $\{\varphi_n^\alpha\}_{n=0}^\infty$. The technique of "local" and "global" parts has also been used for the Riesz transforms, namely in [11] for the system $\{\mathcal{L}_n^\alpha\}_{n=0}^\infty$ and $\alpha > -1$ (again the phenomenon of the interval of p's depending on α appears) and in [22] for the system $\{\varphi_n^\alpha\}_{n=0}^\infty$ and $\alpha \geqslant -1/2$. We also want to mention that some d-dimensional results were proved in [20] about the maximal operators and in [22] about Riesz transforms associated to the system $\{\varphi_n^\alpha\}_{n=0}^\infty$.

The aim of this note is to present a different (and shorter) proof of the results above and then apply this new method to obtain new results. Our method follows the "local" and "global" procedure, but the "local" parts of the kernels are compared directly for the case of the system $\{\varphi_n^\alpha\}_{n=0}^\infty$ with the corresponding local parts of the kernels associated to the Hermite operator in one dimension, see Lemmas 2.11 and 2.13. The novelty of this method is two-fold:

1. To notice the existence of a *pointwise identity* between the kernels of the heat semigroups of Laguerre and Hermite. This identity, formula (2.10), can be used to *transfer results from Hermite in one dimension to Laguerre in one dimension for any index* $\alpha > -1$ (the previous

methods were only valid for special values of α and they need results in higher dimensions for operators associated to Hermite operators, [8] and [10]).

2. The comparison can be used *backwards*, that is, one can obtain results for operators associated to Hermite differential operator from results about operators associated to Laguerre differential operator.

We apply the method to characterize the Banach spaces B for which the Riesz transforms are bounded from L^p_B into itself (see Theorem 4.2) and also to characterize the Köthe Banach lattices B for which the maximal operator of the heat semigroup is bounded from L^p_B into itself (see Theorem 4.1).

We also want to mention that, as a by-product of the proof, we obtain some results that we believe interesting in themselves, namely we prove that the Riesz transforms associated to Laguerre differential operators are principal value operators, see arguments before Theorem 4.2. Also we prove that for principal value operators, satisfying the standard size condition of the kernel, the information about the boundedness of the operators is contained in the boundedness of the local part of the corresponding operator, that is the philosophy behind Theorems 3.9 and 3.11 in the setting of Hermite functions.

In a future work we will use the ideas developed here for studying the L^p-boundedness of the maximal operator associated with the heat semigroup and Riesz transforms for Laguerre expansions in higher dimensions.

The organization of the paper is the following. In Section 2 we present the main computations that we shall need in order to pass from Hermite to Laguerre settings and vice-versa. In Section 3 we discuss some results about Hermite functions that we need along the manuscript. Section 4 is devoted to state the theorems about vector-valued functions that we present as applications of our method. Finally in Section 5 we quickly discuss two different Laguerre systems and we present some theorems for them that are parallel to the results from the system considered in Section 2.

Throughout this paper by C we always denote a suitable positive constant that can change from one line to another.

2. Technical results

Let H be the second order differential operator (Hermite operator)

$$H = -\frac{d^2}{dx^2} + x^2.$$

This operator is selfadjoint with respect to the Lebesgue measure on $\mathbb{R}$, its eigenfunctions are the complete family of Hermite functions $\{h_n\}_{n=0}^{\infty}$, given by

$$h_n(x) = \left(\sqrt{\pi}2^n n!\right)^{-1/2} H_n(x) e^{-x^2/2}, \quad x \in \mathbb{R},$$

where H_n denotes the nth Hermite polynomial. For every $n \in \mathbb{N}$ we have

$$Hh_n = (2n+1)h_n.$$

It is known that the heat semigroup associated with the Hermite operator can be described as the integral

$$e^{-tH} f(x) = \mathcal{W}_t f(x) = \int_{-\infty}^{\infty} \mathcal{W}_t(x,y) f(y)\, dy, \quad f \in L^2(\mathbb{R}), \tag{2.1}$$

being

$$\begin{aligned}\mathcal{W}_t(x,y) &= \sum_{n=0}^{\infty} e^{-(2n+1)t} h_n(x) h_n(y) \\ &= e^{-t} \frac{1}{\sqrt{\pi}} \left(1 - e^{-4t}\right)^{-1/2} e^{-\frac{1}{2}\frac{1+e^{-4t}}{1-e^{-4t}}(x^2+y^2) + \frac{2e^{-2t}}{1-e^{-4t}} xy}, \quad x, y \in \mathbb{R},\ t > 0,\end{aligned} \tag{2.2}$$

(see [26,28]). The Hermite operator can be factorized:

$$H = -\frac{1}{2}\left(\left(\frac{d}{dx} + x\right)\left(\frac{d}{dx} - x\right) + \left(\frac{d}{dx} - x\right)\left(\frac{d}{dx} + x\right)\right).$$

The Riesz transform

$$\mathcal{R}(f)(x) = \left(\frac{d}{dx} + x\right) H^{-1/2} f(x) = \sum_{n=0}^{\infty} \left(\frac{2n}{2n+1}\right)^{1/2} h_{n-1}(x) a_n(f) \tag{2.3}$$

was defined in [26,28] for every $f \in L^2(\mathbb{R})$. Here, for every $n \in \mathbb{N}$,

$$a_n(f) = \int_{-\infty}^{\infty} f(y) h_n(y)\, dy.$$

They can be extended as bounded operators from $L^p(\mathbb{R}, w)$ into itself, for $1 < p < \infty$, $w \in A_p$ and from $L^1(\mathbb{R}, w)$ into $L^{1,\infty}(\mathbb{R}, w)$ for $w \in A_1$. As usual, we write A_p as a shorthand for the Muckenhoupt classes of weights w. In (2.3) the fractional integral $H^{-1/2}$ is defined by the formula $H^{-1/2} = \int_0^\infty e^{-tH} \frac{dt}{\sqrt{t}}$.

It is known that the heat semigroup associated with the Laguerre operator can be described as the integral

$$e^{-t\mathbf{L}_\alpha} f(x) = \int_0^\infty W_t^\alpha(x,y) f(y)\, dy,$$

where

$$
\begin{aligned}
W_t^\alpha(x,y) &= \sum_{n=0}^{\infty} e^{-(2n+1+\alpha)t}\varphi_n^\alpha(x)\varphi_n^\alpha(y) \\
&= 2(xy)^{1/2}\frac{e^{-t}}{1-e^{-2t}} I_\alpha\left(\frac{2xye^{-t}}{1-e^{-2t}}\right) e^{-\frac{1}{2}(x^2+y^2)\frac{1+e^{-2t}}{1-e^{-2t}}} \\
&= \sqrt{2}\left(\frac{e^{-t}}{1-e^{-2t}}\right)^{1/2}\left(\frac{2xye^{-t}}{1-e^{-2t}}\right)^{1/2} I_\alpha\left(\frac{2xye^{-t}}{1-e^{-2t}}\right) e^{-\frac{1}{2}\frac{1+e^{-2t}}{1-e^{-2t}}(x^2+y^2)}, \\
&\quad x,y\in(0,\infty),
\end{aligned}
\tag{2.4}
$$

see [6,25,28]. As usual I_α denotes the modified Bessel function I_α of the first kind and order α. We shall use the following properties of the functions I_α, see [12]:

$$
I_\alpha(z) \sim z^\alpha, \quad z\to 0, \tag{2.5}
$$

$$
z^{1/2}I_\alpha(z) = \frac{1}{\sqrt{2\pi}} e^z\left(1+O\left(\frac{1}{z}\right)\right), \quad z\to\infty, \tag{2.6}
$$

$$
\frac{d}{dz}\left(z^{-\alpha}I_\alpha(z)\right) = z^{-\alpha}I_{\alpha+1}(z), \quad z\in(0,\infty). \tag{2.7}
$$

Remark 2.8. For technical reasons that the reader will find in the following computations, we shall need to consider the operators associated to $\frac{H}{2}$ instead of the operators associated to H. Observe that $\sup_{t>0} e^{-tH} = \sup_{t>0} e^{-t\frac{H}{2}}$, $H^{-1/2} = \frac{1}{\sqrt{2}}(\frac{H}{2})^{-1/2}$ and $(\frac{d}{dx}+x)H^{-1/2} = \frac{1}{\sqrt{2}}(\frac{d}{dx}+x)(\frac{H}{2})^{-1/2}$.

The kernel $\mathcal{W}_{t/2}$ of the operator $e^{-t\frac{H}{2}}$ is given, see (2.2), by

$$
\mathcal{W}_{t/2}(x,y) = e^{-t/2}\frac{1}{\sqrt{\pi}}\left(1-e^{-2t}\right)^{-1/2} e^{-\frac{1}{2}\frac{1+e^{-2t}}{1-e^{-2t}}(x^2+y^2)+\frac{2xye^{-t}}{1-e^{-2t}}}, \quad x,y\in\mathbb{R}. \tag{2.9}
$$

From the formulae (2.4) and (2.9) we deduce the following pointwise relation between the heat kernels. This identity is the crucial keystone in this note:

$$
\begin{aligned}
&W_t^\alpha(x,y) - \mathcal{W}_{t/2}(x,y) \\
&\quad = \left\{\sqrt{2\pi}\left(\frac{2xye^{-t}}{1-e^{-2t}}\right)^{1/2} I_\alpha\left(\frac{2xye^{-t}}{1-e^{-2t}}\right) e^{-\frac{2xye^{-t}}{1-e^{-2t}}} - 1\right\}\mathcal{W}_{t/2}(x,y).
\end{aligned}
\tag{2.10}
$$

Now we shall prove two technical lemmas that will be used in a fundamental way in the proofs of our theorems in Section 4.

Lemma 2.11. *There exists $C>0$ such that*:

(i) $W_t^\alpha(x,y) \leqslant Cy^{\alpha+1/2}x^{-\alpha-3/2}$, $t>0$, $0<y<x/2$;
(ii) $W_t^\alpha(x,y) \leqslant Cx^{\alpha+1/2}y^{-\alpha-3/2}$, $t>0$, $y>2x$;
(iii) $|W_t^\alpha(x,y) - \mathcal{W}_{t/2}(x,y)| \leqslant \frac{C}{y}$, $t>0$, $0<\frac{x}{2}<y<2x$.

Proof. We observe that as $W_t^\alpha(x,y) = W_t^\alpha(y,x)$, $t, x, y \in (0,\infty)$ the inequality in point (ii) will be proved as soon as we prove (i). For the proof of (i) we follow some ideas of [6, Section 5, global case]. We distinguish two cases:

Case 1. $\frac{xye^{-t}}{1-e^{-2t}} \geqslant 1$. We observe that

$$-\frac{1}{2}\left(x^2+y^2\right)\frac{1+e^{-2t}}{1-e^{-2t}} + 2xy\frac{e^{-t}}{1-e^{-2t}} = -\frac{(x-e^{-t}y)^2+(y-e^{-t}x)^2}{2(1-e^{-2t})}. \tag{2.12}$$

Hence, by using the relation (2.6) we get

$$\begin{aligned} W_t^\alpha(x,y) &\leqslant C\left(\frac{e^{-t}}{1-e^{-2t}}\right)^{1/2}\exp\left(-\frac{1}{2}\frac{|x-e^{-t}y|^2+|y-e^{-t}x|^2}{1-e^{-2t}}\right) \\ &\leqslant C\left(\frac{e^{-t}}{1-e^{-2t}}\right)^{1/2} e^{-x^2/(8(1-e^{-2t}))}. \end{aligned}$$

Then, if $-1 < \alpha < -1/2$,

$$W_t^\alpha(x,y) \leqslant \frac{C}{x} \leqslant Cy^{\alpha+1/2}x^{-\alpha-3/2},$$

while in the case $\alpha > -1/2$ we obtain

$$\begin{aligned} W_t^\alpha(x,y) &\leqslant C\left(\frac{xye^{-t}}{1-e^{-2t}}\right)^{\alpha+1/2}\left(\frac{e^{-t}}{1-e^{-2t}}\right)^{1/2} e^{-x^2/(8(1-e^{-2t}))} \\ &\leqslant Cy^{\alpha+1/2}x^{-\alpha-3/2}. \end{aligned}$$

Case 2. $\frac{xye^{-t}}{1-e^{-2t}} \leqslant 1$. Now we use the relation (2.5) and we get

$$\begin{aligned} W_t^\alpha(x,y) &\leqslant C\left(\frac{xye^{-t}}{1-e^{-2t}}\right)^{\alpha+1/2}\left(\frac{e^{-t}}{1-e^{-2t}}\right)^{1/2} e^{-\frac{1}{2}(x^2+y^2)\frac{1+e^{-2t}}{1-e^{-2t}}} \\ &\leqslant C(xy)^{\alpha+1/2}\left(x^2+y^2\right)^{-\alpha-1} \\ &\leqslant Cy^{\alpha+1/2}x^{-\alpha-3/2}. \end{aligned}$$

In order to prove (iii) we shall also distinguish the previous cases. In Case 1, formula (2.10) and estimate (2.6) give

$$\left|W_t^\alpha(x,y) - \mathcal{W}_{t/2}(x,y)\right| = \left(\frac{e^{-t}}{1-e^{-2t}}\right)^{1/2}\exp\left(-\frac{1}{2}\frac{|x-e^{-t}y|^2+|y-e^{-t}x|^2}{1-e^{-2t}}\right) O\left(\frac{1-e^{-2t}}{xye^{-t}}\right).$$

Hence

$$\begin{aligned}\left|W_t^\alpha(x,y)-\mathcal{W}_{t/2}(x,y)\right| &\leqslant C\left(\frac{1-e^{-2t}}{xye^{-t}}\right)\left(\frac{e^{-t}}{1-e^{-2t}}\right)^{1/2}\exp\left(-\frac{1}{2}\frac{|x-e^{-t}y|^2+|y-e^{-t}x|^2}{1-e^{-2t}}\right)\\ &\leqslant C\left(\frac{1-e^{-2t}}{xye^{-t}}\right)^{1/2}\frac{1}{\sqrt{xy}}\leqslant\frac{C}{\sqrt{xy}}.\end{aligned}$$

As for Case 2, by using (2.5) it follows that

$$\begin{aligned}&\left|W_t^\alpha(x,y)-\mathcal{W}_{t/2}(x,y)\right|\leqslant\left|W_t^\alpha(x,y)\right|+\left|\mathcal{W}_{t/2}(x,y)\right|\\ &\quad\leqslant C\left[\left(\frac{e^{-t}}{1-e^{-2t}}\right)^{\alpha+1}(xy)^{\alpha+1/2}+\left(\frac{e^{-t}}{1-e^{-2t}}\right)^{1/2}e^{\frac{2xye^{-t}}{1-e^{-2t}}}\right]e^{-\frac{1}{2}\frac{1+e^{-2t}}{1-e^{-2t}}(x^2+y^2)}\\ &\quad\leqslant C\left[\frac{(xy)^{\alpha+1/2}}{y^{2(\alpha+1)}}+\frac{1}{y}\right]\leqslant\frac{C}{y}. \qquad\Box\end{aligned}$$

Lemma 2.13. *Let* $\alpha>-1$ *and* $\mathbf{R}_\alpha$ *the kernel considered in* (1.4). *Then*:

(i) $|\mathbf{R}_\alpha(x,y)|\leqslant Cy^{\alpha+1/2}x^{-\alpha-3/2}$, $0<y<x/2$.
(ii) $|\mathbf{R}_\alpha(x,y)|\leqslant Cx^{\alpha+3/2}y^{-\alpha-5/2}$, $2x<y$.
(iii) $|\mathbf{R}_\alpha(x,y)-\int_0^\infty(\frac{d}{dx}+x)\mathcal{W}_{t/2}(x,y)|\frac{dt}{\sqrt{t}}\leqslant\frac{C}{y}(1+\frac{(xy)^{1/4}}{|x-y|^{1/2}})$, $0<x/2<y<2x$.

Proof. Consider the kernel

$$L_t^\alpha(x,y):=x^{\alpha+1/2}\frac{d}{dx}\left(x^{-(\alpha+1/2)}W_t^\alpha(x,y)\right)+xW_t^\alpha(x,y).$$

We recall for the reader's convenience that

$$\mathbf{R}_\alpha(x,y)=\int_0^\infty L_t^\alpha(x,y)\frac{dt}{\sqrt{t}}.$$

The property of Bessel functions stated in (2.7) produces the following identity:

$$\begin{aligned}L_t^\alpha(x,y)&=\sqrt{2}\left(\frac{e^{-t}}{1-e^{-2t}}\right)^{1/2}\left(\frac{2xye^{-t}}{1-e^{-2t}}\right)^{1/2}I_{\alpha+1}\left(\frac{2xye^{-t}}{1-e^{-2t}}\right)\left(\frac{2ye^{-t}}{1-e^{-2t}}\right)e^{-\frac{1}{2}\frac{1+e^{-2t}}{1-e^{-2t}}(x^2+y^2)}\\ &\quad-\sqrt{2}\left(\frac{e^{-t}}{1-e^{-2t}}\right)^{1/2}\left(\frac{2xye^{-t}}{1-e^{-2t}}\right)^{1/2}I_\alpha\left(\frac{2xye^{-t}}{1-e^{-2t}}\right)\left(\frac{2xe^{-2t}}{1-e^{-2t}}\right)e^{-\frac{1}{2}\frac{1+e^{-2t}}{1-e^{-2t}}(x^2+y^2)}\\ &=\sqrt{2\pi}\left(\frac{2xye^{-t}}{1-e^{-2t}}\right)^{1/2}I_{\alpha+1}\left(\frac{2xye^{-t}}{1-e^{-2t}}\right)\left(\frac{2ye^{-t}}{1-e^{-2t}}\right)e^{-\frac{2xye^{-t}}{1-e^{-2t}}}\mathcal{W}_{t/2}(x,y)\\ &\quad-\sqrt{2\pi}\left(\frac{2xye^{-t}}{1-e^{-2t}}\right)^{1/2}I_\alpha\left(\frac{2xye^{-t}}{1-e^{-2t}}\right)\left(\frac{2xe^{-2t}}{1-e^{-2t}}\right)e^{-\frac{2xye^{-t}}{1-e^{-2t}}}\mathcal{W}_{t/2}(x,y).\end{aligned}$$

The proof will follow, with the obvious modifications, the pattern of the proof of Lemma 2.11. In order to prove (i) we shall distinguish two cases. In Case 1, that is $\frac{2e^{-t}xy}{1-e^{-2t}}\geqslant 1$, (2.6) and formula (2.12) lead to

$$\left|L_t^\alpha(x,y)\right| \leqslant Cx\left(\frac{e^{-t}}{1-e^{-2t}}\right)^{3/2} e^{-\frac{|x-e^{-t}y|^2+|y-e^{-t}x|^2}{2(1-e^{-2t})}} \leqslant Cx\left(\frac{e^{-t}}{1-e^{-2t}}\right)^{3/2} e^{-\frac{x^2}{8(1-e^{-2t})}}.$$

Then, when $-1<\alpha<-1/2$, we have

$$\begin{aligned}
&\int\limits_{\frac{2e^{-t}xy}{1-e^{-2t}}\geqslant 1} \left|L_t^\alpha(x,y)\right|\frac{1}{\sqrt{t}}\,dt \\
&\quad\leqslant Cx\left(\left(\int\limits_{\{0\leqslant t\leqslant 1,\,\frac{2e^{-t}xy}{1-e^{-2t}}\geqslant 1\}} + \int\limits_{\{1\leqslant t\leqslant\infty,\,\frac{2e^{-t}xy}{1-e^{-2t}}\geqslant 1\}}\right)\left(\frac{e^{-t}}{1-e^{-2t}}\right)^{3/2} e^{-\frac{x^2}{8(1-e^{-2t})}}\frac{1}{\sqrt{t}}\,dt\right) \\
&\quad\leqslant Cx\left(\int_0^1 t^{-2}e^{-c_1x^2/t}\,dt + e^{-c_1x^2}\right) \leqslant C\frac{1}{x} \\
&\quad\leqslant C\frac{1}{x}\left(\frac{y}{x}\right)^{\alpha+1/2} \leqslant Cy^{\alpha+1/2}x^{-\alpha-3/2}.
\end{aligned}$$

If $\alpha>-1/2$, we can write

$$\begin{aligned}
\int\limits_{\frac{2e^{-t}xy}{1-e^{-2t}}\geqslant 1} \left|L_t^\alpha(x,y)\right|\frac{1}{\sqrt{t}}\,dt &\leqslant Cx \int\limits_{\frac{2e^{-t}xy}{1-e^{-2t}}\geqslant 1} \left(\frac{2e^{-t}xy}{1-e^{-2t}}\right)^{\alpha+1/2}\left(\frac{e^{-t}}{1-e^{-2t}}\right)^{3/2} e^{-\frac{x^2}{8(1-e^{-2t})}}\frac{dt}{\sqrt{t}} \\
&\leqslant Cx(xy)^{\alpha+1/2}\int_0^\infty \left(\frac{e^{-t}}{1-e^{-2t}}\right)^{\alpha+2} e^{-\frac{x^2}{8(1-e^{-2t})}}\frac{1}{\sqrt{t}}\,dt \\
&\leqslant Cx(xy)^{\alpha+1/2}\left(\int_0^1 t^{-\alpha-3/2}e^{-c_1x^2/t}\frac{dt}{t} + e^{-c_1x^2}\right) \\
&\leqslant C\frac{x(xy)^{\alpha+1/2}}{x^{2\alpha+3}} \leqslant Cy^{\alpha+1/2}x^{-\alpha-3/2}.
\end{aligned}$$

In Case 2, that is $\frac{2e^{-t}xy}{1-e^{-2t}}\leqslant 1$, the use of (2.5) (observe that $z^{\alpha+1}\leqslant z^\alpha$ for $z\to 0$) implies

$$\left|L_t^\alpha(x,y)\right| \leqslant Cx(xy)^{\alpha+1/2}\left(\frac{e^{-t}}{1-e^{-2t}}\right)^{\alpha+2} e^{-\frac{1}{2}(x^2+y^2)\frac{1+e^{-2t}}{1-e^{-2t}}}.$$

Then we get

$$\int\limits_{\frac{2e^{-t}xy}{1-e^{-2t}}\leqslant 1} \left|L_t^\alpha(x,y)\right|\frac{dt}{\sqrt{t}} \leqslant Cx(xy)^{\alpha+1/2}\left(\left(\int_0^1 + \int_1^\infty\right)\left(\frac{e^{-t}}{1-e^{-2t}}\right)^{\alpha+2} e^{-\frac{1}{2}(x^2+y^2)\frac{1+e^{-2t}}{1-e^{-2t}}}\frac{1}{\sqrt{t}}\,dt\right)$$

$$\begin{aligned}
&\leqslant Cx(xy)^{\alpha+1/2}\left(\int_0^1 t^{-\alpha-3/2}e^{-c_1(x^2+y^2)/t}\,\frac{dt}{t}+e^{-c_1(x^2+y^2)}\right)\\
&\leqslant C\frac{(xy)^{\alpha+1/2}x}{(x^2+y^2)^{\alpha+3/2}}\\
&\leqslant Cy^{\alpha+1/2}x^{-\alpha-3/2}.
\end{aligned}$$

The proof of (i) is finished. To show (ii) we can proceed in a similar way. Assume that $\frac{2e^{-t}xy}{1-e^{-2t}}\geqslant 1$ (Case 1). From (2.6) and (2.14) we get

$$\begin{aligned}
&\int\limits_{\frac{2e^{-t}xy}{1-e^{-2t}}\geqslant 1}\left|L_t^\alpha(x,y)\right|\frac{1}{\sqrt{t}}\,dt\\
&\leqslant Cy\left(\int_0^1\left(\frac{2e^{-t}xy}{1-e^{-2t}}\right)^{\alpha+3/2}t^{-2}e^{-c_1y^2/t}\,dt+\int_1^\infty (xy)^{\alpha+3/2}e^{-c_1y^2}e^{-t}\,dt\right)\\
&\leqslant Cy(xy)^{\alpha+3/2}\left(\int_0^1 t^{-\alpha-5/2}e^{-c_1y^2/t}\,\frac{dt}{t}+e^{-c_1y^2}\right)\\
&\leqslant C\frac{y(xy)^{\alpha+3/2}}{y^{2\alpha+5}}.
\end{aligned}$$

In Case 2, by using (2.14) and (2.5) we get

$$\left|L_t^\alpha(x,y)\right|\leqslant Cy(xy)^{\alpha+3/2}\left(\frac{e^{-t}}{1-e^{-2t}}\right)^{\alpha+3}e^{-\frac{c_1y^2}{1-e^{-2t}}}+Cx(xy)^{\alpha+1/2}\left(\frac{e^{-t}}{1-e^{-2t}}\right)^{\alpha+2}e^{-\frac{c_1y^2}{1-e^{-2t}}}.$$

Then

$$\begin{aligned}
&\int\limits_{\frac{2e^{-t}xy}{1-e^{-2t}}\leqslant 1}\left|L_t^\alpha(x,y)\right|\frac{1}{\sqrt{t}}\,dt\\
&\leqslant C\left(y(xy)^{\alpha+3/2}\left[\int_0^1 t^{-\alpha-5/2}e^{-c_1y^2/t}\,\frac{dt}{t}+e^{-c_1y^2}\right]\right.\\
&\qquad\left.+\,x(xy)^{\alpha+1/2}\left[\int_0^1 t^{-\alpha-3/2}e^{-c_1y^2/t}\,\frac{dt}{t}+e^{-c_1y^2}\right]\right)\\
&\leqslant Cx^{\alpha+3/2}y^{-\alpha-5/2}.
\end{aligned}$$

Thus the proof of (ii) is complete. In order to prove (iii) we observe that, by using (2.7) we have

$$
\begin{aligned}
D_t^\alpha(x,y) &= x^{\alpha+1/2}\frac{d}{dx}\bigl(x^{-(\alpha+1/2)}W_t^\alpha(x,y)\bigr)+xW_t^\alpha(x,y)-\left(\frac{d}{dx}+x\right)\mathcal{W}_{t/2}(x,y)\\
&= -\left(\frac{x(1+e^{-2t})}{1-e^{-2t}}\right)\left\{\sqrt{2\pi}\left(\frac{2xye^{-t}}{1-e^{-2t}}\right)^{1/2} I_\alpha\left(\frac{2xye^{-t}}{1-e^{-2t}}\right)e^{-\frac{2xye^{-t}}{1-e^{-2t}}}-1\right\}\mathcal{W}_{t/2}(x,y)\\
&\quad+\left(\frac{2ye^{-t}}{1-e^{-2t}}\right)\left\{\sqrt{2\pi}\left(\frac{2xye^{-t}}{1-e^{-2t}}\right)^{1/2} I_{\alpha+1}\left(\frac{2xye^{-t}}{1-e^{-2t}}\right)e^{-\frac{2xye^{-t}}{1-e^{-2t}}}-1\right\}\mathcal{W}_{t/2}(x,y)\\
&\quad+x\left\{\sqrt{2\pi}\left(\frac{2xye^{-t}}{1-e^{-2t}}\right)^{1/2} I_\alpha\left(\frac{2xye^{-t}}{1-e^{-2t}}\right)e^{-\frac{2xye^{-t}}{1-e^{-2t}}}-1\right\}\mathcal{W}_{t/2}(x,y)\\
&= -\left(\frac{2xe^{-2t}}{1-e^{-2t}}\right)\left\{\sqrt{2\pi}\left(\frac{2xye^{-t}}{1-e^{-2t}}\right)^{1/2} I_\alpha\left(\frac{2xye^{-t}}{1-e^{-2t}}\right)e^{-\frac{2xye^{-t}}{1-e^{-2t}}}-1\right\}\mathcal{W}_{t/2}(x,y)\\
&\quad+\left(\frac{2ye^{-t}}{1-e^{-2t}}\right)\left\{\sqrt{2\pi}\left(\frac{2xye^{-t}}{1-e^{-2t}}\right)^{1/2} I_{\alpha+1}\left(\frac{2xye^{-t}}{1-e^{-2t}}\right)e^{-\frac{2xye^{-t}}{1-e^{-2t}}}-1\right\}\mathcal{W}_{t/2}(x,y).
\end{aligned}
$$

In Case 1, that is $\frac{2e^{-t}xy}{1-e^{-2t}}\geqslant 1$, (2.6) and formula (2.2) lead to

$$\bigl|D_t^\alpha(x,y)\bigr|\leqslant C\frac{x}{1-e^{-2t}}\mathcal{W}_{t/2}(x,y),$$

and

$$\bigl|D_t^\alpha(x,y)\bigr|\leqslant C\frac{1}{y}\mathcal{W}_{t/2}(x,y).$$

Now, by making the change of variables $e^{-t}=\frac{1-s}{1+s}$ we have

$$
\begin{aligned}
\int\limits_{\frac{2e^{-t}xy}{1-e^{-2t}}\geqslant 1}\bigl|D_t^\alpha(x,y)\bigr|\frac{dt}{\sqrt{t}} &\leqslant C\left(\frac{(xy)^{1/4}}{y}\int\limits_0^{1/2} s^{-5/4}e^{-(x-y)^2/4s}\,ds\right.\\
&\qquad\left.+x\int\limits_{1/2}^{1}\bigl(-(1-s)\log(1-s)\bigr)^{-1/2}e^{-s(x+y)^2/4}\,dsg\right)\\
&\leqslant \frac{C}{y}\left(1+\frac{(xy)^{1/4}}{|x-y|^{1/2}}\right).
\end{aligned}
$$

Now for Case 2 we have by using (2.5)

$$
\begin{aligned}
&\bigl|D_t^\alpha(x,y)\bigr|\\
&\quad\leqslant C\left(x\left(\frac{e^{-t}}{1-e^{-2t}}\right)^{3/2}\left(\frac{e^{-t}xy}{1-e^{-2t}}\right)^{\alpha+1/2}e^{-\frac{1}{2}(x^2+y^2)\frac{1+e^{-2t}}{1-e^{-2t}}}+Cx\left(\frac{e^{-t}}{1-e^{-2t}}\right)\mathcal{W}_{t/2}(x,y)\right)\\
&\quad= D_{t,1}^\alpha(x,y)+D_{t,2}(x,y).
\end{aligned}
$$

The ideas used in this proof (see Case 2 in the proof of (i)) drive to

$$\int_{\frac{2e^{-t}xy}{1-e^{-2t}}\leqslant 1} D_{t,1}(x,y)\,\frac{dt}{\sqrt{t}} \leqslant C\frac{1}{y}.$$

On the other hand, by proceeding as in the Case 1, we obtain that

$$\int_{\frac{2e^{-t}xy}{1-e^{-2t}}\leqslant 1} D_{t,2}(x,y)\,\frac{dt}{\sqrt{t}} \leqslant C\frac{1}{y}. \qquad \square$$

3. Some technical results in the Hermite setting

This section contains some results about operators related with Hermite differential operator. In the present paper the section could be considered a technical part of the paper, in order to have a more smooth presentation of the results about Laguerre operators. Nevertheless in our opinion the section has interest in itself and it presents some interesting results in themselves. Of course one of the crucial points of the section is the behaviour of the "restriction" operators to the half-line. In order to produce a clear presentation of these restrictions we introduce the following notation.

Definition 3.1. Let B be a Banach space. Given a measurable function $f:(0,\infty)\to B$, we shall denote by $f^{\uparrow}$ to the extension function $f^{\uparrow}:\mathbb{R}\to B$ defined by

$$f^{\uparrow}(x)=\begin{cases} f(x), & x>0,\\ 0, & x\leqslant 0.\end{cases}$$

Given a measurable function $g:\mathbb{R}\to B$, we shall denote by $g_{\downarrow}$ to the restriction function $g_{\downarrow}:(0,\infty)\to B$ defined as

$$g_{\downarrow}(x)=g(x), \quad x>0.$$

According to this notation, if S is an operator acting on B-valued measurable functions g on $\mathbb{R}$, the operator $S_{\downarrow}$ is defined on B-valued measurable functions f on $(0,\infty)$ by

$$S_{\downarrow}(f)(x)=S\big(f^{\uparrow}\big)_{\downarrow}(x), \quad x\in(0,\infty). \tag{3.1}$$

Definition 3.2. Let B_1 and B_2 be two Banach spaces and let Ω be either $\mathbb{R}$ or $(0,\infty)$. Given an operator L defined on $L^p_{B_1}(\Omega)$ for some p, $1\leqslant p<\infty$, we say that L is a principal value operator with associated kernel $K(x,y)$, $(K(x,y)\in\mathcal{L}(B_1,B_2),\ x,y\in\Omega,\ x\neq y)$ if

$$L(f)(x)=\lim_{\varepsilon\to 0^+}\int_{\{y\in\Omega,\ |x-y|>\varepsilon\}} K(x,y)f(y)\,dy, \quad \text{a.e. } x\in\Omega,\ f\in L^p_{B_1}(\Omega).$$

For further reference, we state the following proposition, whose proof is left to the reader.

Proposition 3.2. *Let $1 \leqslant p < \infty$ and let B_1 and B_2 be two Banach spaces. Suppose that S is a principal value integral operator defined on $L^p_{B_1}(\mathbb{R}, dx)$ with associated kernel $K(x, y)$, $x, y \in \mathbb{R}$ (see Definition 3.2). Assume that the kernel K satisfies*

(a) $\|K(x, y)\|_{\mathcal{L}(B_1,B_2)} \leqslant C|x-y|^{-1}$, $x \neq y$ *(size condition), and*
(b) *for all $x, y \in \mathbb{R}$, either $K(x, y) = K(-x,-y)$, or $K(x, y) = -K(-x,-y)$ ("signum" condition).*

Then $S_\downarrow$ is a principal value integral operator on $L^p_{B_1}((0,\infty), dx)$ with associated kernel $K(x, y)$, $x, y \in (0,\infty)$.

Moreover the operator S is bounded from $L^p_{B_1}(\mathbb{R}, dx)$ into $L^p_{B_2}(\mathbb{R}, dx)$, if and only if the operator $S_\downarrow$ is bounded from $L^p_{B_1}((0,\infty), dx)$ into $L^p_{B_2}((0,\infty), dx)$. Also, the operator S is bounded from $L^1_{B_1}(\mathbb{R}, dx)$ into $L^{1,\infty}_{B_2}(\mathbb{R}, dx)$, if and only if the operator $S_\downarrow$ is bounded from $L^1_{B_1}((0,\infty), dx)$ into $L^{1,\infty}_{B_2}((0,\infty), dx)$.

Definition 3.3. Let $1 \leqslant p < \infty$ and let B_1 and B_2 be two Banach spaces. Let T be a principal value integral operator with associated kernel $K(x, y)$, $x, y \in (0,\infty)$. We define the "local part" T^{loc} of the operator T as

$$T^{\mathrm{loc}} f(x) = \mathrm{p.v.} \int_0^\infty K(x, y)\chi_{(x/2,2x)}(y) f(y)\, dy, \quad \text{a.e. } x \in (0,\infty),\ f \in L^p_{B_1}\big((0,\infty), dx\big). \tag{3.3}$$

In the proof of the next proposition, and also somewhere else latter in the paper, we shall need some results about Hardy type operators. The strong L^p-boundedness for Hardy operators with power weights were established long time ago and they can be found in most harmonic textbooks, weak type results can be found in [1]. See also [6,18]. In order to be precise, we define them and stated the boundedness results that we shall need.

Given real numbers β and η, let H^β_0 and H^η_∞ the operators of Hardy type defined by

$$H^\beta_0 f(x) = x^{-\beta-1} \int_0^x f(y) y^\beta\, dy,$$

$$H^\eta_\infty f(x) = x^\eta \int_x^\infty f(y) y^{-\eta-1}\, dy.$$

Lemma 3.4. *Let $\beta > -1$, $\eta > -1$.*

(i) *If $1 < p < \infty$ and $\gamma < \beta p + p - 1$, then H^β_0 is of strong type (p, p) on $\mathbb{R}^+$ with measure $x^\gamma\, dx$.*
(ii) *If $\gamma \leqslant \beta$, then H^β_0 is of weak type $(1, 1)$ with respect to the measure $x^\gamma\, dx$.*
(iii) *Let $1 < p < \infty$ and $-\eta p - 1 < \gamma$. Then H^η_∞ is of strong type (p, p) with measure $x^\gamma dx$.*

(iv) *Assume either* $-\eta - 1 \leqslant \gamma$ $(\eta \neq 0)$ *or* $-1 < \gamma$ $(\eta = 0)$. *Then* H_∞^η *is of weak type* $(1,1)$ *with respect to the measure* $x^\gamma\, dx$.

Proposition 3.5. *Let* B_1 *and* B_2 *be two Banach spaces,* T *be an operator defined for measurable functions* $f:(0,\infty) \to B_1$, *and* K *be a kernel satisfying condition* (a) *in Proposition* 3.2.

If T *is a principal value operator on* $L^p_{B_1}((0,\infty),dx)$, *for some* $1 < p < \infty$, *with associated kernel* K, $K(x,y)$, $x,y \in (0,\infty)$, *then the following assertions are equivalent.*

(i) T *is bounded from* $L^p_{B_1}((0,\infty),dx)$ *into* $L^p_{B_2}((0,\infty),dx)$.
(ii) *The operator* T^{loc}, *see* (3.3), *is bounded from* $L^p_{B_1}((0,\infty),dx)$ *into* $L^p_{B_2}((0,\infty),dx)$.

Also, if T *is a principal value operator on* $L^1_{B_1}((0,\infty),dx)$ *with associated kernel* $K(x,y)$, $x,y \in (0,\infty)$, *then the following assertions are equivalent.*

(iii) T *satisfies* $T: L^1_{B_1}((0,\infty),dx) \to L^{1,\infty}_{B_2}((0,\infty),dx)$.
(iv) *The operator* T^{loc} *satisfies* $T^{\mathrm{loc}}: L^1_{B_1}((0,\infty),dx) \to L^{1,\infty}_{B_2}((0,\infty),dx)$.

Proof. Let $f \in L^p_{B_1}((0,\infty),dx)$, $1 \leqslant p < \infty$. We have

$$\begin{aligned}
\big\| T(f)(x) - T^{\mathrm{loc}}(f)(x)\big\|_{B_2} &\leqslant \left\| \int_0^\infty K(x,y)\chi_{(x/2,2x)^c}(y) f(y)\, dy \right\|_{B_2} \\
&\leqslant C\left(\int_0^{x/2} + \int_{2x}^\infty g\right) \frac{1}{|x-y|} \big\| f(y)\big\|_{B_1}\, dy \\
&\leqslant C\left(\frac{1}{x}\int_0^{x/2} \big\| f(y)\big\|_{B_1}\, dy + \int_{2x}^\infty \frac{1}{y}\big\| f(y)\big\|_{B_1}\, dy\right) \\
&\leqslant C\big(H_0^0\big(\big\| f(y)\big\|_{B_1}\big)(x) + H_\infty^0\big(\big\| f(y)\big\|_{B_1}\big)(x)\big), \quad \text{a.e. } x \in (0,\infty).
\end{aligned}$$

Then, we use Lemma 3.4 and we finish the proof of the proposition. □

The notions of "local standard kernel," "local Calderón–Zygmund" and "local A_p weights" were introduced and studied by Nowak and Stempak in [21]. We shall need that notions for functions whose values are taken in some Banach space. In particular we give the following definitions.

Definition 3.4. Let B_1, B_2 be two Banach spaces. A (B_1,B_2)-local standard kernel is a function K defined on $(0,\infty)\times(0,\infty) \setminus \{(x,x){:}\ x \in (0,\infty)\}$ such that, for every $x,y \in (0,\infty)$, $x \neq y$, $K(x,y) \in \mathcal{L}(B_1,B_2)$ and

(i) $\|K(x,y)\|_{\mathcal{L}(B_1,B_2)} \leqslant C|x-y|^{-1}$, $x/2 < y < 2x$.
(ii) $\|K(x,y) - K(x,z)\|_{\mathcal{L}(B_1,B_2)} \leqslant C|y-z||x-y|^{-2}$, if $|x-y| > 2|y-z|$, $x/2 < y,z < 2x$.
(iii) $\|K(x,y) - K(x,z)\|_{\mathcal{L}(B_1,B_2)} \leqslant C|x-z||x-y|^{-2}$, if $|x-y| > 2|x-z|$, $x/2 < y,z < 2x$.

Definition 3.5. Let B_1 and B_2 be two Banach spaces. We say that an operator T defined in $L^2_{B_1}((0,\infty), dx)$ is a (B_1, B_2)-local Calderón–Zygmund operator if:

(i) T is bounded from $L^2_{B_1}((0,\infty), dx)$ into itself.
(ii) There exists a local standard (B_1, B_2)-kernel K such that

$$T(f)(x) = \int_{x/2}^{2x} K(x,y) f(y)\,dy, \quad \text{a.e. } x \notin \operatorname{supp}(f),$$

for all $f \in C_c((0,\infty), B_1)$, where $C_c((0,\infty), B_1)$ denotes the spaces of B_1-valued continuous functions having compact support on $(0,\infty)$.

Definition 3.6. Let w be a nonnegative weight on $(0,\infty)$. We say $w \in A_p^{\text{loc}}$, $1 < p < \infty$, when

$$\sup_{0<u<v<2u} \frac{1}{v-u} \left(\int_u^v w \right)^{1/p} \left(\int_u^v w^{-p'/p} \right)^{1/p'} < \infty.$$

In the case $p = 1$, we say that $w \in A_1^{\text{loc}}$ provided that

$$\sup_{0<u<v<2u} \left(\frac{1}{v-u} \int_u^v w \right) \operatorname{ess\,sup}_{(u,v)} w^{-1} < \infty.$$

It is not hard to see that, for every $\sigma \in \mathbb{R}$, the function $w_\sigma(x) = x^\sigma$, $x \in (0,\infty)$, is in A_p^{loc}, for all $1 \leqslant p < \infty$.

A satisfactory theory of vector-valued local Calderón–Zygmund theory can be developed as in the case of classical Calderón–Zygmund operators, see [23]. In particular the following theorem (vector-valued analogue of [21, Theorem 4.3]) can be proved.

Theorem 3.6. *Let B_1 y B_2 be two Banach spaces. Let T be a (B_1, B_2)-local Calderón–Zygmund operator.*

(i) *Given p, $1 < p < \infty$, and $w \in A_p^{\text{loc}}$, the operator T is bounded from $L^p_{B_1}((0,\infty), w(x)\,dx)$ into $L^p_{B_2}((0,\infty), w(x)\,dx)$.*
(ii) *The operator T is bounded from $L^1_{B_1}((0,\infty), w(x)\,dx)$ into $L^{1,\infty}_{B_2}((0,\infty), w(x)\,dx)$ for any weight $w \in A_1^{\text{loc}}$.*

Given a Banach space B and T a bounded linear operator from $L^p(\mathbb{R})$ into itself, then T can be linearly extended in a natural way to the tensorial product $L^p(\mathbb{R}) \otimes B$, by

$$T\left(\sum_{k=1}^n f_k b_k \right) = \sum_{k=1}^n T(f_k) b_k, \tag{3.7}$$

where $f_k \in L^p(\mathbb{R})$ and $b_k \in B$, $k = 1, \dots, n$. It is said that T admits a bounded extension to $L^p_B(\mathbb{R})$ when the operator defined in (3.7) satisfies

$$\|Tf\|_{L^p_B(\mathbb{R})} \leqslant C\|f\|_{L^p_B(\mathbb{R})}, \quad f \in L^p(\mathbb{R}) \otimes B.$$

If T is positive, that is $Tf \geqslant 0$, when $f \geqslant 0$, then, for any Banach space B, T has a bounded extension to $L^p_B(\mathbb{R})$. There are some particular operators whose vector-valued extension has been studied deeply and extensively, producing sometimes special classes of Banach spaces. Probably one of the most named classes is the so called "UMD" property.

We recall that the UMD property for Banach spaces was introduced by Burkholder in a probabilistic setting. For a Banach space B the UMD property is equivalent to the fact that the Hilbert transform admits a B-valued extension to $L^p_B(\mathbb{R})$ for some (any) $1 < p < \infty$ [4,5]. Recently, see [2], UMD property for a Banach space B has been characterized for the L^p_B-boundedness of the Riesz transforms defined in (2.3).

Let $(\Omega, \mathcal{F}, d\mu)$ be a complete σ-finite measure space. A Banach space B consisting of equivalence classes modulo equality almost everywhere of locally integrable real functions on Ω is called a Köthe function space if the following conditions hold:

(i) If $|x(\omega)| \leqslant |y(\omega)|$ a.e. on Ω, x is measurable and $y \in B$, then $x \in B$ and $\|x\| \leqslant \|y\|$.
(ii) For every $E \in \mathcal{F}$ with $\mu(E) < \infty$, the characteristic function χ_E of E belongs to B.

Every Köthe function space is a Banach lattice under the natural order:

$$x \geqslant 0 \quad \text{if, and only if} \quad x(\omega) \geqslant 0 \quad \text{a.e. } \omega \in \Omega.$$

If B is a Köthe function space, J is a finite subset of the set $\mathbb{Q}_+$ of the positive rational numbers, and f is a locally integrable B-valued function defined on $\mathbb{R}$, we define

$$M_J(f)(x) = \sup_{r \in J} \frac{1}{r} \int_{x-r}^{x+r} \left|f(y)\right| dy, \quad x \in \mathbb{R}.$$

We say that B has the Hardy–Littlewood property when there exists $p \in (1, \infty)$ such that, for every $J \subset \mathbb{Q}_+$, J finite,

$$\left\|M_J(f)\right\|_{L^p_B(\mathbb{R})} \leqslant C\|f\|_{L^p_B(\mathbb{R})},$$

where $C > 0$ is not depending on J. Banach lattices with Hardy–Littlewood property for Köthe function spaces was introduced in [7]. In [9] the Hardy–Littlewood property for a Köthe Banach lattice is characterized by using maximal operators associated with the heat and Poisson semigroups in the Ornstein–Uhlenbeck context. In the following theorem we obtain new characterization of the Hardy–Littlewood property in terms of the L^p_B-boundedness of the maximal operator for the heat semigroup in the Hermite setting.

We shall consider the maximal operator $\mathcal{W}^*$ defined as

$$\mathcal{W}^* f(x) = \sup_{t>0}\left|\mathcal{W}_t f(x)\right|,$$

see (2.1). The operator $\mathcal{W}^*$ is not a linear operator and therefore Definition 3.3 cannot be applied. In this case we give the following definition. Let B be a Köthe function space, and f be a measurable function $f:[0,\infty) \to B$. The "local part" of the maximal operator is defined as

$$\mathcal{W}^*_{\mathrm{loc}} f(x) := \sup_{t>0} \big|(\mathcal{W}_t)^{\mathrm{loc}}_{\downarrow}(f)(x)\big|, \tag{3.8}$$

where $|\cdot|$ denotes absolute value in the lattice and $\mathcal{W}_t$, $(\mathcal{W}_t)_{\downarrow}$ $(\mathcal{W}_t)^{\mathrm{loc}}_{\downarrow}$ are defined as in (2.1), (3.1) and (3.3), respectively.

Theorem 3.9. *Let B be a Köthe function space. The following properties are equivalent.*

- (i) *B has the Hardy–Littlewood property.*
- (ii) *The maximal operator $\mathcal{W}^*$ is bounded from $L^p_B(\mathbb{R}, w(x)\,dx)$ into itself, for every $1 < p < \infty$ and $w \in A_p(\mathbb{R})$.*
- (iii) *The maximal operator $\mathcal{W}^*$ is bounded from $L^1_B(\mathbb{R}, w(x)\,dx)$ into $L^{1,\infty}_B(\mathbb{R}, w(x)\,dx)$, for every $w \in A_1(\mathbb{R})$.*
- (iv) *The maximal operator $\mathcal{W}^*_{\mathrm{loc}}$, defined above in* (3.8)*, is bounded from $L^p_B((0,\infty), dx)$ into itself, for some $1 < p < \infty$.*
- (v) *The maximal operator $\mathcal{W}^*_{\mathrm{loc}}$, defined above in* (3.8)*, is bounded from $L^1_B((0,\infty), dx)$ into $L^{1,\infty}_B((0,\infty), dx)$.*
- (vi) *The maximal operator $\mathcal{W}^*_{\mathrm{loc}}$, defined above in* (3.8)*, is bounded from $L^p_B((0,\infty), x^\sigma\,dx)$ into itself, for every $1 < p < \infty$ and $\sigma \in \mathbb{R}$.*
- (vii) *The maximal operator $\mathcal{W}^*_{\mathrm{loc}}$, defined above in* (3.8)*, is bounded from $L^1_B((0,\infty), x^\sigma\,dx)$ into $L^{1,\infty}_B((0,\infty), x^\sigma\,dx)$, for every $\sigma \in \mathbb{R}$.*

Proof. We will prove that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) and then (ii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (vi) and (iii) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vii).

It is known, see [9, Theorem 1.13], that a Köthe Banach lattice has the Hardy–Littlewood property if, and only if the maximal operator $\sup_t |e^{-t\mathbb{O}} g(x)|$ is bounded from $L^2_B(\mathbb{R}, e^{-x^2}dx)$ into itself where $\mathbb{O} = -\frac{1}{2}\Delta + x.\nabla$ represents the Ornstein–Uhlenbeck operator.

We consider the map $U: L^2(\mathbb{R}, e^{-x^2}\,dx) \to L^2(\mathbb{R}, dx)$ defined by $Ug(x) = g(x)e^{-x^2/2}$. It is clear that $\|Ug\|_{L^2(\mathbb{R},dx)} = \|g\|_{L^2(\mathbb{R},e^{-x^2}dx)}$. Moreover $U \circ e^{-t\mathbb{O}} = e^{-t(H-1)} \circ U$, where H denotes the Hermite operator. Therefore a Köthe Banach lattice has the Hardy–Littlewood property if and only if the maximal operator $\sup_{t>0} |e^{-t(H-1)}|$ is bounded from $L^2_B(\mathbb{R}, dx)$ into itself.

Observe that $\sup_{t>0} |\mathcal{W}_t g(x)| = \sup_{t>0} |e^{-tH}(g)(x)| \leqslant \sup_{t>0} |e^{-t(H-1)}(g)(x)|$. On the other hand,

$$\begin{aligned}
\sup_{t>0}\big|e^{-t(H-1)}(g)(x)\big| &\leqslant \sup_{0<t\leqslant 1}\big|e^{-t(H-1)}(g)(x)\big| + \sup_{t>1}\big|e^{-t(H-1)}(g)(x)\big| \\
&\leqslant e \sup_{0<t\leqslant 1}\big|e^{-tH}(g)(x)\big| + \sup_{t>1}\big|e^{-t(H-1)}(g)(x)\big| \\
&\leqslant e \sup_{t>0}\big|\mathcal{W}_t(g)(x)\big| + \sup_{t>1}\big|e^{-t(H-1)}(g)(x)\big|.
\end{aligned}$$

Let $g = \sum_{k=1}^n c_k h_k$, where $c_k \in B$, $k = 1, \ldots, n$. Hence

$$\begin{aligned}\Big\|\sup_{t\geqslant 1}\big|e^t e^{-tH}g(x)\big|\Big\|_B &= \Big\|\sup_{t\geqslant 1}\Big|\int_{\mathbb{R}}\sum_{k=1}^n e^{-t2k}h_k(x)h_k(z)g(z)\,dz\Big|\Big\|_B\\ &\leqslant \sum_{k=1}^n e^{-2k}\big|h_k(x)\big|\Big\|\int_{\mathbb{R}} h_k(z)g(z)\,dz\Big\|_B\\ &\leqslant \sum_{k=1}^n e^{-2k}\big|h_k(x)\big|\|h_k\|_{L^2(\mathbb{R},dx)}\|g\|_{L^2_B(\mathbb{R},dx)}.\end{aligned}$$

Therefore as $\|h_k\|_{L^2(\mathbb{R},dx)}=1$ we get

$$\Big\|\sup_{t\geqslant 1}\big|e^t e^{-tH}g(x)\big|\Big\|_{L^2_B(\mathbb{R},dx)} \leqslant \sum_{k=1}^n e^{-2k}\|g\|_{L^2_B(\mathbb{R},dx)}.$$

Hence we conclude that a Banach lattice has the Hardy–Littlewood property if and only if (ii) is satisfied for $p=2$. Now we shall sketch the proof of (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii). The proof will be based in a (nowadays) well-known vector-valued argument. Define the operator S as

$$g \to Sg = \{\mathcal{W}_t g\}_{t>0}. \tag{3.10}$$

The fact that (ii) is satisfied for $p=2$ is equivalent to say that the operator S is bounded from $L^2_B(\mathbb{R},dx)$ into $L^2_{B(L^\infty)}(\mathbb{R},dx)$. Moreover, by making computations similar than the ones presented in [26] we can see that S is associated with a Calderón–Zygmund kernel on $\mathbb{R}$. Hence, by using standard vector-valued Calderón–Zygmund theory (ii) follows, and also (iii) (see [7]). For the converse implication we can follow the ideas in [7].

We now prove (ii) $\Rightarrow$ (iv) and (iii) $\Rightarrow$ (v). It is sufficient to apply successively Propositions 3.2, 3.5 and Theorem 3.6 to the vector-valued operator S, defined in (3.10), mapping B-valued measurable functions into $B(L^\infty)$-measurable functions. Observe that the kernel of the operator S is given by $K(x,y)=\{\mathcal{W}_t(x,y)\}_t,\ x,y\in\mathbb{R}$. This kernel clearly satisfies the signum condition (b) in Proposition 3.2. As for the size condition (a) in Proposition 3.2 and the smoothness conditions required in Theorem 3.6 we refer to [26], since following the ideas in that paper it can be proved that, for $x,y\in\mathbb{R}$, $x\neq y$,

$$\big\|K(x,y)\big\|_{L^\infty(0,\infty)} \leqslant C|x-y|^{-1},$$

and

$$\big\|\partial_x K(x,y)\big\|_{L^\infty(0,\infty)} + \big\|\partial_y K(x,y)\big\|_{L^\infty(0,\infty)} \leqslant C|x-y|^{-2}.$$

For the converse implications (iv) $\Rightarrow$ (ii) and (v) $\Rightarrow$ (iii), we apply Propositions 3.5 and 3.2.

To show (ii) $\Leftrightarrow$ (vi) and (iii) $\Leftrightarrow$ (vii) we can argue in a similar way. □

We now state the theorem which characterizes the UMD property for a Banach space B in terms of the L^p_B-boundedness of the local Riesz transform associated with the Hermite operator H.

Theorem 3.11. *Let B be a Banach space. The following properties are equivalent.*

(i) *B has the UMD property.*
(ii) *The Riesz transform $\mathcal{R}$ is bounded from $L^p_B(\mathbb{R}, w(x)\,dx)$ into itself, for every $1 < p < \infty$ and $w \in A_p(\mathbb{R})$.*
(iii) *The Riesz transform $\mathcal{R}$ is bounded from $L^1_B(\mathbb{R}, w(x)\,dx)$ into $L^{1,\infty}_B(\mathbb{R}, w(x)\,dx)$, for every $w \in A_1$.*
(iv) *The operator $\mathcal{R}^{\mathrm{loc}}_{\downarrow}$, see* (3.1) *and* (3.3)*, is bounded from $L^p_B((0,\infty), dx)$ into itself, for some $1 < p < \infty$.*
(v) *The operator $\mathcal{R}^{\mathrm{loc}}_{\downarrow}$, see* (3.1) *and* (3.3)*, is bounded from $L^1_B((0,\infty), dx)$ into $L^{1,\infty}_B((0,\infty), dx)$.*
(vi) *The operator $\mathcal{R}^{\mathrm{loc}}_{\downarrow}$, see* (3.1) *and* (3.3)*, is bounded from $L^p_B((0,\infty), x^\sigma\,dx)$ into itself, for every $1 < p < \infty$ and $\sigma \in \mathbb{R}$.*
(vii) *The operator $\mathcal{R}^{\mathrm{loc}}_{\downarrow}$ is bounded from $L^1_B((0,\infty), x^\sigma\,dx)$ into $L^{1,\infty}_B((0,\infty), x^\sigma\,dx)$, for every $\sigma \in \mathbb{R}$.*

Proof. We follow the lines of the proof for Theorem 3.9. In the present case it is already known, see [2, Theorem 2.3], (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii). The Riesz transform $\mathcal{R}$ is a Calderón–Zygmund operator on $\mathbb{R}$, see [26], that can be described as a principal value operator, see [2]. Hence $\mathcal{R}^{\mathrm{loc}}_{\downarrow}$ can be defined and is a local Calderón–Zygmund operator on $(0,\infty)$. Moreover the kernel K of the Riesz transform satisfies that $K(x,y) = -K(-x,-y)$, for every $x, y \in \mathbb{R}$, $x \neq y$. Then to see that (ii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (vi) and that (iii) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vii) we can apply Propositions 3.2, 3.5, and Theorem 3.6. $\square$

Note that the results established in this section also hold if the Hermite operator H is replaced by the operator $\frac{1}{2}H$. Indeed, the heat semigroup $\{\mathbb{W}_t\}_{t>0}$, associated with the operator $\frac{1}{2}H$ is defined by $\mathbb{W}_t = \mathcal{W}_{t/2}$, $t > 0$.

4. Laguerre setting

In this section we obtain new characterizations for the Köthe function spaces having the Hardy–Littlewood property and for the UMD Banach spaces by using the suitable operators in the Laguerre setting.

The maximal operator associated with the heat semigroup for the Laguerre differential operator $\mathbf{L}_\alpha$ is, as usual, defined by

$$W^{\alpha,*}(f) = \sup_{t>0} \left| W^\alpha_t(f) \right|.$$

Theorem 4.1. *Let B be a Köthe function space. The following statements are equivalent.*

(i) *B has the Hardy–Littlewood property.*
(ii) *The maximal operator $W^{\alpha,*}$ is bounded from $L^p_B((0,\infty), dx)$ into itself for some $1 < p < \infty$, when $\alpha > -1/2$, and such that $2/(2\alpha+3) < p < -2/(2\alpha+1)$, when $-1 < \alpha \leqslant -1/2$.*
(iii) *The maximal operator $W^{\alpha,*}$ is bounded from $L^p_B((0,\infty), x^\sigma\,dx)$ into itself for every $1 < p < \infty$ and $-1 - p(\alpha+1/2) < \sigma < p(\alpha+3/2) - 1$.*

Proof. We have that

$$\begin{aligned}
&\sup_{t>0}\left|\int_0^\infty W_t^\alpha(x,y)f(y)\,dy\right| \\
&\quad\leqslant \sup_{t>0}\left|\int_0^\infty W_t^\alpha(x,y)\chi_{\{y<x/2\}}(y)f(y)\,dy\right| + \sup_{t>0}\left|\int_0^\infty W_t^\alpha(x,y)\chi_{\{2x<y\}}(y)f(y)\,dy\right| \\
&\qquad + \sup_{t>0}\left|\int_0^\infty W_t^\alpha(x,y)\chi_{\{x/2<y<2x\}}(y)f(y)\,dy\right|.
\end{aligned}$$

Then, by using Lemma 2.11, we get

$$\begin{aligned}
\left\|\sup_{t>0}\left|\int_0^\infty W_t^\alpha(x,y)f(y)\,dy\right|\right\|_B &\leqslant CH_0^{\alpha+1/2}\big(\|f\|_B\big)(x) + CH_\infty^{\alpha+1/2}\big(\|f\|_B\big)(x) \\
&\quad + \left\|\sup_{t>0}\left|\int_0^\infty W_t^\alpha(x,y)\chi_{\{x/2<y<2x\}}(y)f(y)\,dy\right|\right\|_B.
\end{aligned}$$

Hence, according to Lemma 3.4 we conclude that the hypothesis (iii) is equivalent to

(iii)′ *The maximal operator* $\mathbf{W}^{\alpha,*}(f) = \sup_{t>0}|W_t^{\alpha,\mathrm{loc}}(f)(x)|$ *is bounded from* $L_B^p((0,\infty), x^\sigma\,dx)$ *into itself for* $1<p<\infty$ *and* $-1-p(\alpha+1/2)<\sigma<p(\alpha+3/2)-1$.

Again by using Lemma 2.11 we see that

$$\left|\mathbf{W}^{\alpha,*}f(x) - \mathcal{W}_{\mathrm{loc}}^{*}f(x)\right| \leqslant C\int_{x/2}^{2x}\frac{1}{y}f(y)\,dy.$$

But the local operator $f \to \int_{x/2}^{2x}\frac{1}{y}f(y)\,dy$ is bounded $L_B^p((0,\infty), x^\gamma\,dx)$ into itself for every $\gamma\in\mathbb{R}$ and every Köthe Banach lattice B. The equivalence (i) $\Leftrightarrow$ (iii)′ follows now from Theorem 3.9.

Analogously (i) $\Leftrightarrow$ (ii) follows from Theorem 3.9 in a similar way. □

We now obtain the characterization of the Banach spaces having the UMD property by using Riesz transforms associated with the Laguerre operators. In the way of proving the corresponding theorem, we shall need some ideas developed in Section 3, in particular we need to know that the Riesz transforms defined in (1.1) are principal value operators. In order to show this property we shall use some results that go back to Muckenhoupt's pioneer paper [17]. Consider the differential operator

$$\varPi_\alpha = -2\left(y\frac{d^2}{dy^2} + (\alpha+1-y)\frac{d}{dy}\right), \quad \alpha>-1,\ y>0.$$

The operator Π_α is selfadjoint with respect to the measure $y^\alpha e^{-y}\,dy$. Its eigenfunctions are the Laguerre polynomials L_k^α, in fact $\Pi_\alpha L_k^\alpha(y) = 2kL_k^\alpha(y)$. In [17] it is proved that the operator $\sqrt{y}\frac{d}{dy}(\Pi_\alpha)^{-1/2}$ is a principal value operator in $L^2((0,\infty), y^\alpha e^{-y}\,dy)$. It is an exercise to repeat the corresponding arguments in [17] and show that the operator $\sqrt{y}\frac{d}{dy}(\Pi_\alpha + \alpha + 1)^{-1/2}$ is a principal value operator in $L^2((0,\infty), y^\alpha e^{-y}\,dy)$. Now we consider the isometry $\Xi_\alpha : L^2((0,\infty), y^\alpha e^{-y}\,dy) \to L^2((0,\infty), dy)$ given by

$$\Xi_\alpha(f)(y) = (2y)^{1/2} y^\alpha e^{-\frac{y^2}{2}} f\bigl(y^2\bigr).$$

This isometry satisfies

$$(\Pi_\alpha + \alpha + 1)f = (\Xi_\alpha)^{-1} \circ \mathbf{L}_\alpha \circ \Xi_\alpha f \quad \text{and}$$
$$\sqrt{y}\frac{d}{dy}(\Pi_\alpha + \alpha + 1)^{-1/2} f = (\Xi_\alpha)^{-1} \circ \mathbf{R}_\alpha \circ \Xi_\alpha f.$$

Therefore the Riesz transforms $\mathbf{R}_\alpha$ defined in (1.1) are principal value operators in the space $L^2((0,\infty), dy)$.

Next we establish our characterization of the UMD Banach spaces.

Theorem 4.2. *Let $\alpha > -1$ and let B be a Banach space. Then the following statements are equivalent.*

(i) *B has the UMD property.*
(ii) *Riesz transforms $\mathbf{R}_\alpha$ admit a bounded extension from $L_B^p((0,\infty), dx)$ into itself, for some $\max\{1, 2/(2\alpha+3)\} < p < \infty$.*
(iii) *Riesz transforms $\mathbf{R}_\alpha$ admit a bounded extension from $L_B^p((0,\infty), x^\sigma\,dx)$ into itself, for every $1 < p < \infty$ and $-p(\alpha+3/2) - 1 < \sigma < p(\alpha+3/2) - 1$.*

Proof. The comments that we made before this theorem allow us to write

$$\mathbf{R}_\alpha(f)(x) = \left(\int_0^{x/2} + \text{p.v.}\int_{x/2}^{2x} + \int_{2x}^{\infty}\right)\mathbf{R}_\alpha(x,y)f(y)\,dy, \qquad f \in L^2\bigl((0,\infty), dx\bigr).$$

Then, by using Lemma 2.13(i) and (ii), and Lemma 3.4, we conclude that $\mathbf{R}_\alpha$ is bounded from $L_B^p((0,\infty), x^\sigma\,dx)$ into itself, with $1 < p < \infty$ and $-p(\alpha+3/2) - 1 < \sigma < p(\alpha+3/2) - 1$, if and only if $\mathbf{R}_\alpha^{\text{loc}}$ is bounded from $L_B^p((0,\infty), x^\sigma\,dx)$ into itself. Even more, the new use of Lemma 2.13(iii) allows us to conclude that $\mathbf{R}_\alpha$ is bounded from $L_B^p((0,\infty), x^\sigma\,dx)$ into itself, with $1 < p < \infty$ and $\sigma \in \mathbb{R}$, if and only if the operator $R_\downarrow^{\text{loc}}$ (defined in Theorem 3.11) admits a bounded extension from $L_B^p((0,\infty), x^\sigma\,dx)$ into itself. Therefore by using Theorem 3.11 we get the present theorem. $\square$

5. Results for other Laguerre systems

Let V and W^α be the operators defined by

$$Vf(y) = (2\sqrt{y})^{-1/2} f(\sqrt{y}), \qquad W^\alpha f(y) = y^{-\frac{\alpha}{2}} (2\sqrt{y})^{-1/2} f(\sqrt{y}),$$

for f a measurable function with domain on $(0, \infty)$. For further reference we state the following lemma, whose simple proof is left to the reader.

Lemma 5.1. *Let $\alpha > -1$.*

(i) *Let $2\delta = \gamma + \frac{p}{2} - 1$, then $\|Vf\|_{L^p(y^\delta\,dy)} = 2^{-\frac{1}{2}+\frac{1}{p}} \|f\|_{L^p(y^\gamma\,dy)}$.*

(ii) *Let $\gamma = (2\alpha + 1)(\frac{p}{2} - 1) + 2\rho$, then $\|W^\alpha f\|_{L^p(y^\rho, d\mu_\alpha)} = 2^{-\frac{1}{2}+\frac{1}{p}} \|f\|_{L^p(y^\gamma, dy)}$, where $d\mu_\alpha(y) = y^\alpha\,dy$.*

The orthonormal system considered in (1.2) is the family of eigenfunctions of the second order (selfadjoint with respect to the Lebesgue measure on $(0, \infty)$) differential operator

$$L_\alpha = -2\left(y\frac{d^2}{dy^2} + \frac{d}{dy} - \frac{y}{4} - \frac{\alpha^2}{4y}\right), \quad y > 0.$$

In fact

$$L_\alpha\big(\mathcal{L}_k^\alpha\big) = (2k + \alpha + 1)\mathcal{L}_k^\alpha. \tag{5.2}$$

For these functions the following derivatives were defined in [10]:

$$D_\alpha = 2\left\{\sqrt{y}\,\frac{d}{dy} + \frac{1}{2}\left(\sqrt{y} - \frac{\alpha}{\sqrt{y}}\right)\right\}.$$

The actions on the corresponding Laguerre functions are given by

$$D_\alpha\big(\mathcal{L}_k^\alpha\big) = -2\sqrt{k}\,\mathcal{L}_{k-1}^{\alpha+1}. \tag{5.3}$$

If we denote by $(D_\alpha)^*$ the formal adjoint of D_α with respect to the Lebesgue measure, it follows that

$$L_\alpha - (\alpha + 1) = \frac{1}{2}(D_\alpha)^* D_\alpha.$$

Accordingly, we can define the Riesz transforms for the Laguerre function expansions as

$$R_\alpha = D_\alpha (L_\alpha)^{-1/2}, \quad \alpha > -1.$$

Analogously the orthonormal system, $\{\ell_k^\alpha\}_{k=0}^\infty$, defined in (1.3) is the family of eigenfunctions of the second order (selfadjoint with respect to the measure $d\mu_\alpha(y) = y^\alpha\, dy$ on $(0,\infty)$) differential operator

$$\mathbb{L}_\alpha = -2\left(y\frac{d^2}{dy^2} + (\alpha+1)\frac{d}{dy} - \frac{y}{4}\right).$$

More explicitly

$$\mathbb{L}_\alpha \ell_k^\alpha = (2k+\alpha+1)\,\ell_k^\alpha. \tag{5.4}$$

The operator $\mathbb{L}_\alpha$ can be "factorized" as

$$\mathbb{L}_\alpha - (\alpha+1) = \frac{1}{2}(\mathbb{D}_\alpha)^*\mathbb{D}_\alpha,$$

where $\mathbb{D}_\alpha = 2(\sqrt{y}\,\frac{d}{dy} + \frac{1}{2}\sqrt{y})$ and $(\mathbb{D}_\alpha)^*$ is the formal adjoint of $\mathbb{D}_\alpha$ with respect to the measure $d\mu_\alpha$. Furthermore,

$$\mathbb{D}_\alpha \ell_k^\alpha(y) = -2\sqrt{k}\sqrt{y}\,\ell_{k-1}^{\alpha+1}(y). \tag{5.5}$$

In this setting the Riesz transform is defined by

$$\mathbb{R}_\alpha = \mathbb{D}_\alpha(\mathbb{L}_\alpha)^{-1/2}, \quad \alpha > -1.$$

Proposition 5.6. *Let $\alpha > -1$ and let f be a finite linear combination of Laguerre functions $\{\varphi_k^\alpha\}$. Then*

(i) $e^{-t\mathbf{L}_\alpha} f = V^{-1}e^{-tL_\alpha}Vf = (W^\alpha)^{-1}\,e^{-t\mathbb{L}_\alpha}W^\alpha f$;
(ii) $\mathbf{R}_\alpha f = V^{-1}R_\alpha\,Vf = (W^\alpha)^{-1}\mathbb{R}_\alpha W^\alpha f$.

Proof. Observe that $\varphi_k^\alpha = V^{-1}\mathcal{L}_k^\alpha = (W^\alpha)^{-1}\ell_k^\alpha$. Hence, in order to prove the proposition we just use the fact

$$\mathbf{D}_\alpha\left(\varphi_k^\alpha\right) = -2\sqrt{k}\varphi_{k-1}^{\alpha+1}$$

jointly with (5.2)–(5.5). $\square$

Theorem 5.7. *Let B a Banach space. Let $\alpha > -1$, $1 < p < \infty$, δ, ρ and σ be real numbers. Then the following are equivalent.*

(i) *The Banach space B is UMD.*
(ii) *The operator $\mathbf{R}_\alpha$ has a bounded extension from $L_B^p((0,\infty), y^\sigma\, dy)$ into itself, for σ satisfying*

$$-p\left(\alpha+\frac{3}{2}\right) - 1 < \sigma < p\left(\alpha+\frac{3}{2}\right) - 1.$$

(iii) *The operator R_α has a bounded extension from $L^p_B((0,\infty), y^\delta\,dy)$ into itself, for δ such that*

$$-1-\frac{\alpha+1}{2}p<\delta<\frac{\alpha p}{2}+p-1.$$

(iv) *The operator $\mathbb{R}_\alpha$ has a bounded extension from $L^p_B((0,\infty), y^\rho\,d\mu_\alpha)$ into itself, for ρ in the range*

$$-1-\alpha-p/2<\rho<(\alpha+1)(p-1).$$

Proof. The equivalence between (i) and (ii) has been proved in Section 4. The equivalence among (ii)–(iv) can be deduced easily from Lemma 5.1 and Proposition 5.6. □

A parallel theorem to Theorem 5.7 can be proved in the case of Hardy–Littlewood property and heat maximal semigroups.

The L^p-behaviour of the Riesz transforms associated with Laguerre expansions in the endpoints of the intervals specified in the last theorem is known, see [11]. It is an interesting question to describe the Banach spaces for which the maximal operator for the heat semigroup and Riesz transforms in the Laguerre setting satisfy the suitable boundedness L^p-properties (weak type or restricted weak type) in those endpoints.

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Gradient estimates for a degenerate parabolic equation with gradient absorption and applications

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Abstract

Qualitative properties of non-negative solutions to a quasilinear degenerate parabolic equation with an absorption term depending solely on the gradient are shown, providing information on the competition between the nonlinear diffusion and the nonlinear absorption. In particular, the limit as $t \to \infty$ of the L^1-norm of integrable solutions is identified, together with the rate of expansion of the support for compactly supported initial data. The persistence of dead cores is also shown. The proof of these results strongly relies on gradient estimates which are first established.

Keywords: Gradient absorption; p-Laplacian; Localization; Gradient estimates; Temporal decay rates

1. Introduction

We investigate the properties of non-negative and bounded continuous solutions to the Cauchy problem

$$\partial_t u - \Delta_p u + |\nabla u|^q = 0, \quad (t,x) \in Q_\infty := (0,\infty) \times \mathbb{R}^N, \tag{1.1}$$

$$u(0) = u_0 \geqslant 0, \quad x \in \mathbb{R}^N, \tag{1.2}$$

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the parameters p and q ranging in $(2,\infty)$ and $(1,\infty)$, respectively, and the p-Laplacian operator Δ_p being defined by

$$\Delta_p u := \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big).$$

When $p>2$, (1.1) is a quasilinear degenerate parabolic equation with a nonlinear absorption term $|\nabla u|^q$ depending solely on the gradient of u, and reduces to the semilinear diffusive Hamilton–Jacobi equation

$$\partial_t v - \Delta v + |\nabla v|^q = 0 \quad \text{in } Q_\infty, \tag{1.3}$$

when $p=2$. Several recent papers have been devoted to the study of properties of non-negative solutions to (1.3) with a particular emphasis on the large time behaviour which turns out to depend strongly on the value of the parameter $q\in(0,\infty)$ [1,4–9,11,19].

One of the keystones of these investigations are optimal gradient estimates of the form $\|\nabla(v^\alpha)(t)\|_\infty \leqslant C(\|v(0)\|_\infty)t^{-\beta}$ for suitable exponents $\alpha\in(0,1)$ and $\beta>0$, both depending on N and q [4,20]. Not only do such estimates provide an instantaneous smoothing effect from $L^\infty(\mathbb{R}^N)$ to $W^{1,\infty}(\mathbb{R}^N)$ but temporal decay estimates as well, the latter being the starting point of a precise study of the large time dynamics. Let us recall here that the proof of the above-mentioned gradient estimates relies on a modification of the Bernstein technique [4,20].

Owing to the nonlinearity of the diffusion term when $p>2$, the availability of similar gradient estimates for solutions to (1.1), (1.2) is unclear and is actually our first result. More precisely, for $p>2$ and $q>1$, we introduce the exponents $\alpha_p\in(0,1)$ and $\beta_{p,q}\in(0,1)$ defined by

$$\frac{1}{\alpha_p} := \frac{p-1}{p-2} - \frac{N-1}{p(N+3)-2(N+1)} \quad \text{and} \quad \beta_{p,q} := \max\left\{\alpha_p, \frac{q-1}{q}\right\}. \tag{1.4}$$

Theorem 1.1. *Consider a non-negative initial condition $u_0\in\mathcal{BC}(\mathbb{R}^N)$. There is a non-negative viscosity solution $u\in\mathcal{BC}([0,\infty)\times\mathbb{R}^N)$ to* (1.1), (1.2) *such that*

$$0\leqslant u(t,x)\leqslant \|u_0\|_\infty, \quad (t,x)\in Q_\infty, \tag{1.5}$$

$$\big|\nabla\big(u^{\alpha_p}\big)(t,x)\big| \leqslant C(p,N)\big\|u(s)\big\|_\infty^{(p\alpha_p+2-p)/p}\,(t-s)^{-1/p}, \tag{1.6}$$

$$\big|\nabla\big(u^{\beta_{p,q}}\big)(t,x)\big| \leqslant C(p,q,N)\big\|u(s)\big\|_\infty^{(q\beta_{p,q}+1-q)/q}(t-s)^{-1/q}, \tag{1.7}$$

and

$$\int_{\mathbb{R}^N}\big(u(t,x)-u(s,x)\big)\vartheta(x)\,dx + \int_s^t\int_{\mathbb{R}^N}\big(|\nabla u|^{p-2}\nabla u\cdot\nabla\vartheta + |\nabla u|^q\vartheta\big)\,dx\,d\tau = 0 \tag{1.8}$$

for $t>s\geqslant 0$ and $\vartheta\in\mathcal{C}_0^\infty(\mathbb{R}^N)$.

Furthermore, this solution is unique if $u_0\in\mathcal{BUC}(\mathbb{R}^N)$.

Let us emphasize that the main contribution of Theorem 1.1 is the estimates (1.6), (1.7), and not the existence of a viscosity solution to (1.1) which could probably be obtained by alternative approaches. But, owing to the poor regularity of the solutions to (1.1), (1.2), we cannot prove

(1.6) and (1.7) directly and instead use an approximation procedure. Indeed, the proof of (1.6) and (1.7) relies on a modification of the Bernstein technique. It requires the study of the partial differential equation solved by $|\nabla\varphi(u)|^2$ for a suitably chosen function φ and thus some regularity which is not available for solutions to (1.1), (1.2). The existence part of Theorem 1.1 is in fact an intermediate step in the proof of (1.6) and (1.7).

It is clear from (1.6) and (1.7) with $s=0$ that they lead to different temporal decay estimates. In fact, as we shall see below, (1.6) results from the diffusive part of (1.1) while (1.7) stems from the absorption term. In particular, it is worth mentioning that (1.6) is also valid for non-negative solutions to the p-Laplacian equation

$$\partial_t w - \Delta_p w = 0 \quad \text{in } Q_\infty, \tag{1.9}$$

which seems to be new for $N \geqslant 2$. When $N=1$, it has been proved in [17, Theorem 2]. Also, (1.7) is true for non-negative viscosity solutions to the Hamilton–Jacobi equation

$$\partial_t h + |\nabla h|^q = 0 \quad \text{in } Q_\infty, \tag{1.10}$$

and can be deduced from [25, Theorem I.1]. For $p=2$, similar gradient estimates have been obtained in [4,20] with $\alpha_2 = \beta_{2,q} = (q-1)/q$.

The previous gradient estimates may be improved for non-negative, radially symmetric, and non-increasing initial data.

Theorem 1.2. *Assume that the initial condition $u_0 \in \mathcal{BC}(\mathbb{R}^N)$ is non-negative, radially symmetric, and non-increasing. There is a non-negative viscosity solution u to* (1.1), (1.2) *satisfying* (1.5), (1.8) *and such that*

$$x \mapsto u(t,x) \textit{ is non-negative, radially symmetric, and non-increasing},$$

$$\left|\nabla\left(u^{(p-2)/(p-1)}\right)(t,x)\right| \leqslant C(p,N)\left\|u(s)\right\|_\infty^{(p-2)/p(p-1)} (t-s)^{-1/p}, \tag{1.11}$$

$$\left|\nabla\left(u^{(q-1)/q}\right)(t,x)\right| \leqslant \frac{(q-1)^{(q-1)/q}}{q}\, t^{-1/q} \quad \textit{if } q \geqslant p-1, \tag{1.12}$$

and

$$\left|\nabla\left(u^{(p-2)/(p-1)}\right)(t,x)\right| \leqslant C(p,q)\left\|u(s)\right\|_\infty^{(p-1-q)/q(p-1)} (t-s)^{-1/q} \quad \textit{if } q \in (1,p-1), \tag{1.13}$$

for $t > s \geqslant 0$.

Theorem 1.2 is proved as Theorem 1.1 for $N=1$. We will thus only give the proof of the latter.

Here again, the gradient estimate (1.11) is valid for non-negative solutions to the p-Laplacian equation (1.9) with radially symmetric and non-increasing initial data and is easily seen to be optimal in that case: indeed, the Barenblatt solution to the p-Laplacian equation (1.9) is given by

$$\mathcal{B}(t,x) := t^{-N\eta}\left(1-\gamma_p\left(\frac{|x|}{t^\eta}\right)^{p/(p-1)}\right)_+^{(p-1)/(p-2)}, \quad (t,x)\in(0,\infty)\times\mathbb{R}^N,$$

with $\eta := 1/(N(p-2)+p)$ (see, e.g., [16, Chapter XI, Eq. (1.6)]) and $\nabla(\mathcal{B}^\vartheta)(t,x)$ is bounded only for $\vartheta \geqslant (p-2)/(p-1)$.

Remark 1.3. Since we are mainly interested in qualitative properties of solutions to (1.1), (1.2), we leave aside the question of uniqueness of such solutions for initial data in $\mathcal{BC}(\mathbb{R}^N)\setminus\mathcal{BUC}(\mathbb{R}^N)$. Nevertheless, since the solutions in Theorems 1.1 and 1.2 are constructed as limits of classical solutions, they still enjoy a comparison principle. More precisely, if u_0 and $\hat{u}_0$ are two non-negative functions in $\mathcal{BC}(\mathbb{R}^N)$ such that $u_0\leqslant\hat{u}_0$, then the corresponding solutions u and $\hat{u}$ to (1.1) with initial data u_0 and $\hat{u}_0$ constructed in Theorem 1.1 satisfy $u(t,x)\leqslant\hat{u}(t,x)$ for all $(t,x)\in Q_\infty$. This fact will be used repeatedly in the sequel.

Several qualitative properties follow from the previous gradient estimates. As a first consequence, we derive temporal decay estimates in $W^{1,\infty}(\mathbb{R}^N)$ for non-negative and integrable solutions to (1.1), (1.2). We set

$$q_* := p - \frac{N}{N+1}, \qquad \xi := \frac{1}{q(N+1)-N}, \qquad \eta := \frac{1}{N(p-2)+p}. \tag{1.14}$$

Proposition 1.4. *Assume that*

$$u_0 \in L^1(\mathbb{R}^N)\cap\mathcal{BC}(\mathbb{R}^N), \quad u_0\geqslant 0, \tag{1.15}$$

and denote by u the corresponding viscosity solution to (1.1), (1.2) *constructed in Theorem* 1.1. *Then $u\in\mathcal{C}([0,\infty);L^1(\mathbb{R}^N))$.*

Let $t>0$. If $q\in(1,q_*)$, then

$$\|u(t)\|_\infty \leqslant C\|u_0\|_1^{q\xi}t^{-N\xi}, \tag{1.16}$$

$$\|\nabla u(t)\|_\infty \leqslant C\|u_0\|_1^{\xi}t^{-(N+1)\xi}, \tag{1.17}$$

while, if $q>q_*$,

$$\|u(t)\|_\infty \leqslant C\|u_0\|_1^{p\eta}t^{-N\eta}, \tag{1.18}$$

$$\|\nabla u(t)\|_\infty \leqslant C\|u_0\|_1^{2\eta}t^{-(N+1)\eta}. \tag{1.19}$$

Recall that the L^∞-norm of non-negative and integrable solutions w to the p-Laplacian equation (1.9) decays as $t^{-N\eta}$ [21, Theorem 3]. However this decay might be enhanced by the nonlinear absorption term and this is indeed the case for $q\in(1,q_*)$. Indeed, $t^{-N\xi}\leqslant t^{-N\eta}$ for $t\geqslant 1$ and $q\in(1,q_*)$. According to Proposition 1.4, we thus expect the nonlinear absorption

term to be negligible as $t \to \infty$ for $q > q_*$ and the large time dynamics to feel the effects of the absorption only for $q \in (1, q_*)$. The next result is a further step in that direction.

It readily follows from (1.1) and the non-negativity of u that $t \mapsto \|u(t)\|_1$ is a non-increasing and non-negative function. Introducing

$$I_1(\infty) := \lim_{t\to\infty} \big\|u(t)\big\|_1 = \inf_{t\geqslant 0}\big\{\big\|u(t)\big\|_1\big\} \in \big[0, \|u_0\|_1\big], \tag{1.20}$$

we study the possible values of $I_1(\infty)$.

Proposition 1.5. *Assume that u_0 satisfies* (1.15) *with $\|u_0\|_1 > 0$ and denote by u the corresponding viscosity solution to* (1.1), (1.2) *constructed in Theorem* 1.1. *Then $I_1(\infty) > 0$ if and only if $q > q_*$, the parameter q_* being defined in* (1.14).

Since $\|w(t)\|_1 = \|w(0)\|_1$ for all $t \geqslant 0$ for non-negative and integrable solutions w to the p-Laplacian equation (1.9), we realize that the absorption term is not strong enough for $q > q_*$ to drive the L^1-norm of $u(t)$ to zero as $t \to \infty$, thus indicating a diffusion-dominated behaviour for large times. For $q \in (p-1, p)$ Proposition 1.5 is already proved in [1, Theorems 1.3 and 1.4] by a different method. A similar result is already available for $p = 2$ [4,7,9].

We next turn to a property which marks a striking difference between the semilinear case $p = 2$ and the quasilinear case $p > 2$ corresponding to *slow diffusion*, namely the finite speed of propagation. Since the support of non-negative and compactly supported solutions w to the p-Laplacian equation (1.9) grows as t^η, it is natural to wonder whether the absorption term will slow down this process.

Theorem 1.6. *Assume that u_0 fulfils* (1.15) *and is compactly supported, and denote by u the corresponding solution to* (1.1), (1.2). *For $t \geqslant 0$ we put*

$$\varrho(t) := \inf\big\{R > 0 \text{ such that } u(t,x) = 0 \text{ for } |x| > R\big\}. \tag{1.21}$$

Then $\varrho(t) < \infty$ for all $t \geqslant 0$ and:

(i) *If $q \in (1, p-1)$ then*

$$\limsup_{t\to\infty} \varrho(t) < \infty. \tag{1.22}$$

(ii) *If $q = p-1$ then*

$$\varrho(t) \leqslant C(1 + \ln t) \quad \text{for } t \geqslant 1. \tag{1.23}$$

(iii) *If $q \in (p-1, q_*)$ then*

$$\varrho(t) \leqslant C t^{(q-p+1)/(2q-p)} \quad \text{for } t \geqslant 1. \tag{1.24}$$

(iv) *If $q \geqslant q_*$ then*

$$\varrho(t) \leqslant C\, t^\eta \quad \text{for } t \geqslant 1. \tag{1.25}$$

Here again, the absorption term seems to have no real effect on the expansion on the support of $u(t)$ for $q > q_*$ as the upper bound (1.25) is exactly the growth rate of the support for non-negative and compactly supported solutions w to the p-Laplacian equation (1.9). But, as soon as q is below q_*, the dynamics starts to feel the effects of the absorption term and the expansion of the support of $u(t)$ slows down. It even stops for $q \in (1, p-1)$. In that case, the support of $u(t)$ remains *localized* in a fixed ball of $\mathbb{R}^N$: such a property is already enjoyed by compactly supported non-negative solutions to second-order degenerate parabolic equations with a sufficiently strong absorption involving the solution only as, for instance, $\partial_t z - \Delta_p z + z^r = 0$ in Q_∞ when $r \in (1, p-1)$ [15,22,27]. It has apparently remained unnoticed for second-order degenerate parabolic equations with an absorption term depending solely on the gradient. In our case, this property is clearly reminiscent of that enjoyed by the solutions h to the Hamilton–Jacobi equation (1.10): namely, the support of $h(t)$ does not evolve through time evolution [2]. Finally, for $q \in (p-1, q_*)$, compactly supported self-similar solutions to (1.1) are constructed in [26] and the boundaries of their support evolve at the speed given by the right-hand side of (1.24).

As a by-product of the proof of Theorem 1.6 we obtain improved decay estimates for the L^1-norm of solutions to (1.1), (1.2) with compactly supported initial data.

Corollary 1.7. *Assume that u_0 fulfils* (1.15) *and is compactly supported.*

(i) *If $q \in (1, p-1)$ then*

$$\|u(t)\|_1 \leqslant C t^{-1/(q-1)}, \quad t \geqslant 2. \tag{1.26}$$

(ii) *If $q = p-1$ then*

$$\|u(t)\|_1 \leqslant C\, t^{-1/(q-1)} (\ln t)^{1/\xi(q-1)} \quad \text{for } t \geqslant 2. \tag{1.27}$$

(iii) *If $q \in (p-1, q_*)$ then*

$$\|u(t)\|_1 \leqslant C\, t^{-((N+1)(q_*-q))/(2q-p)} \quad \text{for } t \geqslant 2. \tag{1.28}$$

(iv) *If $q = q_*$ then*

$$\|u(t)\|_1 \leqslant C (\ln t)^{-1/(q-1)} \quad \text{for } t \geqslant 2. \tag{1.29}$$

For $q \in (p-1, q_*]$, Theorem 1.6 and Corollary 1.7 are already proved in [1, Theorems 1.1 and 1.2] by a completely different approach. In addition, for non-compactly supported initial data, temporal decay estimates involving the behaviour of u_0 for large values of x are obtained in [1, Theorem 1.3] for the L^1-norm of u. Let us also mention that the decay rate of $\|u(t)\|_1$ for $q \in (1, p-1)$ is the same as the one obtained in [2] for non-negative and compactly supported solutions to the Hamilton–Jacobi equation (1.10). The bound (1.26) then provides another clue of the dominance of the absorption term for $q \in (1, p-1)$. That it is indeed true is shown in [24].

For $q \in (1, p-1)$, it follows from Theorem 1.6(i) that the support of the solutions to (1.1), (1.2) with compactly supported initial data remains bounded through time evolution. A natural counterpart of this phenomenon is to study what happens to a solution to (1.1), (1.2) starting from an initial condition vanishing inside a ball of $\mathbb{R}^N$. It turns out that, if the radius of the ball is sufficiently large, the solution still vanishes inside of a smaller ball for all times, a phenomenon which may be called the persistence of *dead cores*.

Proposition 1.8. *Consider a non-negative initial condition* $u_0 \in \mathcal{BC}(\mathbb{R}^N)$ *such that*

$$u_0(x) = 0 \quad \text{if } |x| \leqslant R_0 \tag{1.30}$$

for some $R_0 > 0$, *and denote by* u *the corresponding solution to* (1.1), (1.2) *constructed in Theorem* 1.1. *If* $q \in (1, p-1)$ *there is a constant* $\delta_0 = \delta_0(p,q) > 0$ *such that, if* $R_0 \geqslant \delta_0 \, \|u_0\|_\infty^{(p-1-q)/(p-q)}$ *then*

$$u(t,x) = 0 \quad \text{if } |x| \leqslant R_0 - \delta_0 \|u_0\|_\infty^{(p-1-q)/(p-q)} \text{ and } t \geqslant 0.$$

The proof of Proposition 1.8 is in fact quite similar to that of Theorem 1.6(i).

This paper is organized as follows. Gradient estimates for an approximation of (1.1) are established in Section 2 by a modified Bernstein technique with the help of a trick introduced in [10] to obtain gradient estimates for the porous medium equation. Theorems 1.1 and 1.2 are then proved in Section 3. Sections 4 and 5 are devoted to integrable initial data for which we prove Propositions 1.4 and 1.5. We focus on compactly supported initial data in Section 6 where Theorem 1.6 and Corollary 1.7 are proved. The persistence of dead cores is studied in Section 7 while the proof of a technical lemma from Section 2 is postponed to Appendix A.

2. Gradient estimates

As already mentioned the proof of the gradient estimates (1.6) and (1.7) rely on a modified Bernstein technique: owing to the degeneracy of the diffusion we cannot expect (1.1) to have smooth solutions and we thus need to use an approximation procedure. We first report the following technical lemma.

Lemma 2.1. *Let* a *and* b *be two non-negative functions in* $\mathcal{C}^2([0,\infty))$ *and* u *be a classical solution to*

$$\partial_t u - \operatorname{div}\big(a\big(|\nabla u|^2\big)\nabla u\big) + b\big(|\nabla u|^2\big) = 0 \quad \text{in } Q_\infty. \tag{2.1}$$

Consider next a $\mathcal{C}^3$*-smooth increasing function* φ *and set* $v := \varphi^{-1}(u)$ *and* $w := |\nabla v|^2$. *Then* w *satisfies the following differential inequality*:

$$\partial_t w - \mathcal{A}w - \mathcal{V}\cdot\nabla w + 2\mathcal{R}_1 w^2 + 2\mathcal{R}_2 w \leqslant 0 \quad \text{in } Q_\infty, \tag{2.2}$$

where $\mathcal{A}$, $\mathcal{R}_1$ *and* $\mathcal{R}_2$ *are given by*

$$\mathcal{A}w := a\Delta w + 2a'(\nabla u)^t D^2 w \nabla u, \tag{2.3}$$

$$\mathcal{R}_1 := -a\left(\frac{\varphi''}{\varphi'}\right)' - \left((N-1)\frac{a'^2}{a} + 4a''\right)(\varphi'\varphi'')^2 w^2 - 2a'w\big(2\varphi''^2 + \varphi'\varphi'''\big), \tag{2.4}$$

$$\mathcal{R}_2 := \frac{\varphi''}{\varphi'^2}\big(2b'\varphi'^2 w - b\big), \tag{2.5}$$

while $\mathcal{V}$ *is given by* (A.2) *below. Here and in the following we omit the variable in* a, b *and* φ *and their derivatives.*

Furthermore, if φ is convex, a is non-decreasing and $x \mapsto u(t,x)$ is radially symmetric and non-increasing for each $t \geqslant 0$, then $\mathcal{R}_1$ may be replaced by $\mathcal{R}_1^{\mathrm{r}}$ given by

$$\mathcal{R}_1^{\mathrm{r}} := -a\left(\frac{\varphi''}{\varphi'}\right)' - 4a''(\varphi'\varphi'')^2 w^2 - 2a'w\left(2\varphi''^2 + \varphi'\varphi'''\right). \tag{2.6}$$

The proof of Lemma 2.1 is rather technical and is postponed to Appendix A. We however emphasize that it uses a trick introduced by Bénilan [10] to prove gradient estimates for solutions to the porous medium equation in several space dimensions. It is also worth noticing that $\mathcal{R}_1 = \mathcal{R}_1^{\mathrm{r}}$ for $N = 1$.

Consider next a non-negative function $u_0 \in \mathcal{BC}(\mathbb{R}^N)$. There is a sequence of functions $(u_{0,k})_{k\geqslant 1}$ such that, for each integer $k \geqslant 1$, $u_{0,k} \in \mathcal{BC}^\infty(\mathbb{R}^N)$,

$$0 \leqslant u_{0,k}(x) \leqslant u_{0,k+1}(x) \leqslant u_0(x), \quad x \in \mathbb{R}^N, \tag{2.7}$$

and $(u_{0,k})$ converges uniformly towards u_0 on compact subsets of $\mathbb{R}^N$. In addition, if $u_0 \in W^{1,\infty}(\mathbb{R}^N)$ we may assume that

$$\|\nabla u_{0,k}\|_\infty \leqslant \left(1 + \frac{K_1}{k}\right)\|\nabla u_0\|_\infty, \tag{2.8}$$

for some constant $K_1 > 0$ depending only on the approximation process. Next, since $\xi \mapsto |\xi|^{p-2}$ and $\xi \mapsto |\xi|^q$ are not regular enough for small values of p and q, we set

$$a_\varepsilon(\xi) := \left(\varepsilon^2 + \xi\right)^{(p-2)/2} \quad \text{and} \quad b_\varepsilon(\xi) := \left(\varepsilon^2 + \xi\right)^{q/2} - \varepsilon^q, \quad \xi \geqslant 0, \tag{2.9}$$

for $\varepsilon \in (0, 1/2)$. Then, given

$$0 < \gamma \leqslant \min\left\{\frac{3}{4}, 2\beta_{p,q}, q, \frac{q+2}{2}\right\}, \tag{2.10}$$

the Cauchy problem

$$\partial_t u_{k,\varepsilon} - \operatorname{div}\left(a_\varepsilon\left(|\nabla u_{k,\varepsilon}|^2\right)\nabla u_{k,\varepsilon}\right) + b_\varepsilon\left(|\nabla u_{k,\varepsilon}|^2\right) = 0, \quad (t,x) \in Q_\infty, \tag{2.11}$$

$$u_{k,\varepsilon}(0) = u_{0,k} + \varepsilon^\gamma, \quad x \in \mathbb{R}^N, \tag{2.12}$$

has a unique classical solution $u_{k,\varepsilon} \in \mathcal{C}^{(3+\delta)/2,3+\delta}([0,\infty) \times \mathbb{R}^N)$ for some $\delta \in (0,1)$ [23]. Observing that ε^γ and $\|u_0\|_\infty + \varepsilon^\gamma$ are solutions to (2.11) with $\varepsilon^\gamma \leqslant u_{k,\varepsilon}(0,x) \leqslant \|u_0\|_\infty + \varepsilon^\gamma$, the comparison principle warrants that

$$\varepsilon^\gamma \leqslant u_{k,\varepsilon}(t,x) \leqslant \|u_0\|_\infty + \varepsilon^\gamma, \quad (t,x) \in [0,\infty) \times \mathbb{R}^N. \tag{2.13}$$

We now turn to estimates on the gradient of $u_{k,\varepsilon}$ and first point out that, thanks to the regularity of a_ε, b_ε and $u_{k,\varepsilon}$, we may use Lemma 2.1. We first take $\varphi(r) = \varphi_0(r) := r$ for $r \geqslant 0$ so that $w = |\nabla u_{k,\varepsilon}|^2$ and $\mathcal{R}_1 = \mathcal{R}_2 = 0$. Therefore w satisfies

$$\partial_t w - \mathcal{A}w - \mathcal{V} \cdot \nabla w \leqslant 0 \quad \text{in } Q_\infty.$$

Since $w(0) \leqslant \|\nabla u_{0,k}\|_\infty^2$ the comparison principle ensures that

$$\left\|\nabla u_{k,\varepsilon}(t)\right\|_\infty \leqslant \|\nabla u_{0,k}\|_\infty, \quad t \geqslant 0. \tag{2.14}$$

We now establish gradient estimates similar to (1.6) and (1.7) for $u_{k,\varepsilon}$. We first use the specific choice of a_ε and b_ε to compute $\mathcal{R}_1$ and $\mathcal{R}_2$.

Lemma 2.2. *Introducing $g := (|\nabla u_{k,\varepsilon}|^2 + \varepsilon^2)^{1/2}$, we have*

$$\mathcal{R}_1 = -(p-1)g^{p-2}\left\{\left(\frac{\varphi''}{\varphi'}\right)' + \frac{\alpha_p}{1-\alpha_p}\left(\frac{\varphi''}{\varphi'}\right)^2\right\} + \varepsilon^2\mathcal{R}_{11} \tag{2.15}$$

with

$$\begin{aligned}\mathcal{R}_{11} = {} & (p-2)\left(\frac{\varphi''}{\varphi'}\right)' g^{p-4} + \frac{(p-2)(p(N+3)-2(N+1))}{4}\left(\frac{\varphi''}{\varphi'}\right)^2 g^{p-4} \\ & + \frac{(p-2)(p(N+3)-2(N+7))}{4}\left(\frac{\varphi''}{\varphi'}\right)^2\left(g^2-\varepsilon^2\right)g^{p-6},\end{aligned}$$

and

$$\mathcal{R}_2 = \frac{\varphi''}{\varphi'^2}\left\{(q-1)g^q + \varepsilon^q - q\varepsilon^2 g^{q-2}\right\}. \tag{2.16}$$

After these preliminary computations we are in a position to state and prove the main result of this section.

Proposition 2.3. *There are positive real numbers $C = C(p,N)$ and $D_1(k) = D_1(k,p,N)$ such that, for $\varepsilon \in (0,1/2)$, $x \in \mathbb{R}^N$, and $t \in (0, \varepsilon^{-1/4})$,*

$$\left|\nabla\left(u_{k,\varepsilon}^{\alpha_p}\right)(t,x)\right| \leqslant C\left(1 + D_1(k)\varepsilon^{1/4}\right)^{2/p}\left(\|u_{0,k}\|_\infty + \varepsilon^\gamma\right)^{(p\alpha_p+2-p)/p} t^{-1/p}. \tag{2.17}$$

There are a positive real number $D_2(k) = D_2(k,p,q,N)$ and a positive function $\omega \in \mathcal{C}([0,\infty))$ such that $\omega(\varepsilon) \to 0$ as $\varepsilon \to 0$ and

$$\begin{aligned}\left|\nabla\left(u_{k,\varepsilon}^{\beta_{p,q}}\right)(t,x)\right| \leqslant {} & \frac{\beta_{p,q}}{(q-1)^{1/q}(1-\beta_{p,q})^{1/q}}\left(\frac{1}{q} + D_2(k)\omega(\varepsilon)^{1/2}\right)^{1/q} \\ & \times \left(\|u_{0,k}\|_\infty + \varepsilon^\gamma\right)^{(q\beta_{p,q}+1-q)/q} t^{-1/q}\end{aligned} \tag{2.18}$$

for $t \in (0, \omega(\varepsilon)^{-1/2})$, $x \in \mathbb{R}^N$, and $\varepsilon \in (0, \min\{q-1, 1/2\})$.

The proof of Proposition 2.3 relies on suitable choices of the function φ in $\mathcal{R}_1$ and $\mathcal{R}_2$. To motivate the forthcoming choices, we first note that, if $\varphi(r) = r^{1/\alpha_p}$, then $\mathcal{R}_1 = \varepsilon^2\mathcal{R}_{11}$ and (2.17) will in fact be obtained by choosing a "small perturbation" of $r \mapsto r^{1/\alpha_p}$, namely $\varphi(r) = \varphi_1(r) := (2Kr - r^2)^{1/\alpha_p}$ for K sufficiently large. Such a choice has already been employed for the p-Laplacian equation in one space dimension $N = 1$ for the same purpose [17]. Next, previous

investigations for the case $p=2$ suggest that $\varphi(r)=r^{q/(q-1)}$ is a suitable choice in $\mathcal{R}_2$ [4]. However, with this choice of φ, $\mathcal{R}_1$ might give a non-positive contribution according to the value of p and a suitable choice turns out to be $\varphi(r)=\varphi_2(r):=\beta_{p,q} r^{1/\beta_{p,q}}$.

Proof of Proposition 2.3. We first establish (2.17). Consider $\mu>0$ to be specified later and put

$$K:=\sqrt{1+\mu}M^{\alpha_p}, \quad M:=\|u_{0,k}\|_\infty+\varepsilon^\gamma$$

and $\varphi_1(r):=(2Kr-r^2)^{1/\alpha_p}$ for $r\in[0,K]$. Then v is given by

$$v:=K-\left(K^2-u_{k,\varepsilon}^{\alpha_p}\right)^{1/2} \tag{2.19}$$

and satisfies

$$\frac{\varepsilon^{\gamma\alpha_p}}{2K}\leqslant v\leqslant K-\left(K^2-M^{\alpha_p}\right)^{1/2}\leqslant M^{\alpha_p/2} \tag{2.20}$$

by (2.13). Thanks to the bounds (2.20), we can find μ large enough such that φ_1 enjoys the following properties:

$$0\geqslant\left(\frac{\varphi_1''}{\varphi_1'}\right)'(v)\geqslant-\frac{C_1(\mu)}{v^2}, \tag{2.21}$$

$$0\leqslant\frac{\varphi_1''}{\varphi_1'}(v)\leqslant\frac{C_2(\mu)}{v}, \tag{2.22}$$

$$\left(\frac{\varphi_1''}{\varphi_1'}\right)'(v)+\frac{\alpha_p}{1-\alpha_p}\left(\frac{\varphi_1''}{\varphi_1'}\right)^2(v)\leqslant-\frac{1+\alpha_p}{2\alpha_p}\frac{1}{Kv}. \tag{2.23}$$

We then infer from (2.21) and (2.22) that

$$\mathcal{R}_{11}\geqslant-\frac{C_3(\mu)}{v^2}g^{p-4}.$$

Therefore, by (2.20) and the elementary inequality $g\geqslant|\nabla u_{k,\varepsilon}|$, we have

$$w\mathcal{R}_{11}\geqslant-\frac{|\nabla u_{k,\varepsilon}|^2}{(\varphi_1')^2(v)}\frac{C_3(\mu)}{v^2}g^{p-4}\geqslant-\frac{C_4(\mu)}{Mv^{2/\alpha_p}}g^{p-2}\geqslant-\frac{C_5(\mu)}{\varepsilon^{2\gamma}}g^{p-2}.$$

Combining the previous inequality with (2.15) and (2.23), we obtain

$$w^2\mathcal{R}_1\geqslant\frac{C_6(\mu)M^{-\alpha_p/2}}{v}g^{p-2}w^2-C_5(\mu)\varepsilon^{2(1-\gamma)}g^{p-2}w.$$

Now, we have $g\leqslant\|\nabla u_{0,k}\|_\infty+\varepsilon$ by (2.14) and

$$g^2\geqslant|\nabla u_{k,\varepsilon}|^2=(\varphi_1')^2(v)w\geqslant C_7(\mu)Mv^{2(1-\alpha_p)/\alpha_p}w$$

by (2.20). The previous lower bound for $w^2\mathcal{R}_1$ then gives

$$w^2\mathcal{R}_1 \geqslant \frac{C_8(\mu)M^{(p-2-\alpha_p)/2}}{v^{((p-1)\alpha_p-(p-2))/\alpha_p}} w^{(p+2)/2} - C_9(\mu,k)\varepsilon^{2(1-\gamma)}w.$$

Since $(p-1)\alpha_p \geqslant (p-2)$ and $v \leqslant M^{\alpha_p/2}$ by (2.20), we end up with

$$w^2\mathcal{R}_1 \geqslant C_{10}(\mu)M^{(2(p-2)-p\alpha_p)/2} w^{(p+2)/2} - C_9(\mu,k)\varepsilon^{2(1-\gamma)}w. \tag{2.24}$$

Next, since $q > 1$ and $g \geqslant \varepsilon$, we infer from the monotonicity of φ_1 and (2.22) that $\mathcal{R}_2 \geqslant 0$. Recalling (2.2) and (2.24) we have shown that

$$\mathcal{L}_1 w := \partial_t w - \mathcal{A}w - \mathcal{V}\cdot\nabla w + 2C_{10}(\mu)M^{(2(p-2)-p\alpha_p)/2}w^{(p+2)/2} - 2C_9(\mu,k)\varepsilon^{2(1-\gamma)}w \leqslant 0$$

in Q_∞. It is then straightforward to check that

$$S_1(t) := \left(\frac{1+2C_9(\mu,k)\varepsilon^{1/4}}{pC_{10}(\mu)}\right)^{2/p} M^{(p\alpha_p-2(p-2))/p} t^{-2/p}$$

satisfies $\mathcal{L}_1 S_1 \geqslant 0$ in $(0,\varepsilon^{-1/4}) \times \mathbb{R}^N$. The comparison principle then ensures that $w(t,x) \leqslant S_1(t)$ for $(t,x) \in (0,\varepsilon^{-1/4}) \times \mathbb{R}^N$. The estimate (2.17) then readily follows with the help of (2.20).

To prove (2.18) we take $\varphi_2(r) := \beta_{p,q} r^{1/\beta_{p,q}}$, so that $v = (u/\beta_{p,q})^{\beta_{p,q}}$ satisfies

$$\frac{\varepsilon^{\gamma\beta_{p,q}}}{\beta_{p,q}^{\beta_{p,q}}} \leqslant v \leqslant \frac{M^{\beta_{p,q}}}{\beta_{p,q}^{\beta_{p,q}}} \quad \text{with } M := \|u_{0,k}\|_\infty + \varepsilon^\gamma, \tag{2.25}$$

by (2.13). Concerning $\mathcal{R}_1$, the computations are much simpler than in the previous case and it follows from the definition of $\beta_{p,q}$ and (2.14) that

$$\begin{aligned} w^2\mathcal{R}_1 &\geqslant C_{11}\frac{\beta_{p,q}-\alpha_p}{\alpha_p\beta_{p,q}}\frac{g^{p-2}w^2}{v^2} - C_{12}\varepsilon^{(2\beta_{p,q}-\gamma)/\beta_{p,q}}g^{p-2}w, \\ w^2\mathcal{R}_1 &\geqslant -C_{13}(k)\varepsilon^{(2\beta_{p,q}-\gamma)/\beta_{p,q}}w. \end{aligned} \tag{2.26}$$

For $\mathcal{R}_2$, we first claim that

$$(q-1)g^q + \varepsilon^q - q\varepsilon^2 g^{q-2} \geqslant (q-1-\varepsilon)g^q - C_{14}\big(\varepsilon^{(q+2)/2} + \varepsilon^q\big). \tag{2.27}$$

Indeed, if $q > 2$, it follows from the Young inequality that

$$\begin{aligned} (q-1)g^q + \varepsilon^q - q\varepsilon^2 g^{q-2} &\geqslant (q-1)g^q - \varepsilon g^q - 2(q-2)^{(q-2)/2}\varepsilon^{(q+2)/2} \\ &\geqslant (q-1-\varepsilon)g^q - 2(q-2)^{(q-2)/2}\varepsilon^{(q+2)/2}. \end{aligned}$$

If $q \in (1,2]$, we have

$$(q-1)g^q + \varepsilon^q - q\varepsilon^2 g^{q-2} \geqslant (q-1)g^q + \varepsilon^q - q\varepsilon^q \geqslant (q-1-\varepsilon)g^q - (q-1)\varepsilon^q,$$

which completes the proof of (2.27). We then infer from (2.16), (2.25), and (2.27) that

$$\begin{aligned}
\mathcal{R}_2 &\geqslant \frac{1-\beta_{p,q}}{\beta_{p,q}} \frac{1}{v^{1/\beta_{p,q}}}\left[(q-1-\varepsilon)(\varphi_2')^q(v)w^{q/2} - C_{14}\left(\varepsilon^{(q+2)/2}+\varepsilon^q\right)\right] \\
&\geqslant \frac{1-\beta_{p,q}}{\beta_{p,q}}(q-1-\varepsilon)v^{(q(1-\beta_{p,q})-1)/\beta_{p,q}}w^{q/2} - C_{15}\left(\varepsilon^{(q+2-2\gamma)/2}+\varepsilon^{q-\gamma}\right) \\
&\geqslant \frac{1-\beta_{p,q}}{\beta_{p,q}^{q(1-\beta_{p,q})}}(q-1-\varepsilon)M^{q(1-\beta_{p,q})-1}w^{q/2} - C_{15}\left(\varepsilon^{(q+2-2\gamma)/2}+\varepsilon^{q-\gamma}\right).
\end{aligned}$$

Recalling (2.26) we have thus shown that w satisfies

$$\begin{aligned}
\mathcal{L}_2 w := \partial_t w - \mathcal{A}w - \mathcal{V}\cdot\nabla w + 2\frac{1-\beta_{p,q}}{\beta_{p,q}^{q(1-\beta_{p,q})}}(q-1-\varepsilon)M^{q(1-\beta_{p,q})-1}w^{(q+2)/2} \\
- C_{16}(k)\omega(\varepsilon)w \leqslant 0
\end{aligned}$$

in Q_∞, where $\omega(\varepsilon) := \varepsilon^{(2\beta_{p,q}-\gamma)/\beta_{p,q}} + \varepsilon^{(q+2-2\gamma)/2} + \varepsilon^{q-\gamma} \to 0$ as $\varepsilon \to 0$ by the choice (2.10) of γ. The function

$$S_2(t) := \frac{\beta_{p,q}^{2(1-\beta_{p,q})}}{2^{2/q}(1-\beta_{p,q})^{2/q}(q-1-\varepsilon)^{2/q}}\left(\frac{2+qC_{16}(k)\omega(\varepsilon)^{1/2}}{q}\right)^{2/q} M^{2(1-q(1-\beta_{p,q}))/q}t^{-2/q}$$

satisfies $\mathcal{L}_2 S_2 \geqslant 0$ in $(0,\omega(\varepsilon)^{-1/2}) \times \mathbb{R}^N$. We then deduce from the comparison principle that $w(t,x) \leqslant S_2(t)$ for $(t,x) \in (0,\omega(\varepsilon)^{-1/2}) \times \mathbb{R}^N$. The estimate (2.18) then readily follows. □

3. Existence

We are now in a position to prove Theorem 1.1 and proceed along the lines of [20].

Step 1: $\varepsilon \to 0$. We first let $\varepsilon \to 0$. For that purpose, we observe that the gradient bound (2.14) and (2.11) imply the time equicontinuity of $(u_{k,\varepsilon})_{\varepsilon>0}$.

Lemma 3.1. *For $k \geqslant 1$, $\varepsilon > 0$, $x \in \mathbb{R}^N$, $t_1 \geqslant 0$, and $t_2 > t_1$, we have*

$$\left|u_{k,\varepsilon}(t_2,x) - u_{k,\varepsilon}(t_1,x)\right| \leqslant C\left(\|\nabla u_{0,k}\|_\infty + \|\nabla u_{0,k}\|_\infty^{p-1}\right)(t_2-t_1)^{1/2} + \|\nabla u_{0,k}\|_\infty^q (t_2-t_1).$$

The proof of Lemma 3.1 is similar to that of [20, Lemma 5] to which we refer.

We next fix $k \geqslant 1$. Owing to (2.13), (2.14), and Lemma 3.1, we may apply the Arzelà–Ascoli theorem to obtain a subsequence of $(u_{k,\varepsilon})_{\varepsilon>0}$ (not relabeled) and a non-negative function $u_k \in \mathcal{BC}([0,\infty)\times\mathbb{R}^N)$ such that

$$u_{k,\varepsilon} \to u_k \text{ uniformly on any compact subset of } [0,\infty)\times\mathbb{R}^N. \tag{3.1}$$

Furthermore, as $u_{k,\varepsilon}$ is a classical solution to (2.11), (2.12), the classical stability result for continuous viscosity solutions allows us to conclude that u_k is a viscosity solution to (1.1) with

initial condition $u_{0,k}$ (see, e.g., [14], [13, Theorem 1.4] or [3, Théorème 2.3]). By (3.1) and weak convergence arguments, we next infer from (2.13), (2.17), and (2.18) that

$$0 \leqslant u_k(t,x) \leqslant \|u_0\|_\infty, \tag{3.2}$$

$$\left|\nabla\left(u_k^{\alpha_p}\right)(t,x)\right| \leqslant C\|u_{0,k}\|_\infty^{(p\alpha_p+2-p)/p} t^{-1/p}, \tag{3.3}$$

$$\left|\nabla\left(u_k^{\beta_{p,q}}\right)(t,x)\right| \leqslant \frac{\beta_{p,q}}{(q^2-q)^{1/q}(1-\beta_{p,q})^{1/q}} \|u_{0,k}\|_\infty^{(q\beta_{p,q}+1-q)/q} t^{-1/q} \tag{3.4}$$

for all $(t,x) \in Q_\infty$. Finally, (2.11) also reads

$$\partial_t u_{k,\varepsilon} - \operatorname{div}\left(|\nabla u_{k,\varepsilon}|^{p-2}\nabla u_{k,\varepsilon}\right) = \operatorname{div}(f_{k,\varepsilon}) + g_{k,\varepsilon} \quad \text{in } Q_\infty$$

with

$$f_{k,\varepsilon} := \left\{a_\varepsilon\left(|\nabla u_{k,\varepsilon}|^2\right) - |\nabla u_{k,\varepsilon}|^{p-2}\right\}\nabla u_{k,\varepsilon} \quad \text{and} \quad g_{k,\varepsilon} := -b_\varepsilon\left(|\nabla u_{k,\varepsilon}|^2\right).$$

It follows from the definition of a_ε and (2.14) that $(g_{k,\varepsilon})$ is bounded in $L^\infty(Q_\infty)$ and $(f_{k,\varepsilon})$ converges to zero in $L^\infty(Q_\infty)$ as $\varepsilon \to 0$. We may then apply [12, Theorem 4.1] to conclude that

$$\nabla u_{k,\varepsilon} \to \nabla u_k \quad \text{a.e. in } Q_\infty. \tag{3.5}$$

Consequently, upon extracting a further subsequence, we may assume that

$$\nabla u_{k,\varepsilon} \to \nabla u_k \quad \text{a.e. in } L^r\left((0,T)\times B(0,R)\right) \tag{3.6}$$

for every $r \in [1,\infty)$, $T > 0$, and $R > 0$. It then readily follows that u_k satisfies (1.8) with $u_{0,k}$ instead of u_0.

Step 2: $k \to \infty$. It remains to pass to the limit as $k \to \infty$. To this end we first observe that (2.7) implies that $u_{0,k}(x) - u_{0,k+1}(y) \leqslant \|\nabla u_{0,k}\|_\infty |y-x|$ for $k \geqslant 1$, $x \in \mathbb{R}^N$, and $y \in \mathbb{R}^N$. It then follows from the comparison principle [18, Theorem 2.1] that

$$u_k(t,x) \leqslant u_{k+1}(t,x) \quad \text{for } (t,x) \in Q_\infty \text{ and } k \geqslant 1. \tag{3.7}$$

Therefore, by (2.7), (3.2), and (3.7), the function

$$u(t,x) := \sup_{k\geqslant 1} u_k(t,x) \in \left[0, \|u_0\|_\infty\right] \tag{3.8}$$

is well defined for $(t,x) \in [0,\infty)\times\mathbb{R}^N$. We next readily deduce from (3.2) and (3.3) that, for $\tau > 0$,

$$\left\|\nabla u_k(t)\right\|_\infty \leqslant C\|u_0\|_\infty^{2/p} t^{-1/p} \leqslant C\|u_0\|_\infty^{2/p}\tau^{-1/p} \quad \text{for } t \geqslant \tau. \tag{3.9}$$

Thanks to (3.9) we may argue as in the previous step and conclude that

$$u_k \to u \text{ uniformly on any compact subset of } Q_\infty. \tag{3.10}$$

Using again the stability of continuous viscosity solutions, we deduce from the convergence (3.10) that $(t,x) \mapsto u(t+\tau,x)$ is a viscosity solution to (1.1) with initial condition $u(\tau)$ for each $\tau > 0$. In addition, denoting by $\tilde{u}_k$ the solution to the p-Laplacian equation (1.9) with initial condition $u_{0,k}$, the comparison principle entails that

$$u_k(t,x) \leqslant \tilde{u}_k(t,x) \quad \text{for } (t,x) \in Q_\infty \text{ and } k \geqslant 1. \tag{3.11}$$

Furthermore, $(\tilde{u}_k)_{k\geqslant 1}$ converges uniformly on any compact subset of $[0,\infty) \times \mathbb{R}^N$ towards the solution $\tilde{u}$ to the p-Laplacian equation (1.9) with initial condition u_0 [16, Chapter III]. This property and (3.11) warrant that $u(t,x) \leqslant \tilde{u}(t,x)$ for $(t,x) \in [0,\infty) \times \mathbb{R}^N$. Recalling (3.8), we thus obtain the following inequality:

$$u_k(t,x) \leqslant u(t,x) \leqslant \tilde{u}(t,x) \quad \text{for } (t,x) \in Q_\infty \text{ and } k \geqslant 1. \tag{3.12}$$

We then infer from (3.12) that $(u(\cdot + 1/j))_{j\geqslant 1}$ converges towards u uniformly on any compact subset of $[0,\infty) \times \mathbb{R}^N$ as $j \to \infty$. Using once more the stability of continuous viscosity solutions, we conclude that u is a viscosity solution to (1.1), (1.2). We next argue as in the previous step to deduce from (3.3) and (3.4) that u satisfies (1.6), (1.7) and (1.8) for $t > s > 0$. In addition, $u \in L^\infty(Q_\infty)$ by (1.5) and we deduce from (1.5) and (1.6) that $\|\nabla u(t)\|_\infty \leqslant C\|u_0\|_\infty^{2/p} t^{-1/p}$ for $t > 0$. Consequently, ∇u belongs to $L^{p-1}((0,T) \times B(0,R))$ for all $T > 0$ and $R > 0$. We then let $s \to 0$ in (1.8) to conclude that $\nabla u \in L^q((0,T) \times B(0,R))$ for all $T > 0$ and $R > 0$ which in turn warrants that (1.8) is also valid for $s = 0$.

To complete the proof of Theorem 1.1, it remains to check the uniqueness assertion for $u_0 \in \mathcal{BUC}(\mathbb{R}^N)$ which actually follows at once from [18, Theorem 2.1].

4. Temporal decay estimates

This section is devoted to the proof of Proposition 1.4. Let us start with the following lemma.

Lemma 4.1. *Let u be a solution of* (1.1), (1.2). *If $t > s \geqslant 0$, then*

$$\|\nabla u(t)\|_\infty \leqslant C\|u(s)\|_\infty^{2/p} (t-s)^{-1/p}, \tag{4.1}$$

$$\|\nabla u(t)\|_\infty \leqslant C\|u(s)\|_\infty^{1/q} (t-s)^{-1/q}. \tag{4.2}$$

Proof. We write

$$|\nabla u(t)| = \frac{1}{\gamma} u^{1-\gamma} |\nabla u^\gamma|$$

for $\gamma = \alpha_p$ and $\gamma = \beta_{p,q}$ and use the estimates (1.6) and (1.7). □

Proof of Proposition 1.4. We first prove (1.16). Combining the Gagliardo–Nirenberg inequality, the time monotonicity of $\|u\|_1$ and the previous lemma, we obtain

$$\begin{aligned}\|u(t)\|_\infty^q &\leqslant C\|\nabla u(t)\|_\infty^{qN/(N+1)} \|u(t)\|_1^{q/(N+1)} \\ &\leqslant C\|\nabla u(t)\|_\infty^{qN/(N+1)} \|u_0\|_1^{q/(N+1)}\end{aligned}$$

$$\leqslant C(t-s)^{-N/(N+1)}\big\|u(s)\big\|_\infty^{N/(N+1)}\|u_0\|_1^{q/(N+1)}.$$

Integrating with respect to t over (s,∞), we obtain

$$\tau(s):=\int_s^\infty \frac{\|u(t)\|_\infty^q}{t}\,dt \leqslant C\big\|u(s)\big\|_\infty^{N/(N+1)}\|u_0\|_1^{q/(N+1)}\int_s^\infty \frac{dt}{(t-s)^{N/(N+1)}t}$$
$$\leqslant Cs^{-N/(N+1)}\|u_0\|_1^{q/(N+1)}\big\|u(s)\big\|_\infty^{N/(N+1)},$$

whence

$$\tau(s)\leqslant C\|u_0\|_1^{q/(N+1)}\big(-\tau'(s)\big)^{N/q(N+1)}s^{-(N(q-1))/q(N+1)}.$$

Introducing $\tilde\tau(s)=\tau(s^{1/q})$ gives

$$\frac{d\tilde\tau}{ds}(s)+C\|u_0\|_1^{-q^2/N}\tilde\tau(s)^{q(N+1)/N}\leqslant 0.$$

A direct computation shows that $\tilde\tau(s)\leqslant C\|u_0\|_1^{q^2\xi}s^{-N\xi}$ from which we deduce that

$$\tau(s)\leqslant C\|u_0\|_1^{q^2\xi}s^{-qN\xi}.$$

Now, using the time monotonicity of $\|u\|_\infty$, we obtain

$$Cs^{-qN\xi}\|u_0\|_1^{q^2\xi}\geqslant\tau(s)\geqslant\int_s^{2s}\frac{\|u(t)\|_\infty^q}{t}\,dt\geqslant\int_s^{2s}\frac{\|u(2s)\|_\infty^q}{t}\,dt=\ln(2)\big\|u(2s)\big\|_\infty^q,$$

whence (1.16). The estimate (1.17) then readily follows from (1.16) by (4.2). A similar proof relying on (4.1) gives the estimates (1.18) and (1.19). □

5. Limit values of $\|u(t)\|_1$

In this section we investigate the possible values of the limit as $t\to\infty$ of the L^1-norm of non-negative solutions to (1.1), (1.2) and prove Proposition 1.5. We first show that, if q is small enough, the dissipation mechanism induced by the nonlinear absorption term is sufficiently strong to drive the L^1-norm of u to zero in infinite time.

Proposition 5.1. *If $q\in(1,q_*]$ then*

$$\lim_{t\to\infty}\big\|u(t)\big\|_1=0.$$

Proof. It first follows from the integration of (1.1) over $(0,t)\times\mathbb{R}^N$ that

$$\big\|u(t)\big\|_1+\int_0^t\big\|\nabla u(s)\big\|_q^q\,ds=\|u_0\|_1, \tag{5.1}$$

which readily implies that $t \mapsto \|\nabla u(t)\|_q^q$ belongs to $L^1(0,\infty)$. Consequently,

$$\omega(t) := \int_t^\infty \big\|\nabla u(s)\big\|_q^q \, ds \underset{t\to\infty}{\longrightarrow} 0. \tag{5.2}$$

We next consider a $\mathcal{C}^\infty$-smooth function ϑ in $\mathbb{R}^N$ such that $0 \leqslant \vartheta \leqslant 1$ and

$$\vartheta(x) = 0 \quad \text{if } |x| \leqslant 1 \quad \text{and} \quad \vartheta(x) = 1 \quad \text{if } |x| \geqslant 2.$$

For $R > 0$ and $x \in \mathbb{R}^N$ we put $\vartheta_R(x) = \vartheta(x/R)$. We multiply (1.1) by $\vartheta_R(x)$ and integrate over $(t_1, t_2) \times \mathbb{R}^N$ to obtain

$$\int_{\mathbb{R}^N} u(t_2,x)\vartheta_R(x)\,dx \leqslant \int_{\mathbb{R}^N} u(t_1,x)\vartheta_R(x)\,dx + \frac{1}{R}\int_{t_1}^{t_2}\int_{\mathbb{R}^N} \big|\nabla u(s,x)\big|^{p-2}\nabla\vartheta\left(\frac{x}{R}\right)\nabla u(s,x)\,dx\,ds,$$

which, together with the properties of ϑ, gives

$$\int_{\{|x|\geqslant 2R\}} u(t_2,x)\,dx \leqslant \int_{\{|x|\geqslant R\}} u(t_1,x)\,dx + \frac{1}{R}\int_{t_1}^{t_2}\int_{\mathbb{R}^N} \left|\nabla\vartheta\left(\frac{x}{R}\right)\right| \big|\nabla u(s,x)\big|^{p-1}\,dx\,ds. \tag{5.3}$$

Case 1: $q \in [p-1, q_*]$. By the Hölder inequality we have

$$\begin{aligned}
&\frac{1}{R}\int_{t_1}^{t_2}\int_{\mathbb{R}^N} \left|\nabla\vartheta\left(\frac{x}{R}\right)\right| \big|\nabla u(s,x)\big|^{p-1}\,dx\,ds \\
&\quad\leqslant R^{(N(q-p+1)-q)/q}(t_2-t_1)^{(q-p+1)/q}\|\nabla\vartheta\|_{(q-p+1)/q}\left(\int_{t_1}^{t_2}\big\|\nabla u(s)\big\|_q^q\,ds\right)^{(p-1)/q} \\
&\quad\leqslant C R^{(N(q-p+1)-q)/q}\omega(t_1)^{(p-1)/q}(t_2-t_1)^{(q-p+1)/q}.
\end{aligned}$$

Combining the above inequality with (1.16), (5.3) and the time monotonicity of $\|u\|_1$ we obtain

$$\begin{aligned}
\big\|u(t_2)\big\|_1 &= \int_{\{|x|\leqslant 2R\}} u(t_2,x)\,dx + \int_{\{|x|\geqslant 2R\}} u(t_2,x)\,dx \\
&\leqslant C R^N\big\|u(t_2)\big\|_\infty + \int_{\{|x|\geqslant R\}} u(t_1,x)\,dx \\
&\qquad + C R^{(N(q-p+1)-q)/q}\omega(t_1)^{(p-1)/q}(t_2-t_1)^{(q-p+1)/q} \\
&\leqslant \int_{\{|x|\geqslant R\}} u(t_1,x)\,dx + C R^N(t_2-t_1)^{-N\xi}
\end{aligned}$$

$$+ C R^{(N(q-p+1)-q)/q} \omega(t_1)^{(p-1)/q} (t_2-t_1)^{(q-p+1)/q}.$$

Choosing

$$R = R(t_1,t_2) := \omega(t_1)^{(p-1)/(q+N(p-1))} (t_2-t_1)^{(qN\xi+q-p+1)/(q+N(p-1))}$$

we are led to

$$\begin{aligned}\|u(t_2)\|_1 \leqslant & \int_{\{|x|\geqslant R(t_1,t_2)\}} u(t_1,x)\,dx \\ & + C\omega(t_1)^{(N(p-1))/(q+N(p-1))} (t_2-t_1)^{-qN\xi(N+1)(q_*-q)/(q+N(p-1))}.\end{aligned}$$

Since $\xi > 0$ and $q_* - q \geqslant 0$ we may let $t_2 \to \infty$ in the previous inequality to conclude that

$$\begin{aligned} I_1(\infty) &\leqslant 0 \quad \text{if } q \in [p-1, q_*), \\ I_1(\infty) &\leqslant C\omega(t_1)^{(N(p-1))/(q_*+N(p-1))} \quad \text{if } q = q_*.\end{aligned}$$

We have used here that $R(t_1,t_2) \to \infty$ as $t_2 \to \infty$ and that $u(t_1) \in L^1(\mathbb{R}^N)$. Owing to the non-negativity of $I_1(\infty)$, we readily obtain that $I_1(\infty) = 0$ if $q \in [p-1, q_*)$. When $q = q_*$, we let $t_1 \to \infty$ and use (5.2) to conclude that $I_1(\infty) = 0$ also in that case.

Case 2: $q \in (1, p-1)$. By (1.17) and (5.3) we have

$$\begin{aligned}\int_{\{|x|\geqslant 2R\}} u(t_2,x)\,dx &\leqslant \int_{\{|x|\geqslant R\}} u(t_1,x)\,dx + \frac{1}{R}\|\nabla\vartheta\|_\infty \int_{t_1}^{t_2} \|\nabla u(s)\|_\infty^{p-1-q} \|\nabla u(s)\|_q^q\,ds \\ &\leqslant \int_{\{|x|\geqslant R\}} u(t_1,x)\,dx + \frac{C}{R}\int_{t_1}^{t_2} s^{-(p-1-q)(N+1)\xi} \|\nabla u(s)\|_q^q\,ds \\ &\leqslant \int_{\{|x|\geqslant R\}} u(t_1,x)\,dx + \frac{C}{R} t_1^{-(p-1-q)(N+1)\xi}\omega(t_1).\end{aligned}$$

Taking $t_1 = 1$ and noting that $\omega(t_1) \leqslant \omega(0) \leqslant \|u_0\|_1$, we end up with

$$\int_{\{|x|\geqslant 2R\}} u(t_2,x)\,dx \leqslant \int_{\{|x|\geqslant R\}} u(1,x)\,dx + \frac{C}{R}, \quad t_2 \geqslant 1.$$

We then infer from (1.16) and the above inequality that, if $t_2 \geqslant 1$,

$$\|u(t_2)\|_1 \leqslant C R^N t^{-N\xi} + \int_{\{|x|\geqslant R\}} u(1,x)\,dx + \frac{C}{R}$$

and the choice $R = R(t_2) = t_2^{(N\xi)/(N+1)}$ gives

$$\|u(t_2)\|_1 \leqslant \int_{\{|x| \geqslant R(t_2)\}} u(1, x)\, dx + C t_2^{-(N\xi)/(N+1)}.$$

Since $R(t_2) \to \infty$ as $t_2 \to \infty$ and $u(1) \in L^1(\mathbb{R}^N)$ we may let $t_2 \to \infty$ in the above inequality to establish that $I_1(\infty) = 0$, which completes the proof of Proposition 5.1. □

We next turn to higher values of q and adapt an argument of [4, Theorem 6] to show the positivity of $I_1(\infty)$.

Proposition 5.2. *Assume that $\|u_0\|_1 > 0$ and $q > q_*$. Then $I_1(\infty) > 0$.*

Proof. Since $u_0 \in \mathcal{BC}(\mathbb{R}^N)$ is not identically equal to zero there are $x_0 \in \mathbb{R}^N$ and a radially symmetric and non-increasing continuous function $U_0 \not\equiv 0$ such that $u_0(x) \geqslant U_0(x - x_0)$. Denoting by U the solution to (1.1) with initial condition U_0 it follows from the invariance of (1.1) by translation and the comparison principle that

$$u(t, x) \geqslant U(t, x - x_0), \quad (t, x) \in [0, \infty) \times \mathbb{R}^N. \tag{5.4}$$

Let $\tau > 0$ and $x \in \mathbb{R}^N$. Since

$$\nabla U(\tau, x) = \frac{p-1}{p-2} U(\tau, x)^{1/(p-1)} \nabla\big(U^{(p-2)/(p-1)}\big)(\tau, x)$$

and $q > q_* > p - 1$, we infer from (1.11) and the time monotonicity of $\|u\|_\infty$ that

$$\begin{aligned}
\big|\nabla U(\tau, x)\big|^q &\leqslant \left(\frac{p-1}{p-2}\right)^q U(\tau, x)^{q/(p-1)} \big|\nabla\big(U^{(p-2)/(p-1)}\big)(\tau, x)\big|^q \\
&\leqslant C U(\tau, x) \big\|U(\tau)\big\|_\infty^{(q-p+1)/(p-1)} \left\|U\left(\frac{\tau}{2}\right)\right\|_\infty^{q(p-2)/p(p-1)} \tau^{-q/p} \\
&\leqslant C U(\tau, x) \left\|U\left(\frac{\tau}{2}\right)\right\|_\infty^{(2q-p)/p} \tau^{-q/p},
\end{aligned}$$

whence, by (1.18),

$$\big|\nabla U(\tau, x)\big|^q \leqslant C U(\tau, x) \tau^{-\eta/\xi}. \tag{5.5}$$

Consider now $s \in (0, \infty)$ and $t \in (s, \infty)$. It follows from (1.1) and (5.5) that

$$\begin{aligned}
\big\|U(t)\big\|_1 &= \big\|U(s)\big\|_1 - \int_s^t \int_{\mathbb{R}^N} \big|\nabla U(\tau, x)\big|^q \, dx\, d\tau \\
&\geqslant \big\|U(s)\big\|_1 - C \int_s^t \tau^{-\eta/\xi} \big\|U(\tau)\big\|_1 \, d\tau.
\end{aligned}$$

Owing to the monotonicity of $\tau \mapsto \|U(\tau)\|_1$, we further obtain

$$\big\|U(t)\big\|_1 \geqslant \big\|U(s)\big\|_1 \left(1 - C \int_s^t \tau^{-\eta/\xi}\, d\tau\right).$$

Since $q > q_*$ we have $\eta > \xi$ and the right-hand side of the above inequality has a finite limit as $t \to \infty$. We may then let $t \to \infty$ to obtain

$$\mathcal{I}_1(\infty) := \lim_{t\to\infty} \big\|U(t)\big\|_1 \geqslant \big\|U(s)\big\|_1 \big(1 - C s^{-(\eta-\xi)/\xi}\big), \quad s > 0.$$

Consequently, for s large enough, we have $\mathcal{I}_1(\infty) \geqslant \|U(s)\|_1/2$, while [1, Lemma 4.1] warrants that $\|U(s)\|_1 > 0$ for each $s \geqslant 0$ since $U_0 \not\equiv 0$. Therefore, $\mathcal{I}_1(\infty) > 0$. Recalling (5.4) we realize that $\|u(t)\|_1 \geqslant \|U(t)\|_1$ for each $t \geqslant 0$ so that $I_1(\infty) \geqslant \mathcal{I}_1(\infty) > 0$. $\square$

6. Compactly supported initial data

This section is devoted to the proofs of Theorem 1.6 and Corollary 1.7. Let $u_0 \in L^1(\mathbb{R}^N) \cap \mathcal{BC}(\mathbb{R}^N)$ be a non-negative initial condition with compact support in the ball $B(0, R_0)$ for some $R_0 > 0$. Denoting by u the corresponding solution to (1.1), (1.2) and by v the corresponding solution to the p-Laplacian equation

$$\partial_t v - \Delta_p v = 0, \quad (t,x) \in Q_\infty, \tag{6.1}$$

with initial condition $v(0) = u_0$, the comparison principle ensures that

$$0 \leqslant u(t,x) \leqslant v(t,x), \quad (t,x) \in Q_\infty. \tag{6.2}$$

Since u_0 is compactly supported, so is $v(t)$ for each $t \geqslant 0$ by [16, Lemma 8.1] and $\operatorname{Supp} v(t) \subset B(0, C_1 t^\eta)$. Consequently, $u(t)$ is compactly supported for each $t \geqslant 0$ with $\operatorname{Supp} u(t) \subset B(0, C_1 t^\eta)$. In particular, the support of u does not expand faster than that of v with time. A natural question is then whether the damping term slows down this expansion and the answer depends heavily on the value of q. We shall thus distinguish between three cases in the proof of Theorem 1.6.

We first note that, since u_0 is non-negative continuous and compactly supported, there exists a non-negative continuous radially symmetric and non-increasing function U_0 with compact support such that $0 \leqslant u_0 \leqslant U_0$. Denoting by U the corresponding solution to (1.1) with initial condition $U(0) = U_0$, the function $x \mapsto U(t,x)$ is also radially symmetric and non-increasing for each $t \geqslant 0$ and we deduce from the comparison principle that

$$0 \leqslant u(t,x) \leqslant U(t,x), \quad (t,x) \in [0,\infty) \times \mathbb{R}^N. \tag{6.3}$$

Moreover, by comparison with the p-Laplacian equation, $U(t)$ is also compactly supported for each $t \geqslant 0$ with $\operatorname{Supp} U(t) \subset B(0, \sigma(t))$ for some $\sigma(t) > 0$. Clearly,

$$\varrho(t) \leqslant \sigma(t), \quad t \geqslant 0, \tag{6.4}$$

by (6.3).

It next follows from (1.1) that, if y is a non-negative function in $\mathcal{C}^1([0,\infty))$, we have

$$\begin{aligned}
\frac{d}{dt}\int_{\{|x|\geqslant y(t)\}} U(t,x)\,dx &= \int_{\{|x|\geqslant y(t)\}} \partial_t U(t,x)\,dx - y'(t)\int_{\{|x|=y(t)\}} U(t,x)\,dx \\
&\leqslant \int_{\{|x|\geqslant y(t)\}} \operatorname{div}\bigl(|\nabla U|^{p-2}\nabla U\bigr)(t,x)\,dx \\
&\quad - y'(t)\int_{\{|x|=y(t)\}} U(t,x)\,dx \\
&\leqslant -\int_{\{|x|=y(t)\}} \bigl|\nabla U(t,x)\bigr|^{p-2}\nabla U(t,x)\cdot\frac{x}{|x|}\,dx \\
&\quad - y'(t)\int_{\{|x|=y(t)\}} U(t,x)\,dx \\
&\leqslant \int_{\{|x|=y(t)\}} \bigl\{\bigl|\nabla U(t,x)\bigr|^{p-1} - y'(t)U(t,x)\bigr\}\,dx,
\end{aligned}$$

$$\begin{aligned}
&\frac{d}{dt}\int_{\{|x|\geqslant y(t)\}} U(t,x)\,dx \\
&\quad\leqslant \int_{\{|x|=y(t)\}} \left\{\frac{p-1}{p-2}\bigl|\nabla\bigl(U^{(p-2)/(p-1)}\bigr)(t,x)\bigr|^{p-1} - y'(t)\right\}U(t,x)\,dx. \qquad (6.5)
\end{aligned}$$

The next step is to use the gradient estimates established in Theorem 1.2 to find a suitable function y for which the right-hand side of (6.5) is non-positive. The gradient estimates depending on the value of q, we handle separately the cases $q\in(1,p-1]$ and $q\in(p-1,q_*)$.

Proof of Theorem 1.6: $q\in(1,p-1]$. In that case we infer from (1.13) and (1.16) that

$$\begin{aligned}
\bigl|\nabla\bigl(U^{(p-2)/(p-1)}\bigr)(t,x)\bigr|^{p-1} &\leqslant C\left\|u\left(\frac{t}{2}\right)\right\|_\infty^{(p-1-q)/q} t^{-(p-1)/q} \\
&\leqslant Ct^{-\xi((p-1)(N+1)-N)},
\end{aligned}$$

so that (6.5) becomes

$$\frac{d}{dt}\int_{\{|x|\geqslant y(t)\}} U(t,x)\,dx \leqslant \int_{\{|x|=y(t)\}} \bigl\{Ct^{-\xi((p-1)(N+1)-N)} - y'(t)\bigr\}U(t,x)\,dx.$$

Choosing $y'(t) := Ct^{-\xi((p-1)(N+1)-N)}$ for $t \geqslant 1$ and $y(1) = \sigma(1)$, we conclude that

$$\int_{\{|x| \geqslant y(t)\}} U(t,x)\,dx \leqslant \int_{\{|x| \geqslant \sigma(1)\}} U(1,x)\,dx = 0$$

for $t \geqslant 1$. Consequently, $\sigma(t) \leqslant y(t)$ for $t \geqslant 1$ from which we deduce that $\varrho(t) \leqslant y(t)$ for $t \geqslant 1$ by (6.3). Now, either $q \in (1, p-1)$ and $\xi((p-1)(N+1)-N) > 1$. Therefore $y(t)$ has a finite limit as $t \to \infty$ from which (1.22) readily follows. Or $q = p-1$ and $y(t) = \sigma(1) + C \ln t$ which gives (1.23). □

We next consider the case $q \in (p-1, q_*)$ which turns out to be more complicated as (1.13) is no longer available. We instead use (1.11) which somehow provides less information and thus complicates the proof. We shall also need the following lemma which is an easy consequence of the Poincaré and Hölder inequalities.

Lemma 6.1. *There is a positive constant κ depending only on N and q such that, if $R > 0$ and w is a function in $W_0^{1,q}(B(0,R))$ then*

$$R^{-1/\xi} \|w\|_{L^1(B(0,R))}^q \leqslant \kappa \|\nabla w\|_{L^q(B(0,R))}^q. \tag{6.6}$$

Proof of Theorem 1.6: $q \in (p-1, q_*)$. We fix $t_0 \geqslant 0$. It follows from (1.11) and (1.16) that

$$\begin{aligned}
\frac{p-1}{p-2}\left|\nabla\left(U^{(p-2)/(p-1)}\right)(t,x)\right|^{p-1} &\leqslant C\left\|u\left(\frac{t+t_0}{2}\right)\right\|_\infty^{(p-2)/p}(t-t_0)^{-(p-1)/p} \\
&\leqslant C\left\|u(t_0)\right\|_1^{q\xi(p-2)/p}(t-t_0)^{-(p-1+N\xi(p-2))/p}
\end{aligned}$$

for $t \geqslant t_0$. Since $q > p-1 > N(p-1)/(N+1)$, we have $1 - N\xi(p-2) > 0$ and we choose $y(t) = \sigma(t_0) + pC\|u(t_0)\|_1^{q\xi(p-2)/p}(t-t_0)^{(1-N\xi(p-2))/p}/(1-N\xi(p-2))$ for $t \geqslant t_0$. The previous inequality then reads

$$\frac{p-1}{p-2}\left|\nabla\left(U^{(p-2)/(p-1)}\right)(t,x)\right|^{p-1} \leqslant y'(t), \quad t \geqslant t_0.$$

Combining the latter estimate with (6.5) we realize that

$$\frac{d}{dt}\int_{\{|x| \geqslant y(t)\}} U(t,x)\,dx \leqslant 0 \quad \text{for } t \geqslant t_0,$$

whence

$$\int_{\{|x| \geqslant y(t)\}} U(t,x)\,dx \leqslant \int_{\{|x| \geqslant \sigma(t_0)\}} U(t_0,x)\,dx = 0, \quad t \geqslant t_0.$$

We have thus established that $\sigma(t) \leqslant y(t)$ for $t \geqslant t_0$ from which we readily conclude that

$$\sigma(t) \leqslant \sigma(t_0) + C\left\|U(t_0)\right\|_1^{q\xi(p-2)/p}(t-t_0)^{(1-N\xi(p-2))/p}, \quad t \geqslant t_0. \tag{6.7}$$

We next integrate (1.1) over $\mathbb{R}^N$ and obtain

$$\frac{d}{dt}\left\|U(t)\right\|_1 + \left\|\nabla U(t)\right\|_q^q = 0.$$

Since the support of $U(t)$ is included in $B(0,\sigma(t))$, we infer from Lemma 6.1 that

$$\left\|\nabla U(t)\right\|_q^q = \int\limits_{\{|x|<\sigma(t)\}} \left|\nabla U(t,x)\right|^q dx \geqslant \frac{1}{\kappa\sigma(t)^{1/\xi}}\left(\int\limits_{\{|x|<\sigma(t)\}} U(t,x)\,dx\right)^q = \frac{\|U(t)\|_1^q}{\kappa\sigma(t)^{1/\xi}}.$$

Inserting this lower bound in the previous differential equality gives

$$\frac{d}{dt}\left\|U(t)\right\|_1 + \frac{1}{\kappa}\frac{\|U(t)\|_1^q}{\sigma(t)^{1/\xi}} \leqslant 0. \tag{6.8}$$

Before going on we introduce the following notations:

$$\Sigma(T) := \sup_{t\in[1,T]}\left\{t^{-A}\sigma(t)\right\}, \qquad A := \frac{q-p+1}{2q-p},$$
$$L(T) := \sup_{t\in[1,T]}\left\{t^{B}\left\|U(t)\right\|_1\right\}, \qquad B := \frac{(N+1)(q_*-q)}{2q-p},$$

for $T \geqslant 1$ and notice that $\Sigma(T)$ and $L(T)$ are well defined for each $T \geqslant 1$ while A and B satisfy

$$A + \frac{q\xi(p-2)}{p}B = \frac{1-N\xi(p-2)}{p} \quad \text{and} \quad 1 - \frac{A}{\xi} = (q-1)B. \tag{6.9}$$

Fix $T \geqslant 1$. We infer from (6.8) that, if $t \in [1,T]$,

$$\frac{d}{dt}\left\|U(t)\right\|_1 + \frac{t^{-A/\xi}}{\kappa}\frac{\|U(t)\|_1^q}{t^{-A/\xi}\sigma(t)^{1/\xi}} \leqslant 0,$$
$$\frac{d}{dt}\left\|U(t)\right\|_1 + \frac{1}{\kappa\Sigma(T)^{1/\xi}}\frac{\|U(t)\|_1^q}{t^{A/\xi}} \leqslant 0,$$

which gives

$$\left\|U(t)\right\|_1 \leqslant C\Sigma(T)^{1/((q-1)\xi)}\left(t^{(q-1)B}-1\right)^{-1/(q-1)}, \quad t \in [1,T], \tag{6.10}$$

after integration. Consider next $t \in [1,T]$. Either $t \leqslant 4$ and it follows from (6.7) with $t_0 = 1$ that

$$t^{-A}\sigma(t) \leqslant t^{-A}\sigma(1) + C\left\|U(1)\right\|_1^{q\xi(p-2)/p}(t-1)^{(1-N\xi(p-2))/p}t^{-A} \leqslant C.$$

Or $t \geqslant 4$ and we infer from (6.7) with $t_0 = t/2 \geqslant 2$, (6.9) and (6.10) that

$$t^{-A}\sigma(t) \leqslant t^{-A}\sigma\left(\frac{t}{2}\right) + C\left\|U\left(\frac{t}{2}\right)\right\|_1^{q\xi(p-2)/p} t^{q\xi(p-2)B/p}$$
$$\leqslant 2^{-A}\Sigma(T) + C\Sigma(T)^{(q(p-2))/(p(q-1))}.$$

Consequently,

$$t^{-A}\sigma(t) \leqslant 2^{-A}\Sigma(T) + C\big(1 + \Sigma(T)^{(q(p-2))/(p(q-1))}\big), \quad t \in [1, T],$$

from which we conclude that

$$\Sigma(T) \leqslant 2^{-A}\Sigma(T) + C\big(1 + \Sigma(T)^{(q(p-2))/(p(q-1))}\big).$$

Since $A > 0$ and $q(p-2) < p(q-1)$ the above inequality entails that $\Sigma(T) \leqslant C$ for each $T \geqslant 1$, the constant C being independent of T. Recalling (6.4) we have thus proved that $\varrho(t) \leqslant \sigma(t) \leqslant Ct^A$ for $t \geqslant 1$, hence (1.24).

Furthermore the boundedness of $\Sigma(T)$ and (6.10) ensure that $\|U(t)\|_1 \leqslant C(t-1)^{-B}$ for $t \geqslant 1$ which, together with (6.3), implies that

$$\big\|u(t)\big\|_1 \leqslant Ct^{-B}, \quad t \geqslant 2. \tag{6.11}$$

We have thus also established the assertion (iii) of Corollary 1.7. $\square$

Proof of Corollary 1.7. Assume first that $q \in (1, p-1)$. Then, on the one hand, it follows from (1.22) that there is $\varrho_\infty > 0$ such that $\varrho(t) \leqslant \varrho_\infty$ for $t \geqslant 1$. On the other hand, we may proceed as in the proof of (6.8) to establish that

$$\frac{d}{dt}\big\|u(t)\big\|_1 + \frac{1}{\kappa}\frac{\|u(t)\|_1^q}{\varrho(t)^{1/\xi}} \leqslant 0. \tag{6.12}$$

Therefore,

$$\frac{d}{dt}\big\|u(t)\big\|_1 + \frac{1}{\kappa}\frac{\|u(t)\|_1^q}{\varrho_\infty^{1/\xi}} \leqslant 0, \quad t \geqslant 1,$$

from which (1.26) readily follows.

Similarly, if $q = p-1$, we infer from (1.23) and (6.12) that, for $t \geqslant 2$,

$$\big\|u(t)\big\|_1 \leqslant C\left(\int_1^t (1+\ln s)^{-1/\xi}\,ds\right)^{-1/(q-1)}$$
$$\leqslant C\left(\int_0^{\ln t} (1+s)^{-1/\xi}e^s\,ds\right)^{-1/(q-1)}$$
$$\leqslant C\big((1+\ln t)^{-1/\xi}(t-1)\big)^{-1/(q-1)},$$

which gives (1.27).

Since the case $q \in (p-1, q_*)$ has already been handled in the proof of Theorem 1.6 (recall (6.11)) we are left with the case $q = q_*$. In that particular case, $\xi = \eta$ and we infer from (1.25) and (6.12) that

$$\frac{d}{dt}\left\|u(t)\right\|_1 + \frac{C}{t}\left\|u(t)\right\|_1^q \leqslant 0, \quad t \geqslant 1,$$

which gives (1.29) by integration. □

7. Persistence of dead cores

Proof of Proposition 1.8. We first study the one-dimensional case $N = 1$. We consider a non-negative function $y \in \mathcal{C}^1([0,\infty))$ to be specified later and proceed as in the proof of Theorem 1.6 to deduce from (1.1) that

$$\frac{d}{dt}\int_{-y(t)}^{y(t)} u(t,x)\,dx = \left[\left(\frac{p-1}{p-2}\left|\partial_x\left(u^{(p-2)/(p-1)}\right)(t,x)\right|^{p-1} + y'(t)\right)u(t,x)\right]_{x=-y(t)}^{x=y(t)}. \tag{7.13}$$

On the one hand, we infer from (1.6) that

$$\begin{aligned}\frac{p-1}{p-2}\left|\partial_x\left(u^{(p-2)/(p-1)}\right)(t,x)\right|^{p-1} &\leqslant \frac{p-1}{p-2}C(p,1)^{p-1}\|u_0\|_\infty^{(p-2)/p}t^{-(p-1)/p}\\ &\leqslant c_1\|u_0\|_\infty^{(p-2)/p}t^{-(p-1)/p}.\end{aligned}$$

On the other hand, since $p-1 > q$, we have $\beta_{p,q} = \alpha_p = (p-2)/(p-1)$ and it follows from (1.7) that

$$\begin{aligned}\frac{p-1}{p-2}\left|\partial_x\left(u^{(p-2)/(p-1)}\right)(t,x)\right|^{p-1} &\leqslant \frac{p-1}{p-2}C(p,q,1)^{p-1}\|u_0\|_\infty^{(p-1-q)/q}t^{-(p-1)/q}\\ &\leqslant c_2\|u_0\|_\infty^{(p-1-q)/q}t^{-(p-1)/q}.\end{aligned}$$

Consequently, choosing

$$\begin{cases} y'(t) = -\min\left\{c_1\|u_0\|_\infty^{p-2}t^{-(p-1)/p},\ c_2\|u_0\|_\infty^{(p-1-q)/q}t^{-(p-1)/q}\right\},\\ y(0) = R_0,\end{cases} \tag{7.14}$$

we have

$$\frac{p-1}{p-2}\left|\partial_x\left(u^{(p-2)/(p-1)}\right)(t,x)\right|^{p-1} \leqslant -y'(t). \tag{7.15}$$

We then deduce from (7.13) and (7.15) that

$$\frac{d}{dt}\int_{-y(t)}^{y(t)} u(t,x)\,dx \leqslant 0,$$

whence

$$\int_{-y(t)}^{y(t)} u(t,x)\,dx \leqslant \int_{-R_0}^{R_0} u_0(x)\,dx = 0 \quad \text{for } t \geqslant 0.$$

Now it is actually possible to compute the function y defined by (7.14) and to see that

$$y(t) \geqslant y_\infty := \lim_{s\to\infty} y(s) = R_0 - \delta_0 \|u_0\|_\infty^{(p-1-q)/(p-q)}$$

for some δ_0 depending only on c_1, c_2, p, and q. Then $u(t,x) = 0$ for $x \in [-y_\infty, y_\infty]$ and $t \geqslant 0$, and $y_\infty > 0$ under the assumptions of Proposition 1.8.

In several space dimensions $N \geqslant 2$, consider $\varepsilon \in (0, R_0/2)$ and put

$$u_0^\varepsilon(x_1) := \begin{cases} \|u_0\|_\infty & \text{if } |x_1| \geqslant R_0, \\ \dfrac{\|u_0\|_\infty}{\varepsilon}\big(|x_1| - R_0 + \varepsilon\big) & \text{if } R_0 - \varepsilon \leqslant |x_1| \leqslant R_0, \\ 0 & \text{if } |x_1| \leqslant R_0 - \varepsilon, \end{cases}$$

Clearly, $u_0 \leqslant u_0^\varepsilon$ in $\mathbb{R}^N$ and the comparison principle entails that $u(t, x_1, x_2, \ldots, x_N) \leqslant u^\varepsilon(t, x_1)$ for $(t,x) \in [0,\infty) \times \mathbb{R}^N$, where u^ε denotes the solution to (1.1) with initial condition u_0^ε and $N = 1$. Choosing ε appropriately small provides the expected result in the x_1-direction. We proceed analogously in every direction to complete the proof of Proposition 1.8. □

Appendix A. Proof of Lemma 2.1

Since $\partial_t u = \varphi'(v)\partial_t v$ and $\nabla u = \varphi'(v)\nabla v$ we deduce from (2.1) that

$$\partial_t v - a\Delta v - a\frac{\varphi''}{\varphi'} w - 2a'\varphi'\varphi'' w^2 - 2a'\varphi'^2 (\nabla v)^t D^2 v \nabla v + \frac{b'}{\varphi'} = 0.$$

Observing that

$$(\nabla v)^t D^2 v \nabla v = \frac{1}{2}\nabla v \cdot \nabla w \quad \text{and} \quad \Delta w = 2\nabla v \cdot \nabla \Delta v + 2\sum_{i,j} |\partial_i \partial_j v|^2,$$

elementary, but laborious calculation shows that

$$\partial_t w - \mathcal{A} w + 2a\sum_{i,j} |\partial_i \partial_j v|^2 + 2a'\varphi'\varphi'' w \nabla v \cdot \nabla w - \mathcal{V} \cdot \nabla w + 2\mathcal{S}_1 w^2 + 2\mathcal{R}_2 w = 0$$

with

$$\mathcal{S}_1 := -a\left(\frac{\varphi''}{\varphi'}\right)' - 2a'\varphi'\varphi''\Delta v - 4a''(\varphi'\varphi'')^2 w^2 - 2a' w\big(2\varphi''^2 + \varphi'\varphi'''\big), \tag{A.1}$$

and

$$\begin{aligned}\mathcal{V} := {}& 2\left[a\frac{\varphi''}{\varphi'} + a'\varphi'^2\left(\Delta v + \frac{2\varphi''}{\varphi'} w\right)\right]\nabla v \\ & + 4\varphi'\varphi''\big[\big(a''\varphi'^2 w + 3a'\big) + a''\varphi'^2 w\big] w\nabla v \\ & + 2\big[a''\varphi'^4 \nabla v\cdot\nabla w - b'\varphi'\big]\nabla v + a'\varphi'^2\nabla w. \end{aligned} \tag{A.2}$$

In order to handle the term involving Δv in $\mathcal{S}_1$ we proceed as in [10]: more precisely we have

$$\begin{aligned} & 2a\sum_{i,j}|\partial_i\partial_j v|^2 + 2a'\varphi'\varphi'' w\nabla v\cdot\nabla w - 4a'\varphi'\varphi''\Delta v w^2 \\ & = 4a'\varphi'\varphi'' w\left(\frac{1}{2}\nabla v\cdot\nabla w - w\Delta v\right) + 2a\sum_{i,j}|\partial_i\partial_j v|^2 \\ & = 4a'\varphi'\varphi'' w\left(\sum_{i,j}\partial_i\partial_j v\partial_i v\partial_j v - w\sum_i \partial_i^2 v\right) + 2a\sum_{i,j}|\partial_i\partial_j v|^2 \\ & = \sum_i\big\{2a\big|\partial_i^2 v\big|^2 + 4a'\varphi'\varphi'' w\big(|\partial_i v|^2 - w\big)\partial_i^2 v\big\} \\ & \quad + \sum_{i\neq j}\big\{2a|\partial_i\partial_j v|^2 + 4a'\varphi'\varphi'' w\partial_i\partial_j v\partial_i v\partial_j v\big\} \\ & = 2a\sum_i\left\{\partial_i^2 v + \frac{a'}{a}\varphi'\varphi'' w\big(|\partial_i v|^2 - w\big)\right\}^2 \\ & \quad - 2\sum_i\frac{a'^2}{a}(\varphi'\varphi'')^2 w^2\big(|\partial_i v|^2 - w\big)^2 \\ & \quad + 2a\sum_{i\neq j}\left\{\partial_i\partial_j v + \frac{a'}{a}\varphi'\varphi'' w\partial_i v\partial_j v\right\}^2 \\ & \quad - 2\sum_{i\neq j}\frac{a'^2}{a}(\varphi'\varphi'')^2 w^2|\partial_i v|^2|\partial_j v|^2 \\ & \geqslant -2(N-1)\frac{a'^2}{a}(\varphi'\varphi'')^2 w^2. \end{aligned}$$

Consequently,

$$2a\sum_{i,j}|\partial_i\partial_j v|^2 + 2a'\varphi'\varphi'' w\nabla v\cdot\nabla w + 2\mathcal{S}_1 w^2 \geqslant 2\mathcal{R}_1 w^2,$$

which completes the proof of the first assertion of Lemma 2.1.

In the case where $x \mapsto u(t,x)$ is radially symmetric and non-increasing for each $t \geqslant 0$, we have $u(t,x) = U(t,|x|)$ for $(t,x) \in [0,\infty) \times \mathbb{R}^N$ and $\partial_r U(t,r) \leqslant 0$ for $(t,r) \in [0,\infty) \times [0,\infty)$. Introducing $V = \varphi^{-1}(U)$ we have $v(t,x) = V(t,|x|)$ and the monotonicity of φ warrants that $\partial_r V(t,r) \leqslant 0$. In addition, owing to the non-negativity of a', φ' and φ'', we have

$$\begin{aligned}
&2a'\varphi'\varphi'' w \nabla v \cdot \nabla w - 4a'\varphi'\varphi'' w^2 \Delta v \\
&\quad = 2a'\varphi'\varphi'' w \left[2|\partial_r V|^2 \partial_r^2 V - 2|\partial_r V|^2 \left(\partial_r^2 V + \frac{N-1}{r} \partial_r V \right) \right] \\
&\quad \geqslant 0,
\end{aligned}$$

from which we deduce that

$$2a'\varphi'\varphi'' w \nabla v \cdot \nabla w + 2\mathcal{S}_1 w^2 \geqslant 2\mathcal{R}_1^{\mathrm{r}} w^2,$$

and end the proof of Lemma 2.1. $\square$

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The classification of separable simple C^*-algebras which are inductive limits of continuous-trace C^*-algebras with spectrum homeomorphic to the closed interval [0, 1]

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Abstract

A classification is given of certain separable nuclear C^*-algebras not necessarily of real rank zero, namely, the class of separable simple C^*-algebras which are inductive limits of continuous-trace C^*-algebras whose building blocks have spectrum homeomorphic to the closed interval [0, 1], or to a disjoint union of copies of this space. Also, the range of the invariant is calculated.

Keywords: K-theory; Classification; C^*-algebras; Inductive limits; Real rank one

1. Introduction

It is shown in [10] that an important class of separable simple crossed product C^*-algebras are approximately subhomogeneous. Recall that a C^*-algebra is said to be subhomogeneous if it is isomorphic to a sub-C^*-algebra of $M_n(C_0(X))$ for some natural number n and for some locally

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compact Hausdorff space X. An approximately subhomogeneous C^*-algebra, abbreviated ASH algebra, is an inductive limit of subhomogeneous algebras.

This article contains a partial result in the direction of classifying all simple ASH algebras by their Elliott invariant.

The first result on the classification of C^*-algebras not of real rank zero was the classification by G. Elliott of unital simple approximate interval algebras, abbreviated AI algebras (see [4]). This result was extended to the non-unital case independently by I. Stevens [14] and K. Thomsen [18]. Also, an interesting partial extension of this result to the non-simple case was given by K. Stevens [15]. It is worth mentioning that all these algebras are what are referred to as approximately homogeneous algebras, abbreviated AH algebras, and that the most general classification result for simple AH algebras was obtained by Elliott, Gong and Li in [5].

One of the first isomorphism results for ASH algebras was the proof by H. Su of the classification of C^*-algebras of real rank zero which are inductive limits of matrix algebras over non-Hausdorff graphs; see [17]. The classification of ASH algebras was also considered in [8,12,13]. (This list of contributions is intended to be representative rather than complete for the classification of ASH algebras.)

An important work on the classification of ASH algebras not of real rank zero, and in fact one of the first ones, is due to I. Stevens [16]. The main result of the present paper is a substantial extension of Stevens's work, to the class consisting of all simple C^*-algebras which are inductive limits of continuous-trace C^*-algebras with spectrum homeomorphic to the closed interval [0, 1] (or to a finite disjoint union of closed intervals). In particular, the spectra of the building blocks considered here are the same as for those considered by Stevens. The building blocks themselves are more general.

The isomorphism theorem is proved by applying the Elliott intertwining argument.

Inspired by I. Stevens's work, the proof proceeds by showing an Existence theorem and a Uniqueness theorem for certain special continuous trace C^*-algebras. (As can be seen from the proofs, it is convenient to have a special kind of continuous trace C^*-algebra as the domain algebra in both these theorems. By special we mean having finite-dimensional irreducible representations and such that the dimension of the representation, as a function on the interval, is a finite (lower semicontinuous) step function.)

The present Existence theorem, theorem 5.1, differs in an important way from that of [16, Theorem 29.4.1]. In fact Theorem 29.4.1 of [16] is false, as is shown in Section 5.1 below.

The proof of the present Existence theorem is an eigenvalue pattern perturbation, as shown in Section 5, which is similar to the approach used in [16]. (Indeed, once the statement of [16, Theorem 29.4.1] is corrected, the argument given in [16] does not need to be essentially changed.)

The proof of the present Uniqueness theorem is different from the one in [16]. It uses the finite presentation of special continuous trace C^*-algebras that was given in [6] and [7]. Also the present Uniqueness theorem has the advantage that both the statement and the proof are intrinsic, i.e., there is no need to say that the building blocks are hereditary sub-C^*-algebras of interval algebras as in [16].

In order to apply the Existence and Uniqueness theorems, it is necessary to approximate the general continuous trace C^*-algebras appearing in a given inductive limit decomposition by special continuous trace C^*-algebras, as described in [7, Theorem 4.15]. This is admissible since in [7] (and also more generally in [6]), it is shown that these special C^*-algebras are weakly semiprojective, i.e., have stable relations. (A result of T. Loring [11, Lemma 15.2.2], allows one to conclude that the original inductive limit decomposition can be replaced by an inductive limit of special continuous trace C^*-algebras.)

An important step of the proof of the isomorphism theorem is the pulling back of the invariant from the inductive limit to the finite stages. The invariant has roughly two major components: a stable part and a non-stable part. The pulling back of the stable part is contained in [4] or [16] and is performed in the present situation with respect to the unital hereditary sub-C^*-algebras. The intertwining which is obtained at the level of the stable invariant will approximately respect the non-stable part of the invariant on finitely many elements, as pointed out in [16]. To be able to apply the Existence theorem it is crucial to ensure that the non-stable part of the invariant is exactly preserved on finitely many elements (actually, just a single element). It is possible to obtain an exact preservation of the non-stable invariant on finitely many elements because one can change the given finite stage algebras in the inductive limit decomposition in such a way that a non-zero gap arises at the level of the affine function spaces; see Section 8 below. It is this non-zero gap that will ultimately guarantee (after passing to subsequences in a convenient way) the exact intertwining on finite sets of the non-stable invariant, as shown in Section 9. It is worth mentioning that in the pulling back of the stable invariant, we must ensure, at the same time that the maps at the affine function space level are given by eigenvalue patterns. This is necessary in order to apply the Existence theorem and is possible by the Thomsen–Li theorem.

Now all the hypotheses of the Elliott intertwining argument are fulfilled and in this way the proof of the isomorphism Theorem 3.1 is completed.

I. Stevens's description of the range of the invariant is also extended to include the case of unbounded traces (Theorem 3.2).

To conclude, the class of simple inductive limits of continuous-trace C^*-algebras under consideration is compared with the class of simple AI algebras.

2. The invariant

The invariant is similar to the invariant I. Stevens has used in [16], usually summed up as the Elliott invariant, namely, $(K_0(A), \operatorname{Aff} T^+A, \operatorname{Aff}' A)$, where $K_0(A)$ is a partially ordered abelian group, $\operatorname{Aff} T^+A$ is a partially ordered vector space consisting of linear and continuous functions defined on the cone of traces T^+A, $\operatorname{Aff}' A$ is a certain special subset of $\operatorname{Aff} T^+A$. The special subset $\operatorname{Aff}' A$ is the most important part of the invariant for our purposes, and in an informal way it might be said to be the non-stable part of the $\operatorname{Aff} T^+A$. Formally, the special subset $\operatorname{Aff}' A$ is the convex set obtained as the closure of $\{\hat{a} \in \operatorname{Aff} T^+A \mid a \geqslant 0,\ a \in \operatorname{Ped}(A) \text{ and } \|a\| \leqslant 1\}$ inside $\operatorname{Aff} T^+A$, with respect to the topology naturally associated to a full projection. Here $\hat{a}$ is the linear and continuous function defined by the positive element a from the Pedersen ideal by $\hat{a}(\tau) = \tau(a)$ where $\tau \in T^+A$. As shown in [16, Remarks 30.1.1 and 30.1.2], the information given by $\operatorname{Aff}' A$ is equivalent with that given by the trace-norm map, which is a lower semicontinuous function $\mu : T^+A \to \mathbb{R}$, $\mu(\tau) = \|\tau\|$ and ∞ if τ is unbounded.

It is a crucial fact that the trace-norm map is equivalent to the dimension function in the case of a building block algebra, cf. Section 4 below. The dimension function of a building block (i.e. the function that assigns to each point in the spectrum of the building block the dimension of the irreducible representation) can be viewed as a lower semicontinuous function on the extreme traces normalized on minimal projections in primitive quotients and hence we can compare it with functions from $\operatorname{Aff} T^+A$. Then the subset $\operatorname{Aff}' A$ is the closure of the set of all affine functions smaller than the dimension function. Conversely by taking the supremum over all elements of $\operatorname{Aff}' A$ we recover the dimension function in the case of the building blocks.

3. The results

Using the invariant described above it is possible to prove a complete isomorphism theorem. Namely,

Theorem 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two non-unital simple C^*-algebras which are inductive limits of continuous-trace C^*-algebras with spectrum homeomorphic to* $[0, 1]$. *Assume that*:

1. *There is a order preserving isomorphism $\psi_0 : K_0(\mathcal{A}) \to K_0(\mathcal{B})$.*
2. *There is an isomorphism $\psi_T : \operatorname{Aff} T^+\mathcal{A} \to \operatorname{Aff} T^+\mathcal{B}$, such that*

$$\psi_T(\operatorname{Aff}' \mathcal{A}) = \operatorname{Aff}' \mathcal{A}.$$

3. *The two isomorphisms are compatible*:

$$\widehat{\psi_0([p])} = \psi_T\big(\widehat{[p]}\big), \quad [p] \in K_0(\mathcal{A}).$$

Then there is an isomorphism of the algebras $\mathcal{A}$ and $\mathcal{B}$ that induces the given isomorphism at the level of the invariant.

A description is given of the range of the invariant. More precisely, the following theorem is proved.

Theorem 3.2. *Suppose that G is a simple countable dimension group and V is the cone associated to a metrizable Choquet simplex. Let $\lambda : V \to \operatorname{Hom}^+(G, R)$ be a continuous affine map which takes extreme rays into extreme rays. Let $f : V \to [0, +\infty]$ be an affine lower semicontinuous map, zero at zero and only at zero. Then (G, V, λ, f) is the invariant of some simple non-unital inductive limit of continuous-trace C^*-algebras whose spectrum is the closed interval* $[0, 1]$.

4. Special continuous trace C^*-algebras with spectrum the interval [0, 1]

In this section we will introduce some terminology. A very important piece of data that we shall consider is a map that assigns, to each class of irreducible representations, the dimension of a representation from that class. Roughly speaking, the dimension function can be thought of as the non-stable part of the invariant when restricted to the building blocks.

Definition 4.1. Let A be a C^*-algebra and let $\hat{A}$ denote the spectrum of A. Then the *dimension function* is the map from $\hat{A}$ to $\mathbb{R} \cup +\infty$,

$$\pi \mapsto \dim(H_\pi),$$

where by $\dim(H_\pi)$ we mean the dimension of the irreducible representation π.

It was shown in [7, Theorem 4.13], that the dimension function is a complete invariant for continuous trace C^*-algebras with spectrum the closed interval [0, 1]. Also concrete examples were constructed for each given dimension function, cf. [7, Section 7].

Therefore given a lower semicontinuous integer valued (i.e., a "dimension function") which is finite-valued and bounded we can exhibit a continuous trace C^*-algebra

$$\begin{pmatrix} C_0(A_n) & C_0(A_n) & C_0(A_n) & \dots & C_0(A_n) \\ C_0(A_n) & C_0(A_{n-1}) & C_0(A_{n-1}) & \dots & C_0(A_{n-1}) \\ C_0(A_n) & C_0(A_{n-1}) & C_0(A_{n-2}) & \dots & C_0(A_{n-2}) \\ \vdots & \vdots & \vdots & \ddots & \\ C_0(A_n) & C_0(A_{n-1}) & C_0(A_{n-2}) & \dots & C[0,1] \end{pmatrix} \subseteq M_n \otimes C[0,1].$$

whose dimension function is the given function. Here $A_n \subseteq A_{n-1} \subseteq \cdots \subseteq [0,1]$ and each A_i is an open subset of $[0,1]$. Moreover any trace on such an algebra is of the form $\mathrm{tr} \otimes \nu$, where tr is the usual trace normalized on minimal matrix projections and ν is a finite measure on $[0,1]$. The extreme traces are parameterized by $t \in [0,1]$, and are given as $(\mathrm{tr} \otimes \delta_t)_{t\in[0,1]}$, where δ_t is the normalized point mass at t. Then the trace norm map is equal to the dimension function when restricted to the extreme traces. To see that the trace norm map is equivalent to the special subset $\mathrm{Aff}'()$ of the affine function space $\mathrm{Aff}\, T^+()$ we repeat the proof of I. Stevens from [16, Remarks 30.1.1 and 30.1.2].

Inspired by a construction of I. Stevens in [16] we make

Definition 4.2. A continuous-trace C^*-algebra whose spectrum is $[0,1]$ will be called a special continuous-trace C^*-algebra if its dimension function is a finite-valued *finite step* function: there is a partition of $[0,1]$ into a finite union of intervals such that the dimension function is finite and constant on each such subinterval.

Remark 4.1. Let A be a continuous trace C^*-algebra with spectrum $[0,1]$ and with dimension function $d : [0,1] \to \mathbb{N} \cup \{+\infty\}$. There exists a projection-valued function that if composed with the rank function gives rise to the dimension function d. To see this first we notice that because the Dixmier–Douady invariant of A is trivial, the C^*-algebra A is a continuous field of elementary C^*-algebras over $[0,1]$, where the fibers are hereditary sub-C^*-algebras of the algebra of compact operators. Then take the unit of the hereditary sub-C^*-algebra in each fiber. In this way we construct a projection-valued function which is lower semicontinuous. By composing this constructed projection-valued function with the rank function we get the dimension function d.

Remark 4.2. A priori our definition for a special sub-C^*-algebra is more general than I. Stevens's definition. As it is shown in [7], any special sub-C^*-algebra in our sense is isomorphic to a special sub-C^*-algebra in I. Stevens's sense.

Remark 4.3. It was shown in [7] that special continuous trace C^*-algebras are finite presented and weakly semiprojective. Also a stronger result was proven in [2], namely that special continuous trace C^*-algebras are strongly semiprojective.

5. Balanced inequalities and the Existence theorem

The proof of the isomorphism theorem 3.1 is based on the Elliott intertwining argument. Among the main ingredients of this procedure are the Existence theorem that will be described below as well as the Uniqueness theorem that is presented in Section 6.

It is worth noticing that for the Existence theorem and the Uniqueness theorem we require that the inequalities are balanced, i.e., independent of the choice we make for the normalization of the affine function space. We normalize the affine function spaces with respect to a full projection. Even though we fix a projection in the domain algebra for both the Existence theorem and the Uniqueness theorem, this choice does not make any difference when we apply the theorems to obtain an approximate commuting diagram. As was pointed out to us by Andrew Toms, we only need to consider a compatible family of projections when we go through the whole proof, provided that a corresponding projection is chosen in the codomain algebra. In fact, we can state the theorems without mentioning the choices of the projections as long as their K_0-classes are compatible with respect to the K_0-map under consideration even though they exist and some choices of them will be used during the proof.

To be able to focus on the new aspects of the present Existence theorem as opposed to the Existence theorem for unital continuous trace C^*-algebras proved by Elliott in [4], we will both state the theorem and prove it in terms of so-called eigenvalue pattern maps. In our situation an eigenvalue pattern map is a positive unital map from $C([0,1])$ to $C([0,1])$ which is a finite sum of $*$-homomorphisms from $C([0,1])$ to $C([0,1])$. Using the Gelfand theory each such $*$-homomorphism is given by a continuous function from $[0,1]$ to $[0,1]$. As follows from the intertwining of the invariant and will be explained below, Section 9, one can always obtain a (non-necessarily compatible) eigenvalue patterns maps.

The proof of the Existence theorem is obtained by perturbing an eigenvalue pattern map between the affine function spaces in a such a way that it defines an algebra map between the building blocks.

Theorem 5.1. *Let A be a special building block and by d_A denote the dimension function of A. Let a finite subset F contained in $\operatorname{Aff} T^+A$, and $\epsilon > 0$ be given. There is $f' \in \operatorname{Aff}' A$ such that for any special building block B with dimension function d_B, and maps $k : D(A) \to D(B)$ and $T : \operatorname{Aff} T^+A \to \operatorname{Aff} T^+B$ verifying the conditions:*

1. *k has multiplicity M_k.*
2. *T is given by an eigenvalue pattern and has the property*
$$T(f') \leqslant d_B.$$
3. *k and T are exactly compatible, i.e.,*
$$\widehat{k([r])} = T([\hat{r}]),$$

there is a homomorphism $\psi : A \to B$ such that $k = \psi_0$ and
$$\left\|(T - \psi_T)a\right\|_{\widehat{k(p)}} \leqslant \epsilon \|a\|_{\hat{p}}, \quad a \in F.$$

Remark 5.2. Recall that $\operatorname{Aff} T^+A$ is a Banach space with a norm given by $\|f\|_p = \sup\{|f(\tau)| \mid \tau(p) = 1,\ \tau \in T^+A\}$, where $f \in \operatorname{Aff} T^+A$ and p is a fixed full projection of A. In addition, using the norm we just defined, $\operatorname{Aff} T^+A$ is identified with $C([0,1])$. This identification allows us to compare in the supremum norm the dimension function and elements of $\operatorname{Aff} T^+A$. Also the norm of $\operatorname{Aff} T^+B$ is defined with respect to a projection from B which is Murray–von Neumann equivalent to $k(p)$. Since our inequalities at the level of the affine function spaces are balanced,

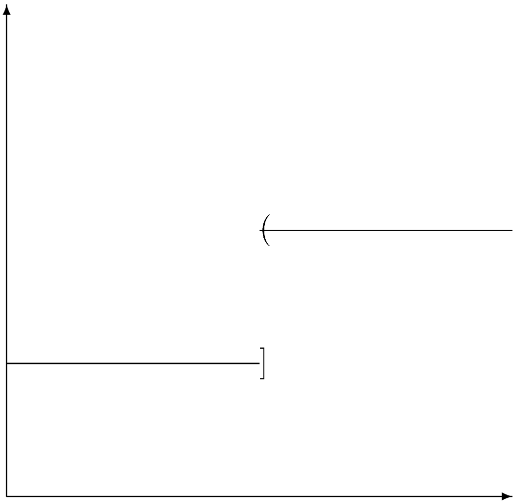

Fig. 1. Dimension function d_A.

which is the only theorem that makes sense, in particular they are independent of the choice of the projection p.

Proof. The idea of the proof is to choose in a clever way a function f' and then change within the given tolerance the eigenvalue functions that appear in the eigenvalue pattern T so that the image of the dimension function d_A under the new eigenvalue pattern is smaller than or equal the dimension function of the algebra B, as desired.

Let $\epsilon > 0$ and a finite set $F \subset \operatorname{Aff} T^+A$ be given. As already mentioned it is a crucial step how f' is chosen. There is no loss in generality if we assume that the dimension function d_A has only one discontinuity point, $t_0 \in [0, 1]$ (see Fig. 1).

Choose f' to be a continuous function such that $f'(t) = d_A(t)$ for $t \in [0, t_0 - \delta] \cup [t_0 + \delta, 1]$, $f'(t) \leqslant d_A(t)$ for $t \in [0, 1]$, and $f'(t_0) = d_A(t_0)$, where $\delta \leqslant \frac{\epsilon}{2M_k^2}$. Hence f' is a continuous function defined on the interval $[0, 1]$ which approximates d_A, namely f' is equal to d_A except on a small neighbourhood around the discontinuity point (see Fig. 2).

Next we proceed by showing how to change the eigenfunctions such that a desired eigenvalue pattern is obtained. We will carry out this procedure in a very special case, namely all the eigenfunctions are assumed to be the identity function.

In Fig. 3 we have the original eigenvalue function λ which is the identity map. We define a new eigenvalue function, see Fig. 4. More precisely the new eigenvalue function $\hat{\lambda} : [0, 1] \to [0, 1]$, $\hat{\lambda}(t) = t$ for $t \in [0, t_0 - \delta) \cup (t_0 + \delta + \delta_{t_0}, 1]$, $\hat{\lambda}(t) = t_0 - \delta$ for $t \in [t_0 - \delta, t_0 + \delta]$, the linear map $\hat{\lambda}(t) = t_0 - \delta + (t - t_0 - \delta)\frac{2\delta + \delta_{t_0}}{\delta_{t_0}}$ for $t \in [t_0 + \delta, t_0 + \delta + \delta_\rho]$, where δ_{t_0} is a strictly positive number such that $t_0 + \delta + \delta_{t_0} \leqslant 1$.

A short computation or a geometric argument shows that the difference $\|\lambda - \hat{\lambda}\|_\infty = 2\delta$.

Moreover the dimension function d_A evaluated on the perturbed eigenvalue $\hat{\lambda}$ is smaller then f' evaluated on the given eigenvalue λ

$$d_A\big(\hat{\lambda}(t)\big) \leqslant f'\big(\lambda(t)\big).$$

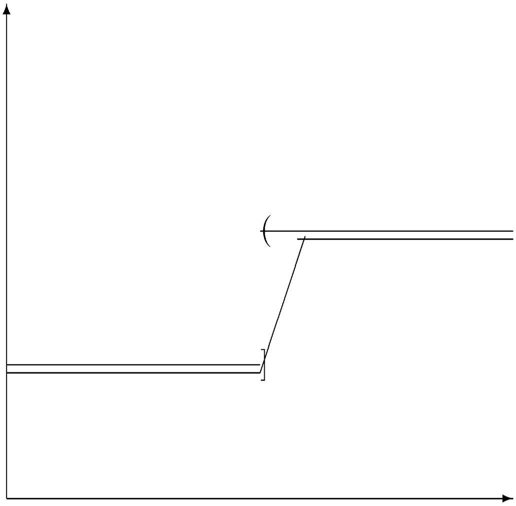

Fig. 2. Graph of f'.

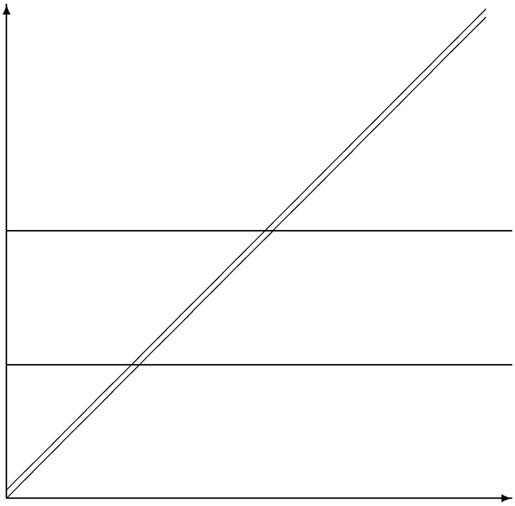

Fig. 3. Eigenfunction λ.

Hence by hypothesis 2 we have

$$\sum_{i=1}^{M_k} d_A \circ \hat{\lambda} \leqslant \sum_{i=1}^{M_k} f' \circ \lambda \leqslant d_B.$$

Here we say that one-dimension function is smaller than another one if the relation holds pointwise.

The change of the eigenvalues is small because of the choice of δ:

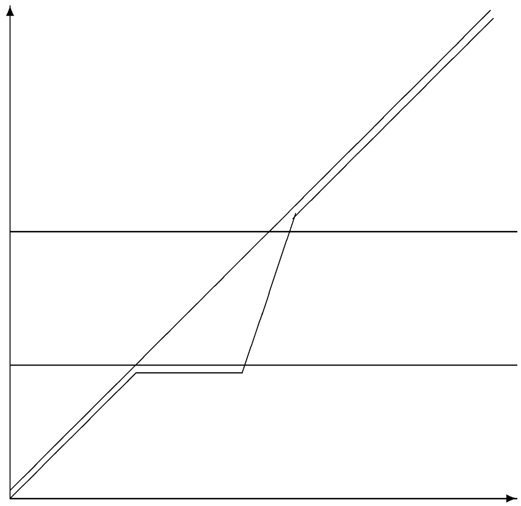

Fig. 4. Eigenfunction $\hat{\lambda}$.

$$
\begin{aligned}
\big\|(T_\lambda - T)(a)\big\|_{\widehat{k(p)}} &= \sum_{i=1}^{M_k} \big\|a \circ (\widehat{\lambda_i} - \lambda_i)\big\|_{\widehat{k(p)}} \\
&= \sum_{i=1}^{M_k} \sup\big\{\big|a \circ (\widehat{\lambda_i} - \lambda_i)(\tau)\big| \;\big|\; \tau\big(k(p)\big) = 1,\ \tau \in T^+A\big\} \\
&= \sum_{i=1}^{M_k} \sup\left\{M_k \left|a \circ (\widehat{\lambda_i} - \lambda_i)\left(\frac{1}{M_k}\tau\right)\right| \;\Big|\; \tau(p) = M_k,\ \tau \in T^+A\right\} \\
&= \sum_{i=1}^{M_k} M_k \big\|a \circ (\widehat{\lambda_i} - \lambda_i)\big\|_{\widehat{p}} \leqslant 2\delta M_k^2 \|a\|_{\widehat{p}} \leqslant \epsilon \|a\|_{\widehat{p}}, \quad a \in F.
\end{aligned}
$$

To obtain the inequality above we used the linearity of the function $a \circ (\widehat{\lambda_i} - \lambda_i)$ and that an extreme trace τ in T^+A has the property that $\tau(k(p)) = 1$ if and only if $\tau(p) = M_k$.

We claim that the argument for the special case shown above can be extended to the case of piecewise linear eigenfunctions which is known to be equivalent to the general case of continuous eigenfunctions that arise in the inductive limits of interval algebras (see for instance [4]). □

5.1. *An exact inequality is necessary between the non-stable part of the invariant*

As mentioned in the introduction, Theorem 29.4.1 of [16] is false. To prove the Existence theorem it is fundamental to have an exact inequality between the non-stable part of the invariant at the level of the affine function space, i.e., $T(f) \leqslant d_B$ for some continuous affine function $f \leqslant d_A$. A weaker inequality is required in the statement of the Existence theorem of [16, Theorem 29.4.1], i.e., $T(f) \leqslant d_B(1 + \delta)$ for some small $\delta > 0$. Therefore it is possible to construct a counterexample to the I. Stevens Existence theorem. This counterexample is already assuming that the positive linear map T is given by an eigenvalue pattern. To reduce the proof of

[16, Theorem 29.4.1] to an eigenvalue pattern problem, one needs an extra assumption in hypothesis 2, for instance a positive gap $\eta > 0$ in the other side of the inequality described above $T(f) + \eta \leqslant d_B(1+\delta)$.

Next we describe the counterexample. Let $d_{\mathcal{A}}$ be the lower semicontinuous function defined on $[0, 1]$ which is equal to 2 on the subintervals $[0, 1/2)$ and $(1/2, 1]$, and equal to 1 at $1/2$. Let ϵ_0 be such that $0 < \epsilon_0 < 1/4$ and $F = \{a_1(t) = t\}$. Let f be a continuous function which approximates $d_{\mathcal{A}}$. Since they can not be equal everywhere around $1/2$, we can assume that $f(t) < 2 = d_{\mathcal{A}}(t)$ for all t in $(1/2 - \eta, 1/2 + \eta)$, where $\eta > 0$ can be chosen as small as needed.

Let $\delta > 0$ be given. There exists a positive integer M_k such that $\frac{1}{2M_k - 1} < \delta$. Then choose T to be defined by M_k eigenvalue functions $(\lambda_i)_{i=1,\dots,M_k}$, all being the identity functions, $\lambda_i(t) = t$, for all $i = 1, \dots, M_k$. Next choose B to be a continuous trace C^*-algebra with dimension function constant equal to $2M_k - 1$.

Note that the hypothesis 2 of the Existence theorem 29.4.1 from [16] holds

$$T(f)(t) = \sum_{i=1}^{M_k} f \circ \lambda_i(t) \leqslant 2M_k \leqslant (1+\delta) d_{\mathcal{B}(t)}.$$

Now we claim that among all perturbations of T which are within the given ϵ_0 with respect to the finite set F, the particular one P which is given by the continuous eigenfunctions $(\mu_i)_{i=1,\dots,M_k}$ that have the property $\mu_i(t) = 1/2$ for $t \in (1/2 - \eta, 1/2 + \eta)$, is the smallest in the sense that the value of $P(d_{\mathcal{A}})$ is the smallest. Here it is important to notice that because $\epsilon_0 < 1/4$ it forces that $(\mu_i)_i(t) = \lambda(t) = t$ for t close to 0 and 1 including 0 and 1. In particular we have $(\mu_i)(0) = \lambda_i(0) = 0$. Therefore

$$P(d_{\mathcal{A}})(0) = \sum_{i=1}^{M_k} d_{\mathcal{A}}\big(\mu_i(0)\big) = 2M_k > 2M_k - 1 = d_{\mathcal{B}}(0).$$

Therefore we cannot perturb the eigenfunctions to obtain a compatible eigenvalue pattern and the Existence theorem as stated in [16] cannot be proved.

6. Uniqueness theorem

It is important to notice that the conclusion of the Existence theorem is part of the hypothesis of the Uniqueness theorem; this makes sense since all inequalities are balanced (i.e. independent of the choice of projection with respect to which the normalization is done).

Theorem 6.1. *Let A be a special continuous-trace C^*-algebra, $F \subset A$ a finite subset and $\epsilon > 0$. Let B be a special continuous-trace C^*-algebra and $\psi, \varphi : A \to B$ be maps with the following properties*:

1. $\varphi_0 = \psi_0 : K_0(A) \to K_0(B)$,
2. *ψ and φ have at least the fraction δ of their eigenvalues in each of the d consecutive subintervals of length $\frac{1}{d}$ of* $[0, 1]$, *for some $d > 0$ such that for $\hat{r_i}$ the functions equal to 0 from 0 to $\frac{i}{d}$, equal to 1 on $[\frac{i+1}{d}, 1]$ and linear in between, for each $0 \leqslant i \leqslant d$, $\|(\varphi_T - \psi_T)(\hat{r_i})\|_{K(p)} < \delta \|\hat{r_i}\|_p$, with respect to the norm of* $\operatorname{Aff} T^+ B$,

3. *Then there is an approximately inner automorphism of the unitization of B, f, such that*

$$\left\|(\psi - f\varphi)(a)\right\| < \epsilon, \quad a \in F.$$

Proof. Because of the isomorphism theorem 4.13 from [7], there is no loss of generality to assume that our building blocks are in a very special form

$$A \cong \begin{pmatrix} C_0(A_1) & C_0(A_1) & C_0(A_1) & \dots & C_0(A_1) \\ C_0(A_1) & C_0(A_2) & C_0(A_2) & \dots & C_0(A_2) \\ C_0(A_1) & C_0(A_2) & C_0(A_3) & \dots & C_0(A_3) \\ \vdots & \vdots & \vdots & \ddots & \\ C_0(A_1) & C_0(A_2) & C_0(A_3) & \dots & C[0,1] \end{pmatrix}.$$

Notice that the cancellation property holds for the unital sub-C^*-algebra of A and any projection of A is Murray–von Neumann equivalent to a projection inside of the unital sub-C^*-algebra. Therefore the cancellation property holds for A. A similar argument shows that the cancellation property holds for any continuous-trace C^*-algebra with the spectrum the closed interval [0, 1].

Since $\varphi_0 = \psi_0$, we can assume that $\varphi(p) = \psi(p)$, where p is the unit of the sub-C^*-algebra $C([0,1])$ of A. In other words the restrictions of the maps to the unital subalgebra share the same unit.

The stable part of the Elliott invariant (i.e., the K_0 group and the affine function space $\operatorname{Aff} T^+$) of A and of $C([0,1])$ is the same. Let us restrict the two maps φ and ψ to the unital sub-C^*-algebra $C([0,1])$. The image of $C([0,1])$ under φ and ψ is up to a unitary a full matrix algebra over the interval. Then using assumptions 1 and 2 we notice that the hypotheses of the Elliott Uniqueness theorem [4, Theorem 6], are fulfilled. Hence we get a partial isometry V of B (a unitary inside of the full matrix sub-C^*-algebra of B) such that

$$\left\|\varphi(f_{A_i} \otimes e_{nn}) - V\psi(f_{A_i} \otimes e_{nn})V^*\right\| \leqslant \epsilon, \quad i \in \{1, \dots, n\}.$$

We want this relation to hold for the case when the domain is A. We follow a strategy already present in the case of full matrix over the interval. An important data that we will use is that the domain algebra A has a finite presentation. In fact we will use the concrete description of this presentation that was given in [7, Section 8]. The set of generators consists of elements of the form $f_{A_i} \otimes e_{in}$ which are certain positive functions tensor the matrix units.

For each i let u_i be a continuous function defined on [0, 1] which is equal to 1 on A_i except near the end points of each open subinterval of A_i and 0, otherwise. One can think of u_i as an approximate unit of the functions f_{A_i}, $i \in \{1, \dots, n\}$, and later estimates depend on the size of the subset of A_i where u_i is not equal to 1.

Define

$$\mathcal{V} = \sum_{i=1}^{n} \varphi(u_i \otimes e_{ni})^* V \psi(u_i \otimes e_{ni}).$$

Then

$$\begin{aligned}
&\mathcal{V}\psi(f_{A_i}\otimes e_{ni})\mathcal{V}^*\\
&=\left(\sum_{k=1}^{n}\varphi(u_k\otimes e_{nk})^*V\psi(u_k\otimes e_{nk})\right)\psi(f_{A_i}\otimes e_{ni})\left(\sum_{l=1}^{n}\psi(u_l\otimes e_{ln})^*V^*\varphi(u_l\otimes e_{nl})\right)\\
&=\varphi(u_n\otimes e_{nn})V\psi(f_{A_i}\otimes e_{ni})\left(\sum_{l=1}^{n}\psi(u_l\otimes e_{ln})^*V^*\varphi(u_l\otimes e_{nl})\right)\\
&=\varphi(u_n\otimes e_{nn})V\psi(f_{A_i}\otimes e_{ni})\left(\sum_{l=1}^{n}\psi(u_l\otimes e_{nl})V^*\varphi(u_l\otimes e_{nl})\right)\\
&=\varphi(u_n\otimes e_{nn})V\psi(f_{A_i}\otimes e_{ni})\psi(u_i\otimes e_{in})V^*\varphi(u_i\otimes e_{ni})\\
&=\varphi(u_n\otimes e_{nn})V\psi(f_{A_i}\otimes e_{nn})V^*\varphi(u_i\otimes e_{ni}).
\end{aligned}$$

Now we have that

$$\varphi(f_{A_i}\otimes e_{ni})=\varphi(u_n\otimes e_{nn})\varphi(f_{A_i}\otimes e_{nn})\varphi(u_i\otimes e_{ni}).$$

Therefore

$$\begin{aligned}
&\left\|\varphi(f_{A_i}\otimes e_{ni})-\mathcal{V}\psi(f_{A_i}\otimes e_{ni})\mathcal{V}^*\right\|\\
&\quad=\left\|\varphi(u_n\otimes e_{nn})\big(\varphi(f_{A_i}\otimes e_{nn})-V\psi(f_{A_i}\otimes e_{nn})V^*\big)\varphi(u_i\otimes e_{ni})\right\|\\
&\quad\leqslant\left\|\varphi(u_n\otimes e_{nn})\right\|\epsilon\left\|\varphi(u_i\otimes e_{ni})\right\|,
\end{aligned}$$

i.e. it can be made as small as needed.

We want to argue that $\mathcal{V}$ gives rise to a partial isometry. Let us calculate

$$\begin{aligned}
\mathcal{V}^*\mathcal{V}&=\sum_{l=1}^{n}\psi(u_l\otimes e_{nl})^*V^*\varphi(u_l\otimes e_{nl})\sum_{i=1}^{n}\varphi(u_i\otimes e_{ni})^*V\psi(u_i\otimes e_{ni})\\
&=\sum_{l=1}^{n}\psi(u_l\otimes e_{ln})V^*\varphi(u_l\otimes e_{nl})\sum_{i=1}^{n}\varphi(u_i\otimes e_{in})V\psi(u_i\otimes e_{ni}).
\end{aligned}$$

Assuming that each u_i is equal to 1 on the open intervals A_i except small neighbourhood around the end points of A_i we get

$$\mathcal{V}^*\mathcal{V}=\sum_{i=1}^{n}\psi(u_i\otimes e_{in})V^*\varphi(u_i\otimes e_{nn})V\psi(u_i\otimes e_{ni})$$

which is very close to

$$\sum_{i=1}^{n}\psi(u_i\otimes e_{in})\psi(u_l\otimes e_{nn})\psi(u_i\otimes e_{ni})=\sum_{i=1}^{n}\psi(u_i\otimes e_{ii})$$

which is the value of the projection-valued map of the hereditary sub-C^*-algebra generated by $\psi(A)$ inside B. In other words $\mathcal{V}^*\mathcal{V}$ is as close as we want to be a projection. It is important to notice that this is true if we are not in a small neighbourhood of the singularity points of the dimension function of the hereditary sub-C^*-algebra generated by $\psi(A)$ (i.e. whenever $u_i = 1$).

Similarly $\mathcal{V}\mathcal{V}^*$ is almost equal to the $\sum_{i=1}^n \varphi(u_i \otimes e_{ii})$ if we are not in a small neighbourhood of the singularity points of the dimension function of the hereditary sub-C^*-algebra generated by $\varphi(A)$. Notice that any singularity point y_0 of the dimension function of the hereditary sub-C^*-algebra generated by $\varphi(A)$ or $\psi(A)$ has the property that there is an eigenfunction λ_i such that $\lambda_i(y_0)$ is a singularity point of the dimension function d_A of A. In addition λ_i is uniform continuous function from [0, 1] to [0, 1]. Hence small neighbourhoods of y_0 correspond to small neighbourhoods of some singularity point of d_A.

From the polar decomposition $\mathcal{V} = \mathcal{W}|\mathcal{V}|$ we get a partial isometry $\mathcal{W}$. We claim that $\mathcal{W}$ still intertwines approximately the two maps φ and ψ, i.e.,

$$\begin{aligned}
&\left\|\varphi(f_{A_i} \otimes e_{ni}) - \mathcal{W}\psi(f_{A_i} \otimes e_{ni})\mathcal{W}^*\right\| < 3\epsilon,\\
&\left\|\mathcal{W}^*\varphi(f_{A_i} \otimes e_{ni})\mathcal{W} - \psi(f_{A_i} \otimes e_{ni})\right\| < 3\epsilon.
\end{aligned}$$

This is true because

$$\begin{aligned}
&\left\|\varphi(f_{A_i} \otimes e_{ni}) - \mathcal{W}\psi(f_{A_i} \otimes e_{ni})\mathcal{W}^*\right\|\\
&\quad = \left\|\varphi(f_{A_i} \otimes e_{ni}) - \mathcal{V}\psi(f_{A_i} \otimes e_{ni})\mathcal{V}^* + \mathcal{W}|\mathcal{V}|\psi(f_{A_i} \otimes e_{ni})|\mathcal{V}|\mathcal{W}^* - \mathcal{W}\psi(f_{A_i} \otimes e_{ni})\mathcal{W}^*\right\|\\
&\quad \leqslant \left\|\varphi(f_{A_i} \otimes e_{ni}) - \mathcal{V}\psi(f_{A_i} \otimes e_{ni})\mathcal{V}^*\right\| + \left\|\mathcal{W}|\mathcal{V}|\psi(f_{A_i} \otimes e_{ni})|\mathcal{V}|\mathcal{W}^* - \mathcal{W}\psi(f_{A_i} \otimes e_{ni})\mathcal{W}^*\right\|\\
&\quad \leqslant \epsilon + \left\||\mathcal{V}|\psi(f_{A_i} \otimes e_{ni})|\mathcal{V}| - \psi(f_{A_i} \otimes e_{ni})\right\|\\
&\quad \leqslant \epsilon + \left\||\mathcal{V}|\psi(f_{A_i} \otimes e_{ni})|\mathcal{V}| - |\mathcal{V}|\psi(f_{A_i} \otimes e_{ni}) + |\mathcal{V}|\psi(f_{A_i} \otimes e_{ni}) - \psi(f_{A_i} \otimes e_{ni})\right\|\\
&\quad \leqslant \epsilon + \epsilon + \epsilon = 3\epsilon
\end{aligned}$$

and similarly we get the other desired inequality.

Hence we have constructed a family of partial isometries $\mathcal{W}$ from the hereditary sub-C^*-algebra generated by $\varphi(A)$ to the hereditary sub-C^*-algebra generated by $\psi(A)$. In addition $\mathcal{W}$ induces an isomorphism between the two above mentioned hereditary sub-C^*-algebras. In particular it implies that the two hereditary sub-C^*-algebras have the same dimension function.

Next we will show how to approximate $\mathcal{W}$ with a unitary in the unitization of the codomain algebra.

Let us start by applying Theorem 4.12 of [7] to the projection-valued function corresponding to the hereditary sub-C^*-algebra generated by $\varphi(A)$. Hence we get a decomposition, possibly infinite, in terms of functions each of which is projection-valued of rank 1 on a certain open subset of [0, 1] and zero otherwise. Notice that the discontinuity points of the dimension function of the hereditary sub-C^*-algebra generated by $\varphi(A)$ correspond to the discontinuity points of the functions appearing in the decomposition and the open sets are increasing in a suitable sense.

Next we apply Lemma 6.2 for each point at singularity in the interval [0, 1], or, in other words, to each function appearing on the decomposition. Thus, we have a family of unitaries that preserves the continuity of the continuous elements of the hereditary sub-C^*-algebra $\varphi(A)$ and at the same time has the property that it still intertwines the two maps. $\square$

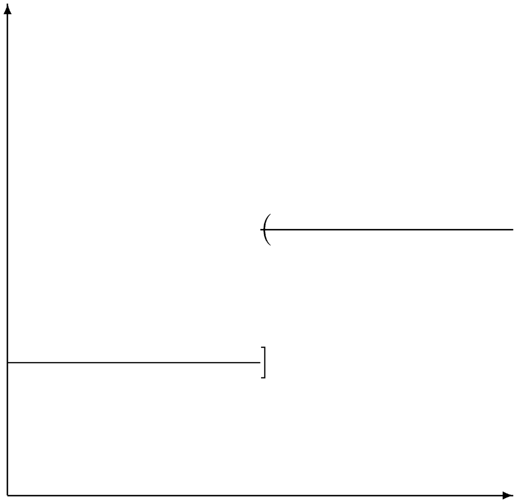

Fig. 5. Dimension function of H_1 and H_2.

In the following lemma the hereditary sub-C^*-algebras H_1 and H_2 are assumed to be continuous bundles over [0, 1] (for more details about continuous bundles of C^*-algebras see [9]).

If A is a continuous bundle of C^*-algebras over [0, 1] then A^t stands for the fiber of A over t.

Lemma 6.2. *Let H_1 and H_2 be hereditary sub-C^*-algebra of $M_2(C[0,1])$ with the same spectrum $[0,1]$ and identical dimension function equal to 1 on the closed interval $[0,t_0]$ and equal to 2 on the half-open interval $(t_0,1]$, $t_0 \in (0,1)$. Let $\mathcal{W} = (W(t))_{t\in[0,1]}$ be a family of partial isometries indexed by the points of $[0,1]$. For each $t \in [0,1]$, $W_t : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ such that $W(t)W(t)^* =$ the unit of H_1^t and $W(t)^*W(t) =$ the unit of H_2^t. Then there exists a family $\mathcal{W}^\perp$ of partial isometries indexed by $[0,1]$ such that $\mathcal{W} + \mathcal{W}^\perp$ is a unitary inside of $M_2(C[0,1])$ and $(W + W^\perp)_t(f)(t) = W_t(f)(t)$ for any continuous function $f \in H_1$ and $t \in [0,1]$.*

Proof. Diagrammatically the dimension function of H_1 and H_2 can be pictured as in Fig. 5.

We construct the family $\mathcal{W}^\perp = (\mathcal{W}_t^\perp)_{t\in[0,1]}$ as follows. Fix a t in $[0,1]$, $t \leqslant t_0$. $W(t)$ is a partial isometry on some dimension-one subspace of $M_2(\mathbb{C})$. Hence $W_t(M) = c(M)M_t$ where $c(M)$ is a constant depending on M and M_t is a projection matrix in $M_2(\mathbb{C})$. Let $W_t^\perp = c(M)(I_2 - M_t)$. Notice that $W_t + W_t^\perp$ is a unitary operator on $M_2(\mathbb{C})$. If $t > t_0$ then $W_t^\perp = 0$.

The family of unitaries $(W_t + W_t^\perp)_{t\in[0,1]}$ is continuous except at the point t_0. Our work below shows that this family can be modified to be continuous overall $[0,1]$.

Extend $(W_t)_{t\in[0,t_0]}$ to be a continuous family $(W_t^1)_{t\in[0,1]}$ of partial isometries on dimension-one subspaces of $M_2(\mathbb{C})$. $W_{t_0}^\perp$ and $\lim_{t\to t_0, t>t_0}(W_t - W_t^1)$ are two partial isometries on the same dimension one subspace of $M_2(\mathbb{C})$, hence they differ by a constant of absolute value one, i.e.

$$W_{t_0}^\perp = c \lim_{t\to t_0,\, t>t_0} \bigl(W_t - W_t^1\bigr).$$

Define the continuous family of unitaries $(U_t)_{t\in[0,1]}$ to be $U_t = W_t + W_t^\perp$ if $t \leqslant t_0$ and $U_t = W_t^1 + c(W_t - W_t^1)$ if $t > t_0$.

Hence the continuous family of unitaries W_t is given by U_t and $(U_t(f)(t) = W_t(f)(t)$ for any continuous function $f \in H_1$ and $t \in [0,1]$. □

7. Inductive limits of special continuous trace C^*-algebras

Next let us show that the Existence theorem and the Uniqueness theorem presented above can be applied, i.e., that the hypotheses of the theorems can be fulfilled. As a first step in this direction let us show that an inductive limit of continuous-trace C^*-algebras with spectrum $[0,1]$ (or disjoint unions of closed intervals) is isomorphic to an inductive limit of special continuous-trace C^*-algebras.

The basic tools in establishing this step are the fact that special continuous trace C^*-algebras are semiprojective (cf. [7, Theorem 6.5]) and a result by T. Loring [11, Lemma 15.2.2] which for the convenience of the reader we state below:

Suppose that A is a C^*-algebra containing a (not necessarily nested) sequence of sub-C^*-algebras A_n with the property that for all $\epsilon > 0$ and for any finite number of elements $x_1, \ldots, x_k$ of A, there exists an integer n such that

$$\{x_1, \ldots, x_k\} \subset_\epsilon A_n.$$

If each A_n is weakly semiprojective and finitely presented, then

$$A \cong \varinjlim (A_{n_k}, \gamma_k)$$

for some subsequence of (A_n) and some maps $\gamma_k : A_{n_k} \to A_{n_{k+1}}$.

Proposition 7.1. *Let A be a simple inductive limit of continuous-trace C^*-algebras whose building blocks have their spectrum homeomorphic to* $[0,1]$. *Then A is an inductive limit of direct sums of special continuous-trace C^*-algebras with spectrum* $[0,1]$.

Proof. In Proposition 5.4 and Theorem 6.5 of [7] it is proved that the class of special continuous trace C^*-algebras with spectrum $[0,1]$ are finitely presented and have weakly stable relations. Each building block from the inductive limit decomposition of A can be approximated by special continuous trace C^*-algebras (cf. [7, Theorem 6.14]). Then A satisfies Loring's hypothesis where the sequence of semiprojective algebras is given by the special algebras from the approximation of the building blocks. Thus the Loring's lemma implies that A is an inductive limit of special continuous trace C^*-algebras. □

8. Getting a non-zero gap at the level of affine function spaces

To be able to exactly intertwine the non-stable part of the invariant it is useful to know that the dimension function of any building block A_m or B_m is taken by the homomorphism $\phi_{m,m+1}$, respectively $\psi_{m,m+1}$, into a function smaller than or equal to the dimension function of A_{m+1} or B_{m+1} such that a non-zero gap arises. In other words we want to exclude the possible cases when the dimension function is taken into the next stage dimension function such that equality holds at a point or at more points. We shall show this in the following lemma. Recall that because of Proposition 7.1, the algebras that we want to classify can be assumed to be inductive limits

of special continuous trace C^*-algebras with spectrum $[0,1]$, i.e., $A \cong \varinjlim(A_n, \phi_{nm})$ and $B \cong \varinjlim(B_n, \psi_{nm})$, where A_n, B_n are special continuous trace C^*-algebras.

Lemma 8.1. *Let $A = \varinjlim(A_n, \phi_{nm})$ be a simple C^*-algebra, where each A_n is a special continuous trace C^*-algebra with spectrum the closed interval $[0,1]$ and the dimension function assumed to be a finite-valued bounded function. Then there exist $\delta_1 > 0$, a subsequence $(A_{n_i})_{n_i \geqslant 0}$ of $(A_n)_n$ and a sequence of maps $\phi_i : A_{n_i} \to A_{n_{i+1}}$ such that*:

1. $A \cong \varinjlim(A_{n_i}, \phi_{n_i m_i})$,
2. $(\phi_{n_1 n_2})_T(\hat{P}_{A_{n_1}}) + \delta_1 < \hat{P}_{A_{n_2}}$,

where the inequality holds pointwise, $(\phi_{nm})_T$ is the induced map at the level of the affine function spaces, $P_{A_{n_1}}$ and $P_{A_{n_2}}$ are the units of the biduals of A_{n_1} and A_{n_2}, and $\hat{P}_{A_{n_1}}$ and $\hat{P}_{A_{n_2}}$ denote the corresponding lower semicontinuous functions.

Proof. Let A be equal to $\varinjlim A_n$ with maps $\phi_{n,m} : A_n \to A_m$.

The plan is to keep the same building blocks and to change slightly the maps with respect to some given finite sets such that the desired property holds. To do this we use the property that the building blocks that appear in the inductive limit decomposition are weakly semiprojective.

Assume that the dimension function of $\phi_{12}(A_1)$ equals the dimension function of A_2 at some point or even everywhere and let $\epsilon > 0$, $F_1 \subset A_1$ be given. Because the largest value of the dimension function of the hereditary sub-C^*-algebra generated by $\phi_{12}(A_1)$ inside A_2 is attained on an open subset U of $[0,1]$, let us construct another dimension function as follows: shrink one of the open intervals of the open set U to get U' and in exchange enlarge the interval adjacent to that discontinuity point. U' is constructed in a such a way that is as close as necessary to the given U.

In this manner we find a sub-C^*-algebra B which is as close as we want to the hereditary sub-C^*-algebra generated by $\phi_{12}(A_1)$ inside of A_2. Next we use that A_1 is weakly semiprojective to find another $*$-homomorphism $\rho_1 : A_1 \to B$ which is close within the given ϵ on the given finite set F_1.

Then there exists some open interval between the dimension function of A_2 and the dimension function of the B. This open interval corresponds to a non-zero ideal I_1 inside of A_2. Now the image of I_1 in the inductive limit is also a non-zero ideal. Since the inductive limit is simple, it implies that the ideal is the whole algebra. We know that there are full projections in the inductive limit. Therefore there is a finite stage in the inductive limit of the ideals coming from I_1 that has a full projection. Assume that the finite stage is inside of A_k. This means that at that stage the image of the ideal I_1 is A_k. Pick a strictly positive element a_1 in I_1. Then the image of a_1 in A_k will be strictly positive at each point from $[0,1]$, $k > 1$. This shows that the image of the dimension function d_B inside the dimension of A_k has a gap of at least 1 everywhere in $[0,1]$.

Because of the normalizations of the affine function, this gap of size 1 will correspond to some strictly non-zero δ_1. To complete the proof we relabel B as A_{n_1}, A_k as A_{n_2}, etc. □

Corollary 8.2. *Let $A = \varinjlim(A_n, \phi_{n,m})$ be a simple C^*-algebra. Then there exist a sequence $(\delta_i)_{i \geqslant 1}$, $\delta_i > 0$, a subsequence of algebras $(A_{n_i})_{i \geqslant 1}$ of $(A_i)_{i \geqslant 1}$ and a sequence of maps $\phi : A_{n_i} \to A_{n_{i+1}}$ such that*:

1. $A \cong \varinjlim(A_{n_i}, \phi_{n_i,m_i})$,
2. $\phi_{T_{n_i,n_{i+1}}}(\hat{P}_{A_{n_i}}) + \delta_i < \hat{P}_{A_{n_{i+1}}}$.

Proof. Follows by successively applying the previous lemma. $\square$

9. Pulling back of the isomorphism between inductive limits at the level of the invariant

Step 1. *(The intertwining between the stable part of the invariant.)* With no loss of generality we assume that the building blocks have the following concrete representation

$$\begin{pmatrix} C_0(A_1) & C_0(A_1) & C_0(A_1) & \dots & C_0(A_1) \\ C_0(A_1) & C_0(A_2) & C_0(A_2) & \dots & C_0(A_2) \\ C_0(A_1) & C_0(A_2) & C_0(A_3) & \dots & C_0(A_3) \\ \vdots & \vdots & \vdots & \ddots & \\ C_0(A_1) & C_0(A_2) & C_0(A_3) & \dots & C[0,1] \end{pmatrix}.$$

One can distinguish a full unital hereditary sub-C^*-algebra

$$\begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & \ddots & 0 & \dots & 0 \\ 0 & \dots & C[0,1] & \dots & C[0,1] \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & \dots & C[0,1] & \dots & C[0,1] \end{pmatrix}.$$

The unital hereditary sub-C^*-algebra has the same stable invariant (i.e., K_0, $\operatorname{Aff} T^+$ and the pairing) as the given C^*-algebra. Moreover the unital hereditary sub-C^*-algebra is a full matrix algebra over the closed interval $[0,1]$. Using this fact we derive an intertwining between the stable invariant, as is shown in [16] or originally in [4].

It is important to mention the method of normalizing the affine function spaces. Pick a full projection $p_1 \in A_1$. Normalize the affine space $\operatorname{Aff} T^+ A_1$ with respect to p_1. Next consider a image of p_1 in A_2 under the map at the dimension range level, call it p_2. Normalize $\operatorname{Aff} T^+ A_2$ with respect to p_2. Note that the map which is induced at the affine level is a contraction. Continue in this way so that we obtain an inductive limit sequence at the level of the affine spaces, with all the maps being contractions:

$$\operatorname{Aff} T^+ A_1 \to \operatorname{Aff} T^+ A_2 \to \cdots \to \operatorname{Aff} T^+ A.$$

Let p_∞ denote the image of p_1 in the inductive limit A and denote by q_∞ a representative of $\phi_0(p_\infty)$ in B. Then there exists $q_1 \in B_1$ such that the image of q_1 is q_∞ in the inductive limit. Normalize the $\operatorname{Aff} T^+ B_1$ with respect to q_1, $\operatorname{Aff} T^+ B_2$ with respect to a image of q_1 in B_2 and so on. Hence we obtain another inductive limit of affine spaces with contractions maps

$$\operatorname{Aff} T^+ A_1 \to \operatorname{Aff} T^+ A_2 \to \cdots \to \operatorname{Aff} T^+ A,$$

$$\operatorname{Aff} T^+ B_1 \to \operatorname{Aff} T^+ B_2 \to \cdots \to \operatorname{Aff} T^+ B.$$

As already mentioned above, we pull back the invariant for the unital hereditary sub-C^*-algebras (i.e. full matrix algebras or the stable invariant). This will give rise to an exact commuting diagram at the K_0-level, an approximate commuting diagram at the affine function spaces level and an exact pairing. The compatibility can be made exact as shown in [3] by noticing that, because of simplicity, non-zero positive elements in both K_0 and $\operatorname{Aff} T^+$ are sent into strictly positive elements and then normalize the affine function spaces in a suitable way.

To summarize, we now have a commutative diagram

$$\begin{tikzcd}
C[0,1] \arrow[r, "\phi_{12}"] \arrow[d, "\tau_1"] & C[0,1] \arrow[r, "\phi_{23}"] \arrow[d, "\tau_2"] & \cdots \arrow[r] & (\operatorname{Aff} T^+A, \operatorname{Aff}' A) \arrow[d, leftrightarrow] \\
C[0,1] \arrow[r, "\psi_{12}"] \arrow[ur, "\tau_1'"'] & C[0,1] \arrow[r, "\psi_{23}"] \arrow[ur, "\tau_2'"'] & \cdots \arrow[ur] & (\operatorname{Aff} T^+B, \operatorname{Aff}' B)
\end{tikzcd}$$

where $\operatorname{Aff} T^+A_i$ and $\operatorname{Aff} T^+B_i$ are identified with $C([0,1])$ and each finite stage algebra A_i and B_i is assumed to have only one direct summand.

For us it is very important to study the pulling back of the non-stable part of the invariant.

Step 2. *(The intertwining of the non-stable part of the invariant.)* As I. Stevens mentioned in [16], at this moment we know that the non-stable part of the invariant is only approximately mapped at a later stage into the non-stable part of the invariant.

To be able to apply the Existence theorem 5.1, one needs to check that hypothesis 2 can be ensured. Otherwise, a counterexample can be given to the Existence theorem, as shown in Section 5.1 above. The special assumption from the hypothesis of the isomorphism theorem, $\phi_T(\operatorname{Aff}' A) \subseteq \operatorname{Aff}' B$, as well as Corollary 8.2 will be used to prove the above mentioned claim.

By applying Corollary 8.2 to the given inductive limits $A = \varinjlim(A_n, \phi_{n,m})$, $B = \varinjlim(B_n, \phi_{n,m})$ we get two sequences $(\delta_i)_{i\geqslant 1}, \delta_i > 0$ and $(\delta_i')_{i\geqslant 1}, \delta_i' > 0$, respectively, and two subsequences of algebras such that after relabeling, we can assume that $\phi_{ii+1}(\hat{P}_{A_i}) + \delta_i < \hat{P}_{A_{i+1}}, \psi_{ii+1}(\hat{P}_{A_i}) + \delta_i < \hat{P}_{A_{i+1}}, \psi_{ii+1}(\hat{P}_{A_i}) + \delta_i < \hat{P}_{A_{i+1}}$ and $\psi_{ii+1}(\hat{P}_{A_i}) + \delta_i < \hat{P}_{A_{i+1}}$ for all $i \geqslant 1$.

Reworking the intertwining of the stable invariant for the new sequences of algebras and the new maps that have gaps δ_i we obtain the following intertwining:

$$\begin{tikzcd}
C([0,1]) \arrow[r, "\phi_{12}"] \arrow[d, "\tau_1"] & C([0,1]) \arrow[r, "\phi_{23}"] \arrow[d, "\tau_2"] & \cdots \arrow[r] & (\operatorname{Aff} T^+A, \operatorname{Aff}' A) \arrow[d, leftrightarrow] \\
C[0,1] \arrow[r, "\psi_{12}"] \arrow[ur, "\tau_1'"'] & C[0,1] \arrow[r, "\psi_{23}"] \arrow[ur, "\tau_2'"'] & \cdots \arrow[ur] & (\operatorname{Aff} T^+B, \operatorname{Aff}' B)
\end{tikzcd}$$

As a consequence of the Thomsen–Li theorem, which in the present case states that the closed convex hull of the set of all unital $*$-homomorphisms of $C([0,1])$ in the strong operator topology is exactly the set of positive of unital operators on $C([0,1])$, we can assume that all the maps $\phi_{ii+1}, \psi_{ii+1}, \tau_i, \tau_i'$ are given by eigenvalue patterns. Because each such map takes the unit, say $\widehat{p}$,

into the unit, $\widehat{K(p)}$, it follows that each map is an average of the eigenvalues, i.e., $\phi_{i,i+1}(f) = \sum_{i=1}^{N_i} \frac{f \circ \lambda_i}{N_i}$, etc.

Let $\hat{P}_{A_1}$ be the image in the affine function space of the unit in the bidual of A_1. Take a continuous function f smaller than $\hat{P}_{A_1}$. It is important to say that there are no extra conditions on f, i.e., f can be any element of the special set $\operatorname{Aff} T' A_1$. Then there exists $\delta_1 > 0$ such that

$$\phi_{12}(\hat{P}_{A_1}) + \delta_1 < \hat{P}_{A_2}.$$

Since $\phi_{12}(f) \leqslant \phi_{12}(\hat{P}_{A_1})$ we have

$$\phi_{12}(f + \delta_1) \leqslant \phi_{12}(\hat{P}_{A_1} + \delta_1) < \hat{P}_{A_2}.$$

Since $\phi_T(\operatorname{Aff}' A) \subseteq \operatorname{Aff}' B$, it follows that there exist a large N and $\epsilon_N \leqslant \delta_1$ such that

$$\tau_N \circ \phi_{N-2N-1} \circ \cdots \circ \phi_{12}(f + \delta_1) < \hat{P}_{B_N} + \epsilon_N.$$

It is important to say that a different choice for f will give rise to possibly different N. This is not a difficulty because we can always pass to subsequence. Equivalently we have

$$\tau_N \circ \phi_{N-2N-1} \circ \cdots \circ \phi_{12}(f) + \delta_1 < \hat{P}_{B_N} + \epsilon_N.$$

Using $\delta_1 \geqslant \epsilon_N$ we conclude

$$\tau_N \circ \phi_{N-2N-1} \circ \cdots \circ \phi_{12}(f) < \hat{P}_{B_N},$$

which is the desired strict inequality from the hypothesis 2 of the Existence theorem 5.1.

10. The isomorphism theorem

To complete the proof of the isomorphism theorem 3.1 for the algebras $\varinjlim A_i = A$ and $\varinjlim B_i = B$, we have to construct an approximate commutative diagram at the algebra level in the following sense, as was defined by Elliott in [3],

> *"for any fixed element in any A_i (or B_i), the difference of the images of this element along two different paths in the diagram, starting at A_i (or B_i) and ending at the same place, converges to zero as the number of steps for which the two paths coincide, starting at the beginning, tends to infinity."*

At this stage because of Step 2 of the previous section, Section 9, we can apply the Existence theorem to generate a sequence of algebra homomorphisms $\nu_1, \nu_2, \ldots$ and $\nu_1', \nu_2', \ldots$ such that $\frac{\|\tau_i(f) - \nu_{i*}(f)\|}{\|f\|} \leqslant \frac{\epsilon}{2^i}$ and $\frac{\|\tau_i'(f) - \nu_{i*}'(f)\|}{\|f\|} \leqslant \frac{\epsilon}{2^i}$ for $f \in F_i$ and $g \in G_i$, where ν_{i*}, ν_{i*}', are the induced affine maps by algebra maps ν_i, ν_i', and F_i and G_i are finite sets.

After relabeling the indices of the inductive limit systems we now have a (not necessarily approximately commutative) diagram of algebra homomorphisms:

$$\begin{tikzcd} A_1 \arrow[r, "\phi_{12}"] \arrow[d, "\tau_1"] & A_2 \arrow[r, "\phi_{23}"] \arrow[d, "\tau_2"] & \cdots \arrow[r] & A \\ B_1 \arrow[r, "\psi_{12}"] \arrow[ur, "\tau_1'"'] & B_2 \arrow[r, "\psi_{23}"] \arrow[ur, "\tau_2'"'] & \cdots \arrow[r] & B \end{tikzcd}$$

that induces an approximately commutative diagram at the level of the invariant.

This will be done with respect to given arbitrary finite sets $F_i \subset A_i$ and $G_i \subset B_i$.

To make the diagram approximately commuting we modify the diagonal maps by composing with approximately inner automorphisms and this will be done with respect to a given arbitrary finite sets $F_i \subset A_i$ and $G_i \subset B_i$ with dense union in A and B, respectively.

Here we notice that we can apply the Uniqueness theorem to the data obtained from the Existence theorem because our inequalities are balanced.

For every $\epsilon > 0$ we find an increasing sequence of integers $1 = M_0 < L_1 < M_2 < L_2 < \cdots$ and unitaries $(U_{M_{i+1}}) \in A^+_{M_{i+1}}$, $(V_i^n)_n \in B^+_{L_i}$ such that for $f \in F_{M_i}$ and $g \in G_{L_i}$ we have

$$\frac{\|U_{M_{i+1}}\tau'_{M_i}(\tau_{M_i}(f))U^*_{M_{i+1}} - \phi_{M_iM_{i+1}}(f)\|}{\|f\|} < \frac{\epsilon}{2^i},$$
$$\frac{\|V_{M_{i+1}}\tau_{L_i}(\tau'_{L_i}(g))V^*_{L_{i+1}} - \phi_{L_iL_{i+1}}(g)\|}{\|g\|} < \frac{\epsilon}{2^i}.$$

In other words passing to suitable subsequences of algebras, it is possible to perturb each of the homomorphisms obtained in the Existence theorem by an approximately inner automorphism, in such a way that the diagram becomes an approximate intertwining, in the sense of [3, Theorem 2.1].

Therefore, by the Elliott approximate intertwining theorem (see [3, Theorem 2.1]), the algebras A and B are isomorphic.

11. The range of the invariant

In this section we prove Theorem 3.2 which answers the question what are the possible values of the invariant from the isomorphism Theorem 3.1. It is useful to notice that the invariant consists of two parts. One part is the stable part, i.e., K_0, $\mathrm{Aff}\, T^+$, $\lambda : T^+ \mapsto S(K_0)$ which was shown by K. Thomsen in [18] to be necessary if one wants to construct an AI-algebra, and the other part which one may call the non-stable part, namely Aff' or equivalently, as shown in [16, Remarks 30.1.1 and 30.1.2], the trace norm map. It is the non-stable part of the invariant that one needs to investigate in its full generality. Next the definition of the trace norm map is introduced.

Definition 11.1. Let $\mathcal{A}$ be a sub-C^*-algebra of a C^*-algebra $\mathcal{B}$. The trace norm map associated to $\mathcal{A}$ is a function $f : T^+(\mathcal{A}) \to (0, \infty]$ such that $f(\tau) = \|\tau|_{\mathcal{A}}\|$, ∞ if τ is unbounded.

Recall the following.

Definition 11.2. $T^+(\mathcal{A})$ is the cone of positive trace functionals on $\mathcal{A}$ with the inherited w^*-topology.

Remark 11.1. The trace norm map is a lower semicontinuous affine map (being a supremum of a sequence of continuous functions).

Remark 11.2. The dimension range can be determined using the values of the trace norm map f, the simplex of tracial states S and dimension group G. A formula for the dimension range D is:

$$D = \{x \in G / v(x) < f(v),\ v \in S,\ v \neq 0\}.$$

I. Stevens has constructed a hereditary sub-C^*-algebra of a simple (unital) AI-algebra which is obtained as an inductive limit of hereditary sub-C^*-algebras of interval algebras, and has as a trace norm map any given affine continuous function; cf. [16, Proposition 30.1.7]. Moreover she showed that any lower semicontinuous map can be realized as a trace norm map in a special case. Our result is a generalization to the case of unbounded trace norm map when restricted to the base of the cone. It is worth mentioning that our approach gives another proof in the case of any lower semicontinuous map as a trace norm map. Still our approach is using the I. Stevens's proof for the case of continuous trace norm map.

Theorem 3.2. *Suppose that G is a simple countable dimension group, V is the cone associated to a metrizable Choquet simplex. Let $\lambda : V \to \mathrm{Hom}^+(G, R)$ be a continuous affine map and taking extreme rays into extreme rays. Let $f : V \to [0, \infty]$ be an affine lower semicontinuous map, zero at zero and only at zero. Then $[G, V, \lambda, f]$ is the Elliott invariant of some simple non-unital inductive limit of continuous trace C^*-algebras whose spectrum is the closed interval* $[0, 1]$ *or a finite disjoint union of closed intervals.*

Proof. The proof is based on I. Stevens's proof in a special case and consists of several steps.

Step 0. We start by constructing a simple stable AI algebra $\mathcal{A}$ with its Elliott invariant: $[(G, D), V, \lambda]$. We know that this is possible (see [14]). By tensoring with the algebra of compact operators we may assume $\mathcal{A}$ is a simple stable AI algebra.

Step 1. We restrict the map f to some base S of the cone $T^+(\mathcal{A})$, where the cone V is naturally identified with $T^+(\mathcal{A})$. Since any lower semicontinuous affine map $f : S \to (0, +\infty]$ is a pointwise limit of an increasing sequence of continuous affine positive maps (see [1]), we can choose $f = \lim f_n$, where f_n are continuous affine and strictly positive functions.

Moreover by considering the sequence of functions $g_n = f_{n+1} - f_n$ if $n > 1$ and $g_1 = f_1$ we get that

$$\sum_{n=1}^{\infty} g_n = f.$$

Step 2. Next we use the results of Stevens [16, Proposition 30.1.7], to realize each such continuous affine map g_n as the norm map of a hereditary sub-C^*-algebra $\mathcal{B}_n$ (which is an inductive limit of special algebra) of the AI algebra $\mathcal{A}$ obtained at Step 0.

Consider the L^∞ direct sum $\bigoplus \mathcal{B}_i$ as a sub-C^*-algebra of $\mathcal{A}$. The trace norm map of the sub-C^*-algebra $\bigoplus \mathcal{B}_i$ of $\mathcal{A}$ is equal to $\sum_{i=1}^{\infty} g_n = f$.

To see that $\bigoplus \mathcal{B}_i$ is a sub-C^*-algebra of $\mathcal{A}$ we use that $\mathcal{A}$ is a stable C^*-algebra:

$$\bigoplus \mathcal{B}_i = \begin{pmatrix} \mathcal{B}_1 & & 0 \\ & \mathcal{B}_2 & \\ 0 & & \ddots \end{pmatrix} \subseteq \mathcal{A} \otimes \mathbb{K} \cong \mathcal{A}.$$

Next denote with $\mathcal{H}$ the hereditary sub-C^*-algebra generated by $\bigoplus \mathcal{B}_i$ inside of $\mathcal{A}$.

To prove that the trace norm map of $\mathcal{H}$ is f is enough to show that the norm of a trace on $\bigoplus \mathcal{B}_i$ is the same as on $\mathcal{H}$.

It suffices to prove that an approximate unit of the sub-C^*-algebra $\bigoplus \mathcal{B}_i$ is still an approximate unit for the hereditary sub-C^*-algebra $\mathcal{H}$.

We shall prove first that the hereditary sub-C^*-algebra generated by $\bigoplus \mathcal{B}_i$ coincides with the hereditary sub-C^*-algebra generated by one of its approximate units. Let $(u_\lambda)_\lambda$ be an approximate unit of $\bigoplus \mathcal{B}_i$. Denote by $\mathcal{U}$ the hereditary sub-C^*-algebra of $\mathcal{H}$ generated by $\{(u_\lambda)_\lambda\}$. We want to prove that $\mathcal{U}$ is equal with $\mathcal{H}$.

Since $(u_\lambda)_\lambda$ is a subset of $\bigoplus \mathcal{B}_i$ we clearly have

$$\mathcal{U} \subseteq \mathcal{H}.$$

For the other inclusion, one can observe that

$$\text{for all } b \in \bigoplus \mathcal{B}_i: \quad b = \lim_{\lambda \to \infty} u_\lambda b u_\lambda.$$

Now each $u_\lambda b u_\lambda$ is an element of the hereditary sub-C^*-algebra generated by $(u_\lambda)_\lambda$ and hence $b \in \mathcal{U}$. Therefore $\bigoplus \mathcal{B}_i \subset \mathcal{U}$ which implies $\mathcal{H} \subseteq \mathcal{U}$.

We conclude that $\mathcal{H} = \mathcal{U}$ and hence the trace norm map of $\mathcal{H}$ is f. Therefore $\mathcal{H}$ is a simple hereditary sub-C^*-algebra of an AI algebra with the prescribed invariant. □

Remark 11.3. The approximate unit $(u_\lambda)_\lambda$ of $\bigoplus \mathcal{B}_i$ is still an approximate unit for the hereditary sub-C^*-algebra $\mathcal{U}$. To see why this is true let us consider the sub-C^*-algebra of $\mathcal{A}$ defined as follows: $\{h \in \mathcal{A} \mid h = \lim_{\lambda \to \infty} u_\lambda h\}$.

This sub-C^*-algebra of $\mathcal{A}$ is a hereditary sub-C^*-algebra. Indeed let $0 \leqslant k \leqslant h$ with $h = \lim_{\lambda \to \infty} u_\lambda h$. We want to prove that $k = \lim_{\lambda \to \infty} u_\lambda k$.

Consider the hereditary sub-C^*-algebra $\overline{h\mathcal{A}h}$ of $\mathcal{A}$ which clearly contains h (because $h^2 = \lim_{\lambda \to \infty} h u_\lambda h$). Therefore $k \in \overline{h\mathcal{A}h}$.

Since $h = \lim_{\lambda \to \infty} u_\lambda h$ we obtain that u_λ is an approximate unit for $\overline{h\mathcal{A}h}$. In particular

$$k = \lim_{\lambda \to \infty} u_\lambda k$$

and hence $\{h \in \mathcal{A} \mid h = \lim_{\lambda \to \infty} u_\lambda h\}$ is a hereditary sub-C^*-algebra of $\mathcal{A}$. Since $\mathcal{U}$ is the smallest hereditary containing $(u_\lambda)_\lambda$ we get that

$$\mathcal{U} \subseteq \left\{ h \in \mathcal{A} \;\middle|\; h = \lim_{\lambda \to \infty} u_\lambda h \right\}$$

and u_λ is an approximate unit for $\mathcal{U}$.

12. Non-AI algebras which are inductive limits of continuous-trace C^*-algebras

In this section we present a necessary and sufficient condition on the invariant for the algebra to be AI. We shall use this in the next section to construct an inductive limit of continuous trace C^*-algebras with spectrum [0, 1] which is not an AI algebra.

With $[G, V, \lambda, f]$ as before we observe that for an AI algebra with Elliott invariant canonically isomorphic to the given invariant the following equality always holds:

$$f(v) = \sup\{v(g)\colon\ g \in D\},$$

where D is the dimension range. This is seen by simply using the fact that any AI algebra has an approximate unit consisting of projections.

Therefore a sufficient condition imposed on the invariant in order to get an inductive limit of continuous trace C^*-algebra with spectrum [0, 1] but not an AI algebra is

$$f(v) \neq \sup\{v(g)\colon\ g \in D\}.$$

This condition is also necessary. Namely assume that we have $f(v) = \sup\{v(g)\colon\ g \in D\}$ and we have constructed a simple C^*-algebra $\mathcal{A}$ which is an inductive limit of continuous trace C^*-algebras with spectrum [0, 1] and with the invariant canonically isomorphic with the tuple $[G, V, \lambda, f]$. Consider $D = \{x \in G\colon\ v(x) < f(v),\ v \in S,\ v \neq 0\}$, where S is a base of the cone V. For the tuple $[G, D, V, S, \lambda]$ we can build (via the range of the invariant for simple AI algebras [14]) a simple AI-algebra $\mathcal{B}$ with the invariant naturally isomorphic with the given tuple.

Note that the trace norm map which is defined starting from the tuple $[K_0(\mathcal{B}), D(\mathcal{B}), T^+\mathcal{B}, \lambda_{\mathcal{B}}]$ is exactly f because of the equality

$$f(v) = \sup\{v(g)\colon\ g \in D\}$$

and $\mathcal{B}$ is an AI algebra.

It is clear that $\mathcal{B}$ is an inductive limit of continuous trace C^*-algebras with spectrum [0, 1] and hence by the isomorphism theorem 3.1 we conclude that $\mathcal{A}$ isomorphic to $\mathcal{B}$. Hence $\mathcal{A}$ is a simple AI algebra as desired and we have proved the following theorem.

Theorem 12.1. *Let $\mathcal{A}$ be a simple C^*-algebra which is an inductive limit of continuous-trace C^*-algebras whose spectrum is homeomorphic to* [0, 1]. *A necessary and sufficient condition for $\mathcal{A}$ to be a simple AI algebra is*

$$f(v) = \sup\{v(g)\colon\ g \in D\}.$$

13. The class of simple inductive limits of continuous trace C^*-algebras with spectrum [0, 1] is much larger than the class of simple AI algebras

To see this consider the simple AI algebra necessarily not of real rank zero with scaled dimension group $(\mathbb{Q}, \mathbb{Q}_+)$ and cone of positive trace functionals a 2-dimensional cone; see [14]. Then the set of possible stably AI algebras, or equivalently the set of possible trace norm maps, may be represented as the extended affine space shown in Fig. 6.

Each off-diagonal point in the diagram is the trace norm map of one of I. Stevens's algebras. The boundary points of the first quadrant are removed (dotted lines) and the points with infinite

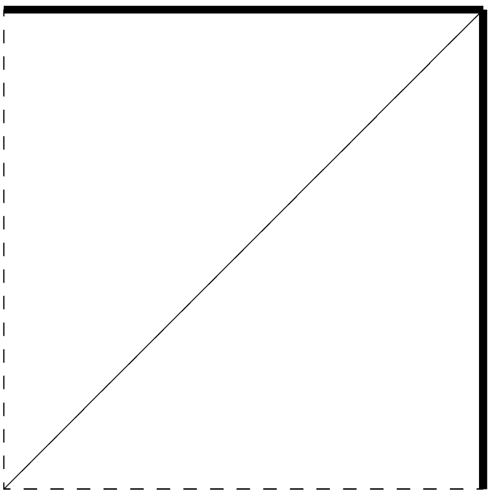

Fig. 6.

coordinates are allowed. The dimension range is embedded in a canonical way in the extended affine space as the main diagonal consisting of the points with rational coordinates.

The two bold lines represent the cases of inductive limits of continuous trace C^*-algebras with unbounded trace norm map (points on these two lines have at least one coordinate infinity).

If the point is off the diagonal and in the first quadrant, by Theorem 12.1 we get that the corresponding C^*-algebra is an inductive limit of continuous trace C^*-algebras which is not AI-algebra. It is clear that the size of the set of points off the diagonal is much larger then the size of the set of points on the diagonal. (For instance in terms of the Lebesgue measure.)

This picture shows that the class of simple AI algebras sits inside the class of inductive limits of continuous trace C^*-algebras in the same way that the main diagonal sits inside the first quadrant.

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Hamiltonian identities for elliptic partial differential equations

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Abstract

New identities for elliptic partial differential equations are obtained. Several applications are discussed. In particular, Young's law for the contact angles in triple junction formation is proven rigorously. Structure of level curves of saddle solutions to Allen–Cahn equation are also carefully analyzed.

Keywords: Hamiltonian identity; Elliptic partial differential equations and systems; Phase transition; Level set; Contact angle; Saddle solutions; Symmetry

1. Introduction and the statement of Hamiltonian type identity

Given a $C^{1,\alpha}$ potential function $H(p)$, $p \in \mathbb{R}^m$, and consider a solution $p(t)$ to a system of second order ordinary differential equation

$$-p''(t) + \nabla_p H\big(p(t)\big) = 0, \quad t \in \mathbb{R}, \tag{1.1}$$

we always have the Hamiltonian identity

$$\frac{1}{2}\big|p'(t)\big|^2 - H\big(p(t)\big) \equiv C, \quad \text{in } \mathbb{R}. \tag{1.2}$$

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Another way of writing the above equation is in the form of first order Hamiltonian system

$$\begin{cases} p' = H_q(p,q), & t \in \mathbb{R}, \\ q' = -H_p(p,q), & t \in \mathbb{R}, \end{cases}$$

where $H(p,q) = \frac{1}{2}|q|^2 - H(p)$. It is a basic and fundamental fact that $H(p,q)$ remains constant in the orbits of the solutions.

On the other hand, consider the case of $m = 1$ and assume that $H \geqslant 0$ and $u(x)$ is a bounded entire solution of the second order elliptic equation

$$-\Delta u(x) + H'\big(u(x)\big) = 0, \quad x \in \mathbb{R}^n. \tag{1.3}$$

Modica proved in [17] a point-wise gradient estimate

$$\frac{1}{2}|\nabla u|^2 - H(u) \leqslant 0, \quad x \in \mathbb{R}^n. \tag{1.4}$$

This inequality may be regarded as a generalization of the Hamiltonian identity to second order partial differential equations with higher spatial dimensions in the case of single equation. It plays an important role in the study of entire solutions, and leads to properties such as monotonicity formula. However, it is only an inequality. This makes one wonder if there exists any identity which could be regarded as a more natural generalization of Hamiltonian identity to partial differential equations in higher dimensions. In particular, we would ask the following questions:

- Is there any identity for partial differential equations which may be a generalization of (1.2)?
- How about systems of partial differential equations?

It is the intention of this article to provide a version of such generalization, which may be called Hamiltonian identity in higher dimensions, and to show some examples of its applications. It would be interesting to see other types of generalizations and applications.

We first state a Hamiltonian identity for partial differential equations on two-dimensional planes, which can be generalized to higher-dimensional spaces. However, due to its simpler formulation and applications, we present it separately.

Consider an entire solution $u \in C^2(\mathbb{R}^2, \mathbb{R}^m)$ to the system of partial differential equations

$$-\Delta u + \nabla_u H\big(u(x)\big) = 0, \quad x \in \mathbb{R}^2. \tag{1.5}$$

Theorem 1.1. *If u is bounded and $u(x_1, x_2)$ converges to $a(x_2), b(x_2)$, respectively, as x_1 tends ∞ and $-\infty$, then the following Hamiltonian identity holds for u:*

$$\int_{-\infty}^{\infty} \left[\frac{1}{2}\left(\left|\frac{\partial u}{\partial x_1}\right|^2 - \left|\frac{\partial u}{\partial x_2}\right|^2\right) + H\big(u(x)\big)\right] dx_1 = C, \quad \forall x_2 \in \mathbb{R}. \tag{1.6}$$

provided that the integral is finite for at least one value of x_2. In general, the identity holds whenever the integral is finite for $x_2 \in \mathbb{R}$ and the limit in the right-hand side of (1.9) *below is zero as N, M go to ∞.*

Proof. Let us define

$$\rho_{N,M}(x_2)=\int_{-M}^{N}\left[\frac{1}{2}\left(\left|\frac{\partial u}{\partial x_1}\right|^2-\left|\frac{\partial u}{\partial x_2}\right|^2\right)+H\big(u(x)\big)\right]dx_1. \tag{1.7}$$

Then, using the equation and integration by parts, we have

$$\begin{aligned}\rho'_{N,M}(x_2)&=\int_{-M}^{N}\big(u_{x_1}\cdot u_{x_1x_2}-u_{x_2}\cdot u_{x_2x_2}+\nabla_u H(u)\cdot u_{x_2}\big)\,dx_1\\&=\int_{-M}^{N}[u_{x_1}\cdot u_{x_1x_2}+u_{x_1x_1}\cdot u_{x_2}]\,dx_1\\&=(u_{x_1}\cdot u_{x_2})(N,x_2)-(u_{x_1}\cdot u_{x_2})(-M,x_2).\end{aligned} \tag{1.8}$$

Without loss of generality, we may assume that the value of x_2 for which the integral in (1.6) is finite is $x_2=0$. We can rewrite the above equality as

$$\rho_{N,M}(x_2)-\rho_{N,M}(0)=\int_{0}^{x_2}\big[(u_{x_1}\cdot u_{x_2})(N,s)-(u_{x_1}\cdot u_{x_2})(-M,s)\big]\,ds. \tag{1.9}$$

Since u is bounded and $H(u)$ is $C^{1,\alpha}$, by the standard elliptic theory we know that u is bounded in $C^2(\mathbb{R}^2,\mathbb{R}^m)$. Furthermore, $u(x_1+N,x_2)$ converges in $C^2_{\mathrm{loc}}(\mathbb{R}^2,\mathbb{R}^m)$ to a solution $u_1(x)$ and $u_1(x_1,x_2)=a(x_2)$. Similarly, $u(x_1-M,x_2)$ converges in $C^2_{\mathrm{loc}}(\mathbb{R}^2,\mathbb{R}^m)$ to a solution $u_2(x)$ and $u_2(x_1,x_2)=b(x_2)$. Therefore $u_{x_1}(x_1,x_2)$ converges to 0 uniformly in any compact set of x_2 as x_1 goes to infinity. The Hamiltonian identity follows immediately by letting N,M in (1.9) go to ∞.

In general, if the right-hand side of (1.9) has zero limit, then the identity (1.6) holds. Therefore, we may write the Hamiltonian identity formally, and verify the limiting procedure in each application. □

The following identity may be regarded as the Hamiltonian identity for higher-dimensional spaces.

Write $x=(x',x_n)\in\mathbb{R}^n$ and consider an entire solution $u\in C^2(\mathbb{R}^n,\mathbb{R}^m)$ to the system of partial differential equations

$$-\Delta u+\nabla_u H\big(u(x)\big)=0,\quad x\in\mathbb{R}^n. \tag{1.10}$$

Theorem 1.2. *The following Hamiltonian identity holds for* u:

$$\int_{\mathbb{R}^{n-1}}\left[\frac{1}{2}\big(|\nabla_{x'}u|^2-|u_{x_n}|^2\big)+H\big(u(x)\big)\right]dx'=C,\quad \forall x_n\in\mathbb{R}, \tag{1.11}$$

provided that the integral is finite for at least one value of x_n and the right-hand side of (1.14) *below tends to zero as R goes to infinity along a sequence.*

Proof. Let us define

$$\rho_R(x_n) = \int\limits_{B_R(0)} \left[\frac{1}{2}\big(|\nabla_{x'}u|^2 - |u_{x_n}|^2\big) + H\big(u(x)\big)\right] dx'. \tag{1.12}$$

Then, using the equation and integration by parts, we have

$$\begin{aligned}
\rho'_R(x_n) &= \int\limits_{B_R(0)} \left[\nabla_{x'}u \cdot \nabla_{x'}u_{x_n} - u_{x_n} \cdot u_{x_n x_n} + \nabla_u H\big(u(x)\big) \cdot u_{x_n}\right] dx' \\
&= \int\limits_{B_R(0)} [\nabla_{x'}u \cdot \nabla_{x'}u_{x_n} + \Delta_{x'}u \cdot u_{x_n}]\, dx' \\
&= \int\limits_{\partial B_R(0)} \left[\frac{\partial u}{\partial \nu_{x'}} \cdot u_{x_n}\right] dS_{x'}.
\end{aligned} \tag{1.13}$$

We may assume that the integral in (1.11) is finite for $x_n = 0$. We can rewrite the above equality as

$$\rho_R(x_n) - \rho_R(0) = \int\limits_0^{x_n} \int\limits_{\partial B_R(0)} \left[\frac{\partial u}{\partial \nu_{x'}}(x', s) \cdot u_{x_n}(x', s)\, dS_{x'}\right] ds. \tag{1.14}$$

The formal identity becomes rigorous, by taking the limit of the above equality as R tends to infinity, under the condition that the limit goes to zero. □

As a special case, the Hamiltonian identity holds with $C = 0$ when the solution belongs to a Sobolev space H^1.

Corollary 1.3. *Assume H is C^2 and $u \in H^1(\mathbb{R}^n, \mathbb{R}^m)$ is a solution to* (1.10). *Then the following Hamiltonian identity holds*:

$$\int\limits_{\mathbb{R}^{n-1}} \left[\frac{1}{2}\big(|\nabla_{x'}u|^2 - |u_{x_n}|^2\big) + H\big(u(x)\big)\right] dx' = 0, \quad \forall x_n \in \mathbb{R}, \tag{1.15}$$

where H is chosen so that $H(0) = 0$.

Proof. We note that u is also a classical solution and $u(x) \to 0$ uniformly as $x \to \infty$, according to the standard theory of elliptic equations. Hence $\nabla H(0) = 0$. Then the integral in (1.15) is finite for at least a sequence of x_n which goes to infinity, since u belongs to $H^1(\mathbb{R}^n, \mathbb{R}^m)$. The same fact also guarantees that the limit condition in Theorem 1.2 holds true and therefore (1.11)

is valid. On the other hand, we know that $\rho(x_n)$ tends to 0 at least along a sequence of x_n tending to infinity. Therefore $C = 0$. □

A typical example of a H^1 solution is the unique positive radial solution of

$$-\Delta u + u - u^p = 0, \quad x \in \mathbb{R}^2,\ 1 < p < \frac{n+2}{N-2}, \tag{1.16}$$

when $H(u) = \frac{1}{2}u^2 - \frac{1}{p+1}u^{p+1}$.

We shall see below that Pohozaev identity can be derived immediately from the above identity. Integrating (1.15) in $\mathbb{R}$ with respect to x_n, we obtain

$$\int_{\mathbb{R}^n} \left[\frac{1}{2}\left(|\nabla_{x'} u|^2 - |u_{x_n}|^2\right) + H\left(u(x)\right)\right] dx = 0. \tag{1.17}$$

Replacing x_n with x_i, we shall obtain $n-1$ similar identities. Sum up all these identities, we derive

$$\int_{\mathbb{R}^n} \left[\frac{n-2}{2}|\nabla u|^2 + nH\left(u(x)\right)\right] dx = 0. \tag{1.18}$$

This is indeed Pohozaev identity in the entire space. We believe that identity (1.15) is a fundamental property of solutions, which gives more detailed information in a lower dimension space and applies to a general class of problems in the whole space.

When a solution u is not in $H^1(\mathbb{R}^n)$, we may still have Hamiltonian identity (1.15) even though Pohozaev identity (1.18) may not hold. A typical example is a solution u of degree $d \geqslant 1$ to the following two-dimensional Ginzburg–Landau equation

$$\Delta u + u\left(1 - |u|^2\right) = 0, \quad u : \mathbb{R}^2 \to \mathbb{R}^2 \approx \mathbb{C}, \tag{1.19}$$

with

$$\int_{\mathbb{R}^2} H(u)\, dx = \frac{1}{4}\int_{\mathbb{R}^2} \left(1 - |u|^2\right)^2 dx = \frac{1}{2}\pi d^2 < \infty. \tag{1.20}$$

Indeed, we can prove

Theorem 1.4. *The solution u of* (1.19) *and* (1.20) *satisfies*

$$\int_{\mathbb{R}} \left[\frac{1}{2}\left(\left|\frac{\partial u}{\partial x_1}\right|^2 - \left|\frac{\partial u}{\partial x_2}\right|^2\right) + \frac{1}{4}\left(1 - \left|u(x)\right|^2\right)^2\right] dx_1 = 0, \quad \forall x_2 \in \mathbb{R}. \tag{1.21}$$

The identity basically follows from (1.6) and the following asymptotic behavior of u at infinity.

Proposition 1.5. *(See [8,20].) Suppose u is a solution to* (1.19) *and* (1.20). *Then there exists $R_0 > 0$ such that $u(x) = f(x)e^{i(d\theta+\psi(x))}$, $\forall x \in B^c_{R_0}$ and*

$$\text{(i)}\qquad f(x) = 1 - \frac{d^2}{2|x|^2} + o\left(\frac{1}{|x|^2}\right), \quad \text{as } |x| \to \infty,$$

$$\text{(ii)}\qquad \left|\nabla f(x)\right| = \frac{d^2}{2|x|^3} + o\left(\frac{1}{|x|^3}\right), \quad \text{as } |x| \to \infty,$$

$$\text{(iii)}\qquad \lim_{|x|\to\infty} \psi(x) = \theta_0, \qquad \int\limits_{|x|\geqslant R_0} \left|\nabla\psi(x)\right|^2 dx < \infty.$$

Proof of Theorem 1.4. We note that Proposition 1.5 leads to

$$|\nabla u|^2 \leqslant 2|\nabla f|^2 + C\left(|\nabla\theta|^2 + |\nabla\psi|^2\right) \leqslant C\left(\frac{1}{|x|^2+1} + |\nabla\psi|^2\right), \quad x \in \mathbb{R}^2. \tag{1.22}$$

Therefore, the integral in (1.21) is finite for almost all $x_2 \in \mathbb{R}$. It is also easy to see that $u(x_1, x_2) \to e^{i\theta_0}$ as $x_1 \to \infty$ and $u(x_1, x_2) \to e^{i(d\pi+\theta_0)}$ as $x_1 \to -\infty$ for any fixed x_2. Therefore, (1.6) holds. By (1.22) and Proposition 1.5, there exists at least a sequence $\{s_n\}$ such that $\lim_{n\to\infty} s_n = \infty$ and

$$\lim_{n\to\infty} \int\limits_{\mathbb{R}} \left|\nabla u(x_1, s_n)\right|^2 dx_1 = 0.$$

It is obvious from (i) of Proposition 1.5 that

$$\lim_{x_2\to\infty} \int\limits_{\mathbb{R}} \left(1 - \left|u(x)\right|^2\right)^2 dx_1 = 0.$$

Hence (1.21) holds. The theorem is proven. □

We note that the solution u to (1.19) and (1.20) does not belong to $H^1(\mathbb{R}^2)$ when $d \geqslant 1$.

In next sections, more applications of the Hamiltonian identity and its modifications shall be discussed. The applications are less obvious and need more analysis. In particular, Section 2 deals with solutions to the vector-valued Allen–Cahn equation in $\mathbb{R}^2$, which needs some preliminaries in the formulation of the problem. Sections 3 and 4 deal with sign changing solutions to the scalar Allen–Cahn equation, which is conceptually easier to understand than Section 2, but contain technically harder analysis. It is arranged that Section 3 consists of the main ideas with simple formulation and Section 4 is devoted to some technical details. The reader may choose either Section 2 or Section 3 to start with.

2. Triple junctions and the Young's law

In the study of multiple phase separation, a vector-valued Allen–Cahn model was proposed by Bronsard and Reitich in [9]. In this model, a physical state of material of multiple phases is

represented by an order parameter (vector-valued function) $v \in \mathbb{R}^2$. The dynamics of the physical state may be modeled by an Allen–Cahn type system of partial differential equation

$$v_t = \epsilon \Delta v - \frac{1}{\epsilon} \nabla_v W(v), \quad x \in \Omega,\ t > 0, \tag{2.1}$$

where $W \in C^{1,\alpha}(\mathbb{R}^2 \to \mathbb{R})$ is a triple well potential satisfying

(H1) there exist three points $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^2$ such that $W(\mathbf{a}) = W(\mathbf{b}) = W(\mathbf{c}) = 0$ and $W(u) > 0$ for $u \in \mathbb{R}^2 \setminus \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$, and $D^2 W(\mathbf{a})$, $D^2 W(\mathbf{b})$ and $D^2 W(\mathbf{c})$ are positive definite;
(H2) there exists $R_0 > 0$ such that $\nabla W(u) \cdot u \geqslant 0$ when $|u| \geqslant R_0$.

Choose any two wells $\mathbf{x}, \mathbf{y} \in \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$, we may consider the minimization problem

$$\mathbf{e_{xy}} = \min\left\{ \int_{\mathbb{R}} \frac{1}{2}|v'|^2 + W(v)\, dt \;\middle|\; v \in H^1_{\mathrm{loc}}(\mathbb{R})^2,\ v(-\infty) = \mathbf{x},\ v(\infty) = \mathbf{y} \right\}. \tag{2.2}$$

It can be shown that $\mathbf{e_{xy}} > 0$ (see, e.g., [23,24]). It is also shown in [1] that there is at least one minimizer $\mathbf{u_{xy}}$ for (2.2) as long as the following partial wetting condition holds:

$$\mathbf{e_{xy}} < \mathbf{e_{xz}} + \mathbf{e_{yz}}, \quad z \in \{\mathbf{a}, \mathbf{b}, \mathbf{c}\} \setminus \{\mathbf{x}, \mathbf{y}\}. \tag{2.3}$$

The minimizer is a heteroclinic solution. See also [3] for more detailed discussion regarding the existence of heteroclinic solutions.

To make our arguments more transparent, we assume in this section that

(H3) the wetting condition (2.3) holds and $\mathbf{u_{xy}}$ is unique up to translation for all $\mathbf{x}, \mathbf{y}$.

We say that a triple well potential W is of symmetry of an equilateral triangle if it satisfies

(H4) three wells $\mathbf{a}, \mathbf{b}, \mathbf{c}$ form an equilateral triangle and the potential W is equivariant under the group action of the isometry group Γ of the triangle.

An example of a triple well potential which satisfies (H1), (H2) and (H4) is

$$W(u) = \left|u^3 - 1\right|^2, \quad u \in \mathbb{R}^2 \approx \mathbb{C}.$$

A special feature of multiple phase separation is the formation of triple junctions, which is analyzed formally in [9]. The finer structure of triple junctions may be demonstrated by an entire solution u to the following system of elliptic equations (vector-valued equation)

$$-\Delta u + \nabla_u W(u) = 0, \quad u : \mathbb{R}^2 \to \mathbb{R}^2, \tag{2.4}$$

with u asymptotically close to $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in three separate sectors of $\mathbb{R}^2$.

Under the conditions (H1), (H2) and (H4), Bronsard, Gui and Schatzman proved rigorously in [10] the existence of such a triple junction solution. To be more precise, a simple version of the main result of [10] may be stated as follows.

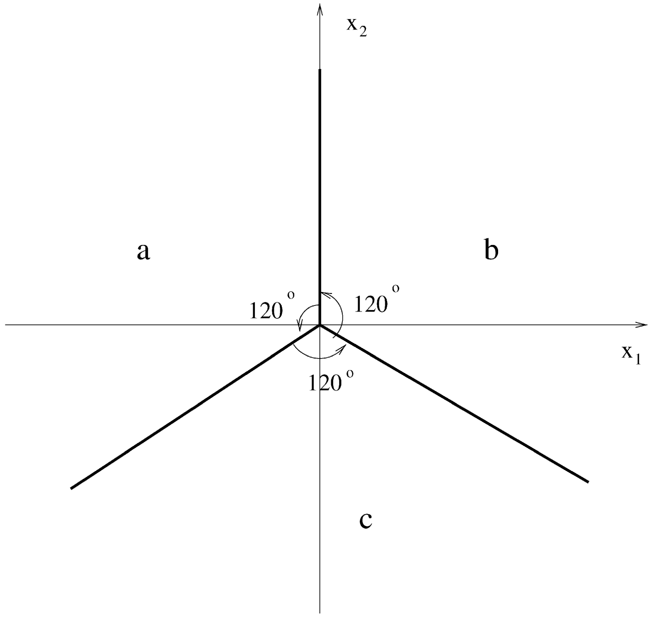

Fig. 1. The triple junction solution u with regions close to **a**, **b**, **c**, respectively.

Theorem 2.1. *(See [10].) Suppose W satisfies* (H1), (H2) *and* (H4). *Then*:

(1) *There exists a nontrivial bounded Γ-equivariant solution* $\mathbf{U}$ *in* $C^3(\mathbb{R}^2, \mathbb{C})$ *to* (2.4). *If we identify $x = (x_1, x_2)$ with $x_1 + ix_2 = re^{i\theta}$, then for any $\eta \in (0, \pi/3)$,* $\mathbf{U}(re^{i\theta})$ *converges to* **a** *uniformly with respect to $\theta \in [(\pi/2) + \eta, (7\pi/6) - \eta]$ as r tends to infinity.*
(2) *The solution* $\mathbf{U}(x_1, x_2)$ *converges to* $\mathbf{u_{ab}}$ *uniformly on* $\mathbb{R}$ *as x_2 goes to infinity.*

In other words, $\mathbf{U}$ *is a solution with a triple junction structure, i.e.,* $\mathbf{U}$ *has three transition layers separating the regions where* $\mathbf{U}$ *is close to* **a**, **b** *or* **c**, *respectively.* (*See Fig.* 1.)

It is natural to ask if there is any other solution to (2.4) which is not necessarily symmetric with respect to Γ, but still displays a triple junction structure. This question seems very difficult to answer now. It would be interesting to ask whether a triple junction solution should be asymptotically symmetric. If we call the angles between the interfaces contact angles, the question would be whether the contact angles of any triple junction solutions must be the same. In physics theory regarding the interfaces of materials, the contact angles near a triple junction are determined by the tensions at the interfaces between the different materials according the Young's law (see [25]):

$$\frac{k_1}{\sin\theta_1} = \frac{k_2}{\sin\theta_2} = \frac{k_3}{\sin\theta_3}, \tag{2.5}$$

where k_i are the surface tension between two materials and θ_i are contact angle of the corresponding two materials. Regarding the limiting problem of (2.1), which is a geometric evolution problem, a formal analysis leads to Young's law, with $\mathbf{e_{xy}}$ being the surface tension between the phases represented by $\mathbf{x}, \mathbf{y}$. See [9,12]. We shall show rigorously below the counterpart of

Young's law for system (2.4). In particular, we answer positively the above question of equal angles for symmetric triple well potential.

Theorem 2.2. *Suppose W satisfies* (H1)–(H3) *and $u \in C^2(\mathbb{R}^2, \mathbb{R}^2)$ is a solution to* (2.4) *with the following triple junction structure*:

(1) *if $\mathbb{R}^2 \approx \mathbb{C}$ is divided into three sections $S_1 := \{x = re^{i\theta} \mid 0 < \theta < \theta_{\mathbf{ca}}\}$, $S_2 := \{x = re^{i\theta} \mid \theta_{\mathbf{ca}} < \theta < \theta_{\mathbf{ca}} + \theta_{\mathbf{ab}}\}$, $S_3 := \{x = re^{i\theta} \mid \theta_{\mathbf{ca}} + \theta_{\mathbf{ab}} < \theta < \theta_{\mathbf{ca}} + \theta_{\mathbf{ab}} + \theta_{\mathbf{bc}} = 2\pi\}$, then $u(re^{i\theta})$ converges to $\mathbf{b}$ in S_1 as the distance $d(x)$ to the boundary of S_1 goes to infinity. Similar statements holds for S_2 and S_3 with limits to $\mathbf{c}$, $\mathbf{a}$, respectively*;
(2) *for any sufficiently small $\delta > 0$, $u(x_1, x_2)$ converges to $\mathbf{u_{ab}}(x_2)$ uniformly in $S_{13}^{\delta} := \{x = re^{i\theta} \mid \theta \in [-\theta_{\mathbf{bc}} + \delta, \theta_{\mathbf{ca}} - \delta]\}$ as x_1 goes to infinity. Similar statement holds for S_{12}^{δ}, S_{23}^{δ} with limiting transitions $\mathbf{u_{ac}}$ and $\mathbf{u_{cb}}$, respectively.*

Then the following Young's law holds:

$$\frac{\mathbf{e_{ab}}}{\sin\theta_{\mathbf{ab}}} = \frac{\mathbf{e_{bc}}}{\sin\theta_{\mathbf{bc}}} = \frac{\mathbf{e_{ca}}}{\sin\theta_{\mathbf{ca}}}. \tag{2.6}$$

Proof. We shall use the Hamiltonian identity in Theorem 1.1 to prove this theorem. We shall first show that in S_1 the solution u is exponentially close to $\mathbf{b}$ in terms of $d(x)$ as $d(x)$ goes to infinity. Similar estimate can be proven for u in S_2, S_3. Since u goes to $\mathbf{b}$ as $d(x)$ goes to infinity in S_1, and $D^2W(\mathbf{b})$ is positive definite, then

$$(u - \mathbf{b}) \cdot \big(\nabla_u W(u) - \nabla_u W(\mathbf{b})\big) \geqslant \mu |u - \mathbf{b}|^2, \quad \text{when } d(x) \geqslant D_0$$

for some positive constants μ and D_0.

Then, from (2.4) we obtain

$$\begin{aligned} \Delta |u - \mathbf{b}|^2 &\geqslant 2\Delta(u - \mathbf{b}) \cdot (u - \mathbf{b}) \\ &\geqslant (u - \mathbf{b}) \cdot \big(\nabla_u W(u) - \nabla_u W(\mathbf{b})\big) \\ &\geqslant \mu |u - \mathbf{b}|^2, \quad \forall d(x) > D_0,\ x \in S_1. \end{aligned} \tag{2.7}$$

Choose exponential function $Ce^{-2\alpha d(x)}$ as a comparison function and apply the maximum principle to the above inequality for $|u - \mathbf{b}|^2$ (see e.g. [14,15]). We obtain

$$\big|u(x) - \mathbf{b}\big| \leqslant Ce^{-\alpha d(x)}, \quad x \in S_1, \tag{2.8}$$

for some positive constant C, α. By the standard theory for elliptic equations, we can obtain

$$\big|\nabla u(x)\big| \leqslant Ce^{-\alpha d(x)}, \quad x \in S_1. \tag{2.9}$$

Then, u satisfies the condition of Theorem 1.1, and we can apply (1.6) to u (with x_1 and x_2 switched) to obtain

$$\rho(x_1) := \int_{-\infty}^{\infty} \left[\frac{1}{2}\left(\left|\frac{\partial u}{\partial x_2}\right|^2 - \left|\frac{\partial u}{\partial x_1}\right|^2\right) + W\big(u(x)\big)\right] dx_2 = C, \quad \forall x_1 \in \mathbb{R}. \tag{2.10}$$

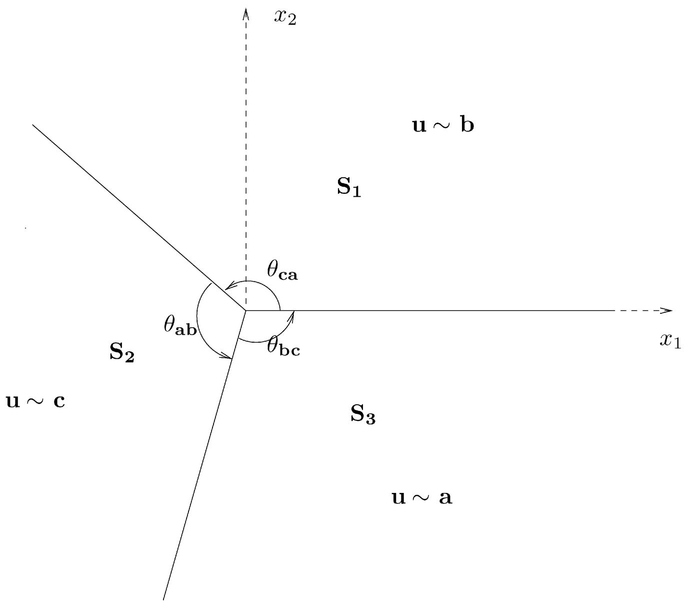

Fig. 2. The three sectors where the value of u is close to **a**, **b**, **c**, respectively.

In the case that all angles $\theta_{\mathbf{ab}}, \theta_{\mathbf{bc}}, \theta_{\mathbf{ca}}$ are in $(\pi/2, \pi)$, by using assumption (2) in Theorem 2.2 it is easy to see that $u(x_1 + s, x_2)$ converges to $\mathbf{u_{ab}}(x_2)$ in $C^1_{\mathrm{loc}}(\mathbb{R}^2)$. Therefore, we derive (see Fig. 2)

$$\lim_{x_1 \to \infty} \rho(x_1) = \mathbf{e_{ab}}.$$

On the other hand, by assumption (2) in Theorem 2.2 we also have

$$\begin{cases} \left\| u(x_1, x_2) - u_{\mathbf{bc}}\big(-x_1 \sin(\theta_{\mathbf{ca}}) - x_2 \cos(\theta_{\mathbf{ca}})\big) \right\|_{C^1(\mathbb{R}^+)} \to 0, \\ \left\| u(x_1, x_2) - u_{\mathbf{ca}}\big(x_1 \sin(\theta_{\mathbf{bc}}) - x_2 \cos(\theta_{\mathbf{bc}})\big) \right\|_{C^1(\mathbb{R}^-)} \to 0 \end{cases} \tag{2.11}$$

as $x_1 \to -\infty$.

Then, in view of the exponential convergence of u to **b**, **c** in S_2, S_3, respectively, we have

$$\begin{aligned} &\lim_{x_1 \to -\infty} \int_0^\infty \left[\frac{1}{2} \left(\left| \frac{\partial u}{\partial x_2} \right|^2 - \left| \frac{\partial u}{\partial x_1} \right|^2 \right) + W\big(u(x)\big) \right] dx_2 \\ &= \int_{-\infty}^\infty \left[\frac{1}{2} \left| \frac{\partial}{\partial x_2} \mathbf{u_{bc}}\big(-x_1 \sin(\theta_{\mathbf{ca}}) - x_2 \cos(\theta_{\mathbf{ca}})\big) \right|^2 \right. \\ &\quad - \frac{1}{2} \left| \frac{\partial}{\partial x_1} \mathbf{u_{bc}}\big(-x_1 \sin(\theta_{\mathbf{ca}}) - x_2 \cos(\theta_{\mathbf{ca}})\big) \right|^2 \\ &\quad \left. + W\big(\mathbf{u_{bc}}\big(-x_1 \sin(\theta_{\mathbf{ca}}) - x_2 \cos(\theta_{\mathbf{ca}})\big)\big) \right] dx_2 \end{aligned}$$

$$= -\frac{1}{2}\mathbf{e_{bc}}\left[\frac{\cos^2(\theta_{\mathbf{ca}})}{\cos(\theta_{\mathbf{ca}})} - \frac{\sin^2(\theta_{\mathbf{ca}})}{\cos(\theta_{\mathbf{ca}})} + \frac{1}{\cos(\theta_{\mathbf{ca}})}\right]$$
$$= -\mathbf{e_{bc}}\cos(\theta_{\mathbf{ca}}). \tag{2.12}$$

Similarly, we have

$$\lim_{x_1\to-\infty}\int_{-\infty}^{0}\left[\frac{1}{2}\left(\left|\frac{\partial u}{\partial x_2}\right|^2 - \left|\frac{\partial u}{\partial x_1}\right|^2\right) + W(u(x))\right]dx_2 = -\mathbf{e_{ca}}\cos(\theta_{\mathbf{bc}}). \tag{2.13}$$

Therefore we obtain

$$\mathbf{e_{ab}} = -\mathbf{e_{bc}}\cos(\theta_{\mathbf{ca}}) - \mathbf{e_{ca}}\cos(\theta_{\mathbf{bc}}). \tag{2.14}$$

If we change the coordinates so that the x_1-axis becomes the direction of $\bar{S}_1 \cap \bar{S}_2$ and $\bar{S}_2 \cap \bar{S}_3$, respectively, and apply the Hamiltonian identity as above, we can also obtain

$$\begin{cases} \mathbf{e_{bc}} = -\mathbf{e_{ca}}\cos(\theta_{\mathbf{ab}}) - \mathbf{e_{ab}}\cos(\theta_{\mathbf{ca}}), \\ \mathbf{e_{ca}} = -\mathbf{e_{ab}}\cos(\theta_{\mathbf{bc}}) - \mathbf{e_{bc}}\cos(\theta_{\mathbf{ab}}). \end{cases} \tag{2.15}$$

In view of $\theta_{\mathbf{ab}} + \theta_{\mathbf{bc}} + \theta_{\mathbf{ca}} = 2\pi$, we derive (2.6) from (2.14) and (2.15) immediately.

Using the above procedure, we can indeed prove that

$$\frac{\pi}{2} < \theta_{\mathbf{ab}},\ \theta_{\mathbf{bc}},\ \theta_{\mathbf{ca}} < \pi. \tag{2.16}$$

This finishes the proof. □

An immediate corollary of Theorem 2.2 is that a triple junction solution for (2.4) with symmetric potential W must have equal contact angles.

3. Saddle solutions to Allen–Cahn equation in $\mathbb{R}^2$

Allen–Cahn equation is a well-known model for bi-phase transition. It is stationary equation in entire space is

$$-\Delta u + F'(u) = 0, \quad |u| < 1,\ x \in \mathbb{R}^n, \tag{3.1}$$

where $F(u)$ is a double well potential with equal depths at $u = 1, -1$, and the scalar function u represents the physical state of a mixture of two materials, with $u \equiv \pm 1$ being two pure phases. A typical double well potential is $F(u) = \frac{1}{4}(1 - u^2)^2$. An entire solution to (3.1) represents a local structure of phase transition near interface or singularities. Regarding monotone solutions of (3.1), i.e., $u_{x_n}(x', x_n) > 0$ in $\mathbb{R}^n$, De Giorgi conjectured in [13] that all such solutions must depend on one direction when $n \leqslant 8$. The conjecture has been proved for $n = 2$ in [14] and $n = 3$ in [4]. For dimensions up to 8, the conjecture is essentially proved in [18], provided that u satisfies the limiting condition

$$\lim_{x_n\to\pm\infty} u(x', x_n) = \pm 1, \quad \forall x' \in \mathbb{R}^{n-1}. \tag{3.2}$$

Related results can also be found in [2,5,7,19,22], etc. Therefore, all monotone entire solutions to Allen–Cahn equation for $n = 2, 3$ or $4 \leqslant n \leqslant 8$ with (3.2) are like $g(x \cdot \nu + a)$ for some $a \in \mathbb{R}^n$ and $\nu \in S^{n-1}$, where g is the unique solution (up to translation) to the corresponding ordinary differential equation

$$-g''(t) + F'\big(g(t)\big) = 0, \quad g'(t) > 0,\ t \in \mathbb{R}. \tag{3.3}$$

We may fix g so that $g(0) = 0$. This solution can also be regarded as a minimizer of

$$\min\left\{E(v) = \int_{-\infty}^{\infty} \frac{1}{2}\big|v'(t)\big|^2 + F\big(v(t)\big)\,dt\colon v \in H^1_{\mathrm{loc}}(\mathbb{R}),\ \lim_{t\to\pm\infty} v = \pm 1\right\} \tag{3.4}$$

with minimum energy

$$\mathbf{e} = \mathbf{e}_F := \int_{-1}^{1} \sqrt{2F(u)}\,du. \tag{3.5}$$

When the potential $F(u)$ is an even function, it is obvious that g is odd.

There are also other types of solutions to (3.1) which are not monotone. In particular, saddle solutions are shown to exist in [11] for some *even* potential F. Indeed, the following slightly more general existence theorem can be proven. For simplicity, below we will only discuss the two-dimensional case $n = 2$ and assume that F is a C^2 function satisfying

$$\begin{cases} F(1) = F(-1) = 0, \quad F(u) > 0,\ \forall u \in (-1, 1), \\ F'(-1) = F'(1) = 0, \quad F''(-1) > 0,\ F''(1) > 0, \\ F(u) \text{ has only one critical point in } (-1, 1). \end{cases} \tag{3.6}$$

We define $Q^1 := \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_1 > 0,\ x_2 > 0\}$ and similarly we can define Q^2, Q^3, Q^4.

Proposition 3.1. *If we assume that F is an even function and satisfies* (3.6)*, then there exists a saddle solution* $\mathbf{u}$ *to* (3.1) *such that*

$$\begin{cases} \mathbf{u}(x_1, x_2) = -\mathbf{u}(x_1, -x_2) = -\mathbf{u}(-x_1, x_2), \quad \forall x \in \mathbb{R}^2; \\ \mathbf{u}(x) > 0, \quad \forall x \in Q^1 \cup Q^3; \qquad \mathbf{u}(x) < 0, \quad \forall x \in Q^2 \cup Q^4. \end{cases} \tag{3.7}$$

It is easy to see that $\mathbf{u}$ is unique and has another symmetry:

$$\mathbf{u}(x_1, x_2) = \mathbf{u}(x_2, x_1) = \mathbf{u}(-x_2, -x_1), \quad x \in \mathbb{R}^2. \tag{3.8}$$

The reader may use the direct variational method or the super–sub-solution method to solve the boundary value problem in $Q^1_R = \{x = (x_1, x_2) \mid x_1 > 0,\ x_2 > 0,\ |x| \leqslant R\}$ with 0 boundary value on both axes and $u = 1$ on the remaining boundary, and hence obtain the desired solution as the limit by taking R to infinity. It can be easily proven that the limiting solution is not trivial by constructing a positive subsolution.

Remark 3.2. In [11], it is assumed that F satisfies an additional condition:

$$\frac{F'(u)}{u} \quad \text{is increasing in } (0,1).$$

This condition can be dropped for both the existence and uniqueness of $\mathbf{u}$, as in Proposition 3.1. See Corollary 3.9 below or [16] for a more detailed proof.

Definition 3.3. We may call a solution of (3.1) a *saddle solution* if its 0-level set consists of exactly two non self-intersecting C^1 curves which intersect each other at most once.

There are two natural questions regarding saddle solutions:

- Does there exist any saddle solution to (3.1) when F is not even?
- Are there any saddle solutions other than $\mathbf{u}$ (and its rotation and translation) when F is even?

If the answer to the second question is affirmative, can we classify all saddle solutions? Or can we show some properties of the solutions such as symmetry?

Regarding the first question, it is claimed in [21] that a saddle solution with 0-level set being the two axes *does* exist. However, existence of such a saddle solution is very counter intuitive. There has been doubt of this result among researchers of Allen–Cahn equation, even though there is no counter example or argument to disprove it. Here we give a rigorous proof that the result is indeed wrong, by using the Hamiltonian identity (1.6). To be more precise, we have proved the following necessary condition for the existence of the above mentioned saddle solutions.

Theorem 3.4. *Suppose F satisfies* (3.6) *and u is a solution to* (3.1) *satisfying*

$$u(x) > 0, \quad \forall x \in Q^1 \cup Q^3; \qquad u(x) < 0, \quad \forall x \in Q^2 \cup Q^4. \tag{3.9}$$

Then $F'(0) = 0$ and

$$\int_0^1 \sqrt{F(u)}\,du = \int_{-1}^0 \sqrt{F(u)}\,du. \tag{3.10}$$

Proof. Let

$$F''(1) = \lambda_1^2, \qquad F''(-1) = \lambda_2^2.$$

For any $\epsilon > 0$, by using comparison functions of the form $Ce^{-\lambda|x_i|}$ in proper region and the maximum principle, we can obtain

$$\begin{cases} \left|u(x_1,x_2) - 1\right| \leqslant C_{1,\epsilon} e^{-(\lambda_1 - \epsilon)\min\{|x_1|,|x_2|\}}, & x \in Q^1 \cup Q^3, \\ \left|u(x_1,x_2) + 1\right| \leqslant C_{1,\epsilon} e^{-(\lambda_2 - \epsilon)\min\{|x_1|,|x_2|\}}, & x \in Q^2 \cup Q^4. \end{cases} \tag{3.11}$$

The standard gradient estimate for elliptic equations lead to

$$\begin{cases} |\nabla u| \leqslant C_{2,\epsilon} e^{-(\lambda_1 - \epsilon)\min\{|x_1|,|x_2|\}}, & x \in Q^1 \cup Q^3, \\ |\nabla u| \leqslant C_{2,\epsilon} e^{-(\lambda_2 - \epsilon)\min\{|x_1|,|x_2|\}}, & x \in Q^2 \cup Q^4, \end{cases} \tag{3.12}$$

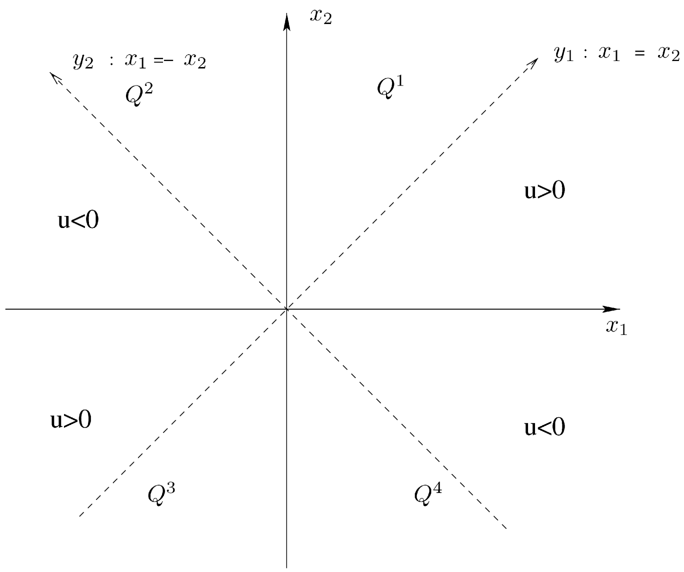

Fig. 3. Applying Hamiltonian identity to saddle solutions.

for some constants $C_{i,\epsilon} > 0$, $i = 1, 2$.

Furthermore, we have

$$\begin{cases} \|u(x_1, x_2) \pm g(x_1)\|_{C^1(\mathbb{R})} \to 0 & \text{as } x_2 \to \infty, \\ \|u(x_1, x_2) \pm g(x_2)\|_{C^1(\mathbb{R})} \to 0 & \text{as } x_1 \to \mp\infty. \end{cases} \tag{3.13}$$

We also note that such a solution is unique and u satisfies (3.8).

We shall prove the (3.10) by applying the Hamiltonian identity to (3.1). For this purpose, we choose a new coordinates (y_1, y_2) so that y_1-axis and y_2-axis coincide with the lines $\{x \mid x_1 = x_2\}$ and $\{x \mid x_1 = -x_2\}$, respectively (see Fig. 3). Now applying Theorem 1.1 (with x_1, x_2 replaced by y_2, y_1, respectively), we obtain

$$\rho(y_1) := \int_{-\infty}^{\infty} \frac{1}{2}\left(\left|\frac{\partial u}{\partial y_2}\right|^2 - \left|\frac{\partial u}{\partial y_1}\right|^2\right) + F\big(u(y)\big)\, dy_2 = C, \quad \forall y_1 \in \mathbb{R}. \tag{3.14}$$

A straightforward computation as in (2.12) leads to

$$\lim_{y_1 \to \infty} \rho(y_1) = \sqrt{2}\mathbf{e}_F = 2\int_{-1}^{1} \sqrt{F(u)}\, du. \tag{3.15}$$

Hence

$$\rho(0) = \int_{-\infty}^{\infty} \left[\frac{1}{2}\left|\frac{\partial u}{\partial y_2}\right|^2 (0, y_2) + F\big(u(0, y_2)\big)\right] dy_2 = 2\int_{-1}^{1} \sqrt{F(u)}\, du. \tag{3.16}$$

Note that in the above equality the derivative u_{y_1} vanishes on y_2-axis due to (3.8). Now we modify F to get an even double potential

$$\tilde{F}(u) = \begin{cases} F(u), & u \leqslant 0, \\ F(-u), & u \geqslant 0. \end{cases} \tag{3.17}$$

It is obvious to see from Eq. (3.1) that $F'(0) = 0$. Hence $\tilde{F}$ is also a $C^{1,\alpha}$ function and satisfies (3.6). By Theorem 3.1, there exists a saddle solution $\tilde{u}$ to (3.1) with F replaced by $\tilde{F}$ and $\tilde{u}$ satisfies (3.7) and (3.8). The application of Hamiltonian identity to $\tilde{u}$ leads to

$$\tilde{\rho}(0) = \int_{-\infty}^{\infty} \left[\frac{1}{2}\left|\frac{\partial \tilde{u}}{\partial y_2}\right|^2 + F\big(\tilde{u}(y)\big)\right] dy_2 = 2\int_{-1}^{1} \sqrt{\tilde{F}(u)}\, du. \tag{3.18}$$

By the uniqueness of $\tilde{u}$ (see Remark 3.2), we know

$$u(x) = \tilde{u}(x), \quad \forall x \in Q^2 \cup Q^4, \tag{3.19}$$

and therefore $\rho(0) = \tilde{\rho}(0)$. Then

$$\int_{-1}^{1} \sqrt{F(u)}\, du = \int_{-1}^{1} \sqrt{\tilde{F}(u)}\, du \tag{3.20}$$

and (3.10) follows immediately from the definition of $\tilde{F}$. □

It remains a question whether F must be an even function in order to have a saddle solution u of (3.1) satisfying (3.9).

Now we discuss the contact angles at infinity for saddle solutions.

Definition 3.5. If the two 0-level curves are asymptotically two intersecting straight lines at infinity, we call the acute angle θ between these two lines the contact angle at infinity.

We have the following partial result.

Theorem 3.6. *Assume that F is a double well potential satisfying* (3.6) *and* (3.10). *Suppose that u is a solution to* (3.1) *with a contact angle θ at infinity. We further assume that u satisfies* (3.8) *and $u(0) = 0$. Then we have*

$$\pi/3 < \theta \leqslant \pi/2. \tag{3.21}$$

Proof. Without loss of generality, we may assume that the angle θ is centered at y_2-axis and let $\theta = 2\alpha$. Following the proof of Theorem 3.4, we can obtain

$$\rho(0) = \int_{-\infty}^{\infty} \left[\frac{1}{2}\left|\frac{\partial u}{\partial y_2}\right|^2 + F\big(u(y)\big)\right] dy_2 = 2\mathbf{e}_F \sin(\alpha). \tag{3.22}$$

On the other hand, in view of $u(0) = 0$ we know

$$\rho(0) = \int_{-\infty}^{\infty} \left[\frac{1}{2} \left| \frac{\partial u}{\partial y_2} \right|^2 (0, y_2) + F\big(u(0, y_2)\big) \right] dy_2 > \mathbf{e}_F. \tag{3.23}$$

Hence $\sin(\alpha) > 1/2$ and the theorem is proven. □

We propose the following conjecture.

Conjecture 3.7. *The contact angle $\theta = 2\alpha$ should be exactly $\pi/2$ under the assumptions of Theorem* 3.6*.*

So far, only for a very special case when F is even and the 0-level set of u consists of two intersecting lines, we can confirm the conjecture. For this purpose, we study positive solutions of Allen–Cahn equation in a sector

$$S_\alpha = \left\{ x = re^{i\theta} \;\middle|\; r > 0,\ -\alpha < \theta < \alpha \right\} \tag{3.24}$$

with condition

$$u(x) > 0, \quad \forall x \in S_\alpha; \qquad u(x) = 0, \quad \forall x \in \partial S_\alpha. \tag{3.25}$$

Similar to the existence of a solution $\mathbf{u}$ in Q^1, it is easy to prove the existence of a solution $\mathbf{u}_\alpha$ to (3.1) with condition (3.25). Furthermore, as for the symmetric saddle solution we have the following estimates for $\mathbf{u}_\alpha$:

$$1 - Ce^{-\kappa r \sin(\alpha - |\theta|)} \leqslant \mathbf{u}_\alpha\big(re^{i\theta}\big) < 1, \quad \forall x = re^{i\theta} \in S_\alpha, \tag{3.26}$$

and

$$\mathbf{u}_\alpha\big(re^{i\theta}\big) - g\big(r \sin\big(\alpha - |\theta|\big)\big) \to 0, \quad \text{uniformly in } S_\alpha \text{ as } r\cos(\alpha - \theta) \to \infty. \tag{3.27}$$

Now we prove a monotonicity property of $\mathbf{u}_\alpha$ in terms of α. Suppose $\alpha > \beta$. For any $\lambda \in [-(\alpha - \beta), \alpha - \beta]$, define

$$\mathbf{u}_\beta^\lambda\big(re^{i\theta}\big) = \mathbf{u}_\beta\big(re^{i(\theta - \lambda)}\big), \quad \forall x \in S_\beta^\lambda, \tag{3.28}$$

where

$$S_\beta^\lambda = \left\{ x = re^{i\theta} \;\middle|\; \theta \in (\lambda - \beta, \lambda + \beta),\ r > 0 \right\}. \tag{3.29}$$

See Fig. 4.

Lemma 3.8. *If $\alpha \geqslant \beta$, then the following inequality holds*:

$$\mathbf{u}_\alpha(x) \geqslant \mathbf{u}_\beta^\lambda(x), \quad \forall x = re^{i\theta} \in S_\beta^\lambda,\ \forall \lambda \in \big[-(\alpha - \beta), \alpha - \beta\big]. \tag{3.30}$$

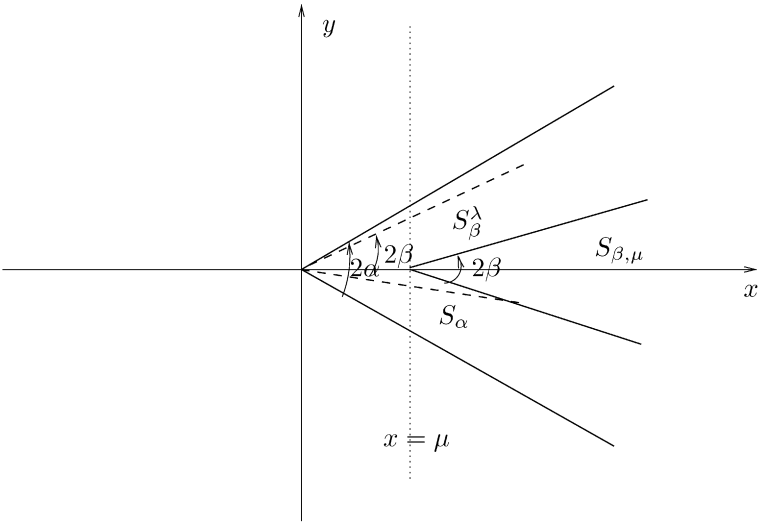

Fig. 4. The translation and rotation of a cone inside a bigger cone.

In particular, if the strict inequality holds if $\alpha > \beta$. In other words, if we rotate the cone S_β inside the cone S_α, the graph of $\mathbf{u}_\beta$ shall always be below that of $\mathbf{u}_\alpha$.

Proof. We first consider a shifted cone $S_{\beta,\mu} := \{x \mid (x_1 - \mu, x_2) \in S_\beta\}$ and $\mathbf{u}_{\beta,\mu}(x) := \mathbf{u}_\beta(x_1 - \mu, x_2)$, $x \in S_{\beta,\mu}$. It is clear that $\mathbf{u}_{\beta,\mu}(x)$ is a solution to (3.1) in $S_{\beta,\mu}$. Below we shall use the sliding plane method to prove $\mathbf{u}_\alpha(x) \geqslant \mathbf{u}_\beta(x)$ in S_β. From (3.6), there is an constant $\delta > 0$ such that $F''(u) > 0$ when $u \in (1-\delta, 1]$. By (3.26) and (3.27), we know that when μ is large enough, $\mathbf{u}_\alpha(x) > 1-\delta$ in $S_{\beta,\mu}$, and $\mathbf{u}_\alpha(x) \geqslant \mathbf{u}_{\beta,\mu}(x)$ as $x \to \infty$ in $S_{\beta,\mu}$ or as $x \to \partial S_{\beta,\mu}$. By the maximum principle, we obtain

$$\mathbf{u}_\alpha(x) \geqslant \mathbf{u}_{\beta,\mu}(x), \quad \forall x \in S_{\beta,\mu}, \tag{3.31}$$

for μ large enough.

Then, we can decrease μ to 0 while still keep (3.31) true by the so-called sliding plane method as follows:

Let

$$\mu_0 := \min\bigl\{\mu \bigm| \text{inequality (3.31) holds}\bigr\}.$$

We claim that $\mu_0 = 0$. If this is not true, then there exist a sequence $\{\mu_n\}_1^\infty$ and a sequence of points $\{\eta_n\}_1^\infty$ such that $\mu_n \leqslant \mu_0$, $\lim_{n\to\infty} \mu_n = \mu_0$ and

$$\mathbf{u}_\alpha(\eta_n) < \mathbf{u}_{\beta,\mu_n}(\eta_n), \quad \forall n.$$

By the asymptotical behavior (3.27) for both $\mathbf{u}_\alpha$ and $\mathbf{u}_\beta$, it is easy to see that $\{\eta_n\}$ is bounded and therefore possesses a convergent subsequence with limit η.

Then $\mathbf{u}_\alpha(\eta) \leqslant \mathbf{u}_{\beta,\mu_0}(\eta)$. Recall that by the definition of μ_0, (3.31) holds with $\mu = \mu_0$. Then, the strong maximum principle implies

$$\mathbf{u}_\alpha(x) = \mathbf{u}_{\beta,\mu_0}(x), \quad \forall x \in S_{\beta,\mu_0}.$$

This is a contradiction due to the zero boundary condition for the solutions. Hence $\mu_0 = 0$ and the lemma holds with $\lambda = 0$.

Then we rotate S_β and apply the above arguments (usually called the rotating plane method) to $\mathbf{u}_\beta^\lambda$ in S_β^λ with λ from 0 to $\alpha - \beta$ or to $-(\alpha - \beta)$. The lemma follows immediately. □

Corollary 3.9. *It is easy to see from the above lemma that u_α is unique, by choosing $\beta = \alpha$ and exchanging the order of two possible solutions.*

Theorem 3.10. *Assume F is even and satisfies* (3.6) *and u is a saddle solution to* (3.1) *with* 0*-level set being two straight lines with contact angle θ. Then $\theta = \pi/2$.*

Proof. Let $\alpha = \theta/2$. We just note that if $\theta < \frac{\pi}{2}$, then $\frac{\pi}{2} - \alpha > \alpha$. By Lemma 3.8 we have

$$\mathbf{u}_{\frac{\pi}{2}-\alpha}(x) > \mathbf{u}_\alpha^{\frac{\pi}{2}-2\alpha}(x), \quad \forall x \in S_\alpha^{\frac{\pi}{2}-2\alpha}. \tag{3.32}$$

By the Hopf's lemma, we deduce

$$\frac{\partial \mathbf{u}_{\frac{\pi}{2}-\alpha}}{\partial \nu}(x) < \frac{\partial \mathbf{u}_\alpha^{\frac{\pi}{2}-2\alpha}}{\partial \nu}(x), \quad \forall x \in \partial S_\alpha^{\frac{\pi}{2}-2\alpha} \cap \partial S_{\frac{\pi}{2}-\alpha}. \tag{3.33}$$

By the uniqueness of $\mathbf{u}_\alpha$, we know that, after a proper rotation, $u(re^{i\alpha}) = \mathbf{u}_\alpha$ in S_α and $u(re^{i\alpha}) = -\mathbf{u}_{\frac{\pi}{2}-\alpha}^{\pi/2}$ in $S_{\frac{\pi}{2}-\alpha}^{\pi/2}$. Then, on $\partial S_\alpha \cap \partial S_{\frac{\pi}{2}-\alpha}^{\pi/2}$ we have

$$\frac{\partial u}{\partial \nu'}(x) > \frac{\partial u}{\partial \nu}(x), \tag{3.34}$$

where ν is the normal of $S_{\frac{\pi}{2}-\alpha}^{\pi/2}$ while ν' is normal of S_α. This is in contrast with u being a classical solution to (3.1). Therefore $\frac{\pi}{2} - \alpha = \alpha$, and hence $\theta = \pi/2$. This finishes the proof. □

4. Further study of saddle solutions

In this section, we consider a saddle solution u to (3.1) satisfying the even symmetry condition (3.8). Here we just assume that F is a double well potential satisfying (3.6). We shall use the Cartesian coordinates (y_1, y_2) with y_1-axis and y_2-axis coinciding with the lines $\{x = (x_1, x_2) \mid x_1 = x_2\}$ and $\{x = (x_1, x_2) \mid x_1 = -x_2\}$, respectively. In the new coordinates, the condition (3.8) becomes

$$u(y_1, y_2) = u(y_1, -y_2) = u(-y_1, y_2), \quad y \in \mathbb{R}^2. \tag{4.1}$$

We assume further that u satisfies the following monotonicity condition

$$u_{y_1}(y) > 0, \quad \text{if } y_1 > 0; \qquad u_{y_2}(y) < 0, \quad \text{if } y_2 > 0. \tag{4.2}$$

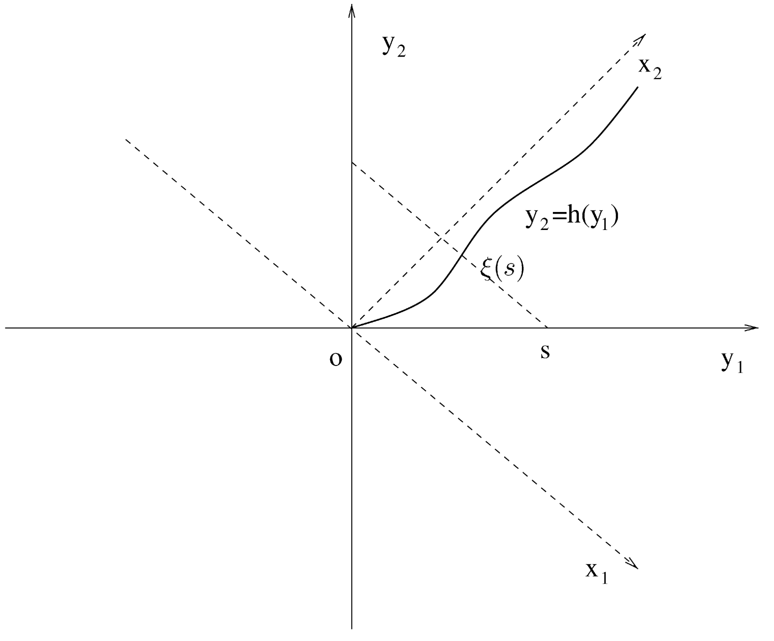

Fig. 5. The γ-level curve of u and the coordinates.

Denote $\gamma = u(0,0)$. We can expand u near $y = (0,0)$ as

$$u(y) = \gamma + ay_1^2 - by_2^2 + o\big(|y|^2\big),$$

where a, b are positive constants. It is easy to see that $F'(\gamma) = 2(b-a)$. Moreover, by the implicit function theorem the γ-level set of u near the origin in the first quadrant $Q_1 = \{y = (y_1, y_2) \mid y_1 > 0,\ y_2 > 0\}$ is a C^2 curve which can be extended to infinity. Indeed, by (4.2) we know that the γ-level set curve can be expressed as the graph of a strictly increasing C^2 function $y_2 = h(y_1)$ which has an inverse function $y_1 = k(y_2)$. In the next several lemmas we shall show that the γ-level curve is asymptotically a straight line. (See Fig. 5.)

Lemma 4.1. *The function $y_2 = h(y_1)$ for the γ-level curve is defined for all $y_1 > 0$ and the following limit holds*

$$\lim_{y_1 \to \infty} h(y_1) = \infty. \tag{4.3}$$

Proof. By the monotonicity property (4.2) of u and the implicit function theorem, we know that the γ-level curve extends to infinity. Hence it suffices to show (4.3) when $h(y_1)$ is defined for all $y_1 > 0$. Suppose (4.3) is not true. We define $u_\infty(y) = \lim_{s\to\infty} u(y_1 + s, y_2)$, $y \in \mathbb{R}^2$ and $A = \lim_{y_1\to\infty} h(y_1)$. Then u_∞ is C^2 in $\mathbb{R}^2$ and satisfies (3.1). Furthermore, we have $\frac{\partial u_\infty}{\partial y_1}(y_1, y_2) = 0$, $y \in \mathbb{R}^2$ and $u_\infty(y_1, A) = \gamma$. Then $u_\infty(y_1, y_2) = g(y_2 + b)$, $y \in \mathbb{R}^2$ for some constant b, where g is the unique solution of the ordinary differential equation (3.3). This contradicts the even symmetry (4.1) of u_∞ in y_2. The lemma is proven. $\square$

Lemma 4.2. *There exists $\beta \in (0, \pi/2)$ such that*

$$\lim_{y_1 \to \infty} h'(y_1) = \tan \beta. \tag{4.4}$$

Proof. We shall use the (x_1, x_2) coordinates as well and write $\bar{u}(x_1, x_2) = u(y_1, y_2)$. Define

$$\bar{\rho}(x_2) = \frac{1}{\sqrt{2}} \int_{-x_2}^{x_2} \left[F\big(\bar{u}(x_1, x_2)\big) + \frac{1}{2}\bar{u}_{x_1}^2(x_1, x_2) - \frac{1}{2}\bar{u}_{x_2}^2(x_1, x_2) \right] dx_1. \tag{4.5}$$

Then

$$\begin{aligned} \sqrt{2}\bar{\rho}'(x_2) &= F\big(\bar{u}(x_2, x_2)\big) + \frac{1}{2}\bar{u}_{x_1}^2(x_2, x_2) - \frac{1}{2}\bar{u}_{x_2}^2(x_2, x_2) + (\bar{u}_{x_1}\bar{u}_{x_2})(x_2, x_2) \\ &\quad + F\big(\bar{u}(-x_2, x_2)\big) + \frac{1}{2}\bar{u}_{x_1}^2(-x_2, x_2) - \frac{1}{2}\bar{u}_{x_2}^2(-x_2, x_2) - (\bar{u}_{x_1}\bar{u}_{x_2})(-x_2, x_2) \\ &= F\big(u(\sqrt{2}x_2, 0)\big) + \frac{1}{2}u_{y_1}^2(\sqrt{2}x_2, 0) + F\big(u(0, \sqrt{2}x_2)\big) + \frac{1}{2}u_{y_2}^2(0, \sqrt{2}x_2). \end{aligned} \tag{4.6}$$

Then

$$\sqrt{2}\bar{\rho}(M/\sqrt{2}) = \int_0^M \left[F\big(u(s, 0)\big) + \frac{1}{2}u_{y_1}^2(s, 0) \right] + \left[F\big(u(0, s)\big) + \frac{1}{2}u_{y_2}^2(0, s) \right] ds. \tag{4.7}$$

On the other hand, we let $(x_1(s), s/\sqrt{2})$ be the intersection of the line $x_2 = s/\sqrt{2}$ with the level set curve $y_2 = h(y_1)$ and write its y-coordinates as $y = \xi(s) = (\xi_1(s), \xi_2(s))$. Define $u^s(y) = u(y + \xi(s))$, $y \in \mathbb{R}^2$. By the standard theory of elliptic equations, for any sequence $\{s_n\}$ there is a subsequence $\{s_{n_k}\}$ (which we will denote by $\{s_k\}$ later) such that $u_k(y) := u^{s_k}(y)$ converges to $u_\infty(y)$ in $C^2_{\mathrm{loc}}(\mathbb{R}^2)$ as $k \to \infty$, where u_∞ is a solution of (3.1). In particular, if $s_n \to \infty$, by (4.3) we deduce $\xi_i(s_n) \to \infty$, $i = 1, 2$. Hence, by (4.2) we obtain $\frac{\partial u_\infty}{\partial y_2}(y) \geqslant 0$, $y \in \mathbb{R}^2$. By the strong maximum principle, we know either $\frac{\partial u_\infty}{\partial y_2} \equiv 0$ in $\mathbb{R}^2$ or $\frac{\partial u_\infty}{\partial y_2}(y) > 0$, $y \in \mathbb{R}^2$. Then by [14, Theorem 1.1] (De Giorgi conjecture for $n = 2$) we conclude that $u_\infty(y) = g(y \cdot \nu + t_0)$, $y \in \mathbb{R}^2$, where t_0 is the constant satisfying $g(t_0) = \gamma$, and $\nu \in \mathbb{R}^2$ is constant unit vector. We write $\nu = (\sin\beta, -\cos\beta)$. Fix a large positive constant M. For any small $\epsilon > 0$, we have

$$\big\| \bar{u}(x_1, s_k) - g\big(\big(x_1 - x_1(s_k)\big)\sin(\pi/4 + \beta) + t_0\big) \big\|_{C^2([x_1(s_k) - M, x_1(s_k) + M])} \leqslant \epsilon \tag{4.8}$$

when k is sufficiently large. Moreover,

$$\begin{cases} \big|\bar{u}(x_1, s_k) - 1\big| \leqslant Ce^{-\mu(x_1 - x_1(s_k))\sin\pi/4}, & x_1 \geqslant x_1(s_k) + M, \\ \big|\bar{u}(x_1, s_k) + 1\big| \leqslant Ce^{-\mu(x_1(s_k) - x_1)\sin\pi/4}, & x_1 \leqslant x_1(s_k) - M, \end{cases} \tag{4.9}$$

where C, μ are positive constants independent of M, k and x_1. The gradient estimates for elliptic equations yield

$$\begin{cases} |\nabla\bar{u}| \leqslant Ce^{-\mu(x_1 - x_1(s_k))\sin\pi/4}, & x_1 \geqslant x_1(s_k) + M, \\ |\nabla\bar{u}| \leqslant Ce^{-\mu(x_1(s_k) - x_1)\sin\pi/4}, & x_1 \leqslant x_1(s_k) - M. \end{cases} \tag{4.10}$$

Using (4.5) and (4.8)–(4.10) and choosing M sufficiently large, we can obtain

$$\left|\bar{\rho}(s_k) - e_F \sin(\pi/4 + \beta)\right| \leqslant \epsilon \tag{4.11}$$

when k is sufficiently large. In view of (4.7), $\bar{\rho}(s)$ is increasing in s and then has a finite limit. Hence we derive that

$$\lim_{s_k \to \infty} \bar{\rho}(s_k) = e_F \sin(\pi/4 + \beta). \tag{4.12}$$

Note that the sequence $\{s_n\}$ is arbitrary and hence β in the above equality does not depend on the choice of the sequence. Therefore we conclude

$$\left\| u\big(y + \xi(s)\big) - g(y_1 \sin\beta - y_2 \cos\beta + t_0)\right\|_{C^2_{\mathrm{loc}}(\mathbb{R}^2)} \to 0, \quad \text{as } s \to \infty. \tag{4.13}$$

Next, we show $\beta \in (0, \pi/2)$. It suffices to show $\beta > 0$, since $\beta < \pi/2$ can be proven similarly. Suppose $\beta = 0$. Following the proof of Theorem 3.4, we derive

$$\int_0^\infty \left[F\big(u(0, y_2)\big) + \frac{1}{2} u_{y_2}^2(0, y_2) \right] dy_2 = e_F.$$

By (4.7) and (4.12), we derive

$$\int_0^\infty \left[F\big(u(s, 0)\big) + \frac{1}{2} u_{y_1}^2(s, 0) \right] ds + \int_0^\infty \left[F\big(u(0, y_2)\big) + \frac{1}{2} u_{y_2}^2(0, y_2) \right] dy_2 = e_F.$$

Hence

$$\int_0^\infty \left[F\big(u(s, 0)\big) + \frac{1}{2} u_{y_1}^2(s, 0) \right] ds = 0.$$

This is a contradiction. The lemma is then proven. $\square$

Next we shall show that the γ-level curve is indeed asymptotically a straight line. We shall prove the following more general lemma regarding solution of (3.1) in a cone.

Lemma 4.3. *Suppose that $u(y_1, y_2)$ is a solution of* (3.1) *in a cone $\mathcal{C} := \{y \in \mathbb{R}^2 \mid |y_1| \leqslant y_2 \tan\alpha_0,\ y_2 \geqslant M > 0\}$ for some $0 < \alpha_0 < \pi$. For some $\gamma \in (-1, 1)$, the γ-level set of u in $\mathcal{C}$ is given by the graph of a function $y_1 = k(y_2)$. Assume*

$$\lim_{y_2 \to \infty} k'(y_2) = 0. \tag{4.14}$$

Then there is a finite number A such that

$$\lim_{y_2 \to \infty} k(y_2) = A. \tag{4.15}$$

Proof. We shall prove the lemma in three steps. First, we show that an energy of u on a line segment $[-y_2\tan\alpha, y_2\tan\alpha_0]$, $\alpha\in(0,\alpha_0)$ is exponentially close to $\mathbf{e}_F$ as y_2 tends to ∞. Second, we construct an optimal approximation of u by a shift $r(y_2)$ of the one-dimensional solution g, and show that the difference is exponentially small in L^2 norm as y_2 goes to infinity. Finally, we deduce that the shift $r(y_2)$ has a finite limit, and then conclude that $k(y_2)$ has a finite limit.

Step 1. Without loss of generality, we assume that $u(y_1,y_2)>\gamma$ when $y_1>k(y_2)$ and $u(y_1,y_2)<\gamma$ when $y_1<k(y_2)$ in $\mathcal{C}$.

It is easy to show by the maximum principle (see, e.g., [14,16]) that for any fixed $\alpha\in(0,\alpha_0)$

$$\begin{cases} \big|u(y)-1\big|\leqslant C_1e^{-\kappa_1(y_1-k(y_2))}, & \text{for } k(y_2)<y_1<y_2\tan\alpha,\\ \big|u(y)+1\big|\leqslant C_1e^{\kappa_1(y_1-k(y_2))}, & \text{for } -k(y_2)>y_1>-y_2\tan\alpha. \end{cases} \tag{4.16}$$

The standard gradient estimates of elliptic equations yield

$$\begin{cases} |\nabla u|\leqslant C_1e^{-\kappa_1(y_1-k(y_2))}, & \text{for } k(y_2)<y_1<y_2\tan\alpha,\\ |\nabla u|\leqslant C_1e^{\kappa_1(y_1-k(y_2))}, & \text{for } -k(y_2)>y_1>-y_2\tan\alpha. \end{cases} \tag{4.17}$$

For any sequence $\{s_n\}$ there exists a subsequence, which we still denote by $\{s_n\}$, such that $u(y_1+k(s_n),y_2+s_n)$ converges in $C^2_{\mathrm{loc}}(\mathbb{R}^2)$ to a solution $u_\infty(y)$ of (3.1) in $\mathbb{R}^2$. Furthermore, we have

$$\begin{cases} u_\infty(0,y_2)=\gamma, & \forall y_2\in\mathbb{R},\\ u_\infty(y_1,y_2)>\gamma, & \text{if } y_1>0,\ \forall y_2\in\mathbb{R},\\ u_\infty(y_1,y_2)<\gamma, & \text{if } y_1<0,\ \forall y_2\in\mathbb{R}. \end{cases} \tag{4.18}$$

By symmetry results in half plane (see [6]), we know that $u_\infty(y)=g(y_1+t_0)$, $y\in\mathbb{R}^2$. Since $\{s_n\}$ is arbitrary, we obtain

$$\big\|u\big(y_1+k(s),y_2+s\big)-g(y_1+t_0)\big\|_{C^2_{\mathrm{loc}}(\mathbb{R}^2)}\to 0,\quad \text{as } s\to\infty.$$

Then,

$$\big\|u(y_1,y_2)-g\big(y_1-k(y_2)+t_0\big)\big\|_{C^2([-y_2\tan\alpha,y_2\tan\alpha])}\to 0,\quad \text{as } y_2\to\infty. \tag{4.19}$$

Define

$$\rho_1(y_2)=\int\limits_{-y_2\tan\alpha}^{y_2\tan\alpha}\left[F\big(u(y_1,y_2)\big)+\frac{1}{2}|u_{y_1}|^2-\frac{1}{2}|u_{y_2}|^2\right]dy_1.$$

It follows easily from (4.19) and (4.16) that

$$\lim_{y_2\to\infty}\rho_1(y_2)=\mathbf{e}_F. \tag{4.20}$$

Combining (4.16), (4.17) and straightforward computations as in (4.6), we obtain

$$\big|\rho_1'(y_2)\big|\leqslant C_2e^{-\kappa_2y_2},\quad y_2\geqslant M, \tag{4.21}$$

for some positive constants C_2, κ_2. Hence we conclude that

$$\left|\rho_1(y_2) - \mathbf{e}_F\right| \leqslant \frac{C_2}{\kappa_2} e^{-\kappa_2 y_2}, \quad y_2 \geqslant M. \tag{4.22}$$

Step 2. We define

$$W(y_2, r) = \left\| u(\cdot, y_2) - g(\cdot + t_0 - r) \right\|^2_{L^2([-y_2 \tan\alpha, y_2 \tan\alpha])}.$$

By (4.16) and (4.19), we know that

$$W\big(y_2, k(y_2)\big) \to 0, \quad \text{as } y_2 \to \infty,$$

and

$$\frac{\partial^2}{\partial r^2} W\big(y_2, k(y_2)\big) = 2 \int \big[-(u-g)g'' + |g'|^2\big]\, dy_1 > 0, \tag{4.23}$$

when y_2 is sufficiently large.

By the implicit function theorem, for y_2 large there exists a unique $r(y_2)$ such that

$$W\big(y_2, r(y_2)\big) = \min_{r \in \mathbb{R}} W(y_2, r).$$

Let

$$v(y_1, y_2) = u(y_1, y_2) - g\big(y_1 + t_0 - r(y_2)\big), \quad y_1 \in [-y_2 \tan\alpha, y_2 \tan\alpha].$$

It follows immediately from (4.19) that

$$\lim_{y_2 \to \infty} \left\| v(\cdot, y_2) \right\| = 0. \tag{4.24}$$

Moreover, the function $r(y_2)$ is differentiable and

$$\lim_{y_2 \to 0} r'(y_2) = 0, \qquad \lim_{y_2 \to 0} r(y_2) - k(y_2) = 0. \tag{4.25}$$

It is also easy to see that

$$\int_{-y_2 \tan\alpha}^{y_2 \tan\alpha} \big[u(y_1, y_2) - g\big(y_1 + t_0 - r(y_2)\big)\big] g'\big(y_1 + t_0 - r(y_2)\big)\, dy_1 = 0. \tag{4.26}$$

Differentiating (4.26) with respect to y_2 leads to

$$\left(\int |g'|^2\, dy_1 - \int (u-g)g''\, dy_1\right) \cdot r'(y_2) + \int u_{y_2} g'\, dy_1 = O\big(e^{-\kappa_2 y_2}\big). \tag{4.27}$$

Now we estimate the energy $\rho_1(y_2)$ in terms of $\|v\|$ as follows:

$$\rho_1(y_2) - \int\limits_{-y_2\tan\alpha}^{y_2\tan\alpha} \left[F\big(g\big(y_1+t_0-r(y_2)\big)\big) + \frac{1}{2}\big|g'\big(y_1+t_0-r(y_2)\big)\big|^2\right] dy_1$$

$$= \int\limits_{-y_2\tan\alpha}^{y_2\tan\alpha} \left[F(u) - F(g) + \frac{1}{2}\big(|u_{y_1}|^2 - |g'|^2\big) - \frac{1}{2}|u_{y_2}|^2\right] dy_1$$

$$= \int\limits_{-y_2\tan\alpha}^{y_2\tan\alpha} \left[F(u) - F(g) - \frac{1}{2}\big(F'(g) + F'(u)\big)(u-g)\right] dy_1$$

$$+ \frac{1}{2}\int u_{y_2y_2}(u-g)\,dy_1 - \frac{1}{2}\int u_{y_2}^2\,dy_1 + O\big(e^{-\kappa_2 y_2}\big)$$

$$= \frac{1}{2}\int u_{y_2y_2}(u-g)\,dy_1 - \frac{1}{2}\int u_{y_2}^2\,dy_1 + O\big(e^{-\kappa_2 y_2}\big) + o\big(\|v\|^2\big). \tag{4.28}$$

In the above estimate, we have used the following estimate:

$$\int\limits_{-y_2\tan\alpha}^{y_2\tan\alpha} \big(|u_{y_1}|^2 - |g'|^2\big)\,dy_1$$

$$= -\int\limits_{-y_2\tan\alpha}^{y_2\tan\alpha} (u_{y_1y_1} + g'')(u-g)\,dy_1 + O\big(e^{-\kappa_2 y_2}\big)$$

$$= \int\limits_{-y_2\tan\alpha}^{y_2\tan\alpha} \big[-\big(F'(u) + F'(g)\big)(u-g) + u_{y_2y_2}(u-g)\big]\,dy_1 + O\big(e^{-\kappa_2 y_2}\big). \tag{4.29}$$

Hence, in view of (4.22) we obtain

$$\int u_{y_2y_2}(u-g)\,dy_1 - \int u_{y_2}^2\,dy_1 = O\big(e^{-\kappa_2 y_2}\big) + o\big(\|v\|^2\big). \tag{4.30}$$

Furthermore, by the spectrum theory we have

$$\int\limits_{-y_2\tan\alpha}^{y_2\tan\alpha} \big[|\phi'|^2 + F''(g)\phi^2\big]\,dy_1 \geqslant \lambda\|\phi\|^2 \tag{4.31}$$

for some positive constant $\lambda > 0$ when ϕ satisfies

$$\int\limits_{-y_2\tan\alpha}^{y_2\tan\alpha} \phi g'\,dy_1 = 0, \quad \phi \in H_0^1\big([-y_2\tan\alpha, y_2\tan\alpha]\big). \tag{4.32}$$

Choosing $\phi(\cdot) = v(\cdot, y_2)$ in (4.31) for any fixed y_2, we obtain

$$\int_{-y_2 \tan\alpha}^{y_2 \tan\alpha} \left[-(u-g)_{y_1 y_1} + F''(g)(u-g)\right](u-g)\,dy_1 \geqslant \lambda\|v\|^2 + O\left(e^{-\kappa_2 y_2}\right). \tag{4.33}$$

Differentiating $\|v\|^2$ twice leads to

$$\begin{aligned}
\frac{d^2}{dy_2^2}\|v\|^2 &= 2\int \left|u_{y_2} + g'r'(y_2)\right|^2 dy_1 \\
&\quad + 2\int \left[u_{y_2 y_2} - g''\left|r'(y_2)\right|^2 + g'r''(y_2)\right](u-g)\,dy_1 + O\left(e^{-\kappa_2 y_2}\right) \\
&= 2\int \left|u_{y_2} + g'r'(y_2)\right|^2 dy_1 + 2\int u_{y_2 y_2}(u-g)\,dy_1 \\
&\quad - 2\left(r'(y_2)\right)^2 \int g''(u-g)\,dy_1 + 2r''(y_2)\int g'(u-g)\,dy_1 + O\left(e^{-\kappa_2 y_2}\right) \\
&\geqslant \int \left|u_{y_2} + g'r'(y_2)\right|^2 dy_1 + \int u_{y_2}^2\,dy_1 + \int u_{y_2 y_2}(u-g)\,dy_1 \quad \big(\text{by (4.30), (4.26)}\big) \\
&\quad + \left(r'(y_2)\right)^2 O\left(\|v\|\right) + o\left(\|v\|^2\right) + O\left(e^{-\kappa_2 y_2}\right) \\
&\geqslant \left(\|g'\|^2 + O\left(\|v\|\right)\right)\left(r'(y_2)\right)^2 + \left(\lambda + o(1)\right)\|v\|^2 + O\left(e^{-\kappa_2 y_2}\right).
\end{aligned} \tag{4.34}$$

Here it is essential to split the term $2\int u_{y_2 y_2}(u-g)\,dy_1$ to two terms: one is replaced by $\int u_{y_2}^2\,dy_1$ using (4.30) and the other is replaced by $\lambda\|v\|^2$ using the following estimate:

$$\begin{aligned}
\int u_{y_2 y_2}(u-g)\,dy_1 &= \int_{-y_2 \tan\alpha}^{y_2 \tan\alpha} \left[\left(F'(u) - u_{y_1 y_1}\right)(u-g)\right] dy_1 \\
&= \int \left[\left(F'(u) - F'(g) - F''(g)(u-g)\right)(u-g)\right] dy_1 \\
&\quad + \int \left[\left(F''(g)(u-g) - (u-g)_{y_1 y_1}\right)(u-g)\right] dy_1 \quad \big(\text{by (4.33)}\big) \\
&\geqslant o\left(\|v\|^2\right) + \lambda\|v\|^2 + O\left(e^{-\kappa_2 y_2}\right).
\end{aligned} \tag{4.35}$$

Therefore we derive a differential inequality

$$\frac{d^2}{dy_2^2}\|v\|^2 \geqslant \frac{\lambda}{2}\|v\|^2 + O\left(e^{-\kappa_2 y_2}\right), \quad y_2 \geqslant M_1, \tag{4.36}$$

where M_1 is a sufficiently large positive constant. By choosing a comparison function of the form $Ce^{-\kappa y_2}$, it is easy to see that

$$\|v\| \leqslant Ce^{-\kappa y_2}, \quad y_2 \geqslant M_1, \tag{4.37}$$

for appropriately chosen constants C and $\kappa < \min\{\kappa_2, \sqrt{\lambda/2}\}$.

Step 3. From (4.37) and (4.30), we derive

$$\int |u_{y_2}|^2\,dy_1 \leqslant Ce^{-\kappa y_2}, \quad y_2 \geqslant M_1. \tag{4.38}$$

Then by (4.27), we obtain

$$\left|r'(y_2)\right| \leqslant Ce^{-\kappa y_2/2}, \quad y_2 \geqslant M_1. \tag{4.39}$$

Therefore

$$\lim_{y_2\to\infty} r(y_2) = A \tag{4.40}$$

for some finite number A. The lemma follows immediately from (4.25). □

Combining Lemmas 4.1–4.3, we can prove the following theorem.

Theorem 4.4. *Assume that u is a solution to the Allen–Cahn equation* (3.1) *where F is a double well potential satisfying* (3.6). *Assume further that u possesses even symmetry* (4.1) *and monotonicity* (4.2). *Then every level set of u approaches asymptotically a slant straight line with the same finite positive slope in the first quadrant as y goes to infinity.*

Proof. We just note that after rotating the coordinates clockwise by an angle $\pi/2 - \beta$, then u satisfies the condition of Lemma 4.3 using Lemma 4.2. Hence we can apply Lemma 4.3 to conclude that the γ-level set of u approaches a straight line of slope $\tan\beta$ in the original coordinate. In view of (4.13), the other level set curves of u are essentially parallel to γ-level curve of u asymptotically, the theorem then follows immediately. □

Remark 4.5. The result in Theorem 4.4 can be generalized to solutions of Allen–Cahn equations in a domain which is a cone at infinity, provided that the level set is a smooth curve contained in a strictly smaller cone near infinity. More details will be provided in a forthcoming paper.

Remark 4.6. The condition that F has only one critical point in $(-1, 1)$ stated in (3.6) can be dropped in most of the discussion. In the case when F has more critical points in $(-1, 1)$, the one-dimensional heteroclinic solution of (3.3) may not be unique up to translation. In the case that F is even, the saddle solution satisfying (3.7) and (3.8) may not be unique either. However, we can state the following:

1. For each heteroclinic solution g_i of (3.3) there exists a pair of critical points $[a_i, b_i]$, which are the limits of g_i at plus and minus infinity, respectively, such that

$$F(u) > F(a_i) = F(b_i), \quad \forall u \in (a_i, b_i). \tag{4.41}$$

If we assume that $F''(a_i) > 0$, $F''(b_i) > 0$ at these points, then there are at most countable many of such pairs.

2. Each saddle solution satisfying (4.1) and (4.2) corresponds to a heteroclinic solution g_i of (3.3) and hence a pair of a_i, b_i.

3. For each pair a_i, b_i, the heteroclinic solution g_i connecting a_i and b_i is indeed unique up to translation. Therefore, the discussion in this section as well as in Section 3 can be carried out with $-1, 1$ replaced by a_i, b_i and g replaced by g_i, except for the uniqueness assertions.

4. In the case F is even and $a_i = -b_i$, there exists a saddle solution as in Proposition 3.1 associated with a_i, b_i.

The proofs of the above statements are either easy or can be modified from the arguments in this paper, we leave them to the reader.

Next we study u more carefully at each side of the γ-level curve. For this purpose, we define

$$\mathbf{e}_\gamma^+ := \int_\gamma^1 \sqrt{2F(u)}\,du, \qquad \mathbf{e}_\gamma^- := \int_{-1}^\gamma \sqrt{2F(u)}\,du. \tag{4.42}$$

We also define

$$\rho_2(y_2) = \int_{k(y_2)}^\infty \left[F\big(u(y_1, y_2)\big) + \frac{1}{2}|u_{y_1}|^2 - \frac{1}{2}|u_{y_2}|^2\right] dy_1.$$

Since $u(k(y_2), y_2) = \gamma$ for $y_2 \geqslant 0$, then

$$u_{y_1}\big(k(y_2), y_2\big) \cdot k'(y_2) + u_{y_2}\big(k(y_2), y_2\big) = 0, \quad \forall y_2 > 0.$$

Hence, by straightforward computations and the Allen–Cahn equation (3.1) we obtain

$$\rho_2'(y_2) = -\left[F(\gamma) - \frac{1}{2}|\nabla u|^2\big(k(y_2), y_2\big)\right]k'(y_2).$$

Hence

$$\begin{aligned}\rho_2(y_2) &= -\int_0^{y_2}\left[F(\gamma) - \frac{1}{2}|\nabla u|^2\big(k(s), s\big)\right]k'(s)\,ds + \rho_2(0)\\ &= -\int_0^{k(y_2)}\left[F(\gamma) - \frac{1}{2}|\nabla u|^2\big(y_1, h(y_1)\big)\right]dy_1 + \rho_2(0).\end{aligned} \tag{4.43}$$

Using (1.6) and computations as in (2.12), we can obtain

$$\rho_2(0) = \mathbf{e}\sin\beta, \qquad \lim_{y_2\to\infty}\rho_2(y_2) = \mathbf{e}_\gamma^+\sin\beta. \tag{4.44}$$

Therefore

$$\int_0^\infty\left[F(\gamma) - \frac{1}{2}|\nabla u|^2\big(y_1, h(y_1)\big)\right]dy_1 = \mathbf{e}_\gamma^-\sin\beta. \tag{4.45}$$

Similarly, we can define

$$\rho_3(y_1) = \int_{h(y_1)}^{\infty} \left[F\big(u(y_1, y_2)\big) + \frac{1}{2}|u_{y_2}|^2 - \frac{1}{2}|u_{y_1}|^2 \right] dy_2$$

and obtain

$$\rho_3'(y_1) = -\left[F(\gamma) - \frac{1}{2}|\nabla u|^2\big(y_1, h(y_1)\big) \right] h'(y_1).$$

Hence

$$\begin{aligned} \rho_3(y_1) &= -\int_0^{y_1} \left[F(\gamma) - \frac{1}{2}|\nabla u|^2\big(s, h(s)\big) \right] h'(s)\, ds + \rho_3(0) \\ &= -\int_0^{h(y_1)} \left[F(\gamma) - \frac{1}{2}|\nabla u|^2\big(k(y_2), y_2\big) \right] dy_2 + \rho_3(0). \end{aligned} \tag{4.46}$$

Using (1.6) and computations as in (2.12), we can also obtain

$$\rho_3(0) = \mathbf{e} \cos\beta, \qquad \lim_{y_1 \to \infty} \rho_3(y_1) = \mathbf{e}_\gamma^+ \cos\beta \tag{4.47}$$

and therefore

$$\int_0^{\infty} \left[F(\gamma) - \frac{1}{2}|\nabla u|^2\big(k(y_2), y_2\big) \right] dy_2 = \mathbf{e}_\gamma^+ \cos\beta. \tag{4.48}$$

Combining (4.48) and (4.45), we can conclude the following result.

Theorem 4.7. *Under the assumptions of Theorem 4.4, if we assume further $u(0) = \gamma$ and the γ-level curve $y_2 = h(y_1)$ is close to a straight line in C^1 norm globally in the first quadrant of $\mathbb{R}^2$, i.e., for some $\beta \in (0, \pi/2)$ and small positive constant ϵ*

$$\big\| h(y_1) - y_1 \tan\beta \big\|_{C^1([0,\infty))} \leqslant \epsilon,$$

then we have

$$(\tan\beta - \epsilon)\tan\beta \leqslant \frac{\mathbf{e}_\gamma^-}{\mathbf{e}_\gamma^+} \leqslant (\tan\beta + \epsilon)\tan\beta. \tag{4.49}$$

In particular, if γ-level curve is a straight line, i.e., $\epsilon = 0$, then

$$\tan^2\beta = \frac{\mathbf{e}_\gamma^-}{\mathbf{e}_\gamma^+}. \tag{4.50}$$

Remark 4.8. Theorems 3.4 and 3.10 are special cases of the above theorem.

In general, we have the following estimate of the contact angle $\theta = 2\beta$ of the γ-level curves.

Theorem 4.9. *Under the assumption of Theorem* 4.7, *the angle β of the γ-level curve with y_1-axis at infinity satisfies*

$$\frac{\mathbf{e}_\gamma^+}{\mathbf{e}} \leqslant \sin\beta \leqslant \frac{\mathbf{e}_\gamma^+}{\mathbf{e}} \sqrt{1 + 2\frac{\mathbf{e}_\gamma^-}{\mathbf{e}_\gamma^+}}. \tag{4.51}$$

In particular,

$$\lim_{\gamma\to -1} \beta = \frac{\pi}{2}, \qquad \lim_{\gamma\to 1} \beta = 0. \tag{4.52}$$

Proof. We just note that $\rho_2(0) \geqslant \mathbf{e}_\gamma^+$ and $\rho_3(0) \geqslant \mathbf{e}_\gamma^-$. Then (4.44) and (4.47) lead to (4.51) immediately. The limits of the angle in terms of γ follows from the following fact:

$$\lim_{\gamma\to -1} \mathbf{e}_\gamma^+ = \mathbf{e}, \qquad \lim_{\gamma\to 1} \mathbf{e}_\gamma^+ = 0. \qquad \square \tag{4.53}$$

Remark 4.10. Theorem 3.6 is a special case of the above theorem.

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Schrödinger operators on graphs and geometry I: Essentially bounded potentials

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Abstract

The inverse spectral problem for Schrödinger operators on finite compact metric graphs is investigated. The relations between the spectral asymptotics and geometric properties of the underlying graph are studied. It is proven that the Euler characteristic of the graph can be calculated from the spectrum of the Schrödinger operator in the case of essentially bounded real potentials and standard boundary conditions at the vertices. Several generalizations of the presented results are discussed.

Keywords: Quantum graph; Trace formula; Euler characteristic

1. Introduction

The theory of differential operators on metric graphs is a rapidly developing area of modern mathematical physics. Interest to these problems can be explained not only by important applications in the theory of quantum wave guides and nanoelectronics, but by discovered interesting phenomena putting these problems in the area between ordinary and partial differential operators. Indeed methods originally developed for both areas are successfully applied to study the problems on metric graphs. The main aim of this paper is to study the relation between the spectrum of a Schrödinger operator on such a graph and geometric properties of the graph. This question

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has already been studied for Laplace operators with standard boundary conditions at the vertices (see (1) below) and it was proven that the spectrum of the Laplace operator determines the total length, the number of connected components and the Euler characteristics of the underlying graph (see Definition 3). To establish the relation between the spectrum and the Euler characteristics one used so-called trace formula (see (2)) connecting the spectrum of the Laplacian with the set of periodic orbits on the graph. Using this relation effective formulas for the Euler characteristic can be proven (see (6) and (7)). The main goal of the current paper is to generalize these results to the case of Schrödinger operators on graphs. Such operators are not uniquely determined by the underlying graphs (like Laplacians with standard boundary conditions), but depend on the choice of real valued potential and may be other than the standard boundary conditions at the vertices. In the current article we are going to confine ourselves to the case of essentially bounded potentials and standard boundary conditions. The case of L_1 potentials and most general symmetric boundary conditions will be considered in the following publication, since these problems can be treated by similar methods.

To obtain a connection between the spectrum of a Schrödinger operator and the Euler characteristic of the graph we use trace formula. A similar formula was first proven by J.-P. Roth [33] using the heat kernel expansion, but we are going to use the trace formula in the form (2) first presented (without a proof) by J.-P. Roth as well. The formula we are going to use was first given by T. Kottos and U. Smilansky, but without paying attention to the fact that the secular equation describing the spectrum using vertex and edge scattering matrices in general does not determine the correct multiplicity of the eigenvalue zero. Correction of this inaccuracy allowed us to prove that two isospectral graphs (the corresponding Laplacians have the same spectra) have the same Euler characteristic and provide an effective formula for it. In the current article we not only prove a slightly different formula for the Euler characteristic, but give an alternative proof of this formula without any use of the trace formula, but only for graphs with the edges being integer multiples of one length to be called the basic length. We hope that this approach provides a new insight in the spectral asymptotics for such operators.

The new formula for the Euler characteristic obtained in this paper is valid not only for Laplacians but for Schrödinger operators with essentially bounded potentials and standard boundary conditions. To prove this fact we show first that the Euler characteristics is determined by the asymptotics of the eigenvalues only. Then it remains to prove that the spectra of a Laplacian and the corresponding Schrödinger operator are asymptotically close and therefore the formula for the Euler characteristic originally proven for Laplacians gives the correct result if the spectrum of the Laplacian is just substituted with the spectrum of the Schrödinger operator.

Current paper extends the recent article [26] and therefore correct references and historical remarks can be found there. On the other hand, it is impossible not to mention that the theory of differential operators on graphs has been grown from the pioneering works by B. Pavlov, N. Gerasimenko [15,16], P. Exner, P. Šeba [13] and Y. Colin de Verdière [11,12]. Recent interest in this subject was initiated by papers by V. Kostrykin and R. Schrader [19–21]. Several articles have been devoted to differential operators on trees—special class of graphs without cycles. It is possible to state that this problem is fully understood now [1,3,4,9,27,34,36,37]. On the other hand, very few papers are devoted to operators on arbitrary graphs (with cycles) and here we would like to mention J. von Below [5], R. Carlson [10], L. Friedlander [14], B. Gutkin and U. Smilansky [18], V. Kostrykin and R. Schrader [22], J. Boman, P. Kurasov, F. Stenberg and M. Nowaczyk [6,24–26,30,31]. The results presented in this article can be considered as a natural generalization of the classical results on inverse spectral and scattering problems for

ordinary differential equations [7,8,28]. Similar questions for discrete operators on graphs have been discussed by S.P. Novikov [29] and Y. Colin de Verdière [12].

2. Graph Laplacians: basic notations and trace formula

We are going to consider here only finite compact metrics graphs, i.e. metric graphs formed by a finite number of compact intervals.

Definition 1. A *finite compact metric graph* $\Gamma = \Gamma(\mathbf{E}, \sigma)$ consists of a finite set $\mathbf{E}$ of compact intervals Δ_j, $j = 1, 2, \ldots, N$, called *edges*, and a partition σ of the set $\mathbf{V} = \{x_j\}$ of endpoints x_j of the edges, $\mathbf{V} = \bigcup_m V_m$. The equivalence classes V_m, $m = 1, 2, \ldots, M$, will be called *vertices*, and the number of elements in V_m will be called the *valence* of V_m.

The distances on the edges and identification of the end points belonging to the same equivalence class induce naturally the distance on Γ and allows one to introduce the space $L_2(\Gamma)$ of square integrable functions on the graph with the standard scalar product $\langle f, g\rangle = \int_\Gamma \overline{f(x)} g(x)\, dx$. This space is independent of the connectivity of the graph, since

$$L_2(\Gamma) = \bigoplus_{n=1}^{N} L_2(\Delta_n).$$

Definition 2. The *Laplace operator* $L(\Gamma)$ is the operator of negative second derivative in $L_2(\Gamma)$ defined on the domain of functions f from the Sobolev space $\bigoplus_{n=1}^{N} W_2^2(\Delta_n)$ satisfying standard boundary conditions at the vertices

$$\begin{cases} \sum_{x_j \in V_m} \partial_n f(x_j) = 0; \\ f \text{ is continuous at} V_m; \end{cases} \quad m = 1, 2, \ldots, M, \tag{1}$$

where $\partial_n f(x_j)$ denotes the normal derivative of the function f at the end point x_j:

$$\partial_n f(x_j) = \begin{cases} f'(x_j) & \text{if } x_j \text{ is the left end point,} \\ -f'(x_j) & \text{if } x_j \text{ is the right end point.} \end{cases}$$

These boundary conditions reflect the connectivity of the graph in the sense, that two graphs formed by the same set of edges determine different standard boundary conditions if their vertex structures are different. It is important that the graph Γ determines the operator $L(\Gamma)$ uniquely. One of important characteristics of graphs is their Euler characteristic.

Definition 3. The *Euler characteristic* of a (not necessarily connected) graph Γ formed by M vertices and N edges is

$$\chi(\Gamma) = M - N.$$

For connected graphs the Euler characteristic determines the number g of generators in the fundamental group on Γ

$$g = 1 - \chi.$$

One of the main tools to study the spectral properties of graph Laplacians is the trace formula, which connects the spectrum of the Laplace operator on a finite compact graph with the set of periodic orbits on it. The first formula of this kind was proven by J.-P. Roth [33]. The formula we are going to use first appeared in the papers by B. Gutkin, T. Kottos and U. Smilansky [18,23], and later by P. Kurasov and M. Nowaczyk [24,26]. We refer to [26] for the proof of the following theorem:

Proposition 1 *(Trace formula, Theorem 2 from [26]). Let Γ be a compact metric graph with Euler characteristic χ and the total length $\mathcal{L}$, and let $L(\Gamma)$ be the corresponding Laplace operator. Then the following two trace formulae establish the relation between the spectrum $\{k_n^2\}$ of $L(\Gamma)$ and the set $\mathcal{P}$ of closed paths on the metric graph Γ*

$$\begin{aligned} u(k) &\equiv 2m_s(0)\delta(k) + \sum_{k_n \neq 0} \big(\delta(k-k_n) + \delta(k+k_n)\big) \\ &= \chi\delta(k) + \frac{\mathcal{L}}{\pi} + \frac{1}{\pi}\sum_{p\in\mathcal{P}} l\big(\mathrm{prim}(p)\big)S(p)\cos kl(p), \end{aligned} \tag{2}$$

and

$$\begin{aligned} \sqrt{2\pi}\,\hat{u}(l) &= 2m_s(0) + \sum_{k_n \neq 0} 2\cos k_n l \\ &= \chi + 2\mathcal{L}\delta(l) + \sum_{p\in\mathcal{P}} l\big(\mathrm{prim}(p)\big)S(p)\big(\delta\big(l-l(p)\big) + \delta\big(l+l(p)\big)\big), \end{aligned} \tag{3}$$

where

- *$m_s(0)$ is the multiplicity of the eigenvalue zero*;[2]
- *p is a closed path on Γ*;
- *$l(p)$ is the length of the closed path p*;
- $\mathrm{prim}(p)$ *is one of the primitive paths for p*;
- *$S(p)$ is the product of all vertex scattering coefficients along the path p.*

By a closed path on a metric graph we understand any continuous closed path, which does not turn back in the interior of any edge, but which may turn back at any vertex. It is clear that if the graph has neither loops nor two edges with the same endpoints, then every closed path is uniquely determined by the sequence of vertices it comes across. Cyclic permutation of the sequence of vertices does not change the closed path, i.e. paths are viewed as geometric sets and do not have the start and end points. By a primitive path $\mathrm{prim}(p)$ we denote any closed continuous path such that the path p can be obtained by repeating the path $\mathrm{prim}(p)$. With each vertex V_m we associate

[2] It is equal to the number C of connected components in accordance with Theorem 1 from [26].

the vertex scattering matrix formed by reflection and transition coefficients which are determined by the valence of the vertex only

$$\left(S_v^m\right)_{ij} = \begin{cases} \frac{2}{v_m}, & i \neq j, \\ \frac{2-v_m}{v_m}, & i = j, \end{cases} \tag{4}$$

where v_m is the valence of V_m, i.e. the number of elements in the equivalence class V_m. Metric graphs can be viewed as billiards where point particles are moving along the edges. Coming to the vertices the particles may be either reflected or transmitted to a neighboring edge. Each closed path is just a periodic trajectory for such a particle. Therefore to each path one associates the product $S(p)$ of all scattering coefficients along this path which represents in some sense the probability for the particle to take this particular trajectory.

Formulas (2) and (3) allow one to prove that two Laplace operators on compact finite metric graphs having the same spectrum have also the same

- number of connected components;
- total length;
- Euler characteristic.

(Uniqueness Theorem 1, [26].) The number of connected components is equal to the multiplicity of the zero eigenvalue (Theorem 1 from [26]). The total length of the graph determines the asymptotics of the eigenvalues (Weyl's law)

$$\lim_{n\to\infty} \sqrt{\lambda_n}/n = \pi/\mathcal{L}. \tag{5}$$

This formula can be proven if one takes into account that the Laplace operator on Γ is a finite rank perturbation (in the resolvent sense) of the orthogonal sum of Laplace operators on separated edges. The same asymptotics holds for Schrödinger operators and can be proven using the same method (see e.g. [35] where this formula is proven even for weighted Laplacians).

To calculate the Euler characteristic the following formula has been derived (see (4.1) from [26]):

$$\chi = 2m_s(0) + \lim_{t\to\infty} \sum_{k_n \neq 0} \frac{2t}{k_n} \sin\frac{k_n}{t}\left(2\cos\frac{k_n}{t} - 1\right), \quad k_n^2 = \lambda_n. \tag{6}$$

In what follows we shall need a modification of this formula.

Theorem 1. *Let Γ be a compact metric graph and $L(\Gamma)$ be the corresponding Laplace operator. Then the Euler characteristic $\chi(\Gamma)$ is uniquely determined by the spectrum $\{\lambda_n\}$ of the Laplace operator $L(\Gamma)$*

$$\begin{aligned} \chi &= 2m_s(0) + 2\lim_{t\to\infty} \sum_{k_n \neq 0} \cos k_n/t \left(\frac{\sin k_n/2t}{k_n/2t}\right)^2 \\ &= 2m_s(0) - 2\lim_{t\to\infty} \sum_{k_n \neq 0} \frac{1 - 2\cos k_n/t + \cos 2k_n/t}{(k_n/t)^2}. \end{aligned} \tag{7}$$

Proof. We present here a proof of this theorem based on trace formula (2), another direct proof for graphs with rationally depended lengths will be given in the following subsection.

Consider the function φ defined as

$$\varphi(l) = \begin{cases} l, & 0 \leqslant l \leqslant 1; \\ 2-l, & 1 \leqslant l \leqslant 2; \\ 0, & \text{otherwise}. \end{cases}$$

This function and any scaled function $\varphi_t(x) = t\varphi(tx)$ satisfy the following properties:

$$\int_{-\infty}^{+\infty} \varphi_t(x)\,dx = 1 \quad \text{and} \quad \varphi_t(0) = 0.$$

It is clear then that the Euler characteristic can be calculated from the Fourier transform of the distribution u using formula (3) and taking the limit $t \to +\infty$, since the length of the shortest orbit is positive

$$\chi(\Gamma) = \sqrt{2\pi} \lim_{t \to +\infty} \hat{u}[\varphi_t].$$

In fact $\hat{u}[\varphi_t]$ is just equal to $\chi(\Gamma)/\sqrt{2\pi}$ for $t > 2/\min\{d_j\}$, where d_j are the lengths of Δ_j. The Fourier transform of the function φ_t can easily be calculated

$$\hat{\varphi}_t(k) = \frac{1}{\sqrt{2\pi}} e^{ik/t} \left(\frac{\sin k/2t}{k/2t} \right)^2.$$

Now the Euler characteristic may be calculated by applying the distribution u to the test function $\hat{\varphi}_t$

$$\chi(\Gamma) = \sqrt{2\pi} \lim_{t \to +\infty} u[\hat{\varphi}_t] = 2m_s(0) + 2 \lim_{t \to +\infty} \sum_{k_n \neq 0} \cos k_n/t \left(\frac{\sin k_2/2t}{k_2/2t} \right)^2.$$

The second formula (7) is a result of direct calculation. □

The eigenvalues of $L(\Gamma)$ satisfy the Weyl asymptotic law, i.e. grow linearly with n and therefore the sum in (7) is absolutely converging in contrast to one in (6).

3. Graphs having basic length

In this section we are going to study a very special class of graphs with rationally depended lengths of edges, namely the graphs with all edge lengths being integer multiples of one and the same length Δ to be called the *basic length*

$$d_j = n_j \Delta, \quad n_j \in \mathbb{N}. \tag{8}$$

Without loss of generality we may assume that Δ is the largest such number and we introduce the *basic frequency*

$$\Omega = \frac{2\pi}{\Delta}. \tag{9}$$

Such graphs will be called *graphs with basic length* Δ in what follows.[3]

The spectrum of the Laplace operator on such a graph appears to be almost periodic with the period Ω if one considers the k-scale instead of the λ-scale ($k^2 = \lambda$).

Lemma 1. *Let k_n^2 be an eigenvalue of the Laplace operator $L(\Gamma)$ on a graph Γ with the basic length Δ. Then $(k_n + \frac{2\pi}{\Delta})^2$ is an eigenvalue of $L(\Gamma)$ as well. The multiplicities of the eigenvalues coincide if both k_n^2 and $(k_n + \frac{2\pi}{\Delta})^2$ are different from zero.*

Proof. To prove the first statement it is enough to show that if $\psi(x)$ is an eigenfunction of $L(\Gamma)$ corresponding to a certain eigenvalue k_n^2, then there exists another function $\tilde{\psi}$, which is an eigenfunction corresponding to $(k_n + \frac{2\pi}{\Delta})^2$. Every eigenfunction ψ can be written as a combination of incoming waves

$$\psi(x) = a_{2j-1} e^{ik_n|x-x_{2j-1}|} + a_{2j} e^{ik_n|x-x_{2j}|}, \quad x \in [x_{2j-1}, x_{2j}].$$

Then the function $\tilde{\psi}$ given on each interval $[x_{2j-1}, x_{2j}]$ by

$$\tilde{\psi}(x) = a_{2j-1} e^{i(k_n+2\pi/\Delta)|x-x_{2j-1}|} + a_{2j} e^{i(k_n+2\pi/\Delta)|x-x_{2j}|} \tag{10}$$

is an eigenfunction corresponding to the eigenvalue $\lambda = (k_n + 2\pi/\Delta)^2$. Really, this function satisfies the necessary differential equation on the edges and is continuous at the vertices. Hence to show that $\tilde{\psi}$ is an eigenfunction it is enough to show that the sum of normal derivatives at each vertex is zero, which is obviously true, since the normal derivatives of ψ and $\tilde{\psi}$ are related by

$$\partial_n \psi(x_j) = \frac{k_n}{k_n + 2\pi/\Delta} \partial_n \tilde{\psi}(x_j).$$

If $k_n + 2\pi/\Delta = 0$, then it is well known that the corresponding point $\lambda = 0$ is always an eigenvalue of the Laplacian (Theorem 1 from [26]).

The second statement that the multiplicities of the two eigenvalues coincide follows from the fact that for $\lambda \neq 0$ every eigenfunction is uniquely determined by the coefficients a_j, $j = 1, 2, \ldots, 2N$. Therefore the transformation $\psi \mapsto \tilde{\psi}$ is one-to-one. $\square$

This lemma allows us to describe the structure of the spectrum of $L(\Gamma)$ for graphs having basic length.

Theorem 2. *Let $L(\Gamma)$ be a Laplace operator on a connected graph Γ with the basic length Δ. Its spectrum is pure discrete*

$$\sigma\big(L(\Gamma)\big) = \big\{\lambda_n = k_n^2,\ k_n \geqslant 0\big\}_{n=0}^{\infty}$$

[3] Similar graphs have been studied recently in [32].

and has the following properties:

(1) *Apart from the point* $k = 0$ *the set* $\{k_n\}_{n=0}^{\infty}$ *is invariant under right shifts by* $\Omega = 2\pi/\Delta$.
(2) *The points* k_n *inside the interval* $(0, 2\pi/\Delta)$ *are symmetric with respect to the center of the interval, i.e. if* $\lambda_n = k_n^2$ *is an eigenvalue, then* $(2\pi/\Delta - k_n)^2$ *is also an eigenvalue.*
(3) *The point* $\lambda_0 = 0$ *has multiplicity* 1.
(4) *The points* $\lambda = (\frac{2\pi}{\Delta} m)^2, m = 1, 2, \dots,$ *have multiplicity* $2 - \chi$, *where* χ *is the Euler characteristic of* Γ.

Proof. The proof of statement (1) follows immediately from Lemma 1. To prove the second statement we use again Lemma 1 for $(-k_n)^2 = k_n^2$ to conclude that if k_n^2 is an eigenvalue, then $(-k_n + 2\pi/\Delta)^2$ is also an eigenvalue. The third statement is valid for arbitrary connected graphs (Theorem 1 from [26]).

Let us establish the last assertion. The proof we present here is a modification of the proof of Theorem 1 from [26]. Without loss of generality we assume that $k = \Omega = 2\pi/\Delta$. Then it is easy to construct $g + 1$ linearly independent eigenfunctions corresponding to this particular k, where g is the number of generators in the fundamental group for Γ. Let us denote by $\psi_0(x, k)$ the eigenfunction given by

$$\psi_0(x, k) = \cos k(x - x_{2j-1}), \quad x \in [x_{2j-1}, x_{2j}].$$

This function satisfies the differential equation on each edge, is continuous at all vertices (in fact it is equal to 1 at all vertices) and all normal derivatives are equal to zero, which implies that their sum at each vertex is also zero and the boundary conditions at all vertices are satisfied.

If Γ is not a tree, then it can be transformed to a tree T by removing exactly $g = 1 - \chi = N - M + 1$ edges denoted by $\Delta_1, \Delta_2, \dots, \Delta_{N-M+1}$ without loss of generality. Let us denote by l_i the shortest nontrivial closed path on $T \cup \Delta_i$ passing through Δ_i. Note that every such path comes across exactly one removed edge. Consider the functions ψ_i defined by

$$\psi_i(x, k) = \begin{cases} \pm \sin k(x - x_{2j-1}), & \text{provided } x \in \Delta_j \text{ and } \Delta_j \subset l_i; \\ 0, & \text{otherwise;} \end{cases}$$

where the sign depends on whether the path l_i runs along Δ_j in the positive $(+)$ or in the negative $(-)$ direction. The function ψ_i is not only continuous along the path l_i but its first derivative is continuous as well.

Each function ψ_i satisfies the eigenfunction equation, is continuous at all vertices (in fact equal to zero there) and the sum of normal derivatives at each vertex is zero (if the vertex is on the path l_i then only two normal derivatives are different from zero but cancel each other, if the vertex is not on the path, then all normal derivatives are zero).

It is clear that the functions $\psi_0, \psi_1, \dots, \psi_g$ are linearly independent and this implies that the multiplicity of the eigenvalue k is not less than $1 + g$. It remains to prove that these functions span the corresponding eigensubspace up.

Let $\psi(x, k)$ be any eigenfunction of $L(\Gamma)$ corresponding to $k = 2\pi/\Delta$. First of all it is clear that this function attains the same value at all vertices, say f_0. The function $\psi(x, k) - f_0\psi_0(x, k)$ is an eigenfunction of $L(\Gamma)$ equal to zero at all vertices. Then the restriction of this function to the edge Δ_i is proportional to $\sin k(x - x_{2i-1})$

$$\psi(x, k) - f_0\psi_0(x, k)|_{\Delta_j} = f_i \sin k(x - x_{2i-1}).$$

Consider the function

$$\hat{\psi}(x,k) = \psi(x,k) - f_0\psi_0(x,k) - \sum_{i=1}^{g} f_i\psi_i(x,k),$$

which is an eigenfunction of $L(\Gamma)$ equal to zero at all vertices and on all edges $\Delta_i, i = 1, 2, \ldots, g$. It follows that this function is supported by the tree T. It is easy to see that $\hat{\psi}(x,k)$ is identically equal to zero. First we note that $\hat{\psi}$ is equal to zero on all loose edges (satisfies zero Cauchy data at the loose end points). Then it follows that $\hat{\psi}$ is zero on all edges connected by at least one end point to only loose edges. Continuing in this way we conclude that $\hat{\psi}(x,k)$ is identically equal to zero, which implies

$$\psi(x,k) = f_0\psi_0(x,k) + \sum_{i=1}^{g} f_i\psi_i(x,k).$$

The last equality means that the spectral multiplicity of $\lambda = (2\pi/\delta)^2$ is equal to $1 + g = 2 - \chi$. The proof for $\lambda = (2\pi m/\delta)^2, m = 2, 3, \ldots,$ follows the same lines. □

Now we are ready to give a direct proof of Theorem 1 formula (7) for the Euler characteristic under the additional condition that Γ has a basic length.

Proof of Theorem 1. (For graphs having basic length.) Assume first that the graph Γ is connected. Let us denote by ω_j^2, $j = 1, 2, \ldots, J$ the eigenvalues of $L(\Gamma)$ inside the interval $(0, \Omega)$ (see (9)). Then the limit on the right-hand side of (7) can be written as follows:

$$\begin{aligned}
&2 - 2\lim_{t\to\infty} \sum_{k_n \neq 0} \frac{1 - 2\cos k_n/t + \cos 2k_n/t}{(k_n/t)^2} \\
&\quad = 2 - 2(1+g)\lim_{t\to\infty} \sum_{m=1}^{\infty} \frac{1 - 2\cos \Omega m/t + \cos 2\Omega m/t}{(\Omega m/t)^2} \\
&\qquad - 2\lim_{t\to\infty} \sum_{m=0}^{\infty}\sum_{j=1}^{J} \frac{1 - 2\cos(\omega_j + \Omega m)/t + \cos 2(\omega_j + \Omega m)/t}{((\omega_j + \Omega m)/t)^2},
\end{aligned} \tag{11}$$

where we used that all points $k = m\Omega, m = 1, 2, \ldots,$ have multiplicity $1 + g$ and the point $k = 0$ has multiplicity 1. The first limit can be calculated using formula

$$\sum_{m=1}^{\infty} \frac{1 - 2\cos m/t + \cos 2m/t}{(m/t)^2} = \frac{1}{2}, \tag{12}$$

since the limit obviously does not depend on the value of Ω. To prove this formula we use the sum ((1.443.3) from [17])

$$\sum_{m=1}^{\infty} \frac{\cos mx}{m^2} = \frac{\pi^2}{6} - \frac{\pi x}{2} + \frac{x^2}{4}$$

to obtain

$$\sum_{m=1}^{\infty}\frac{1-2\cos m/t+\cos 2m/t}{(m/t)^2}=t^2\left(\sum_{m=1}^{\infty}\frac{1}{m^2}-2\sum_{m=1}^{\infty}\frac{\cos m\frac{1}{t}}{m^2}+\sum_{m=1}^{\infty}\frac{\cos m\frac{2}{t}}{m^2}\right)$$
$$=t^2\left\{\frac{\pi^2}{6}-2\left(\frac{\pi^2}{6}-\frac{\pi}{2}\frac{1}{t}+\frac{1}{4}\frac{1}{t^2}\right)+\frac{\pi^2}{6}-\frac{\pi}{2}\frac{2}{t}+\frac{1}{4}\frac{4}{t^2}\right\}=\frac{1}{2}.$$

To calculate the second limit let us use that the points ω_j are situated symmetrically with respect to the center of the interval $(0,\Omega)$

$$\begin{aligned}
&\sum_{m=0}^{\infty}\sum_{j=1}^{J}\frac{1-2\cos(\omega_j+\Omega m)/t+\cos 2(\omega_j+\Omega m)/t}{((\omega_j+\Omega m)/t)^2}\\
&=\frac{1}{2}\sum_{m=0}^{\infty}\sum_{j=1}^{J}\left\{\frac{1-2\cos(m+\omega_j/\Omega)/(t/\Omega)+\cos 2(m+\omega_j/\Omega)/(t/\Omega)}{((m+\omega_j/\Omega)/(t/\Omega))^2}\right.\\
&\quad\left.+\frac{1-2\cos(m+(\Omega-\omega_j)/\Omega)/(t/\Omega)+\cos 2(m+(\Omega-\omega_j)/\Omega)/(t/\Omega)}{((m+(\Omega-\omega_j)/\Omega)/(t/\Omega))^2}\right\}\\
&=\frac{1}{2}\sum_{m\in\mathbb{Z}}\sum_{j=1}^{J}\frac{1-2\cos(m+\omega_j/\Omega)/(t/\Omega)+\cos 2(m+\omega_j/\Omega)/(t/\Omega)}{((m+\omega_j/\Omega)/(t/\Omega))^2}.
\end{aligned}\tag{13}$$

We are going to prove that the last sum is equal to zero using the formula

$$\sum_{m\in\mathbb{Z}}\frac{e^{i(m+\alpha)x}}{(m+\alpha)^2}=\frac{2\pi e^{2\pi i\alpha}}{1-e^{2\pi i\alpha}}x-\frac{(2\pi)^2e^{2\pi i\alpha}}{(1-e^{2\pi i\alpha})^2},\quad \alpha\notin\mathbb{Z}.\tag{14}$$

To prove this formula one may exploit the following idea: find a linear function $f(x)=ax+b$, $0\leqslant x\leqslant 2\pi$, such that the series on the left-hand side of the formula is exactly the Fourier series for f in the orthogonal basis $e^{i(n+\alpha)x}$. The function f is represented by the following almost everywhere converging Fourier series

$$f(x)=\frac{1}{2\pi}\sum_{m\in\mathbb{Z}}f_m e^{i(m+\alpha)x},\tag{15}$$

where

$$f_m=\int_0^{2\pi}(ax+b)e^{-i(m+\alpha)x}\,dx.\tag{16}$$

The function f may be chosen equal to

$$f(x)=\frac{e^{2\pi i\alpha}}{1-e^{2\pi i\alpha}}x-\frac{2\pi e^{2\pi i\alpha}}{(1-e^{2\pi i\alpha})^2}.$$

Then the Fourier coefficients are given by

$$f_m = \int_0^{2\pi} \left(\frac{e^{2\pi i\alpha}}{1-e^{2\pi i\alpha}} x - \frac{2\pi e^{2\pi i\alpha}}{(1-e^{2\pi i\alpha})^2} \right) e^{-i(m+\alpha)x}\, dx = \frac{1}{(m+\alpha)^2}.$$

Thus formula (14) is proven and it implies in particular that

$$\sum_{m\in\mathbb{Z}} \frac{1-2e^{i(m+\alpha)x}+e^{2i(m+\alpha)x}}{(m+\alpha)^2} = 0, \quad \text{provided } \alpha \notin \mathbb{Z}. \tag{17}$$

Really using (14) we have:

$$\begin{aligned}
&\sum_{m\in\mathbb{Z}} \frac{1-2e^{i(m+\alpha)x}+e^{2i(m+\alpha)x}}{(m+\alpha)^2} \\
&\quad = \sum_{m\in\mathbb{Z}} \frac{1}{(m+\alpha)^2} - 2\sum_{m\in\mathbb{Z}} \frac{e^{i(m+\alpha)x}}{(m+\alpha)^2} + \sum_{m\in\mathbb{Z}} \frac{e^{2i(m+\alpha)x}}{(m+\alpha)^2} \\
&\quad = -\frac{(2\pi)^2 e^{2\pi i\alpha}}{(1-e^{2\pi i\alpha})^2} - 2\left(-\frac{(2\pi)^2 e^{2\pi i\alpha}}{(1-e^{2\pi i\alpha})^2} + \frac{2\pi e^{2\pi i\alpha}}{1-e^{2\pi i\alpha}} x \right) - \frac{(2\pi)^2 e^{2\pi i\alpha}}{(1-e^{2\pi i\alpha})^2} + \frac{2\pi e^{2\pi i\alpha}}{1-e^{2\pi i\alpha}} 2x \\
&\quad = 0,
\end{aligned}$$

where to calculate the first sum we used that the series (15) at $x=0$ converges to $\frac{1}{2}(f(+0) + e^{-2\pi 1\alpha} f(2\pi - 0))$. It turns out that

$$\sum_{m\in\mathbb{Z}} \frac{1-2\cos(m+\alpha)x+\cos 2(m+\alpha)x}{(m+\alpha)^2} = 0, \quad \text{provided } \alpha \notin \mathbb{Z}, \tag{18}$$

and therefore the second sum in (11) (the sum (13)) is equal to zero. Finally we get

$$2 - 2\lim_{t\to\infty} \sum_{k_n\neq 0} \frac{1-2\cos k_n/t + \cos 2k_n/t}{(k_n/t)^2} = 2 - 2(1+g)\frac{1}{2} + 0 = 1 - g = \chi.$$

Now it is straightforward to generalize this result to include not connected graphs to get (7). □

Thus we have proven the formula for the Euler characteristic for graphs having basic length without any use of the trace formula. It might be important to find a similar proof for arbitrary graphs. Such alternative to the trace formula approach may provide a new insight on the structure of the spectral asymptotics for $L(\Gamma)$.

We would like to present here few explicit examples illustrating formulas (7).

(1) *Single interval.* Let the graph Γ coincide with the interval $[0,\pi]$ (with separated end points). The Euler characteristic is $\chi = 1$. The spectrum of $L(\Gamma)$ is $\sigma(L) = \{n^2, n = 0, 1, 2, \ldots\}$. Substituting $k_n = n$, $n = 0, 1, 2, \ldots,$ into formula (7) we get

$$\chi = 2 - 2 \lim_{t\to\infty} \sum_{n=1}^{\infty} \frac{1 - 2\cos nt + \cos 2n/t}{(n/t)^2} = 2 - 1 = 1,$$

where we used formula (12).

(2) *Simple circle.* Let the graph Γ coincide with the circle having length π, i.e. it can be treated as the interval $[0,\pi]$ with the end points identified. The Euler characteristic is $\chi = 0$. The spectrum of $L(\Gamma)$ is $\sigma(L) = \{n^2, n = 0, 2, 2, 4, 4, \ldots\}$. Substitution $k_n = n$ into formula (7) gives

$$\chi = 2 - 4 \lim_{t\to\infty} \sum_{n=1}^{\infty} \frac{1 - 2\cos 2nt + \cos 4n/t}{(n/t)^2} = 2 - 2 = 0,$$

where we again used formula (12).

(3) *Symmetric star graph.* Let Γ be the star graph formed by m equal edges of the length π joined at one end point. The Euler characteristic is $\chi = 1$. The spectrum consists of simple eigenvalues n^2, $n = 0, 1, 2, \ldots,$ and eigenvalues $(1/2 + n)^2$, $n = 0, 1, 2, \ldots,$ having multiplicity $m - 1$. The formula (7) gives then

$$\begin{aligned}\chi &= 2 - 2 \lim_{t\to\infty} \sum_{n=1}^{\infty} \frac{1 - 2\cos n/t + \cos 2n/t}{(n/t)^2} \\ &\quad - 2(m-1) \lim_{t\to\infty} \sum_{n=1}^{\infty} \frac{1 - 2\cos(n+1/2)/t + \cos 2(n+1/2)/t}{((n+1/2)/t)^2} \\ &= 2 - 1 - 0 = 1,\end{aligned}$$

where we used formulas (12) and (17).

4. Spectral asymptotics and Euler characteristic

In this section we are going to show that the Euler characteristic is determined entirely by the asymptotics of the spectrum of the Laplace operator. The limit of each term in the series (7) does not depend on k_n

$$\lim_{t\to\infty} \frac{1 - 2\cos k_n/t + \cos 2k_n/t}{(k_n/t)^2} = -1.$$

Taking this into account it is clear that changing of any finite number of eigenvalues does not affect the limit (7). We are going to prove that the same is true even if the number of changed eigenvalues is infinite, but under certain additional restrictions.

Lemma 2. *Let k_n and k_n^0 be two real sequences satisfying the following conditions*

$$\left|k_n - k_n^0\right| = O\left(\frac{1}{n}\right), \quad k_n^0 = \frac{\pi}{\mathcal{L}} n + O(1), \tag{19}$$

and the limit $\lim_{t\to\infty} \sum_{n=1}^{\infty} \cos k_n^0/t \left(\frac{\sin k_n^0/2t}{k_n^0/2t}\right)^2$ *exists. Then the following limits coincide*

$$\lim_{t\to\infty} \sum_{n=1}^{\infty} \cos k_n/t \left(\frac{\sin k_n/2t}{k_n/2t}\right)^2 = \lim_{t\to\infty} \sum_{n=1}^{\infty} \cos k_n^0/t \left(\frac{\sin k_n^0/2t}{k_n^0/2t}\right)^2. \tag{20}$$

Proof. Without loss of generality we assume that $\mathcal{L} = \pi$. It will be convenient to write estimates (19) in the form

$$\left|k_n - k_n^0\right| \leqslant A\frac{1}{n}, \qquad |k_n - n| \leqslant B, \qquad \left|k_n^0 - n\right| \leqslant B, \quad n = 1, 2, \ldots, \tag{21}$$

with certain positive constants A and B. In addition we shall use the following notations

$$a_n(t) \equiv \cos k_n/t \left(\frac{\sin k_n/2t}{k_n/2t}\right)^2, \qquad a_n^0(t) \equiv \cos k_n^0/t \left(\frac{\sin k_n^0/2t}{k_n^0/2t}\right)^2.$$

To prove lemma we are going to establish two estimates which will be suitable for terms with small and large indices respectively.

Estimate 1. (Suitable for small values of n.)

$$\left|a_n(t) - a_n^0(t)\right| \leqslant c\frac{(n+B)^2}{t^2}, \tag{22}$$

where c is a certain positive constant $c > 0$. Consider the function

$$f(\alpha) = \begin{cases} \cos 2\alpha \left(\frac{\sin\alpha}{\alpha}\right)^2, & \alpha \neq 0, \\ 1, & \alpha = 0. \end{cases}$$

The derivatives of f are

$$f'(\alpha) = -2\sin 2\alpha \left(\frac{\sin\alpha}{\alpha}\right)^2 + 2\cos 2\alpha \frac{\sin\alpha}{\alpha}\frac{\alpha\cos\alpha - \sin\alpha}{\alpha^2},$$

$$f''(\alpha) = -4\cos 2\alpha \left(\frac{\sin\alpha}{\alpha}\right)^2 - 8\sin 2\alpha \frac{\sin\alpha}{\alpha}\frac{\alpha\cos\alpha - \sin\alpha}{\alpha^2} + 2\cos 2\alpha \left(\frac{\alpha\cos\alpha - \sin\alpha}{\alpha^2}\right)^2$$
$$+ 2\cos 2\alpha \frac{\sin\alpha}{\alpha}\frac{-\alpha^2\sin\alpha - 2\alpha\cos\alpha + 2\sin\alpha}{\alpha^3},$$

and we see that $f'(0) = 0$ and $f''(\alpha)$ is uniformly bounded. Hence Taylor's formula gives

$$f(\alpha) - f(0) - f'(0)\alpha = f''(\xi)\frac{\alpha^2}{2}$$

and therefore

$$\left|f(\alpha)-1\right| \leqslant \frac{1}{2}\max\left|f''(\alpha)\right|\alpha^2.$$

This implies that

$$\left|a_n(t)-1\right| \leqslant \frac{1}{2}\max\left|f''(\alpha)\right|\frac{(n+B)^2}{4t^2}$$

and similar estimate (22) for the difference $|a_n(t)-a_n^0(t)|$ with $c=\frac{1}{4}\max|f''(\alpha)|$.

Estimate 2. (Suitable for large values of n.)

$$\left|a_n(t)-a_n^0(t)\right| \leqslant d\frac{t}{(n-B)^3}, \quad n>B, \tag{23}$$

where d is a certain positive constant $d>0$. To prove the estimate we use that the function $\alpha^2 f'(\alpha)$ is uniformly bounded. Using the first mean value theorem we get

$$a_n(t)-a_n^0(t)=f(k_n/2t)-f\big(k_n^0/2t\big)=f'(\xi_n/2t)\big(k_n/2t-k_n^0/2t\big),$$

where ξ_n satisfies the same estimate as k_n and k_n^0 (see the second and third estimates in (21))

$$|\xi_n-n| \leqslant B.$$

For $n>B$, it follows that

$$\left|a_n(t)-a_n^0(t)\right| \leqslant \max\left|\alpha^2 f'(\alpha)\right|\frac{1}{(\frac{n-B}{2t})^2}A\frac{1}{2nt} \leqslant d\frac{t}{(n-B)^3}$$

with $d=2A\max|\alpha^2 f'(\alpha)|$, which is exactly estimate (23).

The limits appearing in (20) are equal if the following limit equals to zero

$$\lim_{t\to\infty}\sum_{n=1}^{\infty}\left|\cos k_n/t\left(\frac{\sin k_n 2t}{k_n/2t}\right)^2-\cos k_n^0/t\left(\frac{\sin k_n^0 2t}{k_n^0/2t}\right)^2\right| \equiv \lim_{t\to\infty}\sum_{n=1}^{\infty}\left|a_n(t)-a_n^0(t)\right|. \tag{24}$$

Let us split the infinite series into the finite sum of the first K elements and the remaining infinite series as

$$\sum_{n=1}^{\infty}=\sum_{n=1}^{K}+\sum_{n=K+1}^{\infty}.$$

To prove that the limit is zero it is enough to prove that for any $\epsilon>0$ there exists $t_0=t_0(\epsilon)$, such that for any $t>t_0(\epsilon)$ the number $K=K(\epsilon,t)$ can be chosen in such a way that both the finite and infinite sums are less than $\epsilon/2$.

The two sums can be estimated using (22) and (23) as

$$\sum_{n=1}^{K}\left|a_n(t)-a_n^0(t)\right| \leqslant \sum_{n=1}^{K} c\frac{(K+B)^2}{t^2} \leqslant c\frac{(K+B)^3}{t^2},$$

$$\sum_{n=K+1}^{\infty}\left|a_n(t)-a_n^0(t)\right| \leqslant \sum_{n=K+1}^{\infty} d\frac{t}{(n-B)^3} \leqslant \frac{d}{2}\frac{t}{(K-B)^2}.$$

Each of these sums is less than $\epsilon/2$ if the following two inequalities are satisfied

$$K(\epsilon,t) \leqslant \left(\frac{\epsilon t^2}{2c}\right)^{1/3} - B \quad \text{and} \quad K(\epsilon,t) \geqslant \sqrt{\frac{dt}{\epsilon}} + B.$$

Hence the series in (24) is less than ϵ if

$$\sqrt{\frac{dt}{\epsilon}} + B \leqslant \left(\frac{\epsilon t^2}{2c}\right)^{1/3} - B.$$

For any $\epsilon > 0$ there exists t_0, such that for any $t > t_0$ the last inequality is satisfied and it is possible to choose integer $K(\epsilon,t)$, such that both the finite and infinite sums are less than $\epsilon/2$. For such t we have that the infinite series in (24) is less than ϵ. It follows that the limit in (24) holds. $\square$

This result will be used in the following section to derive formula for the Euler characteristic using the spectra of Schrödinger operators on graphs.

5. Schrödinger operators on compact finite graphs

Consider an arbitrary essentially bounded real valued function q on Γ

$$q \in L_\infty(\Gamma). \tag{25}$$

Then the operator Q of multiplication by the function q is bounded in $L_2(\Gamma)$

$$\|Qf\|^2_{L_2(\Gamma)} = \int_\Gamma \left|q(x)f(x)\right|^2 dx = \sum_{j=1}^{N} \int_{\Delta_j} \left|q(x)f(x)\right|^2 dx$$

$$\leqslant \sum_{j=1}^{N} \|q\|^2_{L_\infty(\Delta_j)} \|f\|^2_{L_2(\Delta_j)} \leqslant \|q\|^2_{L_\infty(\Gamma)} \|f\|^2_{L_2(\Gamma)}.$$

The Schrödinger operator with the potential q can be defined as the operator sum

$$S = L(\Gamma) + Q \tag{26}$$

and it is self-adjoint on the domain of the Laplace operator (sum of a self-adjoint and a bounded self-adjoint operators), i.e. on the set of functions from $W_2^2(\Gamma \setminus \mathbf{V})$ satisfying standard boundary conditions (1).

Theorem 3. *Let Γ be a compact finite metric graph and $L(\Gamma)$ be the corresponding Laplace operator with the spectrum $\lambda_n(L) = k_n^2(L)$, $n = 0, 1, 2, 3, \ldots$. Let q be an $L_\infty(\Gamma)$ function and $S = L(\Gamma) + Q$—the corresponding Schrödinger operator. Then the spectrum of S is pure discrete $\lambda_n(S) = k_n^2(S)$, $n = 0, 1, 2, \ldots$,*[4] *and satisfy the asymptotic formula*

$$k_n(S) = k_n(L) + O(1/n), \quad \text{as } n \to \infty. \tag{27}$$

Proof. The statement that the spectrum of S is pure discrete follows from the fact that any bounded perturbation of an operator with pure discrete spectrum is pure discrete as well [2].

It is clear that the Schrödinger operator S is bounded from below by $-\|Q\|$. Let us denote by $N_\Delta(L)$ and $N_\Delta(S)$ the number of eigenvalues in the interval Δ for the operators L and S respectively. Then the following inequalities hold

$$N_{[\lambda_0(L)-\|Q\|, \lambda_n(L)+\|Q\|]}(S) \geqslant n \quad \text{and} \quad N_{[\lambda_0(S)-\|Q\|, \lambda_m(S)+\|Q\|]}(L) \geqslant m,$$

which imply that

$$\left|\lambda_n(S) - \lambda_n(L)\right| \leqslant \|Q\|. \tag{28}$$

In the k-scale this means that $k_n(S)$ and $k_n(L)$ are asymptotically close in the sense that (27) holds. □

We are able to establish now the formula connecting the spectrum of a Schrödinger operator on a graph with the Euler characteristic of the graph.

Theorem 4. *Let Γ be a finite compact metric graph and $L(\Gamma)$—the corresponding Laplace operator (with standard boundary conditions). Let $q \in L_\infty(\Gamma)$ be a real valued potential and $S = L(\Gamma) + Q$—the corresponding Schrödinger operator, where Q is the operator of multiplication by q. Then the Euler characteristic $\chi(\Gamma)$ of the graph Γ is uniquely determined by the spectrum $\lambda_n(S)$ of the operator S and can be calculated using the limit*

$$\chi(\Gamma) = 2 \lim_{t\to\infty} \sum_{n=0}^{\infty} \cos \sqrt{\lambda_n(S)}/t \left(\frac{\sin \sqrt{\lambda_n(S)}/2t}{\sqrt{\lambda_n(S)}/2t} \right)^2, \tag{29}$$

where we use the following natural convention

$$\lambda_m = 0 \quad \Rightarrow \quad \frac{\sin \sqrt{\lambda_m(S)}/2t}{\sqrt{\lambda_m(S)}/2t} = 1. \tag{30}$$

Proof. Theorem 3 together with equality (20) implies that formula (29) gives the same result if the spectrum $\lambda_n(S)$ of the Schrödinger operator is substituted by the spectrum $\lambda_n(L)$ of the corresponding Laplace operator. Then formula (7) implies that the Euler characteristic of Γ is determined by (29). □

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[4] The first several eigenvalues of S may be negative, but we are interested in the asymptotics of $\lambda_n(S)$ as $n \to \infty$. For large values of n the corresponding k_n are all positive reals.

The last two theorems state that two Schrödinger operators on graphs may have the same spectrum only if the underlying graphs have the same total length and Euler characteristic, in other words, if the graphs have the same size and complexity.

Uniqueness Theorem 1. *If two Schrödinger operators on finite compact metric graphs have the same spectrum, then the underlying graphs have the same*

- *total length*;
- *Euler characteristic,*

provided the potentials are essentially bounded and the boundary conditions at the vertices are standard (*see* (1)).

6. Generalizations and further developments

The results of this paper can be extended to the case of most general boundary conditions at the vertices. This problem appears to be more sophisticated than it may be expected. The main reason is that each vertex scattering matrix in general is not energy independent but tends to a certain limiting matrix S_v^∞. The limiting matrix in its turn corresponds to certain symmetric boundary conditions, but these conditions may be incompatible with the connectivity of the graph Γ. In another words these new boundary conditions may not connect all edges joined at the vertex at the same time as the original conditions (corresponding to the energy dependent scattering matrix) do connect all the edges together.

Let us study the following elementary example. Consider the interval $\Delta_1 = [-\pi, \pi]$ turned into circle by joining together the end points $-\pi$ and π with the help of the following boundary conditions

$$\begin{cases} \psi(-\pi) = -\partial_n \psi(+\pi), \\ \psi(\pi) = -\partial_n \psi(-\pi); \end{cases} \tag{31}$$

which are obviously properly connecting, i.e. connect together the boundary values of the functions from both edges. These boundary conditions can be written in the conventional form [19]

$$A \begin{pmatrix} \psi(\pi) \\ \psi(-\pi) \end{pmatrix} + B \begin{pmatrix} \partial_n \psi(\pi) \\ \partial_n \psi(-\pi) \end{pmatrix} = 0$$

by choosing the matrices A and B equal to $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The corresponding vertex scattering matrix is

$$S_v(k) = -\frac{I - ikB}{I + ikB}$$

and it tends to the unit matrix $S_v^\infty = I$ as $k \to \infty$. The boundary conditions determining $S_v(k) = I$ are just Neumann boundary conditions $\partial_n \psi(+\pi) = 0 = \partial_n \psi(-\pi)$, which obviously disconnect the two endpoints. Therefore it is natural to call the boundary conditions (31) by *not asymptotically properly connecting*. Such boundary conditions result in that the spectral asymptotics for such graphs coincides with the one for graphs with boundary conditions having energy independent scattering matrices and different connectivity matrix, i.e. having different geometry.

We illustrate this idea by calculating the spectra of the operators appearing in the example under consideration. Let us denote by $\tilde{L}$ the second derivative operator $-\frac{d^2}{dx^2}$ defined on the functions from $W_2^2[-\pi,\pi]$ and satisfying boundary conditions (31). These boundary conditions can be written as follows using standard derivative with respect to the variable $x \in [-\pi,\pi]$

$$\begin{cases} \psi(-\pi) = \psi'(+\pi), \\ \psi'(-\pi) = -\psi(\pi). \end{cases} \tag{32}$$

It is easy to see that the boundary conditions are invariant with respect to the change of the coordinate $x \mapsto -x$ and hence the operator $\tilde{L}$ commutes with the symmetry operator $\mathcal{P}\psi(x) = \psi(-x)$. Therefore all eigenfunctions of $\tilde{L}$ are either even or odd. The dispersion equations for even and odd functions can be obtained by substituting the Ansätze $\psi_s(x) = \cos kx$ and $\psi_a(x) \sin kx$ into the boundary conditions

$$\tan k^s\pi = -\frac{1}{k^s}; \tag{33}$$

$$\cot k^a\pi = -\frac{1}{k^a}. \tag{34}$$

It is easy to see that the eigenvalues satisfy the following asymptotic conditions

$$k_n^s = n + O\left(\frac{1}{n}\right), \qquad k_n^a = \frac{2n+1}{2} + O\left(\frac{1}{n}\right), \quad n = 0, 1, 2, \ldots, \tag{35}$$

where $(k_n^s)^2$ and $(k_n^a)^2$ denote the eigenvalues for even and odd eigenfunctions respectively. These eigenvalues should be compared with the eigenvalues $k_n^{s0^2} = (n)^2$ and $k_n^{a0^2} = (\frac{2n+1}{2})^2$ for the Laplace operator on the interval $[-\pi,\pi]$ (with standard, i.e. Neumann boundary conditions at the end points $\psi'(-\pi) = 0 = \psi'(\pi)$). Let us try to reconstruct the Euler characteristic of the graph using formula (29). Lemma 2 implies that

$$\lim_{t\to\infty} \sum_{k_n = k_n^s, k_n^a} \cos k_n/t \left(\frac{\sin k_n/2t}{k_n/2t}\right) = \lim_{t\to\infty} \sum_{k_n^0 = k_n^{s0}, k_n^{a0}} \cos k_n^0/t \left(\frac{\sin k_n^0/2t}{k_n^0/2t}\right) = 1, \tag{36}$$

which is in contradiction to the fact that the Euler characteristic of the circle is equal to 0. It follows that formula (7) in general is not valid for Schrödinger operators on graphs with general (self-adjoint) boundary conditions at the vertices. An extension of this formula for the case of general boundary conditions will be the subject of our following publication.

We decided to leave to the following publication another important generalization of the obtained results concerning Schrödinger operators with L_1 potentials. The proofs for $q \in L_1(\Gamma)$ and general boundary conditions at the vertices use similar methods and will be presented in a forthcoming publication.

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Spectral methods for orthogonal rational functions ☆

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Abstract

We present an operator theoretic approach to orthogonal rational functions based on the identification of a suitable matrix representation of the multiplication operator associated with the corresponding orthogonality measure. Two alternatives are discussed, leading to representations which are linear fractional transformations with matrix coefficients acting on infinite Hessenberg or five-diagonal unitary matrices. This approach permits us to recover the orthogonality measure throughout the spectral analysis of an infinite matrix depending uniquely on the poles and the parameters of the recurrence relation for the orthogonal rational functions. Besides, the zeros of the orthogonal and para-orthogonal rational functions are identified as the eigenvalues of matrix linear fractional transformations of finite Hessenberg or five-diagonal matrices. As an application we use operator perturbation theory results to obtain new relations between the support of the orthogonality measure and the location of the poles and parameters of the recurrence relation for the orthogonal rational functions.

Keywords: Orthogonal rational functions; Unitary Hessenberg and five-diagonal matrices; Linear fractional transformations with operator coefficients; Pairs of operators

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1. Introduction

The connection with Jacobi matrices has led to numerous applications of spectral techniques for self-adjoint operators in the theory of orthogonal polynomials on the real line. The direct extension of these ideas to the orthogonal polynomials on the unit circle yields a connection with unitary Hessenberg matrices (see [3,15,20,32,34]) which has provided some results (see for instance [16–19,34]). Nevertheless, the authentic analogue of the Jacobi matrices for the unit circle is a class of unitary five-diagonal matrices which has been only recently discovered (see [12,36]). This discovery has caused an explosion of applications of spectral methods for unitary operators in the theory of orthogonal polynomials on the unit circle, among which the numerous results appearing in the monograph [32,33] have been only the starting point.

The orthogonal polynomials (OP) are a particular case of a more general kind of orthogonal functions with interest in many pure and applied sciences: the orthogonal rational functions (ORF) with prescribed poles (see [11] and the references therein). An important ingredient in the theory of ORF are the linear fractional transformations $z \to (a_1 z + a_2)(a_3 z + a_4)^{-1}$ of a complex variable z, where a_i are complex numbers. Hence, it is natural to expect the related spectral methods to have a close relationship with the operator version of such transformations, i.e., the maps $T \to (A_1 T + A_2)(A_3 T + A_4)^{-1}$ or $T \to (A_3 T + A_4)^{-1}(A_1 T + A_2)$, acting on linear operators T on a Hilbert space, where the coefficients A_i are now operators on the same Hilbert space. However, there is no spectral approach to the study of ORF at present.

The ORF that appear as a natural generalization of the OP on the real line and the unit circle require the poles to be in the extended real line and in the exterior of the unit circle, respectively. The first situation presents special complications, an indication of this being the fact that the poles can lie on the support of the orthogonality measure. Indeed, considered as ORF, the main difference between the OP on the real line and the unit circle is not the location of the support of the measure, but the relative location of the poles with respect to this support. Actually, the Cayley transform maps the ORF on the unit circle with poles in the exterior of the unit circle onto the ORF on the real line with poles in the lower half plane, so both of them can be thought as generalizations of the OP on the unit circle. The purpose of the paper is to develop for this kind of ORF similar spectral techniques to those recently introduced for the OP on the unit circle. As we will see, the cornerstone of these spectral techniques is a matrix linear fractional transformation of the Hessenberg and five-diagonal unitary matrices associated with the polynomial case.

The paper is structured in the following way. Sections 2 and 3 summarize the results that we will need about ORF and operator linear fractional transformations respectively. Sections 4 and 5 develop the basics for the spectral theory of ORF on the unit circle. Section 4 is devoted to the approach based on Hessenberg matrices, while Section 5 deals with the approach related to five-diagonal matrices. Some applications of this spectral theory are presented in Section 6. Finally, Appendix A discusses the peculiarities of the spectral theory for ORF on the real line.

2. ORF on the unit circle

We use the notation:

$$\mathbb{T} = \{z \in \mathbb{C}:\ |z| = 1\}, \qquad \mathbb{D} = \{z \in \mathbb{C}:\ |z| < 1\}, \qquad \mathbb{E} = \{z \in \mathbb{C}:\ |z| > 1\}.$$

In what follows a measure on the unit circle is a probability Borel measure μ with $\operatorname{supp}\mu \subset \mathbb{T}$. L^2_μ denotes the Hilbert space of μ-square-integrable functions with inner product

$$\langle f, g\rangle_\mu = \int \overline{f(z)}g(z)\,d\mu(z), \qquad f, g \in L^2_\mu.$$

Unless we say the opposite we will suppose that $\operatorname{supp}\mu$ is an infinite set.

For any $\alpha \in \mathbb{D}$, the Möbius transformations ζ_α are defined by

$$\zeta_\alpha(z) = \frac{\varpi^*_\alpha(z)}{\varpi_\alpha(z)}, \qquad \begin{cases} \varpi_\alpha(z) = 1 - \bar{\alpha}z, \\ \varpi^*_\alpha(z) = z\varpi_{\alpha*}(z) = z - \alpha, \end{cases}$$

and, up to factors in $\mathbb{T}$, they are all the automorphisms of $\mathbb{D}$. Indeed, they are automorphisms of $\overline{\mathbb{C}}$ which leave invariant $\mathbb{T}$, $\mathbb{D}$ and $\mathbb{E}$. The inverse transformation of ζ_α is $\tilde{\zeta}_\alpha = \zeta_{-\alpha}$, and $\zeta_{\alpha*} = 1/\zeta_\alpha$ where $f_*(z) = \overline{f(\hat{z})}$, $\hat{z} = 1/\bar{z}$. We distinguish the value $\alpha_0 = 0$ that gives $\zeta_{\alpha_0}(z) = z$.

To get rational functions with fixed poles in $\mathbb{E}$ we introduce a sequence $(\alpha_n)_{n\geqslant 1}$ in $\mathbb{D}$, which defines the finite Blaschke products $(B_n)_{n\geqslant 0}$ given by

$$B_0 = 1; \qquad B_n = \zeta_{\alpha_1}\cdots\zeta_{\alpha_n}, \quad n \geqslant 1. \tag{1}$$

The orthonormalization of $(B_n)_{n\geqslant 0}$ in L^2_μ gives the ORF $(\Phi_n)_{n\geqslant 0}$ with respect to μ associated with $(\alpha_n)_{n\geqslant 1}$ (in short, a sequence of ORF on $\mathbb{T}$). For $n = 0, 1, \ldots, \infty$, the subspace $\mathcal{L}_n = \operatorname{span}\{B_k\}_{k=0}^{n-1}$ consists of those rational functions whose poles, counted with multiplicity, lie on $(\hat{\alpha}_k)_{k=1}^{n-1}$. We denote by $\mathcal{L}$ the closure of $\mathcal{L}_\infty$ in L^2_μ.

The ORF on $\mathbb{T}$ satisfy a recurrence relation which, with an appropriate normalization of $(\Phi_n)_{n\geqslant 0}$, has the form (see [11, Theorem 4.1.3])

$$\Phi_0 = 1; \qquad \begin{pmatrix} \Phi_n \\ \Phi^*_n \end{pmatrix} = e_n \frac{\varpi_{n-1}}{\varpi_n} \begin{pmatrix} 1 & \lambda_n \\ \bar{\lambda}_n & 1 \end{pmatrix} \begin{pmatrix} z_n\zeta_{n-1}\Phi_{n-1} \\ \Phi^*_{n-1} \end{pmatrix}, \quad n \geqslant 1, \tag{2}$$

$$\Phi^*_n = z_1 z_2 \cdots z_n B_n \Phi_{n*}, \quad \lambda_n \in \mathbb{D}, \quad e_n = \sqrt{\frac{\varpi_n(\alpha_n)}{\varpi_{n-1}(\alpha_{n-1})}\frac{1}{1-|\lambda_n|^2}},$$

where, for convenience, α_n is substituted by n when used as a subindex, and $z_n = -|\alpha_n|/\alpha_n$ if $\alpha_n \neq 0$, while $z_n = 1$ otherwise. Notice that we do not follow the standard notation $\zeta_n = z_n\zeta_{\alpha_n}$ and $B_n = z_1\zeta_{\alpha_1}\cdots z_n\zeta_{\alpha_n}$ (see [11]), but $\zeta_n = \zeta_{\alpha_n}$ and $B_n = \zeta_{\alpha_1}\cdots\zeta_{\alpha_n}$.

In fact, concerning the spectral approach to the ORF, it is more convenient to avoid the presence of the factors z_n in recurrence (2), something that we can get using the ORF $(\phi_n)_{n\geqslant 0}$ given by

$$\phi_0 = \Phi_0; \qquad \phi_n = \bar{z}_1\bar{z}_2\cdots\bar{z}_n\Phi_n, \quad n \geqslant 1.$$

Then, (2) is equivalent to

$$\phi_0 = 1; \qquad \begin{pmatrix} \phi_n \\ \phi^*_n \end{pmatrix} = e_n \frac{\varpi_{n-1}}{\varpi_n} \begin{pmatrix} 1 & \gamma_n \\ \bar{\gamma}_n & 1 \end{pmatrix} \begin{pmatrix} \zeta_{n-1}\phi_{n-1} \\ \phi^*_{n-1} \end{pmatrix}, \quad n \geqslant 1, \tag{3}$$

$$\phi_n^* = B_n \phi_{n*}, \quad \gamma_n \in \mathbb{D}, \quad e_n = \sqrt{\frac{\varpi_n(\alpha_n)}{\varpi_{n-1}(\alpha_{n-1})} \frac{1}{1-|\gamma_n|^2}}.$$

In the polynomial case corresponding to $\alpha_n = 0$ for all n, (3) gives the standard recurrence relation. As in the polynomial situation, a Favard-type theorem also holds (see [11, Theorem 8.1.4]): given a sequence $(\gamma_n)_{n\geqslant 1}$ in $\mathbb{D}$, the functions $(\phi_n)_{n\geqslant 0}$ defined by recurrence (3) are orthonormal with respect to some measure on $\mathbb{T}$. The measure is unique if the infinite Blaschke product $B = \prod_{n=1}^{\infty} \zeta_n$ diverges to zero in $\mathbb{D}$, i.e., if $\sum_{n=1}^{\infty}(1-|\alpha_n|) = \infty$.

Notice that, given a measure μ on $\mathbb{T}$ and a sequence $(\alpha_n)_{n\geqslant 1}$ in $\mathbb{D}$, the parameters $(\gamma_n)_{n\geqslant 1}$ are uniquely defined. To see this, suppose that $(\hat{\phi}_n)_{n\geqslant 0}$ is another sequence of ORF satisfying a recurrence like (3), but with parameters $(\hat{\gamma}_n)_{n\geqslant 1}$ instead of $(\gamma_n)_{n\geqslant 1}$. Then, $\hat{\phi}_n = \epsilon_n \phi_n$ with $\epsilon_n \in \mathbb{T}$ and $\epsilon_0 = 1$. Comparing the recurrences for $(\phi_n)_{n\geqslant 0}$ and $(\hat{\phi}_n)_{n\geqslant 0}$ gives

$$\frac{1}{\sqrt{1-|\gamma_n|^2}} \begin{pmatrix} \epsilon_n & 0 \\ 0 & \bar{\epsilon}_n \end{pmatrix} \begin{pmatrix} 1 & \gamma_n \\ \overline{\gamma}_n & 1 \end{pmatrix} \begin{pmatrix} \bar{\epsilon}_{n-1} & 0 \\ 0 & \epsilon_{n-1} \end{pmatrix} = \frac{1}{\sqrt{1-|\hat{\gamma}_n|^2}} \begin{pmatrix} 1 & \hat{\gamma}_n \\ \overline{\hat{\gamma}}_n & 1 \end{pmatrix}.$$

Taking determinants in the above equality we obtain $|\hat{\gamma}_n| = |\gamma_n|$. Therefore, $\epsilon_n = \epsilon_{n-1}$ for $n \geqslant 1$, which yields $\epsilon_n = \epsilon_0 = 1$. Hence, $\hat{\phi}_n = \phi_n$ and $\hat{\gamma}_n = \gamma_n$.

The above results show that any sequence $\boldsymbol{\alpha} = (\alpha_n)_{n\geqslant 1}$ in $\mathbb{D}$ defines a surjective application

$$\begin{aligned} \mathcal{S}_{\boldsymbol{\alpha}} : \mathfrak{P} &\to \mathbb{D}^{\infty} \\ \mu &\to \boldsymbol{\gamma} = (\gamma_n)_{n\geqslant 1} \end{aligned}$$

between the set $\mathfrak{P}$ of probability measures infinitely supported on $\mathbb{T}$ and the set $\mathbb{D}^{\infty}$ of sequences in $\mathbb{D}$. $\mathcal{S}_{\boldsymbol{\alpha}}$ is a bijection when $\sum_{n=1}^{\infty}(1-|\alpha_n|) = \infty$. The study of the application $\mathcal{S}_{\boldsymbol{\alpha}}$ is one of the main interests in a spectral theory for ORF. In the polynomial case, corresponding to $\boldsymbol{\alpha} = 0$, such a spectral theory has revealed to be a powerful tool in the study of $\mathcal{S}_0$.

To develop a spectral theory for ORF it is convenient to write recurrence (3) in a different way. For any $\alpha \in \mathbb{D}$ we define $\eta_\alpha = \varpi_\alpha(\alpha)^{1/2} = \sqrt{1-|\alpha|^2}$. Denoting $\eta_n = \eta_{\alpha_n}$ and introducing the parameters

$$\rho_n = \sqrt{1-|\gamma_n|^2}, \qquad \rho_n^+ = \frac{\eta_{n-1}}{\eta_n}\rho_n, \qquad \rho_n^- = \frac{\eta_n}{\eta_{n-1}}\rho_n, \tag{4}$$

(3) can be rewritten as

$$\begin{cases} \varpi_{n-1}^* \phi_{n-1} = \rho_n^+ \varpi_n \phi_n - \gamma_n \varpi_{n-1} \phi_{n-1}^*, \\ \varpi_n \phi_n^* = \overline{\gamma}_n \varpi_n \phi_n + \rho_n^- \varpi_{n-1} \phi_{n-1}^*, \end{cases} \quad n \geqslant 1. \tag{5}$$

3. Operator Möbius transformations

As we will see, the matrix version of the scalar Möbius transformations ζ_α appears in a natural way in the spectral theory of ORF on $\mathbb{T}$. Analogously to the scalar case, the matrix generalization of the Möbius transformations is closely related to the theory of automorphisms with several complex variables, whose study goes back to the work of E. Cartan. The matrix Möbius transformations appear in [30] and [22] as the automorphisms of a Siegel space. An excellent and

comprehensive treatment of these results is in [21]. More recent and applied approaches to this matter, with relations to the Schur algorithm, the orthogonal polynomials on the unit circle and system theory can be found in [6] or [4] and the references therein. The extension of the matrix Möbius transformations to the case of operators on Hilbert spaces is of interest too. For instance, they appear in the theory of spaces with an indefinite metric (see [25,26]) as a particular case of the so called operator linear fractional transformations. Such an approach to the operator Möbius transformations can be found in [27] or in the most recent survey [5] and its references. In this section we will introduce the operator Möbius transformations summarizing the main properties of interest for us.

Let $(\mathbb{B}_H, \|\cdot\|)$ be the Banach space of everywhere defined bounded linear operators on a separable Hilbert space H. We use the notation

$$\mathbb{D}_H = \big\{T \in \mathbb{B}_H\colon \|T\| < 1\big\}, \qquad \mathbb{T}_H = \big\{T \in \mathbb{B}_H\colon \|T\| = 1\big\}, \qquad \overline{\mathbb{D}}_H = \mathbb{D}_H \cup \mathbb{T}_H.$$

If $A \in \mathbb{D}_H$, $A^\dagger$ is its adjoint and $\eta_A = \sqrt{1 - AA^\dagger}$, the operator Möbius transformation ζ_A is the map $\zeta_A : \overline{\mathbb{D}}_H \to \overline{\mathbb{D}}_H$ defined by

$$\zeta_A(T) = \eta_A \varpi_A(T)^{-1} \varpi_A^*(T) \eta_{A^\dagger}^{-1}, \qquad \begin{cases} \varpi_A(T) = 1 - TA^\dagger, \\ \varpi_A^*(T) = T - A. \end{cases}$$

Notice that η_A is positive with bounded inverse and, as in the scalar case, $\eta_A = \varpi_A(A)^{1/2}$. As we will see, the spectral theory of ORF is related to transformations ζ_A with A normal, so that $\eta_{A^\dagger} = \eta_A$ in such a case. ζ_A leaves invariant $\mathbb{D}_H$, $\mathbb{T}_H$, and the sets of isometries and unitary operators on H. Indeed, up to unitary left and right factors, the operator Möbius transformations are the only operator linear fractional transformations mapping bijectively $\mathbb{D}_H$ onto itself (see [27]).

It is direct to see that $S = \zeta_A(T)$ iff $T = \tilde{\zeta}_A(S)$, where

$$\tilde{\zeta}_A(T) = \eta_A^{-1} \tilde{\varpi}_A^*(T) \tilde{\varpi}_A(T)^{-1} \eta_{A^\dagger}, \qquad \begin{cases} \tilde{\varpi}_A(T) = 1 + A^\dagger T, \\ \tilde{\varpi}_A^*(T) = T + A. \end{cases}$$

Thus, $\tilde{\zeta}_A$ is the inverse of ζ_A. Moreover, using the relation $\eta_A^2 A = A\eta_{A^\dagger}^2$ it is straightforward to verify the identities

$$\zeta_A(T)^\dagger = \zeta_{A^\dagger}\big(T^\dagger\big), \qquad \tilde{\zeta}_A(T)^\dagger = \tilde{\zeta}_{A^\dagger}\big(T^\dagger\big), \tag{6}$$

so, $\tilde{\zeta}_A(T) = \tilde{\zeta}_{A^\dagger}(T^\dagger)^\dagger = \zeta_{-A}(T)$ as in the scalar case. Notice that the equalities $\zeta_A = \tilde{\zeta}_{-A}$ and $\tilde{\zeta}_A = \zeta_{-A}$ provide alternative expressions for ζ_A and $\tilde{\zeta}_A$.

Some formulas for the operator Möbius transformations will be of interest. From the relations $(\eta_A^2)^n A = A(\eta_{A^\dagger}^2)^n$ for $n = 0, 1, 2, \ldots$, and using the functional calculus for self-adjoint operators, we find that $\eta_A A = A\eta_{A^\dagger}$. Thus, if we define

$$T_A = \eta_A^{-1} T \eta_{A^\dagger}$$

for any linear operator T on H, then, for all $T \in \overline{\mathbb{D}}_H$,

$$\zeta_A(T_A) = \varpi_A(T)^{-1} \varpi_A^*(T), \qquad \tilde{\zeta}_A(T) = \tilde{\varpi}_A^*(T_A) \tilde{\varpi}_A(T_A)^{-1}. \tag{7}$$

This, together with the immediate identity

$$\varpi_A^*(T) - \varpi_A(T)S = T\tilde{\varpi}_A(S) - \tilde{\varpi}_A^*(S), \tag{8}$$

yields

$$\varpi_A(T)\bigl(\zeta_A(T_A) - S_A\bigr) = \bigl(T - \tilde{\zeta}_A(S)\bigr)\tilde{\varpi}_A(S_A) \tag{9}$$

for all $T, S \in \overline{\mathbb{D}}_H$. Substituting S by $\zeta_A(S)$ in (9) gives

$$T - S = \varpi_A(T)\eta_A^{-1}\bigl(\zeta_A(T) - \zeta_A(S)\bigr)\eta_{A^\dagger}^{-1}\tilde{\varpi}_{-A}(S), \tag{10}$$

where we have used that $\tilde{\varpi}_A(\zeta_A(S_A)) = \tilde{\varpi}_A(\tilde{\zeta}_{-A}(S_A)) = \tilde{\varpi}_{-A}(S)^{-1}\eta_{A^\dagger}^2$. If we take $A = \alpha$ and $S = z$ with $\alpha \in \mathbb{D}$ and $z \in \overline{\mathbb{D}}$, (10) becomes

$$z - T = \frac{\varpi_\alpha(z)}{\varpi_\alpha(\alpha)}\bigl(\zeta_\alpha(z) - \zeta_\alpha(T)\bigr)\varpi_\alpha(T). \tag{11}$$

In particular, choosing $T = \lambda$ with $\lambda \in \overline{\mathbb{D}}$,

$$\zeta_\alpha(z) - \zeta_\alpha(\lambda) = \frac{\varpi_\alpha(\alpha)}{\varpi_\alpha(z)\varpi_\alpha(\lambda)}(z - \lambda). \tag{12}$$

Notice that (11) and (12) actually hold for any $z, \lambda \in \mathbb{C} \setminus \{\hat{\alpha}\}$.

4. ORF and Hessenberg matrices

We start fixing some notations for linear operators that will be used throughout the rest of the paper. Given a linear operator T on a Hilbert space H, $\sigma(T)$ denotes its spectrum and $\sigma_p(T)$ its point spectrum. $T \restriction H_0$ means the restriction of T to a T-invariant subspace H_0. We will use the notation $T^{(H_0)}$ for the orthogonal truncation of T on an arbitrary subspace H_0, i.e., $T^{(H_0)} = PT \restriction H_0$, where P is the orthogonal projection on H_0.

If ℓ^2 is the Hilbert space of square-summable complex sequences, the Banach spaces $\mathbb{B}_{\mathbb{C}^n}$ and $\mathbb{B}_{\ell^2}$ can be identified with the sets of $n \times n$ complex matrices and infinite bounded complex matrices, respectively. In this identification we associate any bounded square matrix M with the operator $x \to Mx$, where x is a column vector of $\mathbb{C}^n$ or ℓ^2. However, we could also consider the operator $x \to xM$, where x is a row vector of $\mathbb{C}^n$ or ℓ^2. Both operators have the same spectrum, but their eigenvalues can be different in the case of ℓ^2. Nevertheless, we will normally work with normal or finite-dimensional matrices, for which the eigenvalues are the same in both situations, although, even in these cases, the eigenvectors are in general different. So, we will distinguish between right eigenvectors (or just eigenvectors) for $x \to Mx$ and left eigenvectors for $x \to xM$. That is, right eigenvectors are the standard ones while left eigenvectors are the transposed of the eigenvectors of M^T (in particular, when M is normal, right eigenvectors are the adjoints of left eigenvectors). In the subsequent discussions, this convention often permits us to avoid the ^{T} superindex, something convenient because many indices appear later.

The main tool for the operator theoretic approach to the ORF on $\mathbb{T}$ is the unitary multiplication operator

$$\begin{aligned} T_\mu : L^2_\mu &\to L^2_\mu \\ f(z) &\to zf(z). \end{aligned}$$

It satisfies $\sigma(T_\mu) = \operatorname{supp}\mu$ and $\sigma_p(T_\mu) = \{\text{mass points of } \mu\}$. The eigenvectors of a given eigenvalue λ are spanned by the characteristic function $\mathcal{X}_{\{\lambda\}}$ of $\{\lambda\}$. Besides, if E is the spectral measure of T_μ, then $\mu(\cdot) = \langle 1, E(\cdot)1\rangle_\mu$. All these properties are true no matter whether $\operatorname{supp}\mu$ is finite or infinite.

If $(f_n)_{n\geqslant 0}$ is a basis of L^2_μ, the matrix of T_μ with respect to $(f_n)_{n\geqslant 0}$ is the matrix M whose (i,j)th element is $M_{ij} = \langle f_i, T_\mu f_j\rangle_\mu$. In other words,

$$\begin{pmatrix} zf_0(z) & zf_1(z) & \dots \end{pmatrix} = \begin{pmatrix} f_0(z) & f_1(z) & \dots \end{pmatrix} M. \tag{13}$$

The $*$-involution $f \to f_*$ defines an anti-unitary operator on L^2_μ, thus $(f_n)_{n\geqslant 0}$ is a basis of L^2_μ iff $(f_{n*})_{n\geqslant 0}$ is a basis too. Moreover, since M is unitary, taking the $*$-involution in (13) shows that the matrix of T_μ with respect to $(f_{n*})_{n\geqslant 0}$ is the transposed M^T of M. This relation holds when μ is finitely supported too, with the only difference that the basis of L^2_μ is finite.

T_μ is unitarily equivalent to the operator defined by M. On the other hand, if we choose as a basis a sequence of ORF with respect to μ, the matrix M will depend on the corresponding sequences $\boldsymbol{\alpha} = (\alpha_n)_{n\geqslant 1}$ and $\boldsymbol{\gamma} = (\gamma_n)_{n\geqslant 1}$. Therefore, this matrix permits us to recover the orthogonality measure μ starting from the location of the poles and the parameters of the recurrence relation for the ORF. Obviously, the utility for this purpose of the matrix M depends on its simplicity as a function of $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$. Following these ideas, our first aim is to find the matrix representation of a unitary multiplication operator with respect to a basis of ORF.

Theorem 4.1. *Let $\boldsymbol{\alpha}$ be a sequence compactly included in $\mathbb{D}$, μ a measure on $\mathbb{T}$ and $\boldsymbol{\gamma} = \mathcal{S}_{\boldsymbol{\alpha}}(\mu)$. Then, $\mathcal{L}$ is T_μ-invariant and the matrix of the isometric operator $T_\mu \restriction \mathcal{L}$ with respect to the corresponding ORF $(\phi_n)_{n\geqslant 0}$ is $\mathcal{V} = \tilde{\zeta}_{\mathcal{A}}(\mathcal{H})$, where*

$$\mathcal{A} = \mathcal{A}(\boldsymbol{\alpha}) = \begin{pmatrix} \alpha_0 & & & \\ & \alpha_1 & & \\ & & \alpha_2 & \\ & & & \ddots \end{pmatrix},$$

$$\mathcal{H} = \mathcal{H}(\boldsymbol{\gamma}) = \begin{pmatrix} -\gamma_1 & -\rho_1\gamma_2 & -\rho_1\rho_2\gamma_3 & -\rho_1\rho_2\rho_3\gamma_4 & \dots \\ \rho_1 & -\bar{\gamma}_1\gamma_2 & -\bar{\gamma}_1\rho_2\gamma_3 & -\bar{\gamma}_1\rho_2\rho_3\gamma_4 & \dots \\ 0 & \rho_2 & -\bar{\gamma}_2\gamma_3 & -\bar{\gamma}_2\rho_3\gamma_4 & \dots \\ 0 & 0 & \rho_3 & -\bar{\gamma}_3\gamma_4 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

The isometric matrix $\mathcal{V}$ represents the full operator T_μ iff any of the following equivalent conditions are fulfilled:

$$\mathcal{L} = L^2_\mu \quad \Leftrightarrow \quad \mathcal{P} = L^2_\mu \quad \Leftrightarrow \quad \log\mu' \notin L^1_m \quad \Leftrightarrow \quad \boldsymbol{\gamma} \notin \ell^2 \quad \Leftrightarrow \quad \mathcal{V} \textit{ is unitary},$$

with $\mathcal{P}$ the closure of $\operatorname{span}\{z^n\}_{n\geqslant 0}$ *in* L^2_μ, $dm(z) = dz/2\pi iz$ *the Lebesgue measure on* $\mathbb{T}$ *and* $\mu' = d\mu/dm$.

Proof. $\|\mathcal{A}\| < 1$ because $\boldsymbol{\alpha}$ is compactly included in $\mathbb{D}$, thus $\tilde{\zeta}_{\mathcal{A}}(\mathcal{H})$ is a well-defined isometric matrix because $\mathcal{H}$ is isometric (see [13,20,32]).

To prove the theorem notice that the second relation in (5) yields

$$\varpi_n\phi_n^* = \overline{\gamma}_n\varpi_n\phi_n + \sum_{k=0}^{n-1} \rho_n^-\rho_{n-1}^-\cdots\rho_{k+1}^-\overline{\gamma}_k\varpi_k\phi_k, \quad n \geqslant 1, \tag{14}$$

where we set $\gamma_0 = 1$. Identity (14) and the first relation in (5) give

$$\varpi_n^*\phi_n = \sum_{k=0}^{\infty} \hat{h}_{k,n}\varpi_k\phi_k,$$
$$\hat{h}_{k,n} = \begin{cases} -\gamma_{n+1}\rho_n^-\rho_{n-1}^-\cdots\rho_{k+1}^-\overline{\gamma}_k & \text{if } k < n, \\ -\gamma_{n+1}\overline{\gamma}_n & \text{if } k = n, \\ \rho_{n+1}^+ & \text{if } k = n+1, \\ 0 & \text{if } k > n+1. \end{cases} \tag{15}$$

If we define the matrix $\hat{\mathcal{H}} = (\hat{h}_{i,j})$, equality (15) can be written as

$$(\phi_0 \quad \phi_1 \quad \ldots)\big(\varpi_{\mathcal{A}}^* - \varpi_{\mathcal{A}}\hat{\mathcal{H}}\big) = 0. \tag{16}$$

Using (4) we find that the Hessenberg matrix

$$\hat{\mathcal{H}} = \begin{pmatrix} -\gamma_1 & -\rho_1^-\gamma_2 & -\rho_1^-\rho_2^-\gamma_3 & -\rho_1^-\rho_2^-\rho_3^-\gamma_4 & \cdots \\ \rho_1^+ & -\overline{\gamma}_1\gamma_2 & -\overline{\gamma}_1\rho_2^-\gamma_3 & -\overline{\gamma}_1\rho_2^-\rho_3^-\gamma_4 & \cdots \\ 0 & \rho_2^+ & -\overline{\gamma}_2\gamma_3 & -\overline{\gamma}_2\rho_3^-\gamma_4 & \cdots \\ 0 & 0 & \rho_3^+ & -\overline{\gamma}_3\gamma_4 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \tag{17}$$

can be related to $\mathcal{H}$ by $\hat{\mathcal{H}} = \eta_{\mathcal{A}}^{-1}\mathcal{H}\eta_{\mathcal{A}} = \mathcal{H}_{\mathcal{A}}$. From this relation, (7) and (8) we see that (16) is equivalent to

$$\big(\phi_0(z) \quad \phi_1(z) \quad \ldots\big)\big(z - \tilde{\zeta}_{\mathcal{A}}(\mathcal{H})\big) = 0. \tag{18}$$

This equality implies that $\mathcal{L}$ is invariant under T_μ, so the restriction $T_\mu \restriction \mathcal{L}$ is well defined, and $\tilde{\zeta}_{\mathcal{A}}(\mathcal{H})$ is its matrix representation with respect to $(\phi_n)_{n\geqslant 0}$.

$T_\mu \restriction \mathcal{L}$ is an isometry because it is the restriction of a unitary operator, which agrees with the fact that $\tilde{\zeta}_{\mathcal{A}}(\mathcal{H})$ is isometric. Also, $\tilde{\zeta}_{\mathcal{A}}(\mathcal{H})$ and $\mathcal{H}$ are unitary at the same time, that is, when $\boldsymbol{\gamma} \notin \ell^2$ (see [13,32]). Besides, $T_\mu \restriction \mathcal{L}$ is unitary iff $T_\mu\mathcal{L} = \mathcal{L}$. This implies that $T_\mu^n\mathcal{L} = \mathcal{L}$ for any $n \in \mathbb{Z}$, so $\{z^n\}_{n\in\mathbb{Z}} \subset \mathcal{L}$. Hence $\mathcal{L} = L^2_\mu$ because $\operatorname{span}\{z^n\}_{n\in\mathbb{Z}}$ is dense in L^2_μ. Conversely, if $\mathcal{L} = L^2_\mu$, then $T_\mu \restriction \mathcal{L} = T_\mu$ is unitary. Therefore, $\tilde{\zeta}_{\mathcal{A}}(\mathcal{H})$ is unitary iff the ORF $(\phi_n)_{n\geqslant 0}$ is a basis of L^2_μ, i.e., iff $\tilde{\zeta}_{\mathcal{A}}(\mathcal{H})$ represents the full operator T_μ. Finally, it is known that the condition

$\sum_{n=1}^{\infty}(1-|\alpha_n|)=\infty$, which is satisfied for $\boldsymbol{\alpha}$ compactly included in $\mathbb{D}$, ensures that $\mathcal{L}=\mathcal{P}$ (see [11, Theorem 7.2.2]) and so it implies the equivalence between $\mathcal{L}=L^2_\mu$, $\mathcal{P}=L^2_\mu$ and $\log\mu'\notin L^1_m$ (see [11, Corollary 7.2.4]). $\square$

In the polynomial case $\mathcal{A}=0$ and $\mathcal{V}=\mathcal{H}$, so we recover the known Hessenberg representation of $T_\mu\restriction\mathcal{P}$ with respect to the OP basis (see [3,15,20,32,34]).

As a consequence of Theorem 4.1 and the spectral properties of the unitary multiplication operator, we have the following spectral interpretation of the orthogonality measure for ORF.

Theorem 4.2. *Let $\boldsymbol{\alpha}$ be a sequence compactly included in $\mathbb{D}$, μ a measure on $\mathbb{T}$ such that $\log\mu'\notin L^1_m$ and $(\phi_n)_{n\geqslant 0}$ the corresponding ORF. Let $\mathcal{V}=\tilde{\zeta}_{\mathcal{A}}(\mathcal{H})$ with $\mathcal{A}=\mathcal{A}(\boldsymbol{\alpha})$, $\mathcal{H}=\mathcal{H}(\boldsymbol{\gamma})$, $\boldsymbol{\gamma}=\mathcal{S}_{\boldsymbol{\alpha}}(\mu)$. If $\mathcal{E}$ is the spectral measure of $\mathcal{V}$, $\mu=\mathcal{E}_{1,1}$. Besides, $\sigma(\mathcal{V})=\operatorname{supp}\mu$ and $\sigma_p(\mathcal{V})=\{$mass points of $\mu\}$. λ is a mass point iff $(\phi_n(\lambda))_{n\geqslant 0}\in\ell^2$. Given a mass point λ, the related eigenvectors of $\mathcal{V}$ are spanned by $(\phi_0(\lambda)\ \phi_1(\lambda)\ \ldots)^\dagger$ and $\mu(\{\lambda\})=(\sum_{n=0}^{\infty}|\phi_n(\lambda)|^2)^{-1}$.*

Proof. Under the hypothesis of the theorem, $\mathcal{V}$ is the matrix representation of the full operator T_μ with respect to $(\phi_n)_{n\geqslant 0}$. Hence, if E is the spectral measure of T_μ, then $\mu(\cdot)=\langle\phi_0,E(\cdot)\phi_0\rangle_\mu=\mathcal{E}_{1,1}(\cdot)$. Also, $\sigma(\mathcal{V})=\sigma(T_\mu)=\operatorname{supp}\mu$ and $\sigma_p(\mathcal{V})=\sigma_p(T_\mu)=\{$mass points of $\mu\}$. If λ is a mass point, $\mathcal{X}_{\{\lambda\}}$ spans the related eigenvectors of T_μ, so $\langle\phi_{n-1},\mathcal{X}_{\{\lambda\}}\rangle_\mu=\mu(\{\lambda\})\overline{\phi_{n-1}(\lambda)}$ is the nth component of a vector spanning the corresponding eigenvectors of $\mathcal{V}$. This implies that $(\phi_n(\lambda))_{n\geqslant 0}\in\ell^2$. Conversely, if $(\phi_n(\lambda))_{n\geqslant 0}\in\ell^2$, relation (18) shows that $(\phi_0(\lambda)\ \phi_1(\lambda)\ \ldots)$ is a left eigenvector of $\mathcal{V}$ with eigenvalue λ. Due to the unitarity of $\mathcal{V}$, $\lambda\in\mathbb{T}$ and the above statement is equivalent to saying that $(\phi_0(\lambda)\ \phi_1(\lambda)\ \ldots)^\dagger$ is a (right) eigenvector of $\mathcal{V}$ with eigenvalue λ. Therefore, λ is a mass point of μ. Finally, the identity $\mu(\{\lambda\})=(\sum_{n=0}^{\infty}|\phi_n(\lambda)|^2)^{-1}$ follows from $\mu(\{\lambda\})=\langle\mathcal{X}_{\{\lambda\}},\mathcal{X}_{\{\lambda\}}\rangle_\mu=\sum_{n=0}^{\infty}\langle\mathcal{X}_{\{\lambda\}},\phi_n\rangle_\mu\langle\phi_n,\mathcal{X}_{\{\lambda\}}\rangle_\mu=\sum_{n=0}^{\infty}\mu(\{\lambda\})^2|\phi_n(\lambda)|^2$. $\square$

The fact that the representation $\mathcal{V}$ is not a Hessenberg matrix, but a Möbius transformation of a Hessenberg matrix, makes the rational case more complicated than the polynomial one. However, the Hessenberg structure can be kept if we formulate the spectral results in terms of pairs of operators.

Remember that, given a Hilbert space H and two operators $T,S\in\mathbb{B}_H$, the spectrum and point spectrum of the pair (T,S) are respectively the sets

$$\sigma(T,S)=\{\lambda\in\overline{\mathbb{C}}\colon\ T-\lambda S \text{ has no inverse in } \mathbb{B}_H\},$$

$$\sigma_p(T,S)=\{\lambda\in\overline{\mathbb{C}}\colon\ T-\lambda S \text{ is not injective}\}.$$

In the finite-dimensional case both sets coincide. The elements of $\sigma_p(T,S)$ are called eigenvalues of the pair, and the eigenvectors of (T,S) corresponding to an eigenvalue λ are the elements $x\in H\setminus\{0\}$ such that $(T-\lambda S)x=0$. In these definitions $T-\lambda S$ must be substituted by S if $\lambda=\infty$.

With the above terminology, the isometric matrix $\mathcal{V}$ and the Hessenberg pair $(\tilde{\varpi}^*_{\mathcal{A}}(\mathcal{H}_{\mathcal{A}}),\tilde{\varpi}_{\mathcal{A}}(\mathcal{H}_{\mathcal{A}}))$ have the same spectrum and eigenvalues because $\tilde{\varpi}_{\mathcal{A}}(\mathcal{H}_{\mathcal{A}})^{\pm 1}\in\mathbb{B}_{\ell^2}$ when $\boldsymbol{\alpha}$ is compactly included in $\mathbb{D}$. So, Theorem 4.2 can be obviously rewritten substituting $\mathcal{V}$ by the pair $(\tilde{\varpi}^*_{\mathcal{A}}(\mathcal{H}_{\mathcal{A}}),\tilde{\varpi}_{\mathcal{A}}(\mathcal{H}_{\mathcal{A}}))$. Notice that, given an eigenvalue λ, $(\phi_0(\lambda)\ \phi_1(\lambda)\ \ldots)$ is a left eigenvector of the pair, i.e., $(\phi_0(\lambda)\ \phi_1(\lambda)\ \ldots)(\tilde{\varpi}^*_{\mathcal{A}}(\mathcal{H}_{\mathcal{A}})-\lambda\tilde{\varpi}_{\mathcal{A}}(\mathcal{H}_{\mathcal{A}}))=0$. Moreover,

$\mathcal{H}_{\mathcal{A}} = \eta_{\mathcal{A}}^{-1}\mathcal{H}\eta_{\mathcal{A}}$ with $\eta_{\mathcal{A}}^{\pm 1} \in \mathbb{B}_{\ell^2}$. Therefore, Theorem 4.2 also holds substituting $\mathcal{V}$ by the Hessenberg pair $(\tilde{\varpi}_{\mathcal{A}}^{*}(\mathcal{H}), \tilde{\varpi}_{\mathcal{A}}(\mathcal{H}))$, but the left eigenvectors with eigenvalue λ are spanned by $(\phi_0(\lambda)\ \phi_1(\lambda)\ \ldots)\eta_{\mathcal{A}}^{-1}$.

4.1. Zeros of ORF and Hessenberg matrices

In this section we will prove that the zeros of the ORF have a spectral interpretation in terms of Möbius transformations of Hessenberg matrices too. For this purpose we consider the operator multiplication by ζ_n in L^2_μ,

$$\begin{aligned} \zeta_n(T_\mu): L^2_\mu &\to L^2_\mu \\ f &\to \zeta_n f \end{aligned}$$

and the orthogonal truncation $\zeta_n(T_\mu)^{(\mathcal{L}_n)}$. We also remind that the nth ORF has the form $\phi_n = p_n/\pi_n$ with $\pi_n = \varpi_1 \cdots \varpi_n$ and p_n a polynomial of degree n with its zeros on $\mathbb{D}$ (see [11, Corollary 3.2.2]). The following theorem is the starting point for a spectral interpretation of such zeros.

Theorem 4.3. *Let $\boldsymbol{\alpha}$ be a sequence in $\mathbb{D}$, μ a measure on $\mathbb{T}$ and $\phi_n = p_n/\pi_n$ the related nth ORF.*

1. *If Z_n is the set of zeros of ϕ_n, $\zeta_n(Z_n)$ is the set of eigenvalues of $\zeta_n(T_\mu)^{(\mathcal{L}_n)}$ and these eigenvalues have geometric multiplicity* 1.
2. *If $p_n(z) \propto \prod_{k=1}^{n}(z-\lambda_k)$, the characteristic polynomial of $\zeta_n(T_\mu)^{(\mathcal{L}_n)}$ is $\prod_{k=1}^{n}(z-\zeta_n(\lambda_k))$.*

Proof. Let L_n be the orthogonal projection on $\mathcal{L}_n$. $f \in \mathcal{L}_n \setminus \{0\}$ is an eigenvector of $\zeta_n(T_\mu)^{(\mathcal{L}_n)}$ with eigenvalue w iff $(L_n\zeta_n - w)f = 0$, that is, $L_n(\zeta_n - w)f = 0$. This is equivalent to $(\zeta_n - w)f \in \mathcal{L}_n^{\perp \mathcal{L}_{n+1}} = \operatorname{span}\{\phi_n\}$, or, in other words, $f \propto \phi_n(\zeta_n - w)^{-1}$. Writing $w = \zeta_n(\lambda)$ and using (12) we find that this condition can be expressed as $f(z) \propto p_n(z)(z-\lambda)^{-1}/\pi_{n-1}(z)$ with $\lambda \in Z_n$. This proves item 1.

Item 2 is equivalent to assert that the algebraic multiplicity m_w of any eigenvalue $w = \zeta_n(\lambda)$ of $\zeta_n(T_\mu)^{(\mathcal{L}_n)}$ is equal to the multiplicity of λ as a root of p_n. Since the geometric multiplicity of w is 1, $m_w \geqslant k$ iff there exists $f \in \mathcal{L}_n$ such that $(L_n\zeta_n - w)^k f = 0$ and $(L_n\zeta_n - w)^{k-1} f \neq 0$. Analogously to the previous discussion, we find that these two conditions are equivalent to $f \in \operatorname{span}\{\phi_n(\zeta_n - w)^{-j}\}_{j=1}^{k} \setminus \operatorname{span}\{\phi_n(\zeta_n - w)^{-j}\}_{j=1}^{k-1}$, i.e., to $f = p/\pi_{n-1}$ with $p(z) \in \operatorname{span}\{\varpi_n^{j-1}(z)p_n(z)(z-\lambda)^{-j}\}_{j=1}^{k} \setminus \operatorname{span}\{\varpi_n^{j-1}(z)p_n(z)(z-\lambda)^{-j}\}_{j=1}^{k-1}$, as can be seen using (12) again. Hence, by induction on k we find that $m_w \geqslant k$ implies that the multiplicity of λ as a root of p_n is not less than k. Conversely, if the multiplicity of λ as a root of p_n is greater than or equal to k, $f(z) = \phi_n(z)(\zeta_n(z) - w)^{-k} \propto \varpi_n^{k-1}(z)p_n(z)(z-\lambda)^{-k}/\pi_{n-1}(z) \in \mathcal{L}_n$ and $(L_n\zeta_n - w)^k f = 0$, $(L_n\zeta_n - w)^{k-1} f \neq 0$, so $m_w \geqslant k$. $\square$

The next step is to obtain a matrix representation of $\zeta_n(T_\mu)^{(\mathcal{L}_n)}$, so that we can give a matrix version of the above theorem. In what follows the subscript $_n$ on a matrix means the corresponding principal submatrix of order n. In particular, I is the infinite identity matrix, so I_n means the identity matrix of order n. This notation will be used throughout the rest of the paper.

Theorem 4.4. *Let $\boldsymbol{\alpha}$ be a sequence in $\mathbb{D}$, μ a measure on $\mathbb{T}$ and $(\phi_n)_{n\geqslant 0}$ the related ORF. The matrix of $\zeta_n(T_\mu)^{(\mathcal{L}_n)}$ with respect to $(\phi_k)_{k=0}^{n-1}$ is $\zeta_n(\mathcal{V}^{(n)})$, where $\mathcal{V}^{(n)}=\tilde{\zeta}_{\mathcal{A}_n}(\mathcal{H}_n)$ and $\mathcal{A}=\mathcal{A}(\boldsymbol{\alpha})$, $\mathcal{H}=\mathcal{H}(\boldsymbol{\gamma})$, $\boldsymbol{\gamma}=\mathcal{S}_{\boldsymbol{\alpha}}(\mu)$.*

Proof. From the factorization (see [13,20,32])

$$\mathcal{H}_n=\begin{pmatrix}\Theta_1 & \\ & I_{n-2}\end{pmatrix}\begin{pmatrix}I_1 & & \\ & \Theta_2 & \\ & & I_{n-3}\end{pmatrix}\cdots\begin{pmatrix}I_{n-2} & \\ & \Theta_{n-1}\end{pmatrix}\begin{pmatrix}I_{n-1} & \\ & -a_n\end{pmatrix},$$
$$\Theta_n=\begin{pmatrix}-\gamma_n & \rho_n \\ \rho_n & \overline{\gamma}_n\end{pmatrix}, \tag{19}$$

we see that $\|\mathcal{H}_n\|=1$ because Θ_n is unitary. Hence, $\mathcal{V}^{(n)}=\tilde{\zeta}_{\mathcal{A}_n}(\mathcal{H}_n)$ is well defined and $\|\mathcal{V}^{(n)}\|=1$ because $\|\mathcal{A}_n\|<1$. A similar reason shows that $\zeta_n(\mathcal{V}^{(n)})$ is well defined too.

To prove the theorem, let us write the first n equations of (16) as

$$(\phi_0 \quad \dots \quad \phi_{n-1})\left(\varpi^*_{\mathcal{A}_n}-\varpi_{\mathcal{A}_n}\hat{\mathcal{H}}_n\right)=b_n\varpi_n\phi_n, \quad b_n\in\mathbb{C}^n. \tag{20}$$

Identities (7), (8) and $\hat{\mathcal{H}}_n=\eta^{-1}_{\mathcal{A}_n}\mathcal{H}_n\eta_{\mathcal{A}_n}=(\mathcal{H}_n)_{\mathcal{A}_n}$ transform (20) into

$$\left(\phi_0(z) \quad \dots \quad \phi_{n-1}(z)\right)\left(z-\mathcal{V}^{(n)}\right)=c_n\varpi_n(z)\phi_n(z), \quad c_n\in\mathbb{C}^n. \tag{21}$$

Using (11) we get

$$\left(\phi_0(z) \quad \dots \quad \phi_{n-1}(z)\right)\left(\zeta_n(z)-\zeta_n\left(\mathcal{V}^{(n)}\right)\right)=d_n\phi_n(z), \quad d_n\in\mathbb{C}^n.$$

Hence, if L_n is the orthogonal projection on $\mathcal{L}_n$,

$$(L_n\zeta_n\phi_0 \quad \dots \quad L_n\zeta_n\phi_{n-1})=(\phi_0 \quad \dots \quad \phi_{n-1})\,\zeta_n\left(\mathcal{V}^{(n)}\right). \qquad \square$$

Theorems 4.3 and 4.4 give a spectral interpretation of the zeros of ORF.

Theorem 4.5. *Let $\boldsymbol{\alpha}$ be a sequence in $\mathbb{D}$, μ a measure on $\mathbb{T}$, $(\phi_n)_{n\geqslant 0}$ the related ORF and $\mathcal{V}^{(n)}=\tilde{\zeta}_{\mathcal{A}_n}(\mathcal{H}_n)$, with $\mathcal{A}=\mathcal{A}(\boldsymbol{\alpha})$, $\mathcal{H}=\mathcal{H}(\boldsymbol{\gamma})$, $\boldsymbol{\gamma}=\mathcal{S}_{\boldsymbol{\alpha}}(\mu)$.*

1. *The zeros of ϕ_n are the eigenvalues of $\mathcal{V}^{(n)}$. If λ is a zero of ϕ_n, the related left eigenvectors of $\mathcal{V}^{(n)}$ are spanned by $(\phi_0(\lambda)\ \dots\ \phi_{n-1}(\lambda))$.*
2. *$\phi_n=\frac{p_n}{\pi_n}$ with p_n proportional to the characteristic polynomial of $\mathcal{V}^{(n)}$.*

Proof. From Theorems 4.3 and 4.4, the eigenvalues of $\zeta_n(\mathcal{V}^{(n)})$ have geometric multiplicity 1 and $\sigma(\zeta_n(\mathcal{V}^{(n)}))=\zeta_n(Z_n)$, Z_n being the zeros of ϕ_n. Also, the characteristic polynomial of $\zeta_n(\mathcal{V}^{(n)})$ is $\prod_{k=1}^n(z-\zeta_n(\lambda_k))$, where $p_n(z)\propto\prod_{k=1}^n(z-\lambda_k)$. $\sigma(\zeta_n(\mathcal{V}^{(n)}))=\zeta_n(\sigma(\mathcal{V}^{(n)}))$, so, bearing in mind that ζ_n is bijective, $\sigma(\mathcal{V}^{(n)})=Z_n$. Furthermore, given an eigenvalue λ of $\mathcal{V}^{(n)}$, the corresponding eigenvalue $\zeta_n(\lambda)$ of $\zeta_n(\mathcal{V}^{(n)})$ has the same geometric and algebraic multiplicity. Therefore, $\prod_{k=1}^n(z-\lambda_k)$ is the characteristic polynomial of $\mathcal{V}^{(n)}$. Finally, if λ is a zero of ϕ_n, (21) shows that $(\phi_0(\lambda)\ \dots\ \phi_{n-1}(\lambda))$ is a left eigenvector of $\mathcal{V}^{(n)}$ with eigenvalue λ. $\square$

From (7) and (8) we get $\varpi^*_{\mathcal{A}_n}(z) - \varpi_{\mathcal{A}_n}(z)\hat{\mathcal{H}}_n = z\tilde{\varpi}_{\mathcal{A}_n}(\hat{\mathcal{H}}_n) - \tilde{\varpi}^*_{\mathcal{A}_n}(\hat{\mathcal{H}}_n) = (z - \mathcal{V}^{(n)}) \cdot \tilde{\varpi}_{\mathcal{A}_n}(\hat{\mathcal{H}}_n)$. In consequence, $p_n(z) \propto \det(\varpi^*_{\mathcal{A}_n}(z) - \varpi_{\mathcal{A}_n}(z)\hat{\mathcal{H}}_n) = \det(z\tilde{\varpi}_{\mathcal{A}_n}(\hat{\mathcal{H}}_n) - \tilde{\varpi}^*_{\mathcal{A}_n}(\hat{\mathcal{H}}_n))$. The above expressions show that p_n can be calculated as a determinant of a Hessenberg matrix. Furthermore, the last expression implies that the zeros of ϕ_n are the eigenvalues of the Hessenberg pair $(\tilde{\varpi}^*_{\mathcal{A}_n}(\hat{\mathcal{H}}_n), \tilde{\varpi}_{\mathcal{A}_n}(\hat{\mathcal{H}}_n))$. Besides, according to Theorem 4.5, the left eigenvectors of $(\tilde{\varpi}^*_{\mathcal{A}_n}(\hat{\mathcal{H}}_n), \tilde{\varpi}_{\mathcal{A}_n}(\hat{\mathcal{H}}_n))$ corresponding to an eigenvalue λ are spanned by $(\phi_0(\lambda)\ \dots\ \phi_{n-1}(\lambda))$. Since $\hat{\mathcal{H}}_n = \eta^{-1}_{\mathcal{A}_n}\mathcal{H}_n\eta_{\mathcal{A}_n}$, the zeros of ϕ_n can be also understood as the eigenvalues of $(\tilde{\varpi}^*_{\mathcal{A}_n}(\mathcal{H}_n), \tilde{\varpi}_{\mathcal{A}_n}(\mathcal{H}_n))$, the left eigenvectors with eigenvalue λ being spanned by $(\phi_0(\lambda)\ \dots\ \phi_{n-1}(\lambda))\eta^{-1}_{\mathcal{A}_n}$. Indeed, $p_n(z) \propto \det(\varpi^*_{\mathcal{A}_n}(z) - \varpi_{\mathcal{A}_n}(z)\mathcal{H}_n) = \det(z\tilde{\varpi}_{\mathcal{A}_n}(\mathcal{H}_n) - \tilde{\varpi}^*_{\mathcal{A}_n}(\mathcal{H}_n))$.

Apart from the sequence $(\phi_n)_{n\geqslant 0}$ of ORF, another remarkable rational functions arise in the theory of ORF. They are the so called para-orthogonal rational functions (PORF), given by

$$Q^v_n = \phi_n + v\phi^*_n, \quad v \in \mathbb{T}. \tag{22}$$

The interest of the PORF relies on the fact that, contrary to the ORF, they have simple zeros lying on $\mathbb{T}$ which, thus, play an important role in quadrature formulas and rational moment problems (see [23] and [11, Chapters 5 and 10]). These quadrature formulas associate with each PORF Q^v_n a measure μ^v_n supported on its zeros with a mass $(\sum_{k=0}^{n-1}|\phi_k(\lambda)|^2)^{-1}$ at each zero λ. Such quadrature formulas are exact in $\mathcal{L}_{n-1}\mathcal{L}_{n-1*}$, so $(\phi_k)_{k=0}^{n-1}$ is an orthonormal basis of $L^2_{\mu^v_n}$.

A spectral interpretation can be also obtained for the zeros of the PORF. Using (3) in (22) we get

$$Q^v_n = (1 + \overline{\gamma}_n v)e_n\frac{\varpi_{n-1}}{\varpi_n}\big(\zeta_{n-1}\phi_{n-1} + u\phi^*_{n-1}\big), \quad u = \tilde{\zeta}_{\gamma_n}(v), \tag{23}$$

which shows that, like ϕ_n, Q^v_n is obtained from n steps of recurrence (3), but changing in the nth step $\gamma_n \in \mathbb{D}$ by $u = \tilde{\zeta}_{\gamma_n}(v) \in \mathbb{T}$. From (19), such a substitution transforms $\mathcal{H}_n$ into the unitary Hessenberg matrix

$$\mathcal{H}^{\mathrm{u}}_n = \begin{pmatrix} \Theta_1 & \\ & I_{n-2} \end{pmatrix}\begin{pmatrix} I_1 & & \\ & \Theta_2 & \\ & & I_{n-3} \end{pmatrix}\cdots\begin{pmatrix} I_{n-2} & \\ & \Theta_{n-1} \end{pmatrix}\begin{pmatrix} I_{n-1} & \\ & -u \end{pmatrix}.$$

Therefore, $\tilde{\zeta}_{\mathcal{A}_n}(\mathcal{H}^{\mathrm{u}}_n)$ is unitary too. The following result gives a spectral interpretation of the zeros of the Q^v_n in terms of $\mathcal{H}^{\mathrm{u}}_n$, as well as a connection of such a matrix with the unitary multiplication operator $T_{\mu^v_n}$. It can be understood as a limit case of Theorems 4.4 and 4.5.

Theorem 4.6. *Let $\boldsymbol{\alpha}$ be a sequence in $\mathbb{D}$, μ a measure on $\mathbb{T}$, $(\phi_n)_{n\geqslant 0}$ the corresponding ORF and $\mathcal{A} = \mathcal{A}(\boldsymbol{\alpha})$, $\mathcal{H} = \mathcal{H}(\boldsymbol{\gamma})$, $\boldsymbol{\gamma} = \mathcal{S}_{\boldsymbol{\alpha}}(\mu)$. Consider $v \in \mathbb{T}$, $u = \tilde{\zeta}_{\gamma_n}(v)$, $Q^v_n = \phi_n + v\phi^*_n$ and the associated measure μ^v_n.*

1. *The matrix of $T_{\mu^v_n}$ with respect to $(\phi_k)_{k=0}^{n-1}$ is $\mathcal{V}^{(n;\mathrm{u})} = \tilde{\zeta}_{\mathcal{A}_n}(\mathcal{H}^{\mathrm{u}}_n)$.*
2. *The zeros of Q^v_n are the eigenvalues of $\mathcal{V}^{(n;\mathrm{u})}$. If λ is a zero of Q^v_n, the related eigenvectors of $\mathcal{V}^{(n;\mathrm{u})}$ are spanned by $(\phi_0(\lambda)\ \dots\ \phi_{n-1}(\lambda))^\dagger$.*
3. *$Q^v_n = \frac{q^v_n}{\pi_n}$ with q^v_n proportional to the characteristic polynomial of $\mathcal{V}^{(n;\mathrm{u})}$.*

Proof. Using (14) in (23) we find that

$$\varpi_{n-1}^*\phi_{n-1} = \sum_{k=0}^{n-1} \hat{h}_{k,n-1}^{\mathrm{u}} \varpi_k \phi_k + \frac{\rho_n^+}{1+\overline{\gamma}_n v} \varpi_n Q_n^v,$$

$$\hat{h}_{k,n-1}^{\mathrm{u}} = \begin{cases} -u\rho_{n-1}^- \rho_{n-2}^- \cdots \rho_{k+1}^- \overline{\gamma}_k & \text{if } k < n-1, \\ -u\overline{\gamma}_{n-1} & \text{if } k = n-1. \end{cases}$$

This relation and the first $n-1$ equations of (16) lead to the matrix identity

$$(\phi_0 \quad \ldots \quad \phi_{n-1})\big(\varpi_{\mathcal{A}_n}^* - \varpi_{\mathcal{A}_n} \hat{\mathcal{H}}_n^{\mathrm{u}}\big) = b_n \varpi_n Q_n^v, \quad b_n \in \mathbb{C}^n,$$

where $\hat{\mathcal{H}}_n^{\mathrm{u}} = \eta_{\mathcal{A}_n}^{-1} \mathcal{H}_n^{\mathrm{u}} \eta_{\mathcal{A}_n} = (\mathcal{H}_n^{\mathrm{u}})_{\mathcal{A}_n}$. So, (7) and (8) give

$$\big(\phi_0(z) \quad \ldots \quad \phi_{n-1}(z)\big)\big(z - \mathcal{V}^{(n;\mathrm{u})}\big) = c_n \varpi_n(z) Q_n^v(z), \quad c_n \in \mathbb{C}^n. \tag{24}$$

$Q_n^v = 0$ in $L^2_{\mu_n^v}$, thus (24) implies that $\mathcal{V}^{(n;\mathrm{u})}$ is the matrix of $T_{\mu_n^v}$ with respect to $(\phi_k)_{k=0}^{n-1}$. The rest of the statements are a consequence of this one and the properties of the multiplication operators, similarly to the proof of Theorem 4.2. Alternatively, they can be obtained directly from relation (24), the unitarity of $\mathcal{V}^{(n:\mathrm{u})}$ and the fact that Q_n^v has n different zeros. □

Analogously to the comments after Theorem 4.5, if $u = \tilde{\zeta}_{\gamma_n}(v)$, $q_n^v(z) \propto \det(\varpi_{\mathcal{A}_n}^*(z) - \varpi_{\mathcal{A}_n}(z)\mathcal{H}_n^{\mathrm{u}}) = \det(z\tilde{\varpi}_{\mathcal{A}_n}(\mathcal{H}_n^{\mathrm{u}}) - \tilde{\varpi}_{\mathcal{A}_n}^*(\mathcal{H}_n^{\mathrm{u}}))$, which gives q_n^v as a determinant of a Hessenberg matrix. The zeros of Q_n^v are the eigenvalues of the Hessenberg pair $(\tilde{\varpi}_{\mathcal{A}_n}^*(\mathcal{H}_n^{\mathrm{u}}), \tilde{\varpi}_{\mathcal{A}_n}(\mathcal{H}_n^{\mathrm{u}}))$, whose left eigenvectors with eigenvalue λ are spanned by $(\phi_0(\lambda) \ \ldots \ \phi_{n-1}(\lambda))\eta_{\mathcal{A}_n}^{-1}$.

5. ORF and five-diagonal matrices

Apart from the presence of operator Möbius transformations, there are some drawbacks in the spectral theory of ORF previously developed: the appearance of a Hessenberg matrix $\mathcal{H}$ instead of a band one, the complicated dependence of $\mathcal{H} = \mathcal{H}(\boldsymbol{\gamma})$ on the parameters $\boldsymbol{\gamma}$, and the fact that the spectral interpretation of the measure μ works only when $\log \mu' \notin L^1_m$. We will not be able to avoid the operator Möbius transformations because they are linked to the ORF, but the other problems can be overcome by choosing a different basis of ORF in L^2_μ.

The key idea is to use, instead of the ORF $(\phi_n)_{n\geqslant 0}$ with poles in $\mathbb{E}$, other ones whose poles are alternatively in $\mathbb{E}$ and $\mathbb{D}$. For this purpose we define the finite odd and even Blaschke products

$$B_0^{\mathrm{o}} = B_0^{\mathrm{e}} = 1; \qquad B_n^{\mathrm{o}} = \zeta_1\zeta_3\cdots\zeta_{2n-1}, \quad B_n^{\mathrm{e}} = \zeta_2\zeta_4\cdots\zeta_{2n}, \quad n \geqslant 1.$$

Consider the rational functions $(\chi_n)_{n\geqslant 0}$ given by

$$\chi_{2n} = B_{n*}^{\mathrm{e}} \phi_{2n}^*, \qquad \chi_{2n+1} = B_{n*}^{\mathrm{e}} \phi_{2n+1}, \quad n \geqslant 0. \tag{25}$$

Let $\mathcal{M}_n = \operatorname{span}\{\chi_k\}_{k=0}^{n-1}$ for $n = 0, 1, \ldots, \infty$. Then,

$$\mathcal{M}_{2n} = B^{\mathrm{e}}_{n-1*}\mathcal{L}_{2n} = \operatorname{span}\{B^{\mathrm{e}}_{0*}, B^{\mathrm{o}}_{1}, B^{\mathrm{e}}_{1*}, \ldots, B^{\mathrm{e}}_{n-1*}, B^{\mathrm{o}}_{n}\},$$
$$\mathcal{M}_{2n+1} = B^{\mathrm{e}}_{n*}\mathcal{L}_{2n+1} = \operatorname{span}\{B^{\mathrm{e}}_{0*}, B^{\mathrm{o}}_{1}, B^{\mathrm{e}}_{1*}, \ldots, B^{\mathrm{o}}_{n}, B^{\mathrm{e}}_{n*}\},$$

that is, $\mathcal{M}_{2n}$ and $\mathcal{M}_{2n+1}$ are the sets of rational functions whose poles, counted with multiplicity, lie respectively on $(\hat{\alpha}_1, \alpha_2, \hat{\alpha}_3, \alpha_4, \ldots, \alpha_{2n-2}, \hat{\alpha}_{2n-1})$ and $(\hat{\alpha}_1, \alpha_2, \hat{\alpha}_3, \alpha_4, \ldots, \hat{\alpha}_{2n-1}, \alpha_{2n})$. $\mathcal{M}$ will be the closure of $\mathcal{M}_\infty$ in L^2_μ.

The orthonormality conditions $\phi_n \perp \mathcal{L}_n$ and $\langle \phi_n, \phi_n \rangle_\mu = 1$ can be rewritten using ϕ^*_n as $\phi^*_n \perp \zeta_n \mathcal{L}_n$ and $\langle \phi^*_n, \phi^*_n \rangle_\mu = 1$. Hence, the orthonormality of $(\phi_n)_{n\geqslant 0}$ is equivalent to $\chi_{2n} \perp B^{\mathrm{e}}_{n*}\zeta_{2n}\mathcal{L}_{2n} = \mathcal{M}_{2n}$, $\chi_{2n+1} \perp B^{\mathrm{e}}_{n*}\mathcal{L}_{2n+1} = \mathcal{M}_{2n+1}$ and $\langle \chi_n, \chi_n \rangle_\mu = 1$, i.e., to the orthonormality of $(\chi_n)_{n\geqslant 0}$. The sequence $(\chi_n)_{n\geqslant 0}$ is therefore the result of orthonormalizing $(B^{\mathrm{e}}_{0*}, B^{\mathrm{o}}_{1}, B^{\mathrm{e}}_{1*}, B^{\mathrm{o}}_{2}, B^{\mathrm{e}}_{2*}, \ldots)$ in L^2_μ. Hence, relation (25) establishes a connection between the ORF associated with the sequences $(\alpha_n)_{n\geqslant 1}$ and $(\alpha_1, \hat{\alpha}_2, \alpha_3, \hat{\alpha}_4, \ldots)$. We can consider also the ORF associated with the sequence $(\hat{\alpha}_1, \alpha_2, \hat{\alpha}_3, \alpha_4, \ldots)$, i.e., the ORF that arise from the orthonormalization of $(B^{\mathrm{e}}_{0}, B^{\mathrm{o}}_{1*}, B^{\mathrm{e}}_{1}, B^{\mathrm{o}}_{2*}, B^{\mathrm{e}}_{2}, \ldots)$ in L^2_μ. This ORF are $(\chi_{n*})_{n\geqslant 0}$, which are related to $(\phi_n)_{n\geqslant 0}$ by

$$\chi_{2n*} = B^{\mathrm{o}}_{n*}\phi_{2n}, \qquad \chi_{2n+1*} = B^{\mathrm{o}}_{n+1*}\phi^*_{2n+1}, \quad n \geqslant 0. \tag{26}$$

The ORF $(\chi_n)_{n\geqslant 0}$ provide new matrix tools for the analysis of questions concerning the ORF $(\phi_n)_{n\geqslant 0}$. The reason is the different nature of the recurrence satisfied by $(\chi_n)_{n\geqslant 0}$, which, as we will see, is a 5-term linear recurrence relation. This provides a matrix representation of T_μ in terms of five-diagonal instead of Hessenberg matrices, as the following theorem states.

Theorem 5.1. *Let $\boldsymbol{\alpha}$ be a sequence compactly included in $\mathbb{D}$, μ a measure on $\mathbb{T}$ and $\boldsymbol{\gamma} = \mathcal{S}_{\boldsymbol{\alpha}}(\mu)$. Then, the ORF $(\chi_n)_{n\geqslant 0}$ associated with $(\alpha_1, \hat{\alpha}_2, \alpha_3, \hat{\alpha}_4, \ldots)$ is a basis of L^2_μ and the matrix of T_μ with respect to $(\chi_n)_{n\geqslant 0}$ is $\mathcal{U} = \tilde{\zeta}_{\mathcal{A}}(\mathcal{C})$, where $\mathcal{A} = \mathcal{A}(\boldsymbol{\alpha})$ and $\mathcal{C} = \mathcal{C}(\boldsymbol{\gamma})$ with*

$$\mathcal{C}(\boldsymbol{\gamma}) = \begin{pmatrix} -\gamma_1 & -\rho_1\gamma_2 & \rho_1\rho_2 & 0 & 0 & 0 & 0 & \ldots \\ \rho_1 & -\overline{\gamma}_1\gamma_2 & \overline{\gamma}_1\rho_2 & 0 & 0 & 0 & 0 & \ldots \\ 0 & -\rho_2\gamma_3 & -\overline{\gamma}_2\gamma_3 & -\rho_3\gamma_4 & \rho_3\rho_4 & 0 & 0 & \ldots \\ 0 & \rho_2\rho_3 & \overline{\gamma}_2\rho_3 & -\overline{\gamma}_3\gamma_4 & \overline{\gamma}_3\rho_4 & 0 & 0 & \ldots \\ 0 & 0 & 0 & -\rho_4\gamma_5 & -\overline{\gamma}_4\gamma_5 & -\rho_5\gamma_6 & \rho_5\rho_6 & \ldots \\ 0 & 0 & 0 & \rho_4\rho_5 & \overline{\gamma}_4\rho_5 & -\overline{\gamma}_5\gamma_6 & \overline{\gamma}_5\rho_6 & \ldots \\ 0 & 0 & 0 & 0 & 0 & -\rho_6\gamma_7 & -\overline{\gamma}_6\gamma_7 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}.$$

Proof. Since $\boldsymbol{\alpha}$ is compactly included in $\mathbb{D}$, $\tilde{\zeta}_{\mathcal{A}}(\mathcal{C})$ is a well-defined unitary matrix because $\mathcal{C}$ is unitary (see [12,32,36]).

Using (5) and (25) we get $\varpi^*_0\chi_0 = \rho^+_1\varpi_1\chi_1 - \gamma_1\varpi_0\chi_0$ and, for $n \geqslant 1$,

$$\begin{aligned} \varpi^*_{2n-1}\chi_{2n-1} &= B^{\mathrm{e}}_{n-1*}\left(\rho^+_{2n}\varpi_{2n}\phi_{2n} - \gamma_{2n}\varpi_{2n-1}\phi^*_{2n-1}\right) \\ &= B^{\mathrm{e}}_{n*}\rho^+_{2n}\left(\rho^+_{2n+1}\varpi_{2n+1}\phi_{2n+1} - \gamma_{2n+1}\varpi_{2n}\phi^*_{2n}\right) \\ &\quad - B^{\mathrm{e}}_{n-1*}\gamma_{2n}\left(\overline{\gamma}_{2n-1}\varpi_{2n-1}\phi_{2n-1} + \rho^-_{2n-1}\varpi_{2n-2}\phi^*_{2n-2}\right) \\ &= \rho^+_{2n}\rho^+_{2n+1}\varpi_{2n+1}\chi_{2n+1} - \rho^+_{2n}\gamma_{2n+1}\varpi_{2n}\chi_{2n} \end{aligned}$$

$$
\begin{aligned}
&\qquad - \overline{\gamma}_{2n-1}\gamma_{2n}\varpi_{2n-1}\chi_{2n-1} - \rho_{2n-1}^{-}\gamma_{2n}\varpi_{2n-2}\chi_{2n-2}, \\
\varpi_{2n}^{*}\chi_{2n} &= B_{n-1*}^{\mathrm{e}}\left(\overline{\gamma}_{2n}\varpi_{2n}\phi_{2n} + \rho_{2n}^{-}\varpi_{2n-1}\phi_{2n-1}^{*}\right) \\
&= B_{n*}^{\mathrm{e}}\overline{\gamma}_{2n}\left(\rho_{2n+1}^{+}\varpi_{2n+1}\phi_{2n+1} - \gamma_{2n+1}\varpi_{2n}\phi_{2n}^{*}\right) \\
&\quad + B_{n-1*}^{\mathrm{e}}\rho_{2n}^{-}\left(\overline{\gamma}_{2n-1}\varpi_{2n-1}\phi_{2n-1} + \rho_{2n-1}^{-}\varpi_{2n-2}\phi_{2n-2}^{*}\right) \\
&= \overline{\gamma}_{2n}\rho_{2n+1}^{+}\varpi_{2n+1}\chi_{2n+1} - \overline{\gamma}_{2n}\gamma_{2n+1}\varpi_{2n}\chi_{2n} \\
&\quad + \overline{\gamma}_{2n-1}\rho_{2n}^{-}\varpi_{2n-1}\chi_{2n-1} + \rho_{2n-1}^{-}\rho_{2n}^{-}\varpi_{2n-2}\chi_{2n-2}.
\end{aligned}
\tag{27}
$$

This is the 5-term linear recurrence for $(\chi_n)_{n\geqslant 0}$, which can be written as

$$
\begin{pmatrix} \chi_0 & \chi_1 & \dots \end{pmatrix}\left(\varpi_{\mathcal{A}}^{*} - \varpi_{\mathcal{A}}\hat{\mathcal{C}}\right) = 0,
\tag{28}
$$

where $\hat{\mathcal{C}}$ is the five-diagonal matrix

$$
\hat{\mathcal{C}} = \begin{pmatrix}
-\gamma_1 & -\rho_1^{-}\gamma_2 & \rho_1^{-}\rho_2^{-} & 0 & 0 & 0 & \dots \\
\rho_1^{+} & -\overline{\gamma}_1\gamma_2 & \overline{\gamma}_1\rho_2^{-} & 0 & 0 & 0 & \dots \\
0 & -\rho_2^{+}\gamma_3 & -\overline{\gamma}_2\gamma_3 & -\rho_3^{-}\gamma_4 & \rho_3^{-}\rho_4^{-} & 0 & \dots \\
0 & \rho_2^{+}\rho_3^{+} & \overline{\gamma}_2\rho_3^{+} & -\overline{\gamma}_3\gamma_4 & \overline{\gamma}_3\rho_4^{-} & 0 & \dots \\
0 & 0 & 0 & -\rho_4^{+}\gamma_5 & -\overline{\gamma}_4\gamma_5 & -\rho_5^{-}\gamma_6 & \dots \\
0 & 0 & 0 & \rho_4^{+}\rho_5^{+} & \overline{\gamma}_4\rho_5^{+} & -\overline{\gamma}_5\gamma_6 & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots
\end{pmatrix}.
$$

Using (4) we find that $\hat{\mathcal{C}} = \eta_{\mathcal{A}}^{-1}\mathcal{C}\eta_{\mathcal{A}} = \mathcal{C}_{\mathcal{A}}$, so, bearing in mind (7) and (8), (28) becomes equivalent to

$$
\begin{pmatrix} \chi_0(z) & \chi_1(z) & \dots \end{pmatrix}\left(z - \tilde{\zeta}_{\mathcal{A}}(\mathcal{C})\right) = 0,
\tag{29}
$$

which shows that $\mathcal{M}$ is invariant under T_μ and $\tilde{\zeta}_{\mathcal{A}}(\mathcal{C})$ is the matrix representation of $T_\mu \restriction \mathcal{M}$ with respect to $(\chi_n)_{n\geqslant 0}$.

Similar arguments to those given in the proof of Theorem 4.1 prove that $T_\mu \restriction \mathcal{M}$ is unitary iff $\mathcal{M} = L_\mu^2$. However, $T_\mu \restriction \mathcal{M}$ is unitary whenever $\boldsymbol{\alpha}$ is compactly included in $\mathbb{D}$ because in this case $\tilde{\zeta}_{\mathcal{A}}(\mathcal{C})$ is unitary for any sequence $\boldsymbol{\gamma}$ in $\mathbb{D}$. Therefore, $\mathcal{M} = L_\mu^2$, i.e., the ORF $(\chi_n)_{n\geqslant 0}$ is a basis of L_μ^2, which implies that $\tilde{\zeta}_{\mathcal{A}}(\mathcal{C})$ is a matrix of the full operator T_μ. $\square$

Remark 5.2. We know that $(\chi_n)_{n\geqslant 0}$ and $(\chi_{n*})_{n\geqslant 0}$ are bases of L_μ^2 at the same time, and the corresponding matrices of T_μ are related by transposition. Therefore, $(\chi_{n*})_{n\geqslant 0}$ is a basis of L_μ^2 whenever $\boldsymbol{\alpha}$ is compactly included in $\mathbb{D}$ and, in this case, the related matrix of T_μ is $\mathcal{U}^T$. Notice that the second equality in (6) implies that $\mathcal{U}^T = \tilde{\zeta}_{\mathcal{A}}(\mathcal{C}^T)$ because $\mathcal{A}$ is diagonal.

Theorem 5.1 states that, contrary to the case of the ORF $(\phi_n)_{n\geqslant 0}$, $(\chi_n)_{n\geqslant 0}$ and $(\chi_{n*})_{n\geqslant 0}$ are bases of L_μ^2 for any measure μ on $\mathbb{T}$ if $\boldsymbol{\alpha}$ is compactly included in $\mathbb{D}$. Indeed, the completeness of $(\chi_n)_{n\geqslant 0}$ and $(\chi_{n*})_{n\geqslant 0}$ in L_μ^2 holds even under a more general condition for $\boldsymbol{\alpha}$, as the next

theorem shows. Denoting $\mathcal{F}_* = \{f_*\colon\ f \in \mathcal{F}\}$ for any set $\mathcal{F}$ of complex functions, the problem is to find sufficient conditions for the equality $\mathcal{M} = L^2_\mu$ or, equivalently, $\mathcal{M}_* = L^2_\mu$.

Proposition 5.3. *Let $\boldsymbol{\alpha}$ be a sequence in $\mathbb{D}$, μ a measure on $\mathbb{T}$ and $(\chi_n)_{n\geqslant 0}$ the ORF associated with $(\alpha_1, \hat\alpha_2, \alpha_3, \hat\alpha_4, \ldots)$. $(\chi_n)_{n\geqslant 0}$ and $(\chi_{n*})_{n\geqslant 0}$ are bases of L^2_μ if $\sum_{k=1}^\infty (1 - |\alpha_{2k-1}|) = \sum_{k=1}^\infty (1 - |\alpha_{2k}|) = \infty$.*

Proof. Given a sequence $\boldsymbol{\beta} = (\beta_n)_{n\geqslant 1}$ in $\overline{\mathbb{C}}\setminus\mathbb{T}$, let $\mathcal{L}_\infty(\boldsymbol{\beta})$ be the set of rational functions with poles in $\hat{\boldsymbol{\beta}} = (\hat\beta_n)_{n\geqslant 1}$, counted with multiplicity, i.e.,

$$\mathcal{L}_\infty(\boldsymbol{\beta}) = \bigcup_{n\geqslant 1} \frac{\operatorname{span}\{z^k\}_{k=0}^n}{\varpi_{\beta_1}\cdots\varpi_{\beta_n}}, \tag{30}$$

where $\varpi_\infty(z) = z$. Also, let $\mathcal{L}(\boldsymbol{\beta})$ be the closure of $\mathcal{L}_\infty(\boldsymbol{\beta})$ in L^2_μ. Notice that $\mathcal{L}(\boldsymbol{\beta})_* = \mathcal{L}(\hat{\boldsymbol{\beta}})$ and $\mathcal{M} = \mathcal{L}(\alpha_1, \hat\alpha_2, \alpha_3, \hat\alpha_4, \ldots)$. We will show that

(i) $\sum_{k=1}^\infty (1 - |\alpha_{2k-1}|) = \infty \ \Rightarrow\ \{z^j\}_{j\in\mathbb{N}} \subset \mathcal{L}(\alpha_1, \hat\alpha_2, \alpha_3, \hat\alpha_4, \ldots)$,
(ii) $\sum_{k=1}^\infty (1 - |\alpha_{2k}|) = \infty \ \Rightarrow\ \{z^{-j}\}_{j\in\mathbb{N}} \subset \mathcal{L}(\alpha_1, \hat\alpha_2, \alpha_3, \hat\alpha_4, \ldots)$.

This demonstrates the proposition because $\operatorname{span}\{z^j\}_{j\in\mathbb{Z}}$ is dense in L^2_μ. Indeed, we only must prove (i) since it implies (ii). To see this, apply (i) to $(\alpha_2, \alpha_1, \alpha_4, \alpha_3, \ldots)$. We find that $\sum_{k=1}^\infty (1 - |\alpha_{2k}|) = \infty$ ensures $\{z^j\}_{j\in\mathbb{N}} \subset \mathcal{L}(\alpha_2, \hat\alpha_1, \alpha_4, \hat\alpha_3, \ldots) = \mathcal{L}(\hat\alpha_1, \alpha_2, \hat\alpha_3, \alpha_4, \ldots)$, which, using the $*$-involution, becomes $\{z^{-j}\}_{j\in\mathbb{N}} \subset \mathcal{L}(\alpha_1, \hat\alpha_2, \alpha_3, \hat\alpha_4, \ldots)$. $\sum_{k=1}^\infty (1 - |\alpha_{2k-1}|) = \infty$ means that the Blaschke product B^{o} diverges to zero in $\mathbb{D}$. Thus, we must prove that such a divergence implies $\{z^j\}_{j\in\mathbb{N}} \in \mathcal{L}(\alpha_1, \hat\alpha_2, \alpha_3, \hat\alpha_4, \ldots)$.

According to (30), $\{z^j\}_{j\in\mathbb{N}} \subset \mathcal{L}_\infty(\alpha_1, \hat\alpha_2, \alpha_3, \hat\alpha_4, \ldots) = \mathcal{M}_\infty$ if $\alpha_{2k-1} = 0$ for infinitely many values $k \in \mathbb{N}$. Hence, we only need to study the opposite case that, without loss of generality, we can suppose is $\alpha_1 = \alpha_3 = \cdots = \alpha_{2s-1} = 0$ and $\alpha_{2k-1} \neq 0$ for $k > s$. Then $\{z, \ldots, z^s\} \subset \mathcal{M}_\infty$, and it suffices to prove that $\inf_{f\in\mathcal{M}_n} \|z^j - f(z)\|_\infty \xrightarrow{n} 0$ for $j > s$. The L^∞-distance between a polynomial and $\mathcal{M}_n$ can be measured using the following result (see [1, p. 243] or the more recent reference [11, p. 150]): denoting $\mathcal{P}_N = \operatorname{span}\{z^k\}_{k=0}^{N-1}$,

$$\min_{q\in\mathcal{P}_N} \left\| \frac{z^N + q(z)}{(z - w_1)\cdots(z - w_n)} \right\|_\infty = \prod_{k=1}^n \frac{1}{\max\{|w_k|, 1\}}, \quad w_k \in \mathbb{C},\ N \geqslant n.$$

Therefore, if B^{o} diverges, taking $n > s$,

$$\begin{aligned} &\inf_{\substack{f\in\mathcal{M}_{2n}\\ a_k\in\mathbb{C}}} \left\| z^{s+m} + a_{m-1}z^{s+m-1} + \cdots + a_1 z^{s+1} - f(z) \right\|_\infty \\ &\quad = \inf_{q\in\mathcal{P}_{2n+m-1}} \left\| \frac{z^{2n+m-1} + q(z)}{\prod_{k=1}^{n-1}(z - \alpha_{2k}) \prod_{k=s+1}^n (z - \hat\alpha_{2k-1})} \right\|_\infty = \prod_{k=s+1}^n |\alpha_{2k-1}| \xrightarrow{n} 0. \end{aligned}$$

This result implies by induction on m that $z^{s+m} \in \mathcal{M}$ for any $m \in \mathbb{N}$. $\square$

When $\mathcal{A}=0, \mathcal{U}=\mathcal{C}$ and we recover the five-diagonal representation related to the polynomial case (see [12,32,36]).

As in the Hessenberg case, the previous theorem provides a spectral interpretation of the orthogonality measure μ. The arguments are similar to those given in the proof of Theorem 4.2, but now the restriction $\log\mu' \notin L^1_m$ is not necessary.

Theorem 5.4. *Let $\boldsymbol{\alpha}$ be a sequence compactly included in $\mathbb{D}$, μ a measure on $\mathbb{T}$, $(\phi_n)_{n\geqslant 0}$ the corresponding ORF and $(\chi_n)_{n\geqslant 0}$ the ORF associated with $(\alpha_1, \hat{\alpha}_2, \alpha_3, \hat{\alpha}_4, \ldots)$. Let $\mathcal{U} = \tilde{\zeta}_{\mathcal{A}}(\mathcal{C})$ with $\mathcal{A} = \mathcal{A}(\boldsymbol{\alpha})$, $\mathcal{C} = \mathcal{C}(\boldsymbol{\gamma})$, $\boldsymbol{\gamma} = \mathcal{S}_{\boldsymbol{\alpha}}(\mu)$. If $\mathcal{E}$ is the spectral measure of $\mathcal{U}$, $\mu = \mathcal{E}_{1,1}$. Besides, $\sigma(\mathcal{U}) = \operatorname{supp}\mu$ and $\sigma_p(\mathcal{U}) = \{$mass points of $\mu\}$. λ is a mass point iff $(\chi_n(\lambda))_{n\geqslant 0} \in \ell^2$. Given a mass point λ, the related eigenvectors of $\mathcal{U}$ are spanned by $(\chi_0(\lambda)\ \chi_1(\lambda)\ \ldots)^\dagger$ and $\mu(\{\lambda\}) = (\sum_{n=0}^\infty |\chi_n(\lambda)|^2)^{-1} = (\sum_{n=0}^\infty |\phi_n(\lambda)|^2)^{-1}$.*

We can formulate the above theorem in terms of the five-diagonal pair $(\tilde{\varpi}^*_{\mathcal{A}}(\mathcal{C}), \tilde{\varpi}_{\mathcal{A}}(\mathcal{C}))$. Theorem 4.2 implies that $\sigma(\tilde{\varpi}^*_{\mathcal{A}}(\mathcal{C}), \tilde{\varpi}_{\mathcal{A}}(\mathcal{C})) = \operatorname{supp}\mu$ and $\sigma_p(\tilde{\varpi}^*_{\mathcal{A}}(\mathcal{C}), \tilde{\varpi}_{\mathcal{A}}(\mathcal{C})) = \{$mass points of $\mu\}$. Also, given a mass point λ, the left eigenvectors of $(\tilde{\varpi}^*_{\mathcal{A}}(\mathcal{C}), \tilde{\varpi}_{\mathcal{A}}(\mathcal{C}))$ are spanned by $(\chi_0(\lambda)\ \chi_1(\lambda)\ \ldots)\eta_{\mathcal{A}}^{-1/2}$. Furthermore, using the factorization (see [12,32,36])

$$\mathcal{C} = \mathcal{C}_{\mathrm{o}}\mathcal{C}_{\mathrm{e}}, \qquad \mathcal{C}_{\mathrm{o}} = \begin{pmatrix} \Theta_1 & & & \\ & \Theta_3 & & \\ & & \Theta_5 & \\ & & & \ddots \end{pmatrix}, \qquad \mathcal{C}_{\mathrm{e}} = \begin{pmatrix} I_1 & & & \\ & \Theta_2 & & \\ & & \Theta_4 & \\ & & & \ddots \end{pmatrix}, \tag{31}$$

we can write these results using the tridiagonal pair $(\tilde{\varpi}^*_{\mathcal{A}}(\mathcal{C}), \tilde{\varpi}_{\mathcal{A}}(\mathcal{C}))\mathcal{C}_{\mathrm{e}}^\dagger = (\mathcal{C}_{\mathrm{o}} + \mathcal{A}\mathcal{C}_{\mathrm{e}}^\dagger, \mathcal{C}_{\mathrm{e}}^\dagger + \mathcal{A}^\dagger\mathcal{C}_{\mathrm{o}})$ instead of the five-diagonal one.

5.1. Zeros of ORF and five-diagonal matrices

The previous results suggest a possible spectral interpretation of the zeros of ORF and PORF in terms of five-diagonal matrices. Analogously to the Hessenberg case, an important ingredient for this is the orthogonal truncation $\zeta_n(T_\mu)^{(\mathcal{M}_n)}$. $\mathcal{M}_n = B^{\mathrm{e}}_{l*}\mathcal{L}_n$, $l = [(n-1)/2]$, thus the following generalization of Theorem 4.3 is of interest to relate $\zeta_n(T_\mu)^{(\mathcal{M}_n)}$ to the zeros of ϕ_n.

Theorem 5.5. *Let $\boldsymbol{\alpha}$ be a sequence in $\mathbb{D}$, μ a measure on $\mathbb{T}$, $\phi_n = p_n/\pi_n$ the related nth ORF and $h : \mathbb{T} \to \mathbb{T}$ a Borel function.*

1. *If Z_n is the set of zeros of ϕ_n, $\zeta_n(Z_n)$ is the set of eigenvalues of $\zeta_n(T_\mu)^{(h\mathcal{L}_n)}$ and these eigenvalues have geometric multiplicity* 1.
2. *If $p_n(z) \propto \prod_{k=1}^n (z-\lambda_k)$, the characteristic polynomial of $\zeta_n(T_\mu)^{(h\mathcal{L}_n)}$ is $\prod_{k=1}^n (z-\zeta_n(\lambda_k))$.*

Proof. The operator $h(T_\mu)$, i.e., the operator multiplication by h in L^2_μ, is unitary because $h(\mathbb{T}) \subset \mathbb{T}$. When restricted in the following way:

$$\begin{aligned} V : \mathcal{L}_n &\to h\mathcal{L}_n \\ f &\to hf \end{aligned}$$

it yields an isometric isomorphism V. The orthogonal projection on $h\mathcal{L}_n$ is $h(T_\mu)L_n h(T_\mu)^\dagger$, L_n being the orthogonal projection on $\mathcal{L}_n$. So, $\zeta_n(T_\mu)^{(h\mathcal{L}_n)} = V\zeta_n(T_\mu)^{(\mathcal{L}_n)}V^{-1}$ and the result follows directly from Theorem 4.3. $\square$

Taking $h = B^{\mathrm{e}}_{l*}$, $l = [(n-1)/2]$, in the previous theorem we find that it holds for $\zeta_n(T_\mu)^{(\mathcal{M}_n)}$. To give a matrix version of this result we simply need a matrix representation of $\zeta_n(T_\mu)^{(\mathcal{M}_n)}$.

Theorem 5.6. *Let $\boldsymbol{\alpha}$ be a sequence in $\mathbb{D}$, μ a measure on $\mathbb{T}$ and $(\chi_n)_{n\geqslant 0}$ the ORF related to $(\alpha_1, \hat\alpha_2, \alpha_3, \hat\alpha_4, \ldots)$. The matrix of $\zeta_n(T_\mu)^{(\mathcal{M}_n)}$ with respect to $(\chi_k)_{k=0}^{n-1}$ is $\zeta_n(\mathcal{U}^{(n)})$, where $\mathcal{U}^{(n)} = \tilde\zeta_{\mathcal{A}_n}(\mathcal{C}_n)$ and $\mathcal{A} = \mathcal{A}(\boldsymbol{\alpha})$, $\mathcal{C} = \mathcal{C}(\boldsymbol{\gamma})$, $\boldsymbol{\gamma} = \mathcal{S}_{\boldsymbol{\alpha}}(\mu)$.*

Proof. Factorization (31) gives $\mathcal{C}_n = \mathcal{C}_{\mathrm{o}n}\mathcal{C}_{\mathrm{e}n}$, thus $\|\mathcal{C}_n\| = 1$ and $\mathcal{U}^{(n)} = \tilde\zeta_{\mathcal{A}_n}(\mathcal{C}_n)$ is a well-defined matrix such that $\|\mathcal{U}^{(n)}\| = 1$ because $\|\mathcal{A}_n\| < 1$. Hence, $\zeta_n(\mathcal{U}^{(n)})$ is well defined too.

Let us consider first an odd n. The first n equations of (28) read as

$$(\,\chi_0 \quad \ldots \quad \chi_{n-1}\,)\big(\varpi^*_{\mathcal{A}_n} - \varpi_{\mathcal{A}_n}\hat{\mathcal{C}}_n\big) = b_n \varpi_n \chi_n, \quad b_n \in \mathbb{C}^n.$$

Since $\hat{\mathcal{C}}_n = \eta^{-1}_{\mathcal{A}_n}\mathcal{C}_n\eta_{\mathcal{A}_n} = (\mathcal{C}_n)_{\mathcal{A}_n}$, identities (7) and (8) yield

$$\big(\,\chi_0(z) \quad \ldots \quad \chi_{n-1}(z)\,\big)\big(z - \mathcal{U}^{(n)}\big) = c_n \varpi_n(z)\chi_n(z), \quad c_n \in \mathbb{C}^n.$$

Using (11) we get

$$\big(\,\chi_0(z) \quad \ldots \quad \chi_{n-1}(z)\,\big)\big(\zeta_n(z) - \zeta_n\big(\mathcal{U}^{(n)}\big)\big) = d_n \chi_n(z), \quad d_n \in \mathbb{C}^n.$$

Therefore, if M_n is the orthogonal projection on $\mathcal{M}_n$,

$$(\,M_n\zeta_n\chi_0 \quad \ldots \quad M_n\zeta_n\chi_{n-1}\,) = (\,\chi_0 \quad \ldots \quad \chi_{n-1}\,)\,\zeta_n\big(\mathcal{U}^{(n)}\big),$$

proving the theorem for odd n.

On the other hand, if n is even, we consider the orthogonal truncation $\zeta_n(T_\mu)^{(\mathcal{M}_{n*})}$. Since $\mathcal{C}$ is unitary, the $*$-involution on (28) gives

$$\big(\,\chi_{0*}(z) \quad \chi_{1*}(z) \quad \ldots\big)\big(\varpi^*_{\mathcal{A}}(z) - \varpi_{\mathcal{A}}(z)\big(\mathcal{C}^T\big)_{\mathcal{A}}\big) = 0. \tag{32}$$

A similar reasoning starting from the first n equations of this equality proves that the matrix of $\zeta_n(T_\mu)^{(\mathcal{M}_{n*})}$ with respect to $(\chi_{k*})_{k=0}^{n-1}$ is $\zeta_n(\mathcal{U}^{(n)}_*)$, where $\mathcal{U}^{(n)}_* = \tilde\zeta_{\mathcal{A}_n}(\mathcal{C}^T_n)$. From (6), $\mathcal{U}^{(n)}_* = \mathcal{U}^{(n)T}$ because $\mathcal{A}_n$ is diagonal.

The subspace $\mathcal{M}_n$ only depends on the parameters $\alpha_1, \ldots, \alpha_{n-1}$ of the sequence $\boldsymbol{\alpha}$, so the same holds for $\zeta_n(T_\mu)^{(\mathcal{M}_n)}$ and $\zeta_n(T_\mu)^{(\mathcal{M}_{n*})}$. Therefore, concerning the spectral properties of these truncations we can suppose without loss of generality that $\boldsymbol{\alpha}$ is compactly supported on $\mathbb{D}$. Then, the matrix representations of T_μ with respect to $(\chi_k)_{k\geqslant 0}$ and $(\chi_{k*})_{k\geqslant 0}$ are $\mathcal{U}$ and $\mathcal{U}^T$, respectively. The representations of $\zeta_n(T_\mu)^{(\mathcal{M}_n)}$ and $\zeta_n(T_\mu)^{(\mathcal{M}_{n*})}$ with respect to $(\chi_k)_{k=0}^{n-1}$ and $(\chi_{k*})_{k=0}^{n-1}$ are the principal submatrices $(\zeta_n(\mathcal{U}))_n$ and $(\zeta_n(\mathcal{U}^T))_n$, respectively. The fact that $(\zeta_n(\mathcal{U}^T))_n = (\zeta_n(\mathcal{U}))_n^T$ implies that, when n is even, the matrix of $\zeta_n(T_\mu)^{(\mathcal{M}_n)}$ with respect to $(\chi_k)_{k=0}^{n-1}$ is $\zeta_n(\mathcal{U}^{(n)}_*)^T = \zeta_n(\mathcal{U}^{(n)})$. $\square$

Remark 5.7. The proof of the previous theorem also shows that the matrix of $\zeta_n(T_\mu)^{(\mathcal{M}_{n*})}$ with respect to $(\chi_{k*})_{k=0}^{n-1}$ is $\zeta_n(\mathcal{U}^{(n)})^T$.

We have the following immediate consequence of Theorems 5.5 and 5.6.

Theorem 5.8. *Let $\boldsymbol{\alpha}$ be a sequence in $\mathbb{D}$, μ a measure on $\mathbb{T}$, $(\phi_n)_{n\geqslant 0}$ the corresponding ORF, $(\chi_n)_{n\geqslant 0}$ the ORF related to $(\alpha_1, \hat{\alpha}_2, \alpha_3, \hat{\alpha}_4, \ldots)$ and $\mathcal{U}^{(n)} = \tilde{\zeta}_{\mathcal{A}_n}(\mathcal{C}_n)$, with $\mathcal{A} = \mathcal{A}(\boldsymbol{\alpha})$, $\mathcal{C} = \mathcal{C}(\boldsymbol{\gamma})$, $\boldsymbol{\gamma} = \mathcal{S}_{\boldsymbol{\alpha}}(\mu)$.*

1. *The zeros of ϕ_n are the eigenvalues of $\mathcal{U}^{(n)}$. If λ is a zero of ϕ_n, the related left and right eigenvectors of $\mathcal{U}^{(n)}$ are spanned by $X_n(\lambda)$ and $Y_n(\lambda)^T$, respectively, where*

$$X_n = B^{\mathrm{e}}_{[\frac{n-1}{2}]}\left(\chi_0 \quad \ldots \quad \chi_{n-1}\right), \qquad Y_n = B^{\mathrm{o}}_{[\frac{n}{2}]}\left(\chi_{0*} \quad \ldots \quad \chi_{n-1*}\right).$$

2. *$\phi_n = \frac{p_n}{\pi_n}$ with p_n proportional to the characteristic polynomial of $\mathcal{U}^{(n)}$.*

Proof. X_n and Y_n are rational functions with the poles lying on $\mathbb{E}$, so they can be evaluated at any zero λ of ϕ_n since $\lambda \in \mathbb{D}$. Besides, $X_n(\lambda), Y_n(\lambda) \neq 0$ because $B^{\mathrm{e}}_k \chi_{2k} = \phi^*_{2k}$, $B^{\mathrm{o}}_k \chi_{2k-1*} = \phi^*_{2k-1}$ and ϕ^*_n has its zeros in $\mathbb{E}$.

The proof of the theorem is similar to the case of Theorem 4.5, the only difference concerning the identification of the eigenvectors. To obtain the left eigenvectors of $\mathcal{U}^{(n)}$ we start writing the first n equations of (28) as

$$\left(\chi_0 \quad \ldots \quad \chi_{n-1}\right)\left(\varpi^*_{\mathcal{A}_n} - \varpi_{\mathcal{A}_n}\hat{\mathcal{C}}_n\right) = b_n \varpi_n \chi_n + d_n \varpi_{n+1}\chi_{n+1}, \tag{33}$$

$$b_n = \begin{cases} \rho^+_n \left(0 \quad \ldots \quad 0 \quad \rho^+_{n-1} \quad \overline{\gamma}_{n-1}\right) & \text{odd } n, \\ -\rho^+_n \gamma_{n+1}\left(0 \quad \ldots \quad 0 \quad 1\right) & \text{even } n, \end{cases}$$
$$d_n = \begin{cases} 0 & \text{odd } n, \\ \rho^+_n \rho^+_{n+1}\left(0 \quad \ldots \quad 0 \quad 1\right) & \text{even } n. \end{cases}$$

(25), (26), and the first equation of (5) for even n, transform (33) into

$$\left(\chi_0 \quad \ldots \quad \chi_{n-1}\right)\left(\varpi^*_{\mathcal{A}_n} - \varpi_{\mathcal{A}_n}\hat{\mathcal{C}}_n\right) = \rho^+_n \varpi_n B^{\mathrm{e}}_{l*}\phi_n v_n, \qquad v_n \in \mathbb{C}^n, \tag{34}$$

with $l = [\frac{n-1}{2}]$. (7) and (8) imply $(z - \mathcal{U}^{(n)})\tilde{\varpi}_{\mathcal{A}_n}(\hat{\mathcal{C}}_n) = \varpi^*_{\mathcal{A}_n}(z) - \varpi_{\mathcal{A}_n}(z)\hat{\mathcal{C}}_n$ because $\hat{\mathcal{C}}_n = \eta^{-1}_{\mathcal{A}_n}\mathcal{C}_n\eta_{\mathcal{A}_n}$. Therefore, if λ is a zero of ϕ_n, (34) proves that $X_n(\lambda)$ is a left eigenvector of $\mathcal{U}^{(n)}$ with eigenvalue λ.

Proceeding in a similar way with the first n equations of (32) we find that $Y_n(\lambda)$ is a left eigenvector of $\mathcal{U}^{(n)T}$ with eigenvalue λ for any zero λ of ϕ_n. Therefore, $Y_n(\lambda)^T$ is a right eigenvector of $\mathcal{U}^{(n)}$. □

For a unitary matrix, like $\mathcal{V}$ in the case $\boldsymbol{\gamma} \notin \ell^2$, $\mathcal{V}^{(n;\mathrm{u})}$ or $\mathcal{U}$, left and right eigenvectors are related by the †-operation. However, this is not the case of the matrices $\mathcal{V}^{(n)}$ or $\mathcal{U}^{(n)}$. Theorem 4.5 only gives information about the left eigenvectors of $\mathcal{V}^{(n)}$, while Theorem 5.8 provides both, the

left and right eigenvectors of $\mathcal{U}^{(n)}$. Apart from the simplicity of $\mathcal{U}^{(n)}$, this is another advantage of using this matrix instead of $\mathcal{V}^{(n)}$ for the spectral representation of the zeros of ORF.

As in the Hessenberg case, there are other alternatives to express p_n as a determinant. Indeed, from (7), (8) and the identity $\hat{\mathcal{C}}_n = \eta_{\mathcal{A}_n}^{-1}\mathcal{C}_n\eta_{\mathcal{A}_n}$, $p_n(z) \propto \det(\varpi_{\mathcal{A}_n}^*(z) - \varpi_{\mathcal{A}_n}(z)\mathcal{C}_n) = \det(z\tilde{\varpi}_{\mathcal{A}_n}(\mathcal{C}_n) - \tilde{\varpi}_{\mathcal{A}_n}^*(\mathcal{C}_n))$. So p_n can be calculated as a determinant of a five-diagonal matrix. Furthermore, the last expression shows that the zeros of ϕ_n are the eigenvalues of the five-diagonal pair $(\tilde{\varpi}_{\mathcal{A}_n}^*(\mathcal{C}_n), \tilde{\varpi}_{\mathcal{A}_n}(\mathcal{C}_n))$. The associated left eigenvectors with eigenvalue λ are spanned by $X_n(\lambda)\eta_{\mathcal{A}_n}^{-1/2}$.

Besides, the factorization $\mathcal{C}_n = \mathcal{C}_{\mathrm{on}}\mathcal{C}_{\mathrm{en}}$ permits us to express p_n as a determinant of a tridiagonal matrix. If n is odd, $\mathcal{C}_{\mathrm{en}}$ is unitary, thus $p_n(z) \propto \det(z(\mathcal{C}_{\mathrm{en}}^\dagger + \mathcal{A}_n^\dagger\mathcal{C}_{\mathrm{on}}) - (\mathcal{C}_{\mathrm{on}} + \mathcal{A}_n\mathcal{C}_{\mathrm{en}}^\dagger))$ and the zeros of ϕ_n are the eigenvalues of the tridiagonal pair $(\mathcal{C}_{\mathrm{on}} + \mathcal{A}_n\mathcal{C}_{\mathrm{en}}^\dagger, \mathcal{C}_{\mathrm{en}}^\dagger + \mathcal{A}_n^\dagger\mathcal{C}_{\mathrm{on}})$, with the same left eigenvectors as $(\tilde{\varpi}_{\mathcal{A}_n}^*(\mathcal{C}_n), \tilde{\varpi}_{\mathcal{A}_n}(\mathcal{C}_n))$. On the contrary, if n is even, $\mathcal{C}_{\mathrm{on}}$ is unitary. In this situation we can use the fact that $\mathcal{U}^{(n)}$ and $\mathcal{U}^{(n)T} = \tilde{\zeta}_{\mathcal{A}_n}(\mathcal{C}_n^T)$ have the same characteristic polynomial, and the left eigenvectors of one of them are the transposed of the right eigenvectors of the other one. Hence, $p_n(z) \propto \det(\varpi_{\mathcal{A}_n}^*(z) - \varpi_{\mathcal{A}_n}(z)\mathcal{C}_n^T) = \det(z\tilde{\varpi}_{\mathcal{A}_n}(\mathcal{C}_n^T) - \tilde{\varpi}_{\mathcal{A}_n}^*(\mathcal{C}_n^T))$ and, bearing in mind that $\mathcal{C}_n^T = \mathcal{C}_{\mathrm{en}}\mathcal{C}_{\mathrm{on}}$, $p_n(z) \propto \det(z(\mathcal{C}_{\mathrm{on}}^\dagger + \mathcal{A}_n^\dagger\mathcal{C}_{\mathrm{en}}) - (\mathcal{C}_{\mathrm{en}} + \mathcal{A}_n\mathcal{C}_{\mathrm{on}}^\dagger))$. So, the zeros of ϕ_n are the eigenvalues of $(\mathcal{C}_{\mathrm{en}} + \mathcal{A}_n\mathcal{C}_{\mathrm{on}}^\dagger, \mathcal{C}_{\mathrm{on}}^\dagger + \mathcal{A}_n^\dagger\mathcal{C}_{\mathrm{en}})$, and the related left eigenvectors with eigenvalue λ are spanned by $Y_n(\lambda)\eta_{\mathcal{A}_n}^{-1/2}$.

The zeros of the PORF Q_n^v have a spectral interpretation in terms of band matrices too. Such an interpretation has to do with the matrix representation of $T_{\mu_n^v}$ with respect $(\chi_k)_{k=0}^{n-1}$, which is an orthonormal basis of $L^2_{\mu_n^v}$ due to the exactness of the quadrature formulas associated with μ_n^v. Similar arguments to those appearing before Theorem 4.6 show that the zeros of the PORF should be related to the unitary matrix $\mathcal{C}_n^{\mathrm{u}}$ obtained from $\mathcal{C}_n$ when substituting the parameter $\gamma_n \in \mathbb{D}$ by $u \in \mathbb{T}$. More precisely, we have the following result, which is a limit case of Theorems 5.6 and 5.8.

Theorem 5.9. *Let $\boldsymbol{\alpha}$ be a sequence in $\mathbb{D}$, μ a measure on $\mathbb{T}$, $(\phi_n)_{n\geqslant 0}$ the corresponding ORF, $\mathcal{A} = \mathcal{A}(\boldsymbol{\alpha})$, $\mathcal{C} = \mathcal{C}(\boldsymbol{\gamma})$, $\boldsymbol{\gamma} = \mathcal{S}_{\boldsymbol{\alpha}}(\mu)$. Consider $v \in \mathbb{T}$, $u = \tilde{\zeta}_{\gamma_n}(v)$, $Q_n^v = \phi_n + v\phi_n^*$ and the associated measure μ_n^v.*

1. *The matrix of $T_{\mu_n^v}$ with respect to $(\chi_k)_{k=0}^{n-1}$ is $\mathcal{U}^{(n;\mathrm{u})} = \tilde{\zeta}_{\mathcal{A}_n}(\mathcal{C}_n^{\mathrm{u}})$.*
2. *The zeros of Q_n^v are the eigenvalues of $\mathcal{U}^{(n;\mathrm{u})}$. If λ is a zero of Q_n^v, the related eigenvectors of $\mathcal{U}^{(n;\mathrm{u})}$ are spanned by $(\chi_0(\lambda)\ \dots\ \chi_{n-1}(\lambda))^\dagger$.*
3. *$Q_n^v = \frac{q_n^v}{\pi_n}$ with q_n^v proportional to the characteristic polynomial of $\mathcal{U}^{(n;\mathrm{u})}$.*

Proof. As in the case of Theorem 4.6, it suffices to prove item 1. For an odd $n = 2l + 1$, using (23) in a similar computation to that of (27) gives

$$\begin{aligned}\varpi_{n-2}^*\chi_{n-2} = \rho_{n-1}^+\rho_n^+\varpi_n B_{[n/2]*}^{\mathrm{e}}\frac{Q_n^v}{1+\overline{\gamma}_n v} - \rho_{n-1}^+ u\varpi_{n-1}\chi_{n-1} \\ - \overline{\gamma}_{n-2}\gamma_{n-1}\varpi_{n-2}\chi_{n-2} - \rho_{n-2}^-\gamma_{n-1}\varpi_{n-3}\chi_{n-3},\end{aligned}$$

$$\varpi_{n-1}^* \chi_{n-1} = \overline{\gamma}_{n-1} \rho_n^+ \varpi_n B_{[n/2]*}^{\mathrm{e}} \frac{Q_n^v}{1+\overline{\gamma}_n v} - \overline{\gamma}_{n-1} u \varpi_{n-1} \chi_{n-1}$$
$$+ \overline{\gamma}_{n-2} \rho_{n-1}^- \varpi_{n-2} \chi_{n-2} + \rho_{n-2}^- \rho_{n-1}^- \varpi_{n-3} \chi_{n-3}.$$

These relations, combined with the first $n-2$ equations of (28), yield

$$(\chi_0 \quad \dots \quad \chi_{n-1}) \big(\varpi_{\mathcal{A}_n}^* - \varpi_{\mathcal{A}_n} \hat{\mathcal{C}}_n^{\mathrm{u}}\big) = b_n \varpi_n B_{l*}^{\mathrm{e}} Q_n^v, \quad b_n \in \mathbb{C}^n,$$

with $\hat{\mathcal{C}}_n^{\mathrm{u}} = \eta_{\mathcal{A}_n}^{-1} \mathcal{C}_n^{\mathrm{u}} \eta_{\mathcal{A}_n} = (\mathcal{C}_n^{\mathrm{u}})_{\mathcal{A}_n}$. Thus, using (7) and (8) we find that

$$\big(\chi_0(z) \quad \dots \quad \chi_{n-1}(z)\big)\big(z - \mathcal{U}^{(n;\mathrm{u})}\big) = c_n \varpi_n(z) B_{l*}^{\mathrm{e}}(z) Q_n^v(z), \quad c_n \in \mathbb{C}^n,$$

so, $\mathcal{U}^{(n;\mathrm{u})}$ is the matrix of $T_{\mu_n^v}$ with respect to $(\chi_k)_{k=0}^{n-1}$ because $Q_n^v = 0$ in $L^2_{\mu_n^v}$.

If $n = 2l$ is even, proceeding in a similar way with (23) and (28) we get

$$\big(\chi_{0*}(z) \quad \dots \quad \chi_{n-1*}(z)\big)\big(z - \mathcal{U}^{(n;\mathrm{u})^T}\big) = c_n \varpi_n(z) B_{l*}^{\mathrm{o}}(z) Q_n^v(z), \quad c_n \in \mathbb{C}^n,$$

thus, $\mathcal{U}^{(n;\mathrm{u})^T}$ is the matrix of $T_{\mu_n^v}$ with respect to $(\chi_{k*})_{k=0}^{n-1}$. Consequently, the matrix of $T_{\mu_n^v}$ with respect to $(\chi_k)_{k=0}^{n-1}$ is $\mathcal{U}^{(n;\mathrm{u})}$. $\square$

The zeros of a PORF are eigenvalues of a pair of band matrices too. If $u = \tilde{\zeta}_{\gamma_n}(v)$, $q_n^v(z) \propto \det(\varpi_{\mathcal{A}_n}^*(z) - \varpi_{\mathcal{A}_n}(z)\mathcal{C}_n^{\mathrm{u}}) = \det(z\tilde{\varpi}_{\mathcal{A}_n}(\mathcal{C}_n^{\mathrm{u}}) - \tilde{\varpi}_{\mathcal{A}_n}^*(\mathcal{C}_n^{\mathrm{u}}))$ gives q_n^v as a determinant of a five-diagonal matrix. The zeros of Q_n^v are the eigenvalues of the five-diagonal pair $(\tilde{\varpi}_{\mathcal{A}_n}^*(\mathcal{C}_n^{\mathrm{u}}), \tilde{\varpi}_{\mathcal{A}_n}(\mathcal{C}_n^{\mathrm{u}}))$ and, given an eigenvalue λ, $(\chi_0(\lambda) \ \dots \ \chi_{n-1}(\lambda))\eta_{\mathcal{A}_n}^{-1/2}$ spans the related left eigenvectors.

We have also a factorization $\mathcal{C}_n^{\mathrm{u}} = \mathcal{C}_{\mathrm{o}n}^{\mathrm{u}} \mathcal{C}_{\mathrm{e}n}^{\mathrm{u}}$, where $\mathcal{C}_{\mathrm{o}n}^{\mathrm{u}}$ and $\mathcal{C}_{\mathrm{e}n}^{\mathrm{u}}$ are the result of substituting γ_n by u in $\mathcal{C}_{\mathrm{o}n}$ and $\mathcal{C}_{\mathrm{e}n}$, respectively. $\mathcal{C}_{\mathrm{o}n}^{\mathrm{u}}$ and $\mathcal{C}_{\mathrm{e}n}^{\mathrm{u}}$ are both unitary, so $p_n(z) \propto \det(z(\mathcal{C}_{\mathrm{e}n}^{u\dagger} + \mathcal{A}_n^\dagger \mathcal{C}_{\mathrm{o}n}^{\mathrm{u}}) - (\mathcal{C}_{\mathrm{o}n}^{\mathrm{u}} + \mathcal{A}_n \mathcal{C}_{\mathrm{e}n}^{u\dagger}))$ and the zeros of ϕ_n are the eigenvalues of the tridiagonal pair $(\mathcal{C}_{\mathrm{o}n}^{\mathrm{u}} + \mathcal{A}_n \mathcal{C}_{\mathrm{e}n}^{u\dagger}, \mathcal{C}_{\mathrm{e}n}^{u\dagger} + \mathcal{A}_n^\dagger \mathcal{C}_{\mathrm{o}n}^{\mathrm{u}})$, which has the same left eigenvectors as $(\tilde{\varpi}_{\mathcal{A}_n}^*(\mathcal{C}_n^{\mathrm{u}}), \tilde{\varpi}_{\mathcal{A}_n}(\mathcal{C}_n^{\mathrm{u}}))$.

6. Applications

In this section we will present some applications of the spectral theory previously developed for ORF on $\mathbb{T}$. We will use the results involving five-diagonal matrices due to their advantages. The corresponding spectral theory associates with each sequence of ORF a five-diagonal unitary matrix $\mathcal{C}(\boldsymbol{\gamma})$ depending on the parameters $\boldsymbol{\gamma}$ of the recurrence relation, and a diagonal matrix $\mathcal{A}(\boldsymbol{\alpha})$ depending on the sequence $\boldsymbol{\alpha}$ which defines the poles $\hat{\alpha}_n$. These band matrices keep all the information about the ORF since they generate the full sequence of ORF through the associated recurrence. The importance of these matrices is that they act as a short cut that connects directly the parameters $\boldsymbol{\gamma}, \boldsymbol{\alpha}$ to the ORF and the related orthogonality measure.

An essential difference with the polynomial case is that the matrix directly related to the ORF and the orthogonality measure is not the five-diagonal one, but an operator Möbius transform of it, namely, $\mathcal{U}(\boldsymbol{\gamma}, \boldsymbol{\alpha}) = \tilde{\zeta}_{\mathcal{A}(\boldsymbol{\alpha})}(\mathcal{C}(\boldsymbol{\gamma}))$. This introduces important difficulties when trying to apply the spectral theory in the rational case. However, in spite of these difficulties, the matrix tool $\mathcal{U}(\boldsymbol{\gamma}, \boldsymbol{\alpha})$ becomes powerful enough to deal with hard problems even in the rational case. To

understand the scope of the rational spectral theory, we will use it to solve some non-trivial problems about the relation between the behavior of the sequences $\boldsymbol{\gamma}, \boldsymbol{\alpha}$ and the properties of the corresponding orthogonality measure $\mu(\boldsymbol{\gamma}, \boldsymbol{\alpha})$. The answers to these problems are known for OP, but the generalizations to ORF are new.

The strategy will be to apply standard results of perturbation operator theory to the unitary matrix $\mathcal{U}(\boldsymbol{\gamma}, \boldsymbol{\alpha})$, comparing it with another normal matrix, eventually with the form $\mathcal{U}(\boldsymbol{\delta}, \boldsymbol{\beta})$. A useful remark for this is that, for $\boldsymbol{\beta}$ compactly supported in $\mathbb{D}$, $\mathcal{U}(\boldsymbol{\delta}, \boldsymbol{\beta})$ defines a unitary operator for any sequence $\boldsymbol{\delta}$ in $\overline{\mathbb{D}}$ since, then, $\mathcal{C}(\boldsymbol{\delta})$ is unitary. However, $\mathcal{U}(\boldsymbol{\delta}, \boldsymbol{\beta})$ only represents a multiplication operator on $\mathbb{T}$ when $\boldsymbol{\delta}$ lies on $\mathbb{D}$. When $\delta_n \in \mathbb{T}$ for some n we know that $\mathcal{C}(\boldsymbol{\delta})$ decomposes as a direct sum of an $n \times n$ and an infinite matrix (see [13,32,36]). Taking into account that $\mathcal{A}(\boldsymbol{\beta})$ is diagonal, a similar decomposition holds for $\mathcal{U}(\boldsymbol{\delta}, \boldsymbol{\beta})$.

The results of operator theory that we will apply state that two operators T, S on H have some common spectral property provided that the perturbation $T - S$ belongs to certain class of operators. We will deal with two kinds of perturbations: compact and trace class operators. Both are subsets of $\mathbb{B}_H$ that are closed under sum, left and right product by any element of $\mathbb{B}_H$ and also under the †-operation, that is, they are Hermitian ideals of $\mathbb{B}_H$. This fact is the key that permits us to use techniques of band matrices in the spectral theory of ORF, according to the following result.

Proposition 6.1. *Let $\mathfrak{I}$ be a Hermitian ideal of $\mathbb{B}_H$. If $A, B \in \mathbb{D}_H$ are normal and $AB = BA$, the condition $A - B \in \mathfrak{I}$ implies the equivalences*

$$T - S \in \mathfrak{I} \quad \Leftrightarrow \quad \zeta_A(T) - \zeta_B(S) \in \mathfrak{I} \quad \Leftrightarrow \quad \tilde{\zeta}_A(T) - \tilde{\zeta}_B(S) \in \mathfrak{I}, \quad \forall T, S \in \overline{\mathbb{D}}_H.$$

Proof. It suffices to prove the first equivalence because $\tilde{\zeta}_A = \zeta_{-A}$. The identities $T_1T_2 - S_1S_2 = (T_1 - S_1)T_2 + S_1(T_2 - S_2)$ and $T^{-1} - S^{-1} = -T^{-1}(T - S)S^{-1}$ show that $T_i, S_i \in \mathbb{B}_H$, $T_i - S_i \in \mathfrak{I} \Rightarrow T_1S_1 - T_2S_2 \in \mathfrak{I}$ and $T^{-1}, S^{-1} \in \mathbb{B}_H$, $T - S \in \mathfrak{I} \Rightarrow T^{-1} - S^{-1} \in \mathfrak{I}$. Suppose now $A, B \in \mathbb{D}_H$ normal such that $AB = BA$ and $A - B \in \mathfrak{I}$. Then $\eta_A^2 - \eta_B^2 = BB^\dagger - AA^\dagger \in \mathfrak{I}$. The functional calculus for normal operators shows that $\eta_A\eta_B = \eta_B\eta_A$, so $\eta_A - \eta_B = (\eta_A + \eta_B)^{-1}(\eta_A^2 - \eta_B^2) \in \mathfrak{I}$ since $(\eta_A + \eta_B)^{-1} \in \mathbb{B}_H$ because η_A and η_B are positive with bounded inverse. If, besides, $T, S \in \overline{\mathbb{D}}_H$ are such that $T - S \in \mathfrak{I}$, then $\varpi_A(T) - \varpi_B(S) = SB^\dagger - TA^\dagger \in \mathfrak{I}$ and $\varpi_A^*(T) - \varpi_B^*(S) = T - S + B - A \in \mathfrak{I}$. In consequence, $T - S \in \mathfrak{I} \Rightarrow \zeta_A(T) - \zeta_B(S) \in \mathfrak{I}$. Substituting in this result A, B by $-A, -B$ and T, S by $\zeta_A(T), \zeta_B(S)$, respectively, we also find the opposite inclusion. □

$\mathcal{A}(\boldsymbol{\alpha})$ is diagonal, thus we have the following immediate consequence.

Corollary 6.2. *Let $\mathfrak{I}$ be a Hermitian ideal of $\mathbb{B}_{\ell^2}$, $\boldsymbol{\gamma}, \boldsymbol{\delta}$ sequences in $\overline{\mathbb{D}}$ and $\boldsymbol{\alpha}, \boldsymbol{\beta}$ sequences compactly included in $\mathbb{D}$. If $\mathcal{A}(\boldsymbol{\alpha}) - \mathcal{A}(\boldsymbol{\beta}) \in \mathfrak{I}$, then*

$$\mathcal{C}(\boldsymbol{\gamma}) - \mathcal{C}(\boldsymbol{\delta}) \in \mathfrak{I} \quad \Leftrightarrow \quad \mathcal{U}(\boldsymbol{\gamma}, \boldsymbol{\alpha}) - \mathcal{U}(\boldsymbol{\delta}, \boldsymbol{\beta}) \in \mathfrak{I}.$$

The perturbation results that we will use are the invariance of the essential spectrum for normal operators under a compact perturbation (Weyl's theorem: see [35] and [8,31]), and the invariance of the absolutely continuous spectrum for unitary operators under a trace class perturbation (Kato–Birman theorem: see [9,24] and [10]). The compactness of an infinite band matrix is equivalent to stating that all the diagonals converge to zero. Besides, any infinite matrix $(k_{i,j})$

such that $\sum_{i,j} |k_{i,j}| < \infty$ is trace class. The essential and absolutely continuous spectrum of an operator T are denoted $\sigma_e(T)$ and $\sigma_{ac}(T)$, respectively. An important result is Krein's theorem (see [2,16]): given a unitary operator T, $\sigma_e(T) \subset \{\lambda_1, \dots, \lambda_n\}$ iff $(\lambda_1 - T) \cdots (\lambda_n - T)$ is compact.

For any measure μ on $\mathbb{T}$, $\sigma_e(T_\mu)$ is the set $\{\operatorname{supp} \mu\}'$ of limit points of $\operatorname{supp} \mu$ and $\sigma_{ac}(T_\mu)$ is the support of the absolutely continuous part μ_{ac} of μ. Concerning the compactness and trace class character of $\mathcal{U}(\boldsymbol{\gamma}, \boldsymbol{\alpha}) - \mathcal{U}(\boldsymbol{\delta}, \boldsymbol{\beta})$, it is a consequence of the same property for $\mathcal{A}(\boldsymbol{\alpha}) - \mathcal{A}(\boldsymbol{\beta})$ and $\mathcal{C}(\boldsymbol{\gamma}) - \mathcal{C}(\boldsymbol{\delta})$, as follows from Corollary 6.2. The diagonal matrix $\mathcal{A}(\boldsymbol{\alpha}) - \mathcal{A}(\boldsymbol{\beta})$ is compact iff $\lim_n (\alpha_n - \beta_n) = 0$, and is trace class iff $\sum_n |\alpha_n - \beta_n| < \infty$. Besides, the factorization $\mathcal{C}(\boldsymbol{\gamma}) = \mathcal{C}_o(\boldsymbol{\gamma})\mathcal{C}_e(\boldsymbol{\gamma})$ shows that the compactness and trace class character of $\mathcal{C}(\boldsymbol{\gamma}) - \mathcal{C}(\boldsymbol{\delta})$ is a consequence of the same property for $\mathcal{C}_o(\boldsymbol{\gamma}) - \mathcal{C}_o(\boldsymbol{\delta})$ and $\mathcal{C}_e(\boldsymbol{\gamma}) - \mathcal{C}_e(\boldsymbol{\delta})$. The compactness and trace class arguments for $\mathcal{C}(\boldsymbol{\gamma}) - \mathcal{C}(\boldsymbol{\delta})$ in the following applications are taken from [32].

As a first group of applications in the study of the dependence $\mu(\boldsymbol{\gamma}, \boldsymbol{\alpha})$, we will analyze the extreme behaviors corresponding to a sequence $\boldsymbol{\gamma}$ converging to zero or (subsequently) to the unit circle. In what follows $\operatorname{Lim}_n x_n$ means the set of limit points of a sequence $(x_n)_{n \geqslant 0}$ in $\mathbb{C}$.

Theorem 6.3. *Let $\boldsymbol{\alpha}$ be compactly included in $\mathbb{D}$.*

1. $\lim_n \gamma_n = 0 \Rightarrow \operatorname{supp} \mu(\boldsymbol{\gamma}, \boldsymbol{\alpha}) = \mathbb{T}$.
2. $\lim_n |\gamma_n| = 1 \Rightarrow \{\operatorname{supp} \mu(\boldsymbol{\gamma}, \boldsymbol{\alpha})\}' = \operatorname{Lim}_n \tilde{\zeta}_n(-\overline{\gamma}_n \gamma_{n+1})$.
3. $\limsup_n |\gamma_n| = 1 \Rightarrow \mu(\boldsymbol{\gamma}, \boldsymbol{\alpha})$ *singular.*

Proof. For any sequence $\boldsymbol{\alpha}$ in $\mathbb{D}$, the ORF corresponding to the Lebesgue measure m are given by $\phi_0 = 1$ and $\phi_n = \eta_n \frac{\varpi_0^*}{\varpi_n} B_{n-1}$ for $n \geqslant 1$, which satisfy recurrence (3) with $\gamma_n = 0$. Therefore, when $\boldsymbol{\alpha}$ is compactly supported in $\mathbb{D}$, $\mathcal{U}(0, \boldsymbol{\alpha})$ is a representation of T_m. So, $\sigma(\mathcal{U}(0, \boldsymbol{\alpha})) = \operatorname{supp} m = \mathbb{T}$.

Now, suppose a sequence $\boldsymbol{\gamma}$ in $\mathbb{D}$ such that $\lim_n \gamma_n = 0$. Then $\mathcal{C}(\boldsymbol{\gamma}) - \mathcal{C}(0)$ is compact, thus $\mathcal{U}(\boldsymbol{\gamma}, \boldsymbol{\alpha}) - \mathcal{U}(0, \boldsymbol{\alpha})$ is compact too. Hence, Weyl's theorem implies $\{\operatorname{supp} \mu(\boldsymbol{\gamma}, \boldsymbol{\alpha})\}' = \{\operatorname{supp} m\}' = \mathbb{T}$, that is, $\operatorname{supp} \mu(\boldsymbol{\gamma}, \boldsymbol{\alpha}) = \mathbb{T}$.

If $\lim_n |\gamma_n| = 1$, $\mathcal{C}(\boldsymbol{\gamma}) - \mathcal{D}(\boldsymbol{\gamma})$ is compact, where

$$\mathcal{D}(\boldsymbol{\gamma}) = \begin{pmatrix} -\gamma_1 & & & \\ & -\overline{\gamma}_1 \gamma_2 & & \\ & & -\overline{\gamma}_2 \gamma_3 & \\ & & & \ddots \end{pmatrix}.$$

Therefore, $\mathcal{U}(\boldsymbol{\gamma}, \boldsymbol{\alpha}) - \tilde{\zeta}_{\mathcal{A}(\boldsymbol{\alpha})}(\mathcal{D}(\boldsymbol{\gamma}))$ is compact too, and Weyl's theorem states that $\{\operatorname{supp} \mu(\boldsymbol{\gamma}, \boldsymbol{\alpha})\}' = \operatorname{Lim}_n \tilde{\zeta}_n(-\overline{\gamma}_n \gamma_{n+1})$.

Finally, assume that $\limsup_n |\gamma_n| = 1$. Then, there is a subsequence $(\gamma_n)_{n \in \mathcal{I}}$ such that $\lim_{n \in \mathcal{I}} \gamma_n = \gamma \in \mathbb{T}$. Without loss of generality we can suppose $\sum_{n \in \mathcal{I}} |\gamma_n - \gamma|^{1/2} < \infty$, so that $\sum_{n \in \mathcal{I}} (|\gamma_n - \gamma| + \rho_n) < \infty$ because $\rho_n \leqslant \sqrt{2|\gamma_n - \gamma|}$. Let $\boldsymbol{\delta}$ be defined by $\delta_n = \gamma$ if $n \in \mathcal{I}$ and $\delta_n = \gamma_n$ if $n \notin \mathcal{I}$. The condition $\sum_{n \in \mathcal{I}} (|\gamma - \gamma_n| + \rho_n) < \infty$ ensures that $\mathcal{C}_o(\boldsymbol{\gamma}) - \mathcal{C}_o(\boldsymbol{\delta})$ and $\mathcal{C}_e(\boldsymbol{\gamma}) - \mathcal{C}_e(\boldsymbol{\delta})$ are trace class, so the same holds for $\mathcal{U}(\boldsymbol{\gamma}, \boldsymbol{\alpha}) - \mathcal{U}(\boldsymbol{\delta}, \boldsymbol{\alpha})$. The Birman–Krein theorem states that $\operatorname{supp} \mu_{ac}(\boldsymbol{\gamma}, \boldsymbol{\alpha}) = \sigma_{ac}(\mathcal{U}(\boldsymbol{\delta}, \boldsymbol{\alpha}))$. $\delta_n \in \mathbb{T}$ for infinitely many values of n, so $\mathcal{U}(\boldsymbol{\delta}, \boldsymbol{\alpha})$ decomposes as a direct sum of finite matrices. Therefore, $\sigma_{ac}(\mathcal{U}(\boldsymbol{\delta}, \boldsymbol{\alpha})) = \emptyset$. $\square$

We can also obtain general conditions for the invariance of $\{\operatorname{supp}\mu(\boldsymbol{\gamma},\boldsymbol{\alpha})\}'$ and $\operatorname{supp}\mu_{\mathrm{ac}}(\boldsymbol{\gamma},\boldsymbol{\alpha})$.

Theorem 6.4. *Let $\boldsymbol{\alpha}$ be compactly included in $\mathbb{D}$.*

1. $\lim_n(\alpha_n-\beta_n)=\lim_n(\gamma_n-\delta_n)=0 \;\Rightarrow\; \{\operatorname{supp}\mu(\boldsymbol{\gamma},\boldsymbol{\alpha})\}'=\{\operatorname{supp}\mu(\boldsymbol{\delta},\boldsymbol{\beta})\}'$.
2. $\sum_n(|\alpha_n-\beta_n|+|\gamma_n-\delta_n|)<\infty \;\Rightarrow\; \operatorname{supp}\mu_{\mathrm{ac}}(\boldsymbol{\gamma},\boldsymbol{\alpha})=\operatorname{supp}\mu_{\mathrm{ac}}(\boldsymbol{\delta},\boldsymbol{\beta})$.
3. *Let $\delta_n=\lambda_n\gamma_n$, $\lambda_n\in\mathbb{C}$. Then*

$$\lim_n|\lambda_n|=\lim_n\lambda_{n+1}\bar{\lambda}_n=1 \quad\Rightarrow\quad \bigl\{\operatorname{supp}\mu(\boldsymbol{\gamma},\boldsymbol{\alpha})\bigr\}'=\bigl\{\operatorname{supp}\mu(\boldsymbol{\delta},\boldsymbol{\alpha})\bigr\}',$$

$$\sum_n\bigl(\bigl||\lambda_n|^2-1\bigr|+|\lambda_{n+1}\bar{\lambda}_n-1|\bigr)<\infty \quad\Rightarrow\quad \operatorname{supp}\mu_{\mathrm{ac}}(\boldsymbol{\gamma},\boldsymbol{\alpha})=\operatorname{supp}\mu_{\mathrm{ac}}(\boldsymbol{\delta},\boldsymbol{\alpha}).$$

4. $\beta_n=\alpha_{n+N},\ \delta_n=\gamma_{n+N} \;\Rightarrow\; \begin{cases}\{\operatorname{supp}\mu(\boldsymbol{\gamma},\boldsymbol{\alpha})\}'=\{\operatorname{supp}\mu(\boldsymbol{\delta},\boldsymbol{\beta})\}',\\ \operatorname{supp}\mu_{\mathrm{ac}}(\boldsymbol{\gamma},\boldsymbol{\alpha})=\operatorname{supp}\mu_{\mathrm{ac}}(\boldsymbol{\delta},\boldsymbol{\beta}).\end{cases}$

Proof. First, notice that any of the hypothesis of the theorem ensures that $\boldsymbol{\beta}$ is compactly included in $\mathbb{D}$ when $\boldsymbol{\alpha}$ satisfies the same property.

The first two properties follow from the fact that $\mathcal{C}(\boldsymbol{\gamma})-\mathcal{C}(\boldsymbol{\delta})$ is compact when $\lim_n(\gamma_n-\delta_n)=0$ and trace class when $\sum_n|\gamma_n-\delta_n|<\infty$ (see [32, Theorems 4.3.5 and 4.3.6]).

Consider $\delta_n=\lambda_n\gamma_n$ with $\lim_n|\lambda_n|=\lim_n\lambda_{n+1}\bar{\lambda}_n=1$. We can write $\lambda_n=|\lambda_n|e^{i\theta_n}$ with $\theta_n\in[\theta_{n-1}-\pi,\theta_{n-1}+\pi)$, so that $\lim_n|\theta_{n+1}-\theta_n|=0$. Define

$$U=\begin{pmatrix} u_1 &&&& \\ & \bar{u}_2 &&& \\ && u_3 && \\ &&& \bar{u}_4 & \\ &&&& \ddots \end{pmatrix}, \qquad u_n=e^{i\theta_n/2}.$$

$U\mathcal{C}(\boldsymbol{\gamma})U^{\dagger}-\mathcal{C}(\boldsymbol{\delta})$ is compact (see [32, Theorem 4.3.8]), hence $U\mathcal{U}(\boldsymbol{\gamma},\boldsymbol{\alpha})U^{\dagger}-\mathcal{U}(\boldsymbol{\delta},\boldsymbol{\alpha})=\zeta_{\mathcal{A}(\boldsymbol{\alpha})}(U\mathcal{C}(\boldsymbol{\gamma})U^{\dagger})-\zeta_{\mathcal{A}(\boldsymbol{\alpha})}(\mathcal{C}(\boldsymbol{\delta}))$ is compact too, which proves the first part of item 3. Besides, $\sum_n(||\lambda_n|^2-1|+|\lambda_{n+1}\bar{\lambda}_n-1|)<\infty$ ensures that $U\mathcal{C}(\boldsymbol{\gamma})U^{\dagger}-\mathcal{C}(\boldsymbol{\delta})$ is trace class (see [32, Theorem 4.3.9]), which similarly proves the second part of item 3.

Finally, let $\delta_n=\gamma_{n+N}$ and $\beta_n=\alpha_{n+N}$ for some $N\in\mathbb{N}$. Consider the sequences $\tilde{\boldsymbol{\gamma}}$ and $\tilde{\boldsymbol{\alpha}}$ given by

$$\tilde{\gamma}_n=\begin{cases}1 & \text{if } n\leqslant N,\\ \gamma_n & \text{if } n>N,\end{cases}\qquad \tilde{\alpha}_n=\begin{cases}0 & \text{if } n\leqslant N,\\ \alpha_n & \text{if } n>N.\end{cases}$$

$\mathcal{A}(\boldsymbol{\alpha})-\mathcal{A}(\tilde{\boldsymbol{\alpha}})$, $\mathcal{C}_{\mathrm{o}}(\boldsymbol{\gamma})-\mathcal{C}_{\mathrm{o}}(\tilde{\boldsymbol{\gamma}})$ and $\mathcal{C}_{\mathrm{e}}(\boldsymbol{\gamma})-\mathcal{C}_{\mathrm{e}}(\tilde{\boldsymbol{\gamma}})$ are finite rank, therefore $\mathcal{U}(\boldsymbol{\gamma},\boldsymbol{\alpha})-\mathcal{U}(\tilde{\boldsymbol{\gamma}},\tilde{\boldsymbol{\alpha}})$ is compact and trace class. Besides, we have the decomposition $\mathcal{U}(\tilde{\boldsymbol{\gamma}},\tilde{\boldsymbol{\alpha}})=-I_N\oplus\mathcal{U}(\boldsymbol{\delta},\boldsymbol{\beta})$, so $\mathcal{U}(\tilde{\boldsymbol{\gamma}},\tilde{\boldsymbol{\alpha}})$ and $\mathcal{U}(\boldsymbol{\delta},\boldsymbol{\beta})$ have the same essential and absolutely continuous spectrum, which proves item 4. $\square$

Combining the different results of the previous theorem we can obtain a more general one.

Theorem 6.5. *Let $\boldsymbol{\alpha}$ be compactly included in $\mathbb{D}$.*

1. *If* $\lim_n(\alpha_{n+N}-\beta_n)=\lim_n(\lambda_n\gamma_{n+N}-\delta_n)=0$, $\lim_n|\lambda_n|=\lim_n\lambda_{n+1}\bar{\lambda}_n=1$, *then*
$$\{\operatorname{supp}\mu(\boldsymbol{\gamma},\boldsymbol{\alpha})\}'=\{\operatorname{supp}\mu(\boldsymbol{\delta},\boldsymbol{\beta})\}'.$$
2. *If* $\sum_n(|\alpha_{n+N}-\beta_n|+|\lambda_n\gamma_{n+N}-\delta_n|+||\lambda_n|^2-1|+|\lambda_{n+1}\bar{\lambda}_n-1|)<\infty$, *then*
$$\operatorname{supp}\mu_{\mathrm{ac}}(\boldsymbol{\gamma},\boldsymbol{\alpha})=\operatorname{supp}\mu_{\mathrm{ac}}(\boldsymbol{\delta},\boldsymbol{\beta}).$$

A particular case of this theorem is worthwhile to be emphasized.

Corollary 6.6. *Let $\alpha\in\mathbb{D}$, $r\in[0,1]$, $\lambda\in\mathbb{T}$, $\Gamma_{\lambda,r}=\{\lambda e^{i\theta}\colon |\theta|<2\arcsin r\}$.*

1. *If* $\lim_n\alpha_n=\alpha$, $\lim_n|\gamma_n|=r$ *and* $\lim_n\frac{\gamma_{n+1}}{\gamma_n}=\lambda$, *then*
$$\{\operatorname{supp}\mu(\boldsymbol{\gamma},\boldsymbol{\alpha})\}'=\mathbb{T}\setminus\tilde{\zeta}_\alpha(\Gamma_{\lambda,r}).$$
2. *If* $\sum_n(|\alpha_n-\alpha|+||\gamma_n|-r|+|\frac{\gamma_{n+1}}{\gamma_n}-\lambda|)<\infty$, *then*
$$\operatorname{supp}\mu_{\mathrm{ac}}(\boldsymbol{\gamma},\boldsymbol{\alpha})=\mathbb{T}\setminus\tilde{\zeta}_\alpha(\Gamma_{\lambda,r}).$$

Proof. Let us write $\gamma_n=|\gamma_n|v_n$, $v_n\in\mathbb{T}$. Notice that $\boldsymbol{\alpha}$ is compactly included in $\mathbb{D}$ because it is convergent in $\mathbb{D}$. Applying Theorem 6.5 to $\mu(\boldsymbol{\gamma},\boldsymbol{\alpha})$ and $\mu(\boldsymbol{\delta},\boldsymbol{\beta})$, with $\beta_n=\alpha$, $\delta_n=\lambda^n r$ and $\lambda_n=\lambda^n\bar{v}_n$, we find that $\{\operatorname{supp}\mu(\boldsymbol{\gamma},\boldsymbol{\alpha})\}'=\{\operatorname{supp}\mu(\boldsymbol{\delta},\boldsymbol{\beta})\}'$ under the assumptions of item 1, and $\operatorname{supp}\mu_{\mathrm{ac}}(\boldsymbol{\gamma},\boldsymbol{\alpha})=\operatorname{supp}\mu_{\mathrm{ac}}(\boldsymbol{\delta},\boldsymbol{\beta})$ under the hypothesis of item 2. On the other hand, $\mu(\boldsymbol{\delta},\boldsymbol{\beta})=\nu_\alpha$, where $\nu=\mu(\boldsymbol{\delta},0)$ is the measure on $\mathbb{T}$ whose OP have parameters $\lambda^n r$ and ν_α is defined by $\nu_\alpha(\cdot)=\nu(\zeta_\alpha(\cdot))$. Therefore, $\{\operatorname{supp}\nu_\alpha\}'=\tilde{\zeta}_\alpha(\{\operatorname{supp}\nu\}')$, $\operatorname{supp}(\nu_\alpha)_{\mathrm{ac}}=\tilde{\zeta}_\alpha(\operatorname{supp}\nu_{\mathrm{ac}})$ and the corollary follows from the well-known result $\{\operatorname{supp}\nu\}'=\operatorname{supp}\nu_{\mathrm{ac}}=\mathbb{T}\setminus\Gamma_{\lambda,r}$ (see [7]). □

Corollary 6.6 of Theorem 6.5 can be understood also as an example of the following general result. It says that, when $\boldsymbol{\alpha}$ is convergent in $\mathbb{D}$, the analysis of $\{\operatorname{supp}\mu(\boldsymbol{\gamma},\boldsymbol{\alpha})\}'$ and $\operatorname{supp}\mu_{\mathrm{ac}}(\boldsymbol{\gamma},\boldsymbol{\alpha})$ can be related to the much more known polynomial case, corresponding to $\boldsymbol{\alpha}=0$.

Theorem 6.7. *Let $\alpha\in\mathbb{D}$.*

1. $\lim_n\alpha_n=\alpha \ \Rightarrow\ \{\operatorname{supp}\mu(\boldsymbol{\gamma},\boldsymbol{\alpha})\}'=\tilde{\zeta}_\alpha(\{\operatorname{supp}\mu(\boldsymbol{\gamma},0)\}')$.
2. $\sum_n|\alpha_n-\alpha|<\infty \ \Rightarrow\ \operatorname{supp}\mu_{\mathrm{ac}}(\boldsymbol{\gamma},\boldsymbol{\alpha})=\tilde{\zeta}_\alpha(\operatorname{supp}\mu_{\mathrm{ac}}(\boldsymbol{\gamma},0))$.

Proof. Again, $\boldsymbol{\alpha}$ is compactly included in $\mathbb{D}$ because it is convergent in $\mathbb{D}$. So, if $\beta_n=\alpha$, Theorem 6.4 implies that $\{\operatorname{supp}\mu(\boldsymbol{\gamma},\boldsymbol{\alpha})\}'=\{\operatorname{supp}\mu(\boldsymbol{\gamma},\boldsymbol{\beta})\}'$ when $\lim_n\alpha_n=\alpha$, and $\operatorname{supp}\mu_{\mathrm{ac}}(\boldsymbol{\gamma},\boldsymbol{\alpha})=\operatorname{supp}\mu_{\mathrm{ac}}(\boldsymbol{\gamma},\boldsymbol{\beta})$ when $\sum_n|\alpha_n-\alpha|<\infty$. Following the notation in the proof of Corollary 6.6, $\mu(\boldsymbol{\gamma},\boldsymbol{\beta})=\nu_\alpha$ with $\nu=\mu(\boldsymbol{\gamma},0)$. The result follows from the relation between ν and ν_α. □

The importance of the above theorem is due to the numerous known results for the relation between μ and $\boldsymbol{\gamma}$ in the case of OP on $\mathbb{T}$. Theorem 6.7 permits us to translate some of these results to those ORF on $\mathbb{T}$ whose poles converge in $\mathbb{E}$. For instance, Corollary 6.6.1 can be understood

as the translation to this kind of ORF of a result for OP on $\mathbb{T}$ due to Barrios and López (see [7]). This result was generalized later on in [28] as an improvement of a partial extension appearing in [14]. The corresponding translation of this generalization to ORF states that Corollary 6.6.1 holds even if we substitute the condition $\lim_n |\gamma_n| = r$ by the more general one $\liminf_n |\gamma_n| = r$.

All the above results provide only sufficient conditions on the sequences $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ to ensure a certain property for the measure $\mu(\boldsymbol{\gamma}, \boldsymbol{\alpha})$. On the contrary, Krein's theorem permits us to characterize exactly those measures $\mu(\boldsymbol{\gamma}, \boldsymbol{\alpha})$ with a fixed finite set $\{\operatorname{supp} \mu(\boldsymbol{\gamma}, \boldsymbol{\alpha})\}'$. The characterization is in terms of the compactness of a matrix depending on $\boldsymbol{\gamma}$ and $\boldsymbol{\alpha}$. The fact that, contrary to the polynomial case, this matrix is not banded makes difficult to translate its compactness into equivalent conditions for the sequences $\boldsymbol{\gamma}$ and $\boldsymbol{\alpha}$. Nevertheless, in the case of $\{\operatorname{supp} \mu(\boldsymbol{\gamma}, \boldsymbol{\alpha})\}'$ with at most two points we can find explicitly such equivalent conditions.

Theorem 6.8. *Let $\boldsymbol{\alpha}$ be compactly included in $\mathbb{D}$ and $\lambda_1, \lambda_2 \in \mathbb{T}$.*

1. $\{\operatorname{supp} \mu(\boldsymbol{\gamma}, \boldsymbol{\alpha})\}' = \{\lambda_1\}$ *iff* $\lim_n \tilde{\zeta}_n(-\overline{\gamma}_n \gamma_{n+1}) = \lambda_1$.
2. $\{\operatorname{supp} \mu(\boldsymbol{\gamma}, \boldsymbol{\alpha})\}' \subset \{\lambda_1, \lambda_2\}$ *iff*

$$\begin{aligned}
&\lim_n \rho_n \rho_{n+1} = 0,\\
&\lim_n \rho_n \left(\frac{\varpi_n(\lambda_2)}{\varpi_n(\alpha_n)} k_n(\lambda_1) - \frac{\varpi_{n-1}^*(\lambda_1)}{\varpi_{n-1}(\alpha_{n-1})} k_{n-1}(\lambda_2) \right) = 0,\\
&\lim_n \Big(\overline{k_n(\lambda_2)} k_n(\lambda_1) + \big(\rho_n^-\big)^2 \overline{\varpi_{n-1}^*(\lambda_2)} \varpi_{n-1}^*(\lambda_1) + \big(\rho_{n+1}^+\big)^2 \overline{\varpi_{n+1}(\lambda_2)} \varpi_{n+1}(\lambda_1) \Big) = 0,
\end{aligned}$$

where $k_n(z) = \gamma_n \varpi_n^*(z) + \gamma_{n+1} \varpi_n(z)$.

Proof. From Krein's theorem, $\{\operatorname{supp} \mu\}' = \{\lambda_1\}$ iff $\lambda_1 - \mathcal{U}$ is compact. (9) yields $\lambda_1 - \mathcal{U} = \lambda_1 - \tilde{\zeta}_{\mathcal{A}}(\mathcal{C}) = \eta_{\mathcal{A}}^{-1} \varpi_{\mathcal{A}}(\lambda_1)(\zeta_{\mathcal{A}}(\lambda_1) - \mathcal{C}) \tilde{\varpi}_{\mathcal{A}}(\mathcal{C})^{-1} \eta_{\mathcal{A}}$. Bearing in mind that $\eta_{\mathcal{A}}$, $\varpi_{\mathcal{A}}(\lambda_1)$ and $\tilde{\varpi}_{\mathcal{A}}(\mathcal{C})$ are bounded with bounded inverse, the above expression shows that the compactness of $\lambda_1 - \mathcal{U}$ is equivalent to the compactness of $\zeta_{\mathcal{A}}(\lambda_1) - \mathcal{C}$. On the other hand, $\zeta_{\mathcal{A}}(\lambda_1) - \mathcal{C}$ is compact iff $\lim_n \rho_n = 0$ and $\lim_n(\zeta_n(\lambda_1) + \overline{\gamma}_n \gamma_{n+1}) = 0$. However, the first of these conditions is a consequence of the second one because $|\zeta_n(\lambda_1) + \overline{\gamma}_n \gamma_{n+1}| \geqslant 1 - |\gamma_n|$ since $\lambda_1 \in \mathbb{T}$. Also, taking into account (9), $\varpi_n(\lambda_1)(\zeta_n(\lambda_1) + \overline{\gamma}_n \gamma_{n+1}) = (\lambda_1 - \tilde{\zeta}_n(-\overline{\gamma}_n \gamma_{n+1})) \tilde{\varpi}_n(-\overline{\gamma}_n \gamma_{n+1})$. Therefore, $\lim_n(\zeta_n(\lambda_1) + \overline{\gamma}_n \gamma_{n+1}) = 0$ iff $\lim_n(\lambda_1 - \tilde{\zeta}_n(-\overline{\gamma}_n \gamma_{n+1})) = 0$ because $2 > |\varpi_n(\lambda)|$, $|\tilde{\varpi}_n(-\overline{\gamma}_n \gamma_{n+1})| \geqslant 1 - |\alpha_n|$ and $\boldsymbol{\alpha}$ is compactly supported in $\mathbb{D}$.

As for the case of two limit points, from Krein's theorem, $\{\operatorname{supp} \mu\}' \subset \{\lambda_1, \lambda_2\}$ iff $(\lambda_2 - \mathcal{U}) \cdot (\lambda_1 - \mathcal{U})$ is compact. To express this condition as the compactness of a band matrix we use the previous expression for the factor $\lambda_1 - \mathcal{U}$, but for $\lambda_2 - \mathcal{U}$ we use the equality $\lambda_2 - \mathcal{U} = \lambda_2 - \zeta_{-\mathcal{A}}(\mathcal{C}) = \eta_{\mathcal{A}} \varpi_{-\mathcal{A}}(\mathcal{C})^{-1}(\zeta_{\mathcal{A}}(\lambda_2) - \mathcal{C}) \tilde{\varpi}_{-\mathcal{A}}(\lambda_2) \eta_{\mathcal{A}}^{-1}$, obtained from (9) and the identity $\tilde{\zeta}_{\mathcal{A}} = \zeta_{-\mathcal{A}}$. We find that $(\lambda_2 - \mathcal{U})(\lambda_1 - \mathcal{U})$ is compact iff the 9-diagonal matrix $(\zeta_{\mathcal{A}}(\lambda_2) - \mathcal{C}) \varpi_{\mathcal{A}}(\lambda_2) \varpi_{\mathcal{A}}(\mathcal{A})^{-1} \varpi_{\mathcal{A}}(\lambda_1)(\zeta_{\mathcal{A}}(\lambda_1) - \mathcal{C})$ is compact. This compactness condition can be equivalently formulated using a simpler band matrix obtained multiplying the above one on the left and the right by the unitary matrices $\mathcal{C}_{\mathrm{o}}^{\dagger}$ and $\mathcal{C}_{\mathrm{e}}^{\dagger}$, respectively. Taking into account the identity $\varpi_{\mathcal{A}}^*(z) = z \varpi_{\mathcal{A}}(z)^{\dagger}$, $z \in \mathbb{T}$, we find in this way that $\{\operatorname{supp} \mu\}' \subset \{\lambda_1, \lambda_2\}$ iff the five-diagonal matrix $K(\lambda_2)^{\dagger} \varpi_{\mathcal{A}}(\mathcal{A})^{-1} K(\lambda_1)$ is compact, where $K(z) = \varpi_{\mathcal{A}}^*(z) \mathcal{C}_{\mathrm{e}}^{\dagger} - \varpi_{\mathcal{A}}(z) \mathcal{C}_{\mathrm{o}}$. Now, it is just a matter of calculating the diagonals of $K(\lambda_2)^{\dagger} \varpi_{\mathcal{A}}(\mathcal{A})^{-1} K(\lambda_1)$ to obtain the conditions given in the theorem. $\square$

The implication $\lim_n \tilde{\zeta}_n(-\overline{\gamma}_n\gamma_{n+1}) = \lambda_1 \in \mathbb{T} \Rightarrow \{\operatorname{supp}\mu\}' = \{\lambda_1\}$ was in fact a consequence of Theorem 6.3.2. Krein's theorem adds the opposite implication. Concerning the case of two limit points notice that, although the third condition is symmetric under the exchange of λ_1 and λ_2, the second one does not show explicitly such a symmetry. However, a detailed analysis of the second condition reveals that it is symmetric too.

It seems that there is no simple way to generalize the arguments given in the proof of Theorem 6.8 to the case of more than two limit points. The reason is that, for $n \geqslant 3$, the identities for the Möbius transformations are not enough to reduce the compactness of $(\lambda_1 - \mathcal{U}) \cdots (\lambda_n - \mathcal{U})$ to the compactness of a band matrix. So, contrary to the polynomial situation (see [16] and [13,28]), the characterization in terms of the sequences $\boldsymbol{\gamma}$ and $\boldsymbol{\alpha}$ of those measures on $\mathbb{T}$ whose support has a finite set of more than two limit points remains as an open problem in the rational case.

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Appendix A. ORF on the real line

In what follows, a measure on the real line will be probability Borel measure μ supported on an infinite subset $\operatorname{supp}\mu$ of $\overline{\mathbb{R}}$. When ∞ is not a mass point of μ we will refer to μ as a measure on $\mathbb{R}$. Notice that we are considering all these measures as measures on $\overline{\mathbb{R}}$, no matter whether they have a mass point at ∞ or not. This means that $\infty \in \operatorname{supp}\mu$ when ∞ is a mass point of μ or when μ is a measure on $\mathbb{R}$ with unbounded standard support, so that $\operatorname{supp}\mu$ is always closed in $\overline{\mathbb{R}}$.

Analogously to the case of the unit circle, for any measure μ on the real line it is possible to consider ORF in L^2_μ with poles in the lower half plane $\mathbb{L} = \{z \in \mathbb{C}\colon \operatorname{Im}(z) < 0\}$. For this purpose we introduce for any $\alpha \in \mathbb{U} = \{z \in \mathbb{C}\colon \operatorname{Im}(z) > 0\}$ the linear fractional transformation

$$\zeta_\alpha(z) = \frac{\varpi^*_\alpha(z)}{\varpi_\alpha(z)}, \qquad \begin{cases} \varpi_\alpha(z) = z - \bar{\alpha}, \\ \varpi^*_\alpha(z) = z - \alpha, \end{cases}$$

which maps $\overline{\mathbb{R}}$, $\mathbb{U}$ and $\mathbb{L}$ onto $\mathbb{T}$, $\mathbb{D}$ and $\mathbb{E}$, respectively, and has the inverse

$$\tilde{\zeta}_\alpha(z) = \frac{\tilde{\varpi}^*_\alpha(z)}{\tilde{\varpi}_\alpha(z)}, \qquad \begin{cases} \tilde{\varpi}_\alpha(z) = 1 - z, \\ \tilde{\varpi}^*_\alpha(z) = \alpha - \bar{\alpha} z. \end{cases}$$

Notice that $\varpi^*_\alpha = \varpi_{\alpha *}$, where the $*$-involution is now defined by $f_*(z) = \overline{f(\bar{z})}$, but nothing similar holds for $\tilde{\varpi}^*_\alpha$. Besides, for the distinguished value $\alpha_0 = i$, $\zeta = \zeta_{\alpha_0}$ is the Cayley transform and $\tilde{\zeta} = \tilde{\zeta}_{\alpha_0}$ its inverse.

Any sequence $\boldsymbol{\alpha}$ in $\mathbb{U}$ defines the products $(B_n)_{n\geqslant 0}$ as in (1), but with the new meaning for ζ_{α_n}. The orthonormalization in L^2_μ of $(B_n)_{n\geqslant 0}$ leads to a sequence $(\phi_n)_{n\geqslant 0}$ of ORF with respect to

μ with poles in $(\bar{\alpha}_n)_{n\geqslant 1}$, which will be called a sequence of ORF on the real line. The study of ORF on the real line can be carried out in a completely analogous way to the case of the unit circle, so most of the results described for the last ones translate directly to the first ones with an obvious change of the meaning in the notations. In particular, the sequence $(\phi_n)_{n\geqslant 0}$ can be chosen such that it satisfies a recurrence like (3) depending on a sequence $\boldsymbol{\gamma}$ in $\mathbb{D}$. This establishes a surjective application $\mathcal{S}_{\boldsymbol{\alpha}} : \mathfrak{P} \to \mathbb{D}^\infty$, where $\mathfrak{P}$ means now the set of probability measures on $\overline{\mathbb{R}}$. This application is a bijection when $B = \prod_{n=1}^{\infty} \zeta_n$ diverges to zero in $\mathbb{U}$, but this is equivalent now to $\sum_{n=1}^{\infty} \operatorname{Im} \alpha_n/(1+|\alpha_n|^2) = \infty$.

Following the same strategy as in the case of the unit circle, we can develop a spectral theory for ORF on the real line. The starting point is again (5), but now the factors η_α, $\alpha \in \mathbb{U}$, are defined by $\eta_\alpha = (\varpi_\alpha(\alpha)/2i)^{1/2} = \sqrt{\operatorname{Im}\alpha}$. Both, the expressions for the unit circle and the real line, can be combined in $\eta_\alpha = (\varpi_\alpha(\alpha)/\varpi_{\alpha_0}(\alpha_0))^{1/2}$. The form (5) of the recurrence is the key tool to obtain the matrix representation with respect to the ORF for the multiplication operator T_μ, where μ is the corresponding orthogonality measure on the real line. If $\operatorname{supp}\mu$ is bounded, T_μ is an everywhere defined self-adjoint operator on L^2_μ. In general, T_μ is a densely defined self-adjoint operator on L^2_μ when the function z is finite μ-a.e. (see [29, p. 259]), i.e., when ∞ is not a mass point of μ. In this case, $\sigma_p(T_\mu) = \{\text{mass points of } \mu\}$ and $\sigma(T_\mu) = \operatorname{supp}\mu$ under the convention that $\infty \in \sigma(T_\mu)$ when T_μ has an unbounded standard spectrum. A way to deal with measures with a mass point at ∞ is to work with the operator multiplication by ζ in L^2_μ

$$\begin{aligned} S_\mu : L^2_\mu &\to L^2_\mu \\ f &\to \zeta f. \end{aligned}$$

This operator is unitary for any measure μ on $\overline{\mathbb{R}}$ and verifies the identities $\sigma_p(S_\mu) = \zeta(\text{mass points of } \mu)$ and $\sigma(S_\mu) = \zeta(\operatorname{supp}\mu)$. The matrix representations of T_μ and S_μ with respect to the corresponding ORF are related to the operator analogs of the new linear fractional transformations ζ_α.

Let us introduce the notations $\operatorname{Re} T = \frac{1}{2}(T + T^\dagger)$ and $\operatorname{Im} T = \frac{1}{2i}(T - T^\dagger)$ for an operator T on a Hilbert space H. The operator linear fractional transformations of interest for ORF on the real line are

$$\zeta_A(T) = \eta_A \varpi_A(T)^{-1} \varpi_A^*(T) \eta_A^{-1}, \qquad \begin{cases} \varpi_A(T) = T - A^\dagger, \\ \varpi_A^*(T) = T - A, \end{cases}$$

$$\tilde{\zeta}_A(T) = \eta_A^{-1} \tilde{\varpi}_A^*(T) \tilde{\varpi}_A(T)^{-1} \eta_A, \qquad \begin{cases} \tilde{\varpi}_A(T) = 1 - T, \\ \tilde{\varpi}_A^*(T) = \eta_A A \eta_A^{-1} - \eta_A A^\dagger \eta_A^{-1} T, \end{cases}$$

where $\eta_A = (\varpi_A(A)/2i)^{1/2} = \sqrt{\operatorname{Im} A}$ and $A \in \mathbb{B}_H$ is such that $\operatorname{Im} A \geqslant \varepsilon$ for some positive number ε (in short, $\operatorname{Im} A > 0$), so that η_A is bounded with bounded inverse. When A is normal, as it is the case related to ORF on the real line, $\tilde{\varpi}_A^*(T) = A - A^\dagger T$.

ζ_A is the composition of the Cayley transform ζ with an operator transformation depending on A which maps onto theirselves $\overline{\mathbb{U}}_H = \{T \in \mathbb{B}_H\colon \operatorname{Im} T \geqslant 0\}$, $\mathbb{U}_H = \{T \in \mathbb{B}_H\colon \operatorname{Im} T > 0\}$ and the set of self-adjoint operators on H. More precisely, taking into account that $\eta_A^{-1} \varpi_A(T) \eta_A^{-1} = \eta_A^{-1}(T - \operatorname{Re} A)\eta_A^{-1} + i$ and $\eta_A^{-1} \varpi_A^*(T) \eta_A^{-1} = \eta_A^{-1}(T - \operatorname{Re} A)\eta_A^{-1} - i$, we obtain

$$\zeta_A(T) = \zeta\big(\eta_A^{-1}(T - \operatorname{Re} A)\eta_A^{-1}\big). \tag{A.1}$$

Due to the properties of the Cayley transform, ζ_A is a bijection of $\overline{\mathbb{U}}_H$ onto $\{T \in \overline{\mathbb{D}}_H\colon 1 \notin \sigma(T)\}$ which maps $\mathbb{U}_H$ onto $\mathbb{D}_H$. Furthermore, ζ_A is also a bijection between the set of (bounded or unbounded) self-adjoint operators and the set of unitary operators whose point spectrum does not contain 1. $\tilde{\zeta}_A$ is the inverse of ζ_A. Relation (A.1) expresses ζ_A as a product of two commutative factors. This provides an alternative representation of ζ_A, namely, $\zeta_A(T) = \eta_A^{-1}\varpi_A^*(T)\varpi_A(T)^{-1}\eta_A$, giving rise to another expression for $\tilde{\zeta}_A$ too. From the above result we get $\zeta_A(T)^\dagger = \zeta_{A^\dagger}(T^\dagger)$ and $\tilde{\zeta}_A(T)^\dagger = \tilde{\zeta}_{A^\dagger}(T^\dagger)$, as in the case of the unit circle.

Finally, if $\mathfrak{I}$ is a Hermitian ideal of $\mathbb{B}_H$, similar arguments to those given in the proof of Theorem 6.1 prove that, for any normal operators $A, B \in \mathbb{U}_H$ such that $AB = BA$, the condition $A - B \in \mathfrak{I}$ implies

$$\begin{aligned} T - S \in \mathfrak{I} \quad &\Leftarrow \quad \zeta_A(T) - \zeta_B(S) \in \mathfrak{I}, \quad \forall T, S \text{ self-adjoint}, \\ T - S \in \mathfrak{I} \quad &\Rightarrow \quad \tilde{\zeta}_A(T) - \tilde{\zeta}_B(S) \in \mathfrak{I}, \quad \forall T, S \text{ unitary},\ 1 \notin \sigma_p(T) \cup \sigma_p(S). \end{aligned}$$

Both implications are equivalent because ζ_A and $\tilde{\zeta}_A$ are mutually inverse transformations. The opposite implications cannot be ensured because $\mathfrak{I}$ is supposed to be an Hermitian ideal of $\mathbb{B}_H$, while the self-adjoint operators involved can be unbounded. These results, although weaker than the ones obtained for the unit circle, are enough to apply perturbative techniques to the spectral theory of ORF on the real line, even if the support of the orthogonality measure is unbounded.

With all these operator tools at hand we can develop the spectral theory for ORF on the real line following the same steps as in the case of the unit circle. In fact, the results for the unit circle are formulated throughout the paper in such a way that the translation to the real line is just a matter of changing the meaning of the symbols according to the previous discussion, together with some other obvious modifications. Nevertheless, two of the main results need a special discussion. The first one concerns the representation of the self-adjoint multiplication operator T_μ for a measure μ on $\mathbb{R}$, and the other one is related to the representation of the self-adjoint multiplication operator $T_{\mu_n^v}$ corresponding to the finitely supported measure μ_n^v associated with the PORF Q_n^v.

Following the same steps as in Theorem 5.1, we would find that, if μ is a measure on $\mathbb{R}$, for any sequence $\boldsymbol{\alpha}$ compactly included in $\mathbb{U}$, the matrix representation of T_μ with respect to the ORF $(\chi_n)_{n\geqslant 0}$ associated with $(\alpha_1, \bar{\alpha}_2, \alpha_3, \bar{\alpha}_4, \ldots)$ is $\mathcal{U} = \tilde{\zeta}_{\mathcal{A}}(\mathcal{C})$, where $\mathcal{A} = \mathcal{A}(\boldsymbol{\alpha})$, $\mathcal{C} = \mathcal{C}(\boldsymbol{\gamma})$ and $\boldsymbol{\gamma} = \mathcal{S}_{\boldsymbol{\alpha}}(\mu)$. However, since the matrix $\mathcal{C}$ is unitary, we can assure that $\tilde{\zeta}_{\mathcal{A}}(\mathcal{C})$ is a well-defined (self-adjoint) operator only when 1 is not an eigenvalue of $\mathcal{C}$. That is, in the case of the real line, the matrix representation $\mathcal{U} = \tilde{\zeta}_A(\mathcal{C})$ is valid provided that $1 \notin \sigma_p(\mathcal{C})$. To understand the meaning of this condition we will relate $\mathcal{C}$ to the matrix representation with respect to $(\chi_n)_{n\geqslant 0}$ of S_μ. When $1 \notin \sigma_p(\mathcal{C})$ the matrix of $S_\mu = \zeta(T_\mu)$ is $\zeta(\mathcal{U})$, but, as we will see, an expression for the matrix representation of S_μ can be obtained for any measure μ on the real line, even if it has a mass point at ∞. This discussion will lead also to a relation between the operator linear fractional transformations in the real line and the unit circle.

Since we are going to consider at the same time the linear fractional transformations used on the real line and on the unit circle, in what follows we will distinguish between both cases with a superscript $^{\mathbb{R}}$ or $^{\mathbb{T}}$, respectively. Let $A \in \mathbb{U}_H$. Then, $B = \zeta(A) \in \mathbb{D}_H$. A direct computation gives $\operatorname{Im} A = (1 - B)^{-1}(1 - BB^\dagger)(1 - B^\dagger)^{-1}$. Therefore, $\eta_A^{\mathbb{R}} = |\eta_B^{\mathbb{T}}(1 - B^\dagger)^{-1}|$ and, using the polar decomposition,

$$\eta_B^{\mathbb{T}}\big(1 - B^\dagger\big)^{-1} = U\eta_A^{\mathbb{R}}, \quad U \text{ unitary}. \tag{A.2}$$

If we change A by $-A^\dagger$, then B changes to $B^\dagger$, thus,

$$\eta^{\mathbb{T}}_{B^\dagger}(1-B)^{-1} = V\eta^{\mathbb{R}}_A, \quad V \text{ unitary}. \tag{A.3}$$

When A is normal, B is normal too and $\eta^{\mathbb{R}}_A = |1-B|^{-1}\eta^{\mathbb{T}}_B$, so $U = V^\dagger = \xi_B = (1-B)|1-B|^{-1}$. In the general case, using (A.2) and (A.3), we find that $\zeta^{\mathbb{T}}_B(\zeta(T)) = U\zeta^{\mathbb{R}}_A(T)V^\dagger$, hence

$$\zeta\big(\tilde{\zeta}^{\mathbb{R}}_A(T)\big) = \tilde{\zeta}^{\mathbb{T}}_B\big(UTV^\dagger\big). \tag{A.4}$$

Denoting $w = \zeta(z)$ and $S = UTV^\dagger$, a straightforward calculation gives

$$\varpi^{*\mathbb{R}}_A(z) - \varpi^{\mathbb{R}}_A(z)T^{\mathbb{R}}_A = \frac{2i}{1-w}\big(\varpi^{*\mathbb{T}}_B(w) - \varpi^{\mathbb{T}}_B(w)S^{\mathbb{T}}_B\big)(1-B)^{-1}, \tag{A.5}$$

where $T^{\mathbb{R}}_A = (\eta^{\mathbb{R}}_A)^{-1}T\eta^{\mathbb{R}}_A$ and $S^{\mathbb{T}}_B = (\eta^{\mathbb{T}}_B)^{-1}S\eta^{\mathbb{T}}_{B^\dagger}$. Since Eqs. (7) and (8) hold for the real line too, the above equality can be written equivalently as

$$z\tilde{\varpi}^{\mathbb{R}}_A\big(T^{\mathbb{R}}_A\big) - \tilde{\varpi}^{*\mathbb{R}}_A\big(T^{\mathbb{R}}_A\big) = \frac{2i}{1-w}\big(w\tilde{\varpi}^{\mathbb{T}}_B\big(S^{\mathbb{T}}_B\big) - \tilde{\varpi}^{*\mathbb{T}}_B\big(S^{\mathbb{T}}_B\big)\big)(1-B)^{-1}. \tag{A.6}$$

Using (A.2) and (A.3) we obtain $S^{\mathbb{T}}_B = (1-B^\dagger)^{-1}T^{\mathbb{R}}_A(1-B)$. Taking this relation into account, a direct computation yields

$$1-\tilde{\zeta}^{\mathbb{T}}_B(S) = 1-\tilde{\varpi}^{*\mathbb{T}}_B\big(S^{\mathbb{T}}_B\big)\varpi^{\mathbb{T}}_B\big(S^{\mathbb{T}}_B\big)^{-1} = \big(\eta^{\mathbb{R}}_A\big)^{-1}(1-T)V^\dagger\tilde{\varpi}^{\mathbb{T}}_B(S)^{-1}\eta^{\mathbb{T}}_{B^\dagger},$$

which implies that, for any $T \in \overline{\mathbb{D}}_H$,

$$1 \in \sigma\big(\tilde{\zeta}^{\mathbb{T}}_B(S)\big) \quad\Leftrightarrow\quad 1 \in \sigma(T), \qquad 1 \in \sigma_p\big(\tilde{\zeta}^{\mathbb{T}}_B(S)\big) \quad\Leftrightarrow\quad 1 \in \sigma_p(T). \tag{A.7}$$

Assume now that $\boldsymbol{\alpha}$ is compactly included in $\mathbb{U}$ and μ is a measure on $\mathbb{R}$ such that $1 \notin \sigma_p(\mathcal{C})$. From (A.4) we see that the matrix representation $\zeta(\tilde{\zeta}^{\mathbb{R}}_{\mathcal{A}}(\mathcal{C}))$ of S_μ can be expressed alternatively as $\tilde{\zeta}^{\mathbb{T}}_{\mathcal{B}}(\xi_{\mathcal{B}}\mathcal{C}\xi_{\mathcal{B}})$, with $\mathcal{B} = \zeta(\mathcal{A})$. Nevertheless, contrary to $\zeta(\tilde{\zeta}^{\mathbb{R}}_{\mathcal{A}}(\mathcal{C}))$, $\tilde{\zeta}^{\mathbb{T}}_{\mathcal{B}}(\xi_{\mathcal{B}}\mathcal{C}\xi_{\mathcal{B}})$ is always a well defined (unitary) matrix, no matter whether 1 is an eigenvalue of $\mathcal{C}$ or not, because $\xi_{\mathcal{B}}\mathcal{C}\xi_{\mathcal{B}}$ is unitary and $\tilde{\zeta}^{\mathbb{T}}_{\mathcal{B}}$ maps unitary operators into unitary operators. Actually, we are going to prove that, if $\boldsymbol{\alpha}$ is compactly included in $\mathbb{U}$, $\tilde{\zeta}^{\mathbb{T}}_{\mathcal{B}}(\xi_{\mathcal{B}}\mathcal{C}\xi_{\mathcal{B}})$ is the matrix representation of S_μ with respect to $(\chi_n)_{n\geqslant 0}$ for any measure μ on $\overline{\mathbb{R}}$. Following similar arguments to those given in the proof of Theorem 5.1 we find that, for any measure μ on $\overline{\mathbb{R}}$, the ORF $(\chi_n)_{n\geqslant 0}$ satisfy Eq. (28) too, but substituting $\hat{\mathcal{C}} = \mathcal{C}^{\mathbb{T}}_{\mathcal{A}}$ by $\hat{\mathcal{C}} = \mathcal{C}^{\mathbb{R}}_{\mathcal{A}}$, and $\varpi^{\mathbb{T}}_{\mathcal{A}}$, $\varpi^{*\mathbb{T}}_{\mathcal{A}}$ by $\varpi^{\mathbb{R}}_{\mathcal{A}}$, $\varpi^{*\mathbb{R}}_{\mathcal{A}}$, respectively. Applying (A.5) and using (7) and (8) we conclude that, for $\boldsymbol{\alpha}$ compactly included in $\mathbb{U}$,

$$\begin{pmatrix}\chi_0(z) & \chi_1(z) & \dots\end{pmatrix}\big(\zeta(z) - \tilde{\zeta}^{\mathbb{T}}_{\mathcal{B}}(\xi_{\mathcal{B}}\mathcal{C}\xi_{\mathcal{B}})\big) = 0, \quad \mathcal{B} = \zeta(\mathcal{A}),$$

which means that $\tilde{\zeta}^{\mathbb{T}}_{\mathcal{B}}(\xi_{\mathcal{B}}\mathcal{C}\xi_{\mathcal{B}})$ is the matrix of S_μ with respect to $(\chi_n)_{n\geqslant 0}$. As a consequence of this result and (A.7), we have the equivalences

$$1 \in \sigma(\mathcal{C}) \quad \Leftrightarrow \quad 1 \in \sigma\big(\tilde{\zeta}_{\mathcal{B}}^{\mathbb{T}}(\xi_{\mathcal{B}}\mathcal{C}\xi_{\mathcal{B}})\big) \quad \Leftrightarrow \quad 1 \in \sigma(S_\mu) \quad \Leftrightarrow \quad 1 \in \zeta(\operatorname{supp}\mu),$$
$$1 \in \sigma_p(\mathcal{C}) \quad \Leftrightarrow \quad 1 \in \sigma_p\big(\tilde{\zeta}_{\mathcal{B}}^{\mathbb{T}}(\xi_{\mathcal{B}}\mathcal{C}\xi_{\mathcal{B}})\big) \quad \Leftrightarrow \quad 1 \in \sigma_p(S_\mu) \quad \Leftrightarrow \quad 1 \in \zeta(\text{mass points of } \mu).$$

Thus, we have reached the following result.

Theorem A.1. *Let $\boldsymbol{\alpha}$ be a sequence compactly included in $\mathbb{U}$, μ a measure on $\overline{\mathbb{R}}$ and $\mathcal{C} = \mathcal{C}(\boldsymbol{\gamma})$ with $\boldsymbol{\gamma} = \mathcal{S}_{\boldsymbol{\alpha}}(\mu)$.*

$$1 \in \sigma(\mathcal{C}) \quad \Leftrightarrow \quad \infty \in \operatorname{supp}\mu, \qquad 1 \in \sigma_p(\mathcal{C}) \quad \Leftrightarrow \quad \infty \text{ is a mass point of } \mu.$$

Therefore, μ is a measure on $\mathbb{R}$ iff its related sequence $\boldsymbol{\gamma}$ satisfies $1 \notin \sigma_p(\mathcal{C})$. In consequence, $\mathcal{U} = \tilde{\zeta}_{\mathcal{A}}^{\mathbb{R}}(\mathcal{C})$ provides a well-defined matrix representation of T_μ for any measure μ on $\mathbb{R}$. Moreover, the measures on $\mathbb{R}$ with bounded support are characterized by the fact that $\boldsymbol{\gamma}$ is such that $1 \notin \sigma(\mathcal{C})$.

In the case of an arbitrary measure μ on $\overline{\mathbb{R}}$, including the possibility of a mass point at ∞, we can study the relation $\mu(\boldsymbol{\gamma}, \boldsymbol{\alpha})$ throughout the spectral analysis of the matrix representation $\tilde{\zeta}_{\mathcal{B}}^{\mathbb{T}}(\xi_{\mathcal{B}}\mathcal{C}\xi_{\mathcal{B}})$ of S_μ or, alternatively, we can deal with a pair of operators. More precisely, relation (A.6) implies that the spectra of $\tilde{\zeta}_{\mathcal{B}}^{\mathbb{T}}(\xi_{\mathcal{B}}\mathcal{C}\xi_{\mathcal{B}})$ and the pair $(\tilde{\varpi}_{\mathcal{A}}^{*\mathbb{R}}(\mathcal{C}), \tilde{\varpi}_{\mathcal{A}}^{\mathbb{R}}(\mathcal{C}))$ are related by the Cayley transform, so

$$\operatorname{supp}\mu = \sigma\big(\tilde{\varpi}_{\mathcal{A}}^{*\mathbb{R}}(\mathcal{C}), \tilde{\varpi}_{\mathcal{A}}^{\mathbb{R}}(\mathcal{C})\big) = \sigma\big(\mathcal{A}\mathcal{C}_{\mathrm{e}}^\dagger - \mathcal{A}^\dagger\mathcal{C}_{\mathrm{o}}, \mathcal{C}_{\mathrm{e}}^\dagger - \mathcal{C}_{\mathrm{o}}\big).$$

Also, the eigenvalues of the pair are the mass points of μ and the related left eigenvectors with eigenvalue λ are spanned by $(\chi_0(\lambda) \;\ldots\; \chi_{n-1}(\lambda))\eta_{\mathcal{A}_n}^{-1/2}$. That is, while the spectral methods that use linear fractional transformations $\tilde{\zeta}_{\mathcal{A}}^{\mathbb{R}}$ of five-diagonal matrices only work for measures on $\mathbb{R}$, their formulation in terms of pairs of band matrices is valid for any measure on $\overline{\mathbb{R}}$.

Similar results hold too for the finitely supported measures associated with the PORF. Given an arbitrary measure μ on $\overline{\mathbb{R}}$, consider the measure μ_n^v supported on the zeros of the PORF $Q_n^v = \phi_n + v\phi_n^*$, $v \in \mathbb{T}$. As in the case of the unit circle, Q_n^v has n different zeros, but now they lie on $\overline{\mathbb{R}}$. Besides, if $u = \tilde{\zeta}_{\gamma_n}$, the matrix representation $\mathcal{U}^{(n;\mathrm{u})} = \tilde{\zeta}_{\mathcal{A}_n}(\mathcal{C}_n^{\mathrm{u}})$ of $T_{\mu_n^v}$ with respect to $(\chi_k)_{k=0}^{n-1}$ is well defined whenever $1 \notin \sigma(\mathcal{C}_n^{\mathrm{u}})$. Concerning this condition, an analogous argument to that of the measure μ proves that $1 \in \sigma(\mathcal{C}_n^{\mathrm{u}}) \Leftrightarrow \infty \in \operatorname{supp}\mu_n^v$, i.e., the matrix representation $\mathcal{U}^{(n;\mathrm{u})}$ of $T_{\mu_n^v}$ is valid for any measure μ_n^v, except for the value $v = -\phi_n^*(\infty)/\phi_n(\infty)$ which locates a zero of Q_n^v at ∞. Nevertheless, analogously to the previous discussion, the spectral interpretation of the PORF in terms of pairs of band matrices given for the unit circle after Theorem 5.9 holds for any PORF on the real line too.

Concerning the applications of the spectral theory for ORF on the real line, we know that, if $\mathfrak{I}$ is an ideal of $\mathbb{B}_{\ell^2}$, for any sequences $\boldsymbol{\alpha}, \boldsymbol{\beta}$ compactly included in $\mathbb{U}$ and any sequences $\boldsymbol{\gamma}, \boldsymbol{\delta}$ in $\overline{\mathbb{D}}$ such that $1 \notin \sigma_p(\mathcal{C}(\boldsymbol{\gamma})) \cup \sigma_p(\mathcal{C}(\boldsymbol{\delta}))$, $\mathcal{A}(\boldsymbol{\alpha}) - \mathcal{A}(\boldsymbol{\beta})$, $\mathcal{C}(\boldsymbol{\gamma}) - \mathcal{C}(\boldsymbol{\delta}) \in \mathfrak{I} \Rightarrow \mathcal{U}(\boldsymbol{\gamma}, \boldsymbol{\alpha}) - \mathcal{U}(\boldsymbol{\delta}, \boldsymbol{\beta}) \in \mathfrak{I}$. This permits us to extend to the real line the applications discussed in Section 6.

Equation (A.4) provides a connection between the real line and the unit circle representations. Let $\boldsymbol{\alpha}$ be a sequence compactly included in $\mathbb{U}$, and consider the sequence $\boldsymbol{\beta}$ in $\mathbb{D}$ given by $\beta_n = \zeta(\alpha_n)$. Following the previous notation we also have $\alpha_0 = i$, so $\beta_0 = 0$. Consider two sequences $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$ in $\mathbb{D}$ related by $\delta_n = \xi_0^2\xi_1^2\cdots\xi_{n-1}^2\gamma_n$, $\xi_n = \frac{1-\beta_n}{|1-\beta_n|}$. We have the identities $\mathcal{C}_{\mathrm{o}}(\boldsymbol{\delta}) = \Lambda^\dagger\xi_{\mathcal{B}}\mathcal{C}_{\mathrm{o}}(\boldsymbol{\gamma})\Upsilon$ and $\mathcal{C}_{\mathrm{e}}(\boldsymbol{\delta}) = \Upsilon^\dagger\mathcal{C}_{\mathrm{e}}(\boldsymbol{\gamma})\xi_{\mathcal{B}}\Lambda$, where $\mathcal{B} = \zeta(\mathcal{A})$, $\mathcal{A} = \mathcal{A}(\boldsymbol{\alpha})$,

$$\Upsilon = \begin{pmatrix} \vartheta_0 & & \\ & \vartheta_1 & \\ & & \ddots \end{pmatrix}, \quad \vartheta_0 = 1, \ \vartheta_n = \begin{cases} \bar{\xi}_0^2 \bar{\xi}_2^2 \cdots \bar{\xi}_{n-1}^2, & \text{odd } n, \\ \xi_1^2 \xi_3^2 \cdots \xi_{n-1}^2, & \text{even } n, \end{cases}$$

$$\Lambda = \begin{pmatrix} \lambda_0 & & \\ & \lambda_1 & \\ & & \ddots \end{pmatrix}, \quad \lambda_0 = 1, \ \lambda_n = \begin{cases} \vartheta_{n-1} \xi_n, & \text{odd } n, \\ \vartheta_{n-1} \bar{\xi}_n, & \text{even } n. \end{cases} \tag{A.8}$$

Therefore, $\mathcal{C}(\boldsymbol{\delta}) = \Lambda^\dagger \xi_{\mathcal{B}} \mathcal{C}(\boldsymbol{\gamma}) \xi_{\mathcal{B}} \Lambda$ and, thus, Eq. (A.4) implies that

$$\zeta\big(\mathcal{U}^{\mathbb{R}}(\boldsymbol{\gamma}, \boldsymbol{\alpha})\big) = \Lambda \mathcal{U}^{\mathbb{T}}(\boldsymbol{\delta}, \boldsymbol{\beta}) \Lambda^\dagger. \tag{A.9}$$

Relation (A.9) can be understood taking into account that the ORF on the real line and the unit circle are related by the Cayley transform. More precisely, $\phi_n(z)$ are ORF on the real line iff $\phi_n(\tilde{\zeta}(z))$ are ORF on the unit circle. If μ is the orthogonality measure on $\overline{\mathbb{R}}$, the corresponding measure ν on $\mathbb{T}$ is given by $\nu(\cdot) = \mu(\tilde{\zeta}(\cdot))$. Also, the parameters α_n and β_n associated respectively with the poles of $\phi_n(z)$ and $\phi_n(\tilde{\zeta}(z))$ are related by $\beta_n = \zeta(\alpha_n)$. Moreover, $\phi_n(z)$ satisfies the analogue of recurrence (3) on the real line with coefficients γ_n iff $\widehat{\phi}_n(z) = \xi_0^2 \xi_1^2 \cdots \xi_{n-1}^2 \xi_n \phi_n(\tilde{\zeta}(z))$ satisfies such a recurrence on the unit circle with coefficients $\delta_n = \xi_0^2 \xi_1^2 \cdots \xi_{n-1}^2 \gamma_n$. If χ_n and $\widehat{\chi}_n$ are the associated ORF given by the corresponding version of (25) on $\overline{\mathbb{R}}$ and $\mathbb{T}$, respectively, then $\widehat{\chi}_n(z) = \lambda_n \chi_n(\tilde{\zeta}(z))$ with λ_n as in (A.8). Therefore, if $\boldsymbol{\alpha}$ is compactly included in $\mathbb{U}$, the matrix $\mathcal{U}^{\mathbb{R}}(\boldsymbol{\gamma}, \boldsymbol{\alpha})$ of T_μ with respect to $(\chi_n)_{n\geqslant 0}$ and the matrix $\mathcal{U}^{\mathbb{T}}(\boldsymbol{\delta}, \boldsymbol{\beta})$ of T_ν with respect to $(\widehat{\chi}_n)_{n\geqslant 0}$ are related by (A.9).

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Weak spectral synthesis in commutative Banach algebras

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Abstract

Let A be a semisimple and regular commutative Banach algebra with structure space $\Delta(A)$. Generalizing the notion of spectral sets in $\Delta(A)$, the considerably larger class of weak spectral sets was introduced and studied in [C.R. Warner, Weak spectral synthesis, Proc. Amer. Math. Soc. 99 (1987) 244–248]. We prove injection theorems for weak spectral sets and weak Ditkin sets and a Ditkin–Shilov type theorem, which applies to projective tensor products. In addition, we show that weak spectral synthesis holds for the Fourier algebra $A(G)$ of a locally compact group G if and only if G is discrete.

Keywords: Commutative Banach algebra; Structure space; Weak spectral set; Weak Ditkin set; Injection theorem; Ditkin–Shilov theorem; Projective tensor product; Fourier algebra

Introduction

Let A be a regular and semisimple commutative Banach algebra with structure space $\Delta(A)$ and Gelfand transform $a \to \widehat{a}$. For any subset M of A, the hull $h(M)$ of M is defined by $h(M) = \{\varphi \in \Delta(A)\colon\ \varphi(M) = \{0\}\}$. Associated to each closed subset E of $\Delta(A)$ are two distinguished ideals with hull equal to E, namely

$$k(E) = \{a \in A\colon\ \widehat{a}(\varphi) = 0 \quad \text{for all } \varphi \in E\}$$

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and

$$j(E) = \{a \in A\colon \widehat{a} \text{ has compact support disjoint from } E\}.$$

Then $k(E)$ is the largest ideal with hull E and $j(E)$ is the smallest such ideal. Recall that E is a *spectral set* (or *set of synthesis*) if $k(E) = \overline{j(E)}$ (equivalently, $k(E)$ is the only closed ideal with hull equal to E). One says that *spectral synthesis holds* for A if every closed subset of $\Delta(A)$ is a spectral set. Moreover, E is a *Ditkin set* if $a \in \overline{aj(E)}$ for every $a \in k(E)$.

In connection with the union problem, that is, the question of when the union of two sets of synthesis is a set of synthesis, Warner [19] introduced and studied the class of weak spectral sets. Motivation certainly also arose from the fact that such sets appeared earlier in the work of Varopoulos [17,18] and others. The definition, although this is not the original one, can be given as follows. A closed subset E of $\Delta(A)$ is called a *weak spectral set* if there exists $n \in \mathbb{N}$ such that $a^n \in \overline{j(E)}$ for every $a \in k(E)$. Adopting the notation of [19], we let $\xi(E)$ denote the smallest such number n. When this happens for each E, we say that *weak spectral synthesis holds* for A. Generalizing the notion of a Ditkin set, we call E a *weak Ditkin set* if there exists $n \in \mathbb{N}$ such that $a^n \in \overline{a^n j(E)}$ for all $a \in k(E)$, and $\eta(E)$ will then stand for the minimal such n. So E is a spectral set (Ditkin set) if and only if $\xi(E) = 1$ ($\eta(E) = 1$). Subsequent to [19], the study of weak spectral sets and of the weak synthesis problem gained considerable attention [7,11,12,20], the more so because there are many commutative Banach algebras for which weak spectral synthesis holds, whereas spectral synthesis fails (compare Section 1).

The purpose of this paper is to investigate weak spectral sets and weak Ditkin sets under various aspects. We start in Section 1 by mentioning some examples of algebras for which spectral synthesis fails, but weak synthesis holds. These examples are followed by some preliminary results concerning countable unions, localness and a sufficient condition for a weak spectral set to be weak Ditkin. In Section 2 the setting is that of a closed ideal I of A together with the embedding i of $\Delta(A/I)$ into $\Delta(A)$. So-called injection theorems for spectral sets and Ditkin sets relate either of these properties of closed subsets E of $\Delta(A/I)$ to the corresponding property of their images $i(E)$ in $\Delta(A)$. We prove injection theorems for weak spectral sets (Theorem 2.2) and weak Ditkin sets (Theorem 2.5), at the same time providing estimates for the values of ξ and η.

The classical Ditkin–Shilov (or Helson–Reiter) theorem states that if singletons in $\Delta(A)$ are Ditkin sets, then every closed subset of $\Delta(A)$ with scattered boundary is a spectral set. A remarkable extension was obtained in [1, Theorem 1.2]. Here we establish, under somewhat weaker hypotheses, an analogous result for weak spectral sets (Theorem 3.1). Like [1, Theorem 1.2], Theorem 3.1 admits an application to projective tensor products (Theorem 3.5). Again, there are upper bounds for $\xi(E)$. Modifying Varopoulos' proof [18] of Malliavin's celebrated theorem [9], it was shown in [12, Theorem 3.1] that weak spectral synthesis fails for the Fourier algebra $A(G)$ of every non-discrete abelian locally compact group G. We conclude the paper by extending this result to arbitrary locally compact groups (Theorem 4.3).

1. Preliminaries, examples and some basic properties

Let A be a semisimple and regular commutative Banach algebra. In [19] a closed subset E of $\Delta(A)$ was defined to be a weak spectral set if every element of the quotient algebra $k(E)/\overline{j(E)}$ is nilpotent. Then, as shown in [19, Theorem 1.2] and [3, footnote 7, p. 885], there exists $n \in \mathbb{N}$ such that $x^n \in \overline{j(E)}$ for all $x \in k(E)$. So this latter property can equally well be taken as the definition. One of the important features of the class of weak spectral sets is that it is closed under

the formation of finite unions. Actually, for any two weak spectral sets E_1 and E_2, $\xi(E_1 \cup E_2) \leqslant \xi(E_1) + \xi(E_2)$ [19, Theorem 2.2] (see [11, Corollary 3.11] for a different approach). Although the class of weak spectral sets shares this finite union property with the class of Ditkin sets, in contrast to the latter a closed countable union of weak spectral sets need not be a spectral set [19, Theorem 2.6].

We point out that, throughout this paper, the most important fundamental tool is the *local membership principle* which we briefly recall for the readers convenience. Let I be a closed ideal in A. An element $x \in A$ is said to belong locally to I at $\varphi \in \Delta(A)$ (at infinity) if there exist a neighbourhood V of φ in $\Delta(A)$ (a compact subset K of $\Delta(A)$) and an element y of I such that $\widehat{x}(\psi) = \widehat{y}(\psi)$ for all $\psi \in V$ ($\psi \in \Delta(A) \setminus K$). If x belongs locally to I at every point of $\Delta(A)$ and at infinity, then $x \in I$. As general references to spectral synthesis we mention [4,13,14].

In this section, we first present three illustrative examples and then give some basic results on weak spectral sets and weak Ditkin sets. Concerning notation, we make the following convention. If $E \subseteq \Delta(A)$ is a singleton, say $\{\varphi\}$, we write $k(\varphi)$ and $j(\varphi)$ in place of $k(\{\varphi\})$ and $j(\{\varphi\})$, respectively.

Example 1.1. (1) For $n \in \mathbb{N}$, the unit sphere S^{n-1} in $\mathbb{R}^n = \Delta(L^1(\mathbb{R}^n))$ is a weak spectral set with $\xi(S^{n-1}) = \lfloor \frac{n+1}{2} \rfloor$ [18, Theorem 3].

(2) For each $n \in \mathbb{N}$, $\mathbb{T}^\infty = \Delta(L^1(\widehat{\mathbb{T}^\infty}))$ contains a weak spectral set E with $\xi(E) = n$ [20, Corollary 2.5(d)].

(3) Let $C^n[0,1]$ be the algebra of all n-times continuously differentiable functions on $[0,1]$ and identify $\Delta(C^n[0,1])$ with $[0,1]$. Then $\xi(E) = n+1$ for every non-empty proper closed subset E of $[0,1]$ [7, Example 2.4(i)].

Example 1.2. Let X be a compact metric space with metric d and let $0 < \alpha \leqslant 1$. Then $\mathrm{Lip}_\alpha X$ is the space of all complex-valued functions f on X such that

$$p_\alpha(f) = \sup\left\{\frac{|f(x) - f(y)|}{d(x,y)^\alpha}: x, y \in X,\ x \neq y\right\}$$

is finite. With pointwise multiplication and the norm $\|f\| = \|f\|_\infty + p_\alpha(f)$, $\mathrm{Lip}_\alpha X$ is a commutative Banach algebra. These Lipschitz algebras were first extensively studied by Sherbert [15]. All the facts relevant to us, can now also be found in Section 4.4 of the monograph [5]. The map $x \to \varphi_x$, where $\varphi_x(f) = f(x)$ for $f \in \mathrm{Lip}_\alpha X$, is a homeomorphism from X onto $\Delta(\mathrm{Lip}_\alpha X)$. With this identification of X and $\Delta(\mathrm{Lip}_\alpha X)$, we have $k(E)^2 = \overline{j(E)}$ for every closed subset E of X, whereas E is a spectral set if and only if E is open in X. Consequently, if X is not discrete, then spectral synthesis fails for $\mathrm{Lip}_\alpha X$, but weak spectral synthesis holds. Actually, inspection of the proof of Theorem 4.4.31 of [5] shows that $f^2 \in \overline{f^2 j(E)}$ for every $f \in k(E)$, so that $\eta(E) \leqslant 2$. Moreover, if E is open and closed in X, then $1_E \in \mathrm{Lip}_\alpha X$, and hence E is a Ditkin set. Thus $\mathrm{Lip}_\alpha X$ provides an example of a semisimple and regular commutative Banach algebra A for which $\xi(E) = \eta(E) \leqslant 2$ for each closed subset E of $\Delta(A)$.

Example 1.3. Let M be the Mirkil algebra which was introduced in [10] as a counterexample to discrete spectral synthesis. Identifying $[-\pi, \pi)$ with the torus $\mathbb{T}$, M is defined to be the space of all functions $f \in L^2(\mathbb{T})$ such that f is continuous on $[-\pi/2, \pi/2]$. With convolution and the norm $\|f\| = \|f\|_2 + \|f|_{[-\pi/2,\pi/2]}\|_\infty$, M is a regular and semisimple commutative Banach algebra, and $\Delta(M)$ can be identified with $\mathbb{Z}$. Every subset E of $\mathbb{Z}$ is a weak Ditkin set with $\eta(E) \leqslant 2$

[20, Theorem 1.4]. On the other hand, no finite subset of $\mathbb{Z}$ is a Ditkin set [10, Section 5], [20, Corollary 2.2]. The Mirkil algebra also serves as a counterexample to the union question. In fact, the sets $4\mathbb{Z}$ and $4\mathbb{Z}+2$ are spectral sets, whereas their union, $2\mathbb{Z}$, is not [2, Theorem] (see also the exposition in Section 4.5 of [5]). Since translates of Ditkin sets in $\mathbb{Z}$ are Ditkin sets [20, Theorem 2.1], it follows that

$$\xi(4\mathbb{Z}) = \xi(4\mathbb{Z}+2) = 1, \qquad \xi(2\mathbb{Z}) = 2 \quad \text{and} \quad \eta(4\mathbb{Z}) = \eta(4\mathbb{Z}+2) = \eta(2\mathbb{Z}) = 2.$$

This in particular shows that the inequality $\xi(E_1 \cup E_2) \leqslant \xi(E_1) + \xi(E_2)$ mentioned above cannot be improved.

As is shown by the sets $4\mathbb{Z}$ and $4\mathbb{Z}+2$ in Example 1.3, in general spectral sets need not be Ditkin sets. However, if $E \subseteq \Delta(A)$ is a spectral set satisfying some additional hypothesis, which might be termed bounded regularity, then E is a Ditkin set. The analogous conclusion (Lemma 1.4), which will be used in Section 3, holds for weak spectral sets. We include the simple proof.

Lemma 1.4. *Let $E \subseteq \Delta(A)$ be a weak spectral set. Suppose that there exists a constant $C > 0$ such that for every compact subset K of $\Delta(A)$ which is disjoint from E, there exists $a \in j(E)$ such that $\|a\| \leqslant C$ and $\widehat{a} = 1$ on K. Then E is a weak Ditkin set and $\eta(E) = \xi(E)$.*

Proof. Let $n = \xi(E)$ and let $x \in k(E)$ and $\epsilon > 0$. There exists $y \in j(E)$ with $\|x^n - y\| \leqslant \epsilon$. By hypothesis, there exists $a \in j(E)$ such that $\|a\| \leqslant C$ and $\widehat{a} = 1$ on $\operatorname{supp} \widehat{y}$. Then $y = ya$ since A is semisimple and $\widehat{ya} = \widehat{y}$. It follows that

$$\|x^n - x^n a\| \leqslant \|x^n - y\| + \|ya - x^n a\| \leqslant (1 + C)\epsilon.$$

This shows that $x^n \in \overline{x^n j(E)}$, and hence $\eta(E) \leqslant \xi(E)$. □

A closed countable union of Ditkin sets is a Ditkin set. Refined arguments allow to show the following version for weak Ditkin sets.

Proposition 1.5. *Suppose that $\emptyset$ is a weak Ditkin set. Let E be a closed subset of $\Delta(A)$ which is a union of closed subsets E_i, $i \in \mathbb{N}$. If $\eta(E_i) \leqslant N$ for some $N \in \mathbb{N}$ and all i, then $\eta(E) \leqslant N\eta(\emptyset)$.*

Proof. Let $m = \eta(\emptyset)$. Then there exists a sequence $(u_n)_n$ in A such that $x^m u_n \to x^m$ and each $\widehat{u_n}$ has compact support. It suffices to show that $(x^m u_n)^N \in \overline{(x^m u_n)^N j(E)}$ for all n. It therefore suffices to show that if $y \in k(E)$ is such that $\widehat{y}$ has compact support then $y^N \in \overline{y^N j(E)}$. Since A is semisimple and regular, this will follow once we have verified that y^N belongs locally to $\overline{y^N j(E)}$ at every point of $\Delta(A)$.

To that end, fix $\varphi \in \Delta(A)$ and a compact neighbourhood U of φ in $\Delta(A)$. There exists $u \in A$ such that $\widehat{u} = 1$ in a neighbourhood of φ and $\operatorname{supp} \widehat{u} \subseteq U$. We claim that $y^N u \in \overline{y^N j(E)}$. Let $\epsilon > 0$ be given. Since all E_i are weak Ditkin sets with $\eta(E_i) \leqslant N$, we can construct by induction a sequence $(z_i)_i$ such that $z_i \in j(E_i)$, $\|y^N - y^N z_1\| \leqslant \frac{\epsilon}{2\|u\|}$ and

$$\|y^N - y^N z_i\| \leqslant \frac{\epsilon}{2^i \|u\| \cdot \prod_{l=1}^{i-1} \|y_l\|}$$

for $i \geqslant 2$. For each i, let V_i be an open set containing E_i such that $\widehat{z_i}$ vanishes on V_i. Now, since $E \cap U$ is compact, $E \cap U \subseteq \bigcup_{i=1}^n V_i$ for some $n \in \mathbb{N}$. Let $z = z_1 \cdot \ldots \cdot z_n$, then $z \in j(E \cap U)$ and hence $uz \in j(E)$ since $\widehat{u}$ vanishes on $\Delta(A) \setminus U$. Finally,

$$\begin{aligned}
\|y^N u - y^N uz\| &\leqslant \|y^N u - y^N uz_1\| + \sum_{i=2}^n \|y^N uz_1 \cdot \ldots \cdot z_{i-1} - y^N uz_1 \cdot \ldots \cdot z_i\| \\
&\leqslant \|u\| \left(\|y^N - y^N z_1\| + \sum_{i=2}^n \|y^N - y^N z_i\| \prod_{l=1}^{i-1} \|z_l\| \right) \\
&\leqslant \epsilon \sum_{i=1}^n \frac{1}{2^i} < \epsilon.
\end{aligned}$$

Since $y^N uz \in \overline{y^N j(E)}$ and $\epsilon > 0$ was arbitrary, it follows that $y^N u \in \overline{y^N j(E)}$. This shows that y^N belongs locally to $\overline{y^N j(E)}$ at φ because $\widehat{u} = 1$ near φ. □

It is well known that the property of being a spectral set is local in the sense that if A satisfies Ditkin's condition at infinity and E is a closed subset of $\Delta(A)$ such that every point of E has a closed relative neighbourhood in E which is a spectral set for A, then E is a spectral set. The corresponding result for weak spectral sets does not hold (see Example 1.7 below). We have, however, the following

Proposition 1.6. *Let A be a regular and semisimple commutative Banach algebra satisfying Ditkin's condition at infinity and let E be a compact subset of $\Delta(A)$. Suppose that each point of E has a closed relative neighbourhood in E which is a weak spectral set for A. Then E is a weak spectral set for A.*

Proof. Let $x \in k(E)$. Since A is semisimple and x belongs locally to $\overline{j(E)}$ at every point of $\Delta(A) \setminus E$, it suffices to show that there exists $n \in \mathbb{N}$ such that x^n belongs locally to $\overline{j(E)}$ at every point φ of E.

By hypothesis, there exist a closed subset E_φ of E and an open neighbourhood U_φ of φ in $\Delta(A)$ such that $U_\varphi \cap E \subseteq E_\varphi$ and $x^{n_\varphi} \in \overline{j(E_\varphi)}$. A being regular, there exists $u_\varphi \in A$ such that $\operatorname{supp} \widehat{u_\varphi} \subseteq U_\varphi$ and $\widehat{u_\varphi} = 1$ in a neighbourhood of φ in $\Delta(A)$. Since E_φ is a weak spectral set, there exist $y_\varphi \in j(E_\varphi)$ and $n_\varphi \in \mathbb{N}$ such that $\|x^{n_\varphi} - y_\varphi\| < \epsilon / \|u_\varphi\|$. Then $\|u_\varphi x^{n_\varphi} - u_\varphi y_\varphi\| < \epsilon$ and $\widehat{u_\varphi y_\varphi}$ vanishes in a neighbourhood of E since $\widehat{y_\varphi} = 0$ in a neighbourhood of E_φ and $\widehat{u_\varphi} = 0$ in a neighbourhood of $\Delta(A) \setminus U_\varphi$ and $E \subseteq E_\varphi \cup (\Delta(A) \setminus U_\varphi)$. So $y_\varphi u_\varphi \in j(E)$. Since $\epsilon > 0$ was arbitrary, it follows that $x^{n_\varphi} u_\varphi \in \overline{j(E)}$. Finally, $\widehat{x^{n_\varphi} u_\varphi} = \widehat{x^{n_\varphi}}$ in a neighbourhood of φ and hence x^{n_φ} belongs locally to $\overline{j(E)}$ at φ.

Thus for each $\varphi \in E$, there exist an open neighbourhood V_φ of $\varphi \in \Delta(A)$ and $z_\varphi \in \overline{j(E)}$ such that $\widehat{x^{n_\varphi}} = \widehat{z_\varphi}$ on V_φ. Since E is compact, there exist $\varphi_1, \ldots, \varphi_m \in E$ such that $E \subseteq \bigcup_{i=1}^m V_{\varphi_i}$. Let $n = \max\{n_{\varphi_i} \colon 1 \leqslant i \leqslant m\}$. Then x^n belongs locally to $\overline{j(E)}$ at every point of E. □

If the appropriate minor modifications are made, the proof of Proposition 1.6 applies to closed subsets E provided that the values $\xi(E_\varphi)$, $\varphi \in E$, are bounded.

Example 1.7. Let $(A_n)_n$ be a sequence of unital and semisimple commutative Banach algebras and A their c_0-direct sum. Then $\Delta(A)$ is the topological sum of the sets $\Delta(A_n)$ and A satisfies Ditkin's condition at infinity. Now suppose that for every $n \in \mathbb{N}$ there exists $\varphi_n \in \Delta(A_n)$ such that $\{\varphi_n\}$ is a weak spectral set with $\xi(\varphi_n) \geqslant n$, and let $E = \overline{\{\varphi_n\colon n \in \mathbb{N}\}}$. Then each singleton $\{\varphi_n\}$ is open in E and there exists $x_n \in k(\varphi_n)$ such that $x_n^k \notin \overline{j(\varphi_n)}$ for all $k < n$. Viewing x_n as an element of A, we have $x_n \in k(E)$ and $x_n^k \notin \overline{j(E)}$ for all $k < n$. It follows that $\xi(E) \geqslant n$ for all $n \in \mathbb{N}$.

As an example of such sequences $(A_n)_n$ and $(\varphi_n)_n$, simply take $A_n = C^n[0,1]$ and, after identifying $\Delta(A_n)$ with the interval $[0,1]$, $\varphi_n = 0$. In fact, $\xi(t) = n+1$ for each $t \in [0,1]$ (Example 1.1(3)).

2. Injection theorems for weak spectral sets and weak Ditkin sets

Let G be a locally compact abelian group and H a closed subgroup of G. Then $L^1(G/H)$ is a quotient of $L^1(G)$ and $\widehat{G/H} = \Delta(L^1(G/H))$ embeds canonically into $\widehat{G} = \Delta(L^1(G))$. Then a closed subset of $\widehat{G/H}$ is a spectral set (Ditkin set) for $L^1(G/H)$ if and only if it is a spectral set (Ditkin set) for $L^1(G)$ (see [13, Theorems 7.3.15 and 7.4.13]). For the obvious reason, these results are referred to as *injection theorems*. The same problem naturally arises in the general context of a regular and semisimple commutative Banach algebra A and a closed ideal I of A, and it is worthwhile to consider weak spectral sets and weak Ditkin sets rather than just spectral sets and Ditkin sets. In this section, we establish such injection theorems. However, as the reader might expect, some additional hypotheses, which are automatically satisfied in the group algebra situation, have to be placed on A and I.

We start with a lemma which is needed to prove the injection theorem for weak spectral sets. In what follows, if I is a closed ideal of A, then $i : \Delta(A/I) \to \Delta(A)$ will denote the embedding defined by $i(\varphi)(x) = \varphi(x+I)$ for $\varphi \in \Delta(A/I)$ and $x \in A$.

Lemma 2.1. *Let A be a regular commutative Banach algebra and I a closed ideal of A such that $h(I) = \Delta(A)$. Let E be a closed subset of $\Delta(A/I)$ and suppose that $k(E)^n \subseteq \overline{j(E)}$ and $I^m \subseteq \overline{j(i(E))}$ for some $n, m \in \mathbb{N}$. Then*

$$k\bigl(i(E)\bigr)^{nm} \subseteq \overline{j\bigl(i(E)\bigr)}.$$

Proof. It suffices to show that given $x \in k(i(E))$ such that $\|x\| < 1$ and $0 < \epsilon < 1 - \|x\|$, there exists $y \in \overline{j(i(E))}$ with $\|x^{nm} - y\| < \epsilon m$. By hypothesis, there exists $y \in A$ such that $\widehat{y+I}$ has compact support disjoint from E and $\|(x^n + I) - (y + I)\| < \epsilon$. Since $\widehat{y+I}(\varphi) = \widehat{y}(i(\varphi))$ for all $\varphi \in \Delta(A/I)$ and since i is a homeomorphism from $\Delta(A/I)$ onto $h(I) = \Delta(A)$, we have $y \in j(i(E))$. Choose $z \in I$ so that $\|x^n - (y+z)\| < \epsilon$. Then $\|y + z\| \leqslant 1$ and

$$\begin{aligned} \bigl\|x^{nm} - (y+z)^m\bigr\| &= \left\| \bigl(x^n - (y+z)\bigr) \sum_{j=0}^{m-1} x^{nj}(y+z)^{m-1-j} \right\| \\ &< \epsilon \sum_{j=0}^{m-1} \|x\|^{nj} \|y+z\|^{m-1-j} \\ &\leqslant \epsilon m. \end{aligned}$$

Since both z^m and y are contained in $\overline{j(i(E))}$, it follows that

$$(y+z)^m = z^m + \sum_{j=1}^{m} \binom{m}{j} y^{j-1} z^{m-j} \in \overline{j\big(i(E)\big)}.$$

So $u = (y+z)^m$ has the desired properties. □

Theorem 2.2. *Let A be a regular and semisimple commutative Banach algebra, I a closed ideal of A and E a closed subset of $\Delta(A/I)$.*

- (i) *If $i(E)$ is a weak spectral set for A, then E is a weak spectral set for A/I and $\xi(E) \leqslant \xi(i(E))$.*
- (ii) *If E is a weak spectral set for A/I and $h(I)$ is a weak spectral set for A, then $i(E)$ is a weak spectral set for A and $\xi(i(E)) \leqslant \xi(E)\xi(h(I))$.*

Proof. (i) Let $n = \xi(i(E))$ and let $x \in A$ be such that $x + I \in k(E)$. Then $x \in k(i(E))$ and hence given $\epsilon > 0$, there exists $y \in j(i(E))$ such that $\|x^n - y\| < \epsilon$. It follows that $\|(y+I) - (x^n + I)\| < \epsilon$ and $\widehat{y+I}$ has compact support disjoint from E. Since $\epsilon > 0$ was arbitrary, $(x+I)^n \in \overline{j(E)}$.

(ii) Note first that $\overline{j(h(I))} \subseteq I$ since A is semisimple and regular. Let $A_1 = A/\overline{j(h(I))}$ and $I_1 = I/\overline{j(h(I))}$. Then $h(I_1) = \Delta(A_1)$, so that Lemma 2.1 applies to A_1 and its ideal I_1. Hence, with i_1 denoting the embedding $\Delta(A_1/I_1) \to \Delta(A_1)$, $k(i_1(E))^{nm} \subseteq \overline{j(i_1(E))}$. Let $q : A \to A_1$ denote the quotient homomorphism and $i_2 : \varphi \to \varphi \circ q$ the embedding of $\Delta(A_1)$ into $\Delta(A)$. Since $i(E) \subseteq h(I)$, we have $\overline{j(h(I))} \subseteq \overline{j(i(E))}$, and since $i = i_2 \circ i_1$, it follows that

$$\begin{aligned}
\overline{j\big(i(E)\big)} &= q^{-1}\Big(q\Big(\overline{j\big(i(E)\big)}\Big)\Big) \supseteq q^{-1}\Big(\overline{j\big(i_1(E)\big)}\Big) \\
&\supseteq q^{-1}\big(k\big(i_1(E)\big)^{nm}\big) \supseteq \big(q^{-1}\big(k\big(i_1(E)\big)\big)\big)^{nm} \\
&= k\big(i(E)\big)^{nm},
\end{aligned}$$

as was to be shown. □

We continue with two consequences of Theorem 2.2.

Corollary 2.3. *Let A be a semisimple and regular commutative Banach algebra and suppose that $\emptyset$ is a Ditkin set. Let $E \subseteq \Delta(A)$ be a weak spectral set for A and F an open and closed subset of E. Then F is a weak spectral set and $\xi(F) \leqslant \xi(E)$.*

Proof. Let $I = k(E)$ and let F' and E' denote the sets in $\Delta(A/I)$ corresponding to F and E', respectively. Then $\Delta(A/I) = E'$ and F' is open and closed in $\Delta(A/I)$. Since A/I is semisimple and regular and satisfies Ditkin's condition at infinity, it follows that F' is a spectral set for A/I. Theorem 2.2(ii) now implies that $F = i(F')$ is a weak spectral set and

$$\xi(F) = \xi\big(i(F')\big) \leqslant \xi(F')\xi\big(h(I)\big) = \xi(E),$$

since $\xi(F') = 1$ and $E = h(I)$. □

For a locally compact group G, let $A(G)$ denote the Fourier algebra of G as introduced and studied extensively by Eymard [6]. $A(G)$ is a regular and semisimple commutative Banach algebra whose spectrum can be identified with G. In fact, the map $t \to \varphi_t$, where $\varphi_t(u) = u(t)$ for $u \in A(G)$, is a homeomorphism from G onto $\Delta(A(G))$ [6, Théorème 3.34]. Recall that when G is abelian, $A(G)$ is isometrically isomorphic (by means of the Fourier transform) to $L^1(\widehat{G})$, the L^1-algebra of the dual group of G. An injection theorem for spectral sets of Fourier algebras was shown in [8, Theorem 3.4]. The following corollary generalizes [19, Corollary 2.5(c)].

Corollary 2.4. *Let G be a locally compact group, H a closed subgroup of G and $i : H = \Delta(A(H)) \to G = \Delta(A(G))$. Then $\xi(E) \leqslant \xi(i(E))$ for any closed subset E of H.*

Proof. Let $I = \{u \in A(G)\colon u|_H = 0\}$. Then the map $u + I \to u|_H$ is an isometric isomorphism between $A(G)/I$ and $A(H)$. The statement now follows from Theorem 2.2 and the fact that H is a set of synthesis for $A(G)$ [16, Theorem 3]. □

We now proceed with the injection theorem for weak Ditkin sets. Note that the hypothesis in part (ii) of the following theorem implies that $h(I)$ is a weak Ditkin set with $\eta(h(I)) \leqslant m$. When $A = L^1(G)$ and I is the kernel of the quotient homomorphism $L^1(G) \to L^1(G/H)$, this condition, with $m = 1$, is always satisfied [13, Lemma 7.4.14].

Theorem 2.5. *Let A be a regular and semisimple commutative Banach algebra, I a closed ideal of A and E a closed subset of $\Delta(A/I)$.*

(i) *If $i(E)$ is a weak Ditkin set for A, then E is a weak Ditkin set for A/I and $\eta(E) \leqslant \eta(i(E))$.*
(ii) *Suppose that there exist $m \in \mathbb{N}$ and a constant $C > 0$ with the following property: For every $a \in A$ and $\epsilon > 0$, there exists $b \in A$ such that $\|a^m - a^m b\| \leqslant C\|a^m + I\| + \epsilon$ and $\widehat{b}$ vanishes in a neighbourhood of $h(I)$. If E is a weak Ditkin set for A/I, then $i(E)$ is a weak Ditkin set for A and $\eta(i(E)) \leqslant m^2\eta(E)^2$.*

Proof. In the following, q denotes the quotient homomorphism from A onto A/I.

(i) Let $n = \eta(i(E))$ and let $x \in A$ be such that $q(x) \in k(E)$. Then, given $\epsilon > 0$, there exists $y \in j(i(E))$ such that $\|x^n - x^n y\| \leqslant \epsilon$. It follows that $\|q(x)^n - q(x)^n q(y)\| \leqslant \epsilon$ and $\widehat{q(y)}$ has compact support and vanishes in a neighbourhood of E in $\Delta(A/I)$. Since $\epsilon > 0$ was arbitrary, $q(x)^n \in \overline{q(x)^n j(E)}$.

(ii) Let $n = \eta(E)$ and let $x \in k(i(E))$, $x \neq 0$, and $\epsilon > 0$ be given. Choose $\delta > 0$ such that

$$\delta \cdot \sum_{j=0}^{nm-1} \|x\|^{n(nm-j)}\big(\|x\|^n + \delta\big)^j \leqslant \epsilon.$$

Then $q(x) \in k(E)$ and hence there exists $u \in A$ such that

$$\big\|q\big(x^n - x^n u\big)\big\| = \big\|q(x)^n - q(x)^n q(u)\big\| \leqslant \delta$$

and $\widehat{q(u)}$ vanishes in a neighbourhood of E in $\Delta(A/I)$. For arbitrary Banach algebra elements s and t and $k \in \mathbb{N}$, we have

$$\|s^k - t^k\| = \left\|(s-t)\sum_{j=0}^{k-1} s^{k-j}t^j\right\| \leqslant \|s-t\| \cdot \sum_{j=0}^{k-1} \|s\|^{k-j}\big(\|s\| + \|t-s\|\big)^j.$$

Setting $s = q(x^n)$, $t = q(x^n u)$, and $k = nm$ and using that $\|q(x^n - x^n u)\| \leqslant \delta$, it follows from the choice of δ that

$$\big\|q\big(x^{n^2m} - x^{n^2m}u^{nm}\big)\big\| \leqslant \delta \cdot \sum_{j=0}^{nm-1} \|x\|^{n(nm-j)}\big(\|x\|^n + \delta\big)^j \leqslant \epsilon.$$

By Theorem 2.2(ii), $i(E)$ is a weak spectral set for A and

$$\xi\big(i(E)\big) \leqslant \xi(E)\xi\big(h(I)\big) \leqslant \eta(E)\eta\big(h(I)\big) = nm.$$

Since $u \in k(i(E))$, there exists $v \in A$ such that $\|u^{nm} - v\| \leqslant \epsilon/\|x\|^{n^2m}$ and $\widehat{v}$ has compact support and vanishes in a neighbourhood of $i(E)$ in $\Delta(A)$. Then

$$\begin{aligned}
\big\|q\big(x^{n^2m} - x^{n^2m}v\big)\big\| &\leqslant \big\|q\big(x^{n^2m} - x^{n^2m}u^{nm}\big)\big\| + \big\|q\big(x^{n^2m}\big)q\big(u^{nm} - v\big)\big\| \\
&\leqslant \big\|q\big(x^{n^2m} - x^{n^2m}u^{nm}\big)\big\| + \|x\|^{n^2m}\big\|u^{nm} - v\big\| \\
&\leqslant 2\epsilon.
\end{aligned}$$

Now let $a = x^{n^2m} - x^{n^2m}v \in A$. By hypothesis, there exists $w \in j(h(I))$ such that $\|a^m - a^m w\| \leqslant C\|q(a^m)\| + \epsilon$. Write

$$\begin{aligned}
a^m - a^m w &= \sum_{j=0}^{m} \binom{m}{j}(-1)^j x^{n^2m^2}\big(v^j - v^j w\big) \\
&= x^{n^2m^2} - x^{n^2m^2}\left(w - \sum_{j=1}^{m}\binom{m}{j}(-1)^{j-1}\big(v^j - v^j w\big)\right).
\end{aligned}$$

Finally, let

$$y = w - \sum_{j=1}^{m}\binom{m}{j}(-1)^{j-1}\big(v^j - v^j w\big) \in A.$$

Then $\widehat{y}$ has compact support since both $\widehat{v}$ and $\widehat{w}$ have compact support. Moreover, $\widehat{y}$ vanishes in a neighbourhood of $i(E)$ in $\Delta(A)$ since $\widehat{v}$ does so and $\widehat{w}$ vanishes in a neighbourhood of $h(I)$. So $y \in j(i(E))$ and

$$\big\|x^{n^2m^2} - x^{n^2m^2}y\big\| \leqslant C\big\|q\big(a^m\big)\big\| + \epsilon \leqslant C(2\epsilon)^m + \epsilon.$$

Since $\epsilon > 0$ was arbitrary, we conclude that $x^{n^2m^2} \in \overline{x^{n^2m^2} j(i(E))}$. This finishes the proof of the theorem. $\square$

3. A Ditkin–Shilov type theorem and an application to projective tensor products

The classical Ditkin–Shilov (or Wiener–Ditkin) theorem asserts that if A is a semisimple and regular commutative Banach algebra such that singletons in $\Delta(A)$ are Ditkin sets, then every closed subset of $\Delta(A)$ with scattered boundary is a set of synthesis.

We remind the reader that a topological space X is called scattered if every non-empty closed subset of X has an isolated point in the relative topology. Clearly, a countable locally compact Hausdorff space is scattered. Conversely, if X is a second countable locally compact Hausdorff space and X is scattered, then X is countable.

In [1, Theorem 1.2] Atzmon established, for unital A, a generalization which admits applications to projective tensor products. The first purpose of this section is to extend Theorem 1.2 of [1] to weak spectral sets and not necessarily unital A. In the sequel, for a closed subset E of a topological space, $\partial(E)$ will denote the boundary of E.

Theorem 3.1. *Let A be a regular and semisimple commutative Banach algebra, T a locally compact Hausdorff space and $f : \Delta(A) \to T$ a continuous, surjective and proper mapping. Suppose that for each $t \in T$, every closed subset of $f^{-1}(t)$ is a weak Ditkin set and that*

$$N = \sup\{\eta(F)\colon\ F \subseteq f^{-1}(t),\ F \text{ closed},\ t \in T\}$$

is finite. Let E be a closed subset of $\Delta(A)$ such that $f(\partial(E))$ is scattered. Then E is a weak spectral set and $\xi(E) \leqslant N$.

Proof. Let $a \in k(E)$ and let S denote the set of all $t \in T$ with the property that a^N does not at all points of $f^{-1}(t)$ belong locally to $\overline{j(E)}$. Since a belongs locally to $\overline{j(E)}$ at every point of $\Delta(A) \setminus \partial(E)$, we have $f^{-1}(t) \cap \partial(E) \neq \emptyset$ for every $t \in S$ and hence $S \subseteq f(\partial(E))$.

We first observe that S is closed in T. To see this, let $(s_\alpha)_\alpha$ be a net in S converging to some $t \in T$ and, towards a contradiction, suppose that $t \notin S$. For each α, choose $\varphi_\alpha \in f^{-1}(s_\alpha)$ such that a^N does not belong locally to $\overline{j(E)}$ at φ_α. Fix a compact neighbourhood U of t. Since $f^{-1}(U)$ is compact, after passing to a subnet if necessary, we can assume that $\varphi_\alpha \in f^{-1}(U)$ for all α and $\varphi_\alpha \to \varphi$ for some $\varphi \in \Delta(A)$. Then $\varphi \in f^{-1}(t)$ since $f(\varphi_\alpha) \to t$. Hence a^N belongs locally to $\overline{j(E)}$ at φ and since $\varphi_\alpha \to \varphi$, the same is true at φ_α for large α. This contradiction shows that S is closed in T.

Suppose that $S \neq \emptyset$. Then S has an isolated point s since $f(\partial(E))$ is scattered and S is closed. Choose an open subset V of T such that $V \cap S = \{s\}$. Since $C = \partial(E) \cap f^{-1}(s)$ is compact, we find an open neighbourhood W of C such that $\overline{W} \subseteq f^{-1}(V)$. Moreover, A being regular, there exists $u \in A$ such that $\widehat{u} = 1$ in a neighbourhood of C and $\widehat{u} = 0$ on $\Delta(A) \setminus \overline{W}$.

By hypothesis, C is a weak Ditkin set and $\eta(C) \leqslant N$. Therefore, there exists a sequence $(u_n)_n$ in $j(C)$ such that $\|a^N - a^N u_n\| \to 0$ and hence $\|a^N u - a^N u u_n\| \to 0$. Now, a^N belongs locally to $\overline{j(E)}$ at every point of $\Delta(A) \setminus f^{-1}(S)$ and of $\Delta(A) \setminus \partial(E)$ and $\widehat{uu_n}$ has compact support and vanishes on open sets containing C and $\Delta(A) \setminus f^{-1}(V)$, respectively. Since

$$\partial(E) \cap f^{-1}(S) \cap f^{-1}(V) = C,$$

this means that $a^N uu_n$ belongs locally to $\overline{j(E)}$ at every point of $\Delta(A)$ and at infinity. Semisimplicity of A implies that $a^N uu_n \in \overline{j(E)}$ and therefore $a^N u \in \overline{j(E)}$. Since $\widehat{u} = 1$ in a neighbourhood of C, it follows that a^N belongs locally to $\overline{j(E)}$ at every point of $f^{-1}(s)$. This contradiction shows that $S = \emptyset$ and finishes the proof of the theorem. $\square$

One situation in which Theorem 3.1 applies is as follows. Let A be a unital, regular and semisimple commutative Banach algebra. Let I be a closed ideal of A and let $X = \Delta(I) \cup \{\omega\}$, the one-point compactification of $\Delta(I)$. Define $f : \Delta(A) \to X$ by $f(\varphi) = \varphi|_I$ for $\varphi \in \Delta(A) \setminus h(I)$ and $f(\varphi) = \omega$ for $\varphi \in h(I)$. Then f is continuous. In fact, every compact subset C of $\Delta(I)$ is closed in $\Delta(A)$ since A is regular, and hence $f^{-1}(X \setminus C) = \Delta(A) \setminus C$ is open in $\Delta(A)$. We leave the reformulation of Theorem 3.1 in this situation to the reader. A more important application concerns projective tensor products. In preparation for this, we need the following two lemmas.

In passing we remind the reader that the structure space of the projective tensor product $A \widehat{\otimes} B$ of two commutative Banach algebras A and B identifies naturally with the product space $\Delta(A) \times \Delta(B)$. More precisely, given $\varphi \in \Delta(A)$ and $\psi \in \Delta(B)$, there is a unique homomorphism $\varphi \widehat{\otimes} \psi : A \widehat{\otimes} B \to \mathbb{C}$ such that $(\varphi \widehat{\otimes} \psi)(a \otimes b) = \varphi(a)\psi(b)$ for all $a \in A$ and $b \in B$, and the map $(\varphi, \psi) \to \varphi \widehat{\otimes} \psi$ is a homeomorphism from $\Delta(A) \times \Delta(B)$ onto $\Delta(A \widehat{\otimes} B)$.

Lemma 3.2. *Let F be a closed subset of $\Delta(B)$ and $\varphi \in \Delta(A)$. Then*

$$k\big(\{\varphi\} \times F\big) = k(\varphi) \widehat{\otimes} B + A \widehat{\otimes} k(F).$$

Proof. Since obviously $k(\varphi) \widehat{\otimes} B + A \widehat{\otimes} k(F) \subseteq k(\{\varphi\} \times F)$, we only have to show the reverse inclusion. Let $u = \sum_{i=1}^{\infty} a_i \otimes b_i \in k(\{\varphi\} \times F)$, $a_i \in A$, $b_i \in B$, $\sum_{i=1}^{\infty} \|a_i\| \cdot \|b_i\| < \infty$. Choose $e \in A$ such that $\varphi(e) = 1$ and write $a_i = x_i + \lambda_i e$, where $x_i \in k(\varphi)$ and $\lambda_i \in \mathbb{C}$. Then, since $|\lambda_i| = |\varphi(a_i)| \leqslant \|a_i\|$ and $\|x_i\| = \|a_i - \lambda_i e\| \leqslant \|a_i\|(1 + \|e\|)$,

$$u = \sum_{i=1}^{\infty} x_i \otimes b_i + e \otimes \sum_{i=1}^{\infty} \lambda_i b_i.$$

Now, for each $\psi \in F$,

$$\psi\left(\sum_{i=1}^{\infty} \lambda_i b_i\right) = \psi\left(\sum_{i=1}^{\infty} \varphi(a_i) b_i\right) = (\varphi \otimes \psi)(u) = 0.$$

Thus $\sum_{i=1}^{\infty} \lambda_i b_i \in k(F)$ and hence, since $x_i \in k(\varphi)$, $u \in k(\varphi) \widehat{\otimes} B + A \widehat{\otimes} k(F)$. $\square$

Lemma 3.3. *Let F and φ be as in Lemma* 3.2. *Then*

$$\xi\big(\{\varphi\} \times F\big) \leqslant \xi(\varphi) + \xi(F) - 1.$$

Proof. It suffices to show that if $n = \xi(\varphi) < \infty$ and $m = \xi(F) < \infty$, then $u^{n+m-1} \in \overline{j(\{\varphi\} \times F)}$ for every $u \in k(\{\varphi\} \times F)$. First, let u be of the form $u = \sum_{i=1}^{\infty} x_i \otimes y_i$, $x_i \in k(\varphi)$, $y_i \in B$. Since

$x^n \in \overline{j(\varphi)}$ for every $x \in k(\varphi)$ and every n-fold product of elements of $k(\varphi)$ is a linear combination of nth powers of elements in $k(\varphi)$, it follows that

$$u^n = \sum_{i_1,\dots,i_n=1}^{\infty} (x_{i_1} \cdot \ldots \cdot x_{i_n}) \otimes (y_{i_1} \cdot \ldots \cdot y_{i_n}) \in \overline{j(\varphi)} \mathbin{\widehat{\otimes}} B.$$

Secondly, let $z \in k(F)$ and $v = e \otimes z$, where $e \in A$ is such that $\varphi(e) = 1$. Then $v^m = e^m \otimes z^m \in A \mathbin{\widehat{\otimes}} \overline{j(F)}$. By the proof of Lemma 3.2, each element of $k(\{\varphi\} \times F)$ is of the form $u + v$, where u and v are as above. Then

$$\begin{aligned}(u+v)^{n+m-1} &= \sum_{i=0}^{n+m-1} \binom{n+m-1}{i} u^i v^{n+m-1} \\ &= u^n \sum_{i=n}^{n+m-1} \binom{n+m-1}{i} u^{i-n} v^{n+m-1} + v^m \sum_{i=0}^{n-1} \binom{n+m-1}{i} u^i v^{n-1-i} \\ &\subseteq \overline{j(F)} \mathbin{\widehat{\otimes}} B + A \mathbin{\widehat{\otimes}} \overline{j(F)}.\end{aligned}$$

Now, by Lemma 1.5 of [7], $\overline{j(\varphi)} \mathbin{\widehat{\otimes}} B + A \mathbin{\widehat{\otimes}} \overline{j(F)} \subseteq \overline{j(\{\varphi\} \times F)}$, and this finishes the proof. □

By simply taking spectral sets for $\{\varphi\}$ and F, it is clear that the estimate in Lemma 3.3 cannot be improved.

We say that a closed subset E of $\Delta(A)$ satisfies *condition* (D) (D referring to Ditkin) if there exists a constant $C > 0$ such that for every neighbourhood U of E, there exists $b \in A$ such that $\|b\| \leqslant C$, supp $\widehat{b} \subseteq U$ and $\widehat{b} = 1$ in a neighbourhood of E. The relevance of this notion is due to the fact that for unital A, (D) is equivalent to the condition appearing in Lemma 1.4, as we point out next.

Remark 3.4. Suppose that A has an identity e. For a closed subset E of $\Delta(A)$, the following are equivalent.

(i) E satisfies condition (D).
(ii) There exists a constant $c > 0$ such that for every compact subset K of $\Delta(A)$ which is disjoint from E, there exists $a \in j(E)$ such that $\|a\| \leqslant c$ and $\widehat{a} = 1$ on K.

To see this, suppose first that (ii) holds and let U be an open set containing E. Choose an open set V such that $E \subseteq V$ and $\overline{V} \subseteq U$ and let $K = \Delta(A) \setminus V$. By (i), there exists $b \in j(E)$ so that $\widehat{b} = 1$ on K and $\|b\| \leqslant c$ (with c only depending on E). Let $a = e - b$, then $\|a\| \leqslant 1 + c$, supp $\widehat{a} \subseteq \overline{V} \subseteq U$ and $\widehat{a} = 1$ in a neighbourhood of E.

(i) ⇒ (ii) is even easier.

Theorem 3.5. *Let A and B be regular commutative Banach algebras such that both are unital and $A \mathbin{\widehat{\otimes}} B$ is semisimple. Suppose that the following conditions* (1) *and* (2) *are satisfied.*

(1) *Every closed subset of $\Delta(B)$ is a weak spectral set and satisfies condition* (D) *and*

$$N = \sup\{\xi(F):\ F \subseteq \Delta(B) \text{ closed}\} < \infty.$$

(2) *Each singleton* $\{\varphi\}$, $\varphi \in \Delta(A)$, *is a weak spectral set and satisfies condition* (D) *and*

$$M = \sup\big\{\xi(\varphi)\colon\ \varphi \in \Delta(A)\big\} < \infty.$$

Let E *be a closed subset of* $\Delta(A) \times \Delta(B)$ *such that the set*

$$\big\{\varphi \in \Delta(A)\colon\ \partial(E) \cap \big(\{\varphi\} \times \Delta(B)\big) \neq \emptyset\big\}$$

is scattered. Then E *is a weak spectral set and* $\xi(E) \leqslant N + M - 1$.

Proof. We apply Theorem 3.1 to $A \widehat{\otimes} B$, $T = \Delta(A)$ and the map

$$f : \Delta(A \widehat{\otimes} B) = \Delta(A) \times \Delta(B) \to \Delta(A), \quad (\varphi, \psi) \to \varphi.$$

By hypothesis, $f(\partial(E))$ is scattered. It follows from conditions (1) and (2) and Lemma 3.3 that for every closed subset F of $\Delta(B)$ and $\varphi \in \Delta(A)$, $\{\varphi\} \times F$ is a weak spectral set and

$$\xi\big(\{\varphi\} \times F\big) \leqslant \xi(\varphi) + \xi(F) - 1 \leqslant N + M - 1.$$

Now, let c and d denote the constants occuring in condition (D) for $\{\varphi\}$ and F, respectively, and let U be an open neighbourhood of φ in $\Delta(A)$ and V an open neighbourhood of F in $\Delta(B)$. There exist $a \in A$ and $b \in B$ with the following properties: $\|a\| \leqslant c$, $\|b\| \leqslant d$, $\operatorname{supp} \widehat{a} \subseteq U$, $\operatorname{supp} \widehat{b} \subseteq V$, $\widehat{a} = 1$ near φ and $\widehat{b} = 1$ in a neighbourhood of F. Then the element $x = a \otimes b$ of $A \widehat{\otimes} B$ satisfies $\|x\| \leqslant cd$, $\operatorname{supp} \widehat{x} \subseteq U \times V$ and $\widehat{x} = 1$ in a neighbourhood of $\{\varphi\} \times F$. It follows from Remark 3.4 and Lemma 1.4 that $\{\varphi\} \times F$ is a weak Ditkin set and $\eta(\{\varphi\} \times F) = \xi(\{\varphi\} \times F) \leqslant N + M - 1$. Thus Theorem 3.1 applies and yields that E is a weak spectral set and $\xi(E) \leqslant N + M - 1$. □

Theorem 3.5 applies, for instance, to the projective tensor product of any two of the Banach algebras $C(X)$, X a compact Hausdorff space, $A(G)$ for a compact group G, $C^n[0,1]$ and $\operatorname{Lip}_\alpha X$ (see Section 1).

4. Weak spectral synthesis for Fourier algebras

It was shown in [12, Theorem 3.1] that weak spectral synthesis fails in $A(G) = L^1(\widehat{G})$ for every non-discrete locally compact abelian group G. Employing this result as well as a deep theorem due to Zelmanov [21], we settle in this final section the weak synthesis problem for the Fourier algebras of arbitrary locally compact groups.

It is easy to see that weak spectral synthesis holds for $A(G)$ when G is discrete. In fact, $A(G)$ is Tauberian [6, Corollaire 3.38] and hence by [19, Corollary 2.4] (see also [11, Corollary 3.10]),

$$\xi(E) \leqslant 1 + \xi\big(\partial(E)\big) = 1 + \xi(\emptyset) = 2$$

for every subset E of G. Alternatively, one can appeal to the following simple observation.

Lemma 4.1. *Let* A *be a semisimple and regular commutative Banach algebra. Suppose that* A *is Tauberian, and let* E *be an open and closed subset of* $\Delta(A)$. *Then* E *is a weak spectral set and* $\xi(E) \leqslant 2$.

Proof. Let $x \in k(E)$. Since A is Tauberian, $x^2 \in xA \subseteq \overline{xj(\emptyset)}$. It therefore suffices to show that $xy \in \overline{j(E)}$ for every $y \in j(\emptyset)$. Now, xy belongs locally to $\overline{j(E)}$ at infinity, at every point of E since E is open, and at every point of $\Delta(A) \setminus E$ anyway. Thus $xy \in \overline{j(E)}$ since A is semisimple and regular. □

The next lemma will be needed to carry out a projective limit argument in the proof of Theorem 4.3 below.

Lemma 4.2. *Let K be a compact normal subgroup of G and identify G and G/K with $\Delta(A(G))$ and $\Delta(A(G/K))$, respectively. Let $q : G \to G/K$ denote the quotient homomorphism. Then $\xi(E) \leqslant \xi(q^{-1}(E))$ for every closed subset E of G/K.*

Proof. We can assume that $n = \xi(q^{-1}(E)) < \infty$. Let $u \in A(G/K)$ such that $u|_E = 0$ and $\epsilon > 0$. Then $u \circ q \in A(G)$ and $u \circ q|_{q^{-1}(E)} = 0$. By hypothesis, there exists $v \in A(G)$ such that $\|(u \circ q)^n - v\|_{A(G)} < \epsilon$ and v vanishes on some open set U which contains $q^{-1}(E)$ and is such that $G \setminus U$ is compact. Since $q^{-1}(E) = q^{-1}(E)K$ and K is compact, a simple topological argument shows that we can assume that $U = UK$. For $v \in A(G)$ and $k \in K$, let $R_k v \in A(G)$ be defined by $R_k v(x) = v(xk)$, $x \in G$. Then the element $\int_K R_k v\, dk$ of $A(G)$ vanishes on U. Define $w \in A(G/K)$ by

$$w(xK) = \int_K R_k v(x)\, dk, \quad x \in G.$$

Then w vanishes on the neighbourhood $q(U)$ of E, $G/K \setminus q(U)$ is compact and

$$\begin{aligned} \|u^n - w\|_{A(G/K)} &= \|u^n \circ q - w \circ q\|_{A(G)} = \|(u \circ q)^n - w \circ q\|_{A(G)} \\ &= \left\| \int_K R_k(u \circ q)^n\, dk - \int_K R_k v\, dk \right\|_{A(G)} \\ &\leqslant \int_K \big\| R_k\big((u \circ q)^n - v\big) \big\|_{A(G)}\, dk \\ &= \|(u \circ q)^n - v\|_{A(G)}. \end{aligned}$$

So $w \in j(E)$ and $\|u^n - w\|_{A(G/K)} < \epsilon$. □

Theorem 4.3. *Let G be an arbitrary locally compact group. Then weak spectral synthesis holds for $A(G)$ if and only if G is discrete.*

Proof. Since $A(G)$ is Tauberian, by Lemma 4.1 we only have to show that G is discrete whenever weak synthesis holds for $A(G)$. As mentioned above, this conclusion is true if G is abelian [12, Theorem 3.1], and our proof is a reduction to this case.

In the first instance, assume that G is a connected Lie group and that the radical R of G, the maximal connected solvable closed normal subgroup of G, is non-trivial. Then R contains a non-trivial connected abelian closed normal subgroup H, namely the last non-trivial member of

the commutator series of R. By Corollary 2.4, weak synthesis holds for $A(H)$ and hence H has to be discrete. This contradiction shows that $R = \{e\}$. So G is semisimple. If G is non-trivial, it contains an infinite compact subgroup K, and by a theorem of Zelmanov [21, Theorem 2], K in turn contains an infinite (closed) abelian subgroup H. As above, this leads to a contradiction. Consequently, G is trivial.

Now, drop the hypothesis that G be a Lie group. The connected group G is a projective limit of Lie groups G/K_α. Since K_α is compact, by Lemma 4.2 weak spectral synthesis holds for G/K_α. By the preceding paragraph, $K_\alpha = G$ for all α and hence $G = \{e\}$.

Turning to the general case, let G_0 denote the connected component of the identity of G. Then weak synthesis holds for G_0 and then $G_0 = \{e\}$ by what we have already shown. So G is totally disconnected. Finally, fix a compact open subgroup K of G and suppose that K is infinite. Applying Zelmanov's theorem again leads to a contradiction. Thus G is discrete. $\square$

The reader should compare the preceding theorem with Proposition 2.2 of [8], where it was shown that:

(i) local spectral synthesis holds for $A(G)$ if and only if G is discrete;
(ii) spectral synthesis holds for $A(G)$ if and only if G is discrete and $u \in \overline{uA(G)}$ for every $u \in A(G)$.

So weak spectral synthesis and local spectral synthesis are equivalent for Fourier algebras. It is not unlikely that these properties already force spectral synthesis to hold for $A(G)$ since no example seems to be known of a discrete group G for which the condition $u \in \overline{uA(G)}$ (the existence of an approximate identity in $A(G)$ in the weakest possible sense) is not satisfied.

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Noncommutative Berezin transforms and multivariable operator model theory ☆

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Abstract

In this paper, we initiate the study of a class $\mathbf{D}_p^m(\mathcal{H})$ of noncommutative domains of n-tuples of bounded linear operators on a Hilbert space $\mathcal{H}$, where $m \geqslant 2$, $n \geqslant 2$, and p is a positive regular polynomial in n noncommutative indeterminates. These domains are defined by certain positivity conditions on p, i.e.,

$$\mathbf{D}_p^m(\mathcal{H}) := \{X := (X_1, \ldots, X_n)\colon\ (1-p)^k(X, X^*) \geqslant 0 \text{ for } 1 \leqslant k \leqslant m\}.$$

Each such a domain has a universal model $(W_1, \ldots, W_n)$ of weighted shifts acting on the full Fock space $F^2(H_n)$ with n generators. The study of $\mathbf{D}_p^m(\mathcal{H})$ is close related to the study of the weighted shifts $W_1, \ldots, W_n$, their joint invariant subspaces, and the representations of the algebras they generate: the domain algebra $\mathcal{A}_n(\mathbf{D}_p^m)$, the Hardy algebra $F_n^\infty(\mathbf{D}_p^m)$, and the C^*-algebra $C^*(W_1, \ldots, W_n)$. A good part of this paper deals with these issues.

The main tool, which we introduce here, is a noncommutative Berezin type transform associated with each n-tuple of operators in $\mathbf{D}_p^m(\mathcal{H})$. The study of this transform and its boundary behavior leads to Fatou type results, functional calculi, and a model theory for n-tuples of operators in $\mathbf{D}_p^m(\mathcal{H})$. These results extend to noncommutative varieties $\mathcal{V}_{p,\mathcal{Q}}^m(\mathcal{H}) \subset \mathbf{D}_p^m(\mathcal{H})$ generated by classes $\mathcal{Q}$ of noncommutative polynomials. When $m \geqslant 2$, $n \geqslant 2$, $p = Z_1 + \cdots + Z_n$, and $\mathcal{Q} = 0$, the elements of the corresponding variety $\mathcal{V}_{p,\mathcal{Q}}^m(\mathcal{H})$ can be seen as multivariable noncommutative analogues of Agler's m-hypercontractions.

Our results apply, in particular, when $\mathcal{Q}$ consists of the noncommutative polynomials $Z_iZ_j - Z_jZ_i$, $i, j = 1, \ldots, n$. In this case, the model space is a symmetric weighted Fock space $F_s^2(\mathbf{D}_p^m)$, which is identified with a reproducing kernel Hilbert space of holomorphic functions on a Reinhardt domain in $\mathbb{C}^n$, and the universal model is the n-tuple $(M_{\lambda_1}, \ldots, M_{\lambda_n})$ of multipliers by the coordinate functions. In this particular

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case, we obtain a model theory for commuting n-tuples of operators in $\mathbf{D}_p^m(\mathcal{H})$, recovering several results already existent in the literature.

Keywords: Multivariable operator theory; Noncommutative domain; Noncommutative variety; Dilation theory; Model theory; Weighted shift; Wold decomposition; Fock space; von Neumann inequality; Berezin transform; Creation operators

0. Introduction

Let $\mathbb{F}_n^+$ be the unital free semigroup on n generators $g_1, \ldots, g_n$ and the identity g_0, and consider a polynomial $q = q(Z_1, \ldots, Z_n) = \sum c_\alpha Z_\alpha$ in noncommutative indeterminates $Z_1, \ldots, Z_n$, where we denote $Z_\alpha := Z_{i_1} \ldots Z_{i_k}$ if $\alpha = g_{i_1} \ldots g_{i_k} \in \mathbb{F}_n^+$, $i_1, \ldots, i_k \in \{1, \ldots, n\}$, and $Z_{g_0} := I$. We associate with q the operator

$$q(X, X^*) := \sum c_\alpha X_\alpha X_\alpha^*,$$

where $X := (X_1, \ldots, X_n) \in B(\mathcal{H})^n$ and $B(\mathcal{H})$ is the algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$. Let $p = p(Z_1, \ldots, Z_n) = \sum a_\alpha Z_\alpha$, $a_\alpha \in \mathbb{C}$, be a positive regular polynomial, i.e., $a_\alpha \geqslant 0$, $a_{g_0} = 0$, and $a_{g_i} > 0$, $i = 1, \ldots, n$. Given $m, n \in \{1, 2, \ldots\}$, we define the noncommutative domain

$$\mathbf{D}_p^m(\mathcal{H}) := \bigl\{X := (X_1, \ldots, X_n) \in B(\mathcal{H})^n\colon\ (1-p)^k(X, X^*) \geqslant 0 \text{ for } 1 \leqslant k \leqslant m\bigr\}.$$

In the last fifty years, these domains have been studied in several particular cases. Most of all, we should mention that the study of the closed operator unit ball

$$\bigl[B(\mathcal{H})\bigr]_1^- := \bigl\{X \in B(\mathcal{H})\colon\ I - XX^* \geqslant 0\bigr\}$$

(which corresponds to the case $m = 1$, $n = 1$, and $p = Z$) has generated the celebrated Sz.-Nagy–Foias theory of contractions on Hilbert spaces and has had profound implications in function theory, interpolation, prediction theory, scattering theory, and linear system theory (see [11,21, 22,51], etc.). The case when $m = 1$, $n \geqslant 2$, and $p = Z_1 + \cdots + Z_n$, corresponds to the closed operator ball

$$\bigl[B(\mathcal{H})^n\bigr]_1^- := \bigl\{(X_1, \ldots, X_n) \in B(\mathcal{H})^n\colon\ I - X_1X_1^* - \cdots - X_nX_n^* \geqslant 0\bigr\}$$

and its study has generated a *free* analogue of Sz.-Nagy–Foias theory (see [10,13,19,23,30–43,46, 47], etc.). The commutative case, which corresponds to the subvariety of $[B(\mathcal{H})^n]_1$ determined by the commutators $Z_iZ_j - Z_jZ_i$, $i, j = 1, \ldots, n$, was considered by Drurry [20], extensively studied by Arveson [7,8], and considered by the author [39] in connection with noncommutative Poisson transforms. More general subvarieties in $[B(\mathcal{H})^n]_1$, determined by classes of noncommutative polynomials, were considered by the author in [43,46]. The study of the unit ball $[B(\mathcal{H})^n]_1$ was extended, in [45], to noncommutative domains $\mathbf{D}_p^m(\mathcal{H})$ (respectively subvarieties) when $m = 1$, $n \geqslant 1$, and p is any positive regular noncommutative polynomial (respectively free holomorphic function in the sense of [44]).

In this paper, we initiate the study of noncommutative domains $\mathbf{D}_p^m(\mathcal{H})$, when $m \geqslant 2$, $n \geqslant 2$, and p is any positive regular noncommutative polynomial. What makes the case $m \geqslant 2$ quite different from the case $m = 1$ is that $\mathbf{D}_p^m(\mathcal{H})$ is not a ball-like domain, when $m \geqslant 2$. This can be seen even in the single variable case ($n = 1$) (see [1,2,26,27]). We introduce a class of noncommutative Berezin transforms associated with any n-tuple of operators in $\mathbf{D}_p^m(\mathcal{H})$. The study of these transforms and their boundary behavior leads to Fatou type results, functional calculi, and a model theory for n-tuples of operators in $\mathbf{D}_p^m(\mathcal{H})$. Our results extend to noncommutative varieties $\mathcal{V}_{p,\mathcal{Q}}^m(\mathcal{H})$ generated by classes $\mathcal{Q}$ of noncommutative polynomials, i.e.,

$$\mathcal{V}_{p,\mathcal{Q}}^m(\mathcal{H}) := \bigl\{(X_1, \ldots, X_n) \in \mathbf{D}_p^m(\mathcal{H}) \colon\ q(X_1, \ldots, X_n) = 0,\ q \in \mathcal{Q}\bigr\}.$$

In Section 1, we associate with each $m, n \in \{1, 2, \ldots\}$ and each positive regular noncommutative polynomial $p = p(Z_1, \ldots, Z_n) = \sum a_\alpha Z_\alpha$, a noncommutative domain $\mathbf{D}_p^m(\mathcal{H}) \subset B(\mathcal{H})^n$ and a unique n-tuple $(W_1, \ldots, W_n)$ of weighted shifts acting on the full Fock space $F^2(H_n)$ with n generators. They will play the role of the *universal model* for the elements of $\mathbf{D}_p^m(\mathcal{H})$. We also introduce the n-tuple $(\Lambda_1, \ldots, \Lambda_n)$ associated with $\mathbf{D}_p^m(\mathcal{H})$, which turns out to be the universal model associated with the noncommutative domain $\mathbf{D}_{\tilde{p}}^m(\mathcal{H})$, where $\tilde{p} = \tilde{p}(Z_1, \ldots, Z_n) = \sum a_{\tilde{\alpha}} Z_\alpha$ and $\tilde{\alpha}$ denotes the reverse of $\alpha = g_{i_1} \cdots g_{i_k}$, i.e., $\tilde{\alpha} := g_{i_k} \cdots g_{i_1}$.

In Section 2, we introduce a *noncommutative Berezin transform* $\mathbf{B}_T$ associated with each n-tuple of operators $T := (T_1, \ldots, T_n) \in \mathbf{D}_p^m(\mathcal{H})$ with the joint spectral radius $r_p(T_1, \ldots, T_n) < 1$. More precisely, the map $\mathbf{B}_T : B(F^2(H_n)) \to B(\mathcal{H})$ is defined by

$$\begin{aligned}
&\bigl\langle \mathbf{B}_T[g]x, y\bigr\rangle \\
&\quad := \Biggl\langle \Biggl(I - \sum_{|\alpha| \geqslant 1} \bar{a}_{\tilde{\alpha}} \Lambda_\alpha^* \otimes T_{\tilde{\alpha}}\Biggr)^{-m} \bigl(g \otimes \Delta_{T,m,p}^2\bigr) \Biggl(I - \sum_{|\alpha| \geqslant 1} a_{\tilde{\alpha}} \Lambda_\alpha \otimes T_{\tilde{\alpha}}^*\Biggr)^{-m} (1 \otimes x),\ 1 \otimes y \Biggr\rangle,
\end{aligned}$$

where $\Delta_{T,m,p} := [(1 - p)^m (T, T^*)]^{1/2}$ and $x, y \in \mathcal{H}$. We remark that in the particular case when: $m = 1$, $n = 1$, $p = Z$, $\mathcal{H} = \mathbb{C}$, and $T = \lambda \in \mathbb{D}$, we recover the Berezin transform [12] of a bounded linear operator on the Hardy space $H^2(\mathbb{D})$, i.e.,

$$\mathbf{B}_\lambda[g] = \bigl(1 - |\lambda|^2\bigr)\langle g k_\lambda, k_\lambda\rangle, \quad g \in B\bigl(H^2(\mathbb{D})\bigr),$$

where $k_\lambda(z) := (1 - \bar{\lambda} z)^{-1}$ and $z, \lambda \in \mathbb{D}$. The noncommutative Berezin transform will play an important role in this paper.

First, we show that the Berezin transform has an extension $\widetilde{\mathbf{B}}_T : B(F^2(H_n)) \to B(\mathcal{H})$ to any n-tuple $T \in \mathbf{D}_p^m(\mathcal{H})$. This is used to prove that the restriction of $\widetilde{\mathbf{B}}_T$ to the operator system $\mathcal{S} := \overline{\operatorname{span}}\{W_\alpha W_\beta^*;\ \alpha, \beta \in \mathbb{F}_n^+\}$ is a unital completely contractive linear map such that

$$\widetilde{\mathbf{B}}_T\bigl[W_\alpha W_\beta^*\bigr] = T_\alpha T_\beta^*, \quad \alpha, \beta \in \mathbb{F}_n^+,$$

when $T := (T_1, \ldots, T_n) \in \mathbf{D}_p^m(\mathcal{H})$ is a *pure* n-tuple of operators (i.e. $p^k(T, T^*) \to 0$ strongly as $k \to \infty$). We obtain a similar result for n-tuple of operators with the radial property, i.e., $(rT_1, \ldots, rT_n) \in \mathbf{D}_p^m(\mathcal{H})$ for any $r \in (\delta, 1]$ and some $\delta \in (0, 1)$. In this case, we show that

$$\Psi(g) := \lim_{r \to 1} \mathbf{B}_{rT}[g], \quad g \in \mathcal{S},$$

exists in the norm operator topology and defines a unital completely contractive map $\Psi : \mathcal{S} \to B(\mathcal{H})$ such that $\Psi(W_\alpha W_\beta^*) = T_\alpha T_\beta^*$, $\alpha, \beta \in \mathbb{F}_n^+$.

In Section 3, we introduce the Hardy algebra $F_n^\infty(\mathbf{D}_p^m)$ (respectively $R_n^\infty(\mathbf{D}_p^m)$) associated with the noncommutative domain $\mathbf{D}_p^m$ and prove some basic properties. We mention that an n-tuple of operators $T := (T_1, \ldots, T_n) \in \mathbf{D}_p^m(\mathcal{H})$ is called *completely non-coisometric* (c.n.c.) if there is no vector $h \in \mathcal{H}$, $h \neq 0$, such that $\langle p^k(T, T^*)h, h\rangle = \|h\|^2$ for any $k = 1, 2, \ldots.$ The main result of Section 3 is an $F_n^\infty(\mathbf{D}_p^m)$-functional calculus for (c.n.c.) n-tuples of operators in the noncommutative domain $\mathbf{D}_p^m(\mathcal{H})$. More precisely, we show that if $T := (T_1, \ldots, T_n)$ is a c.n.c. n-tuple of operators in a noncommutative domain $\mathbf{D}_p^m(\mathcal{H})$ with the radial property, then

$$\Phi(g) := \text{SOT-}\lim_{r\to 1} g(rT_1, \ldots, rT_n), \quad g = g(W_1, \ldots, W_n) \in F_n^\infty\big(\mathbf{D}_p^m\big),$$

exists in the strong operator topology and defines a map $\Phi : F_n^\infty(\mathbf{D}_p^m) \to B(\mathcal{H})$ with the following properties:

(i) $\Phi(g) = \text{SOT-}\lim_{r\to 1} \mathbf{B}_{rT}[g]$, where $\mathbf{B}_{rT}$ is the Berezin transform at $rT \in \mathbf{D}_p^m(\mathcal{H})$;
(ii) Φ is WOT-continuous (respectively SOT-continuous) on bounded sets;
(iii) Φ is a unital completely contractive homomorphism.

In Section 4, we find all the eigenvectors for $W_1^*, \ldots, W_n^*$, where $(W_1, \ldots, W_n)$ is the universal model associated with the noncommutative domain $\mathbf{D}_p^m$. As consequences, we identify the w^*-continuous multiplicative linear functional on the Hardy algebra $F_n^\infty(\mathbf{D}_p^m)$ and find the joint right spectrum of $(W_1, \ldots, W_n)$. We introduce the symmetric weighted Fock space $F_s^2(\mathbf{D}_p^m)$ and identify it with $H^2(\mathbf{D}_{f,\circ}^1(\mathbb{C}))$, the reproducing kernel Hilbert space with reproducing kernel $K_p : \mathbf{D}_{p,\circ}^1(\mathbb{C}) \times \mathbf{D}_{p,\circ}^1(\mathbb{C}) \to \mathbb{C}$ defined by

$$K_p(\mu, \lambda) := \frac{1}{(1 - \sum a_\alpha \mu_\alpha \bar{\lambda}_\alpha)^m} \quad \text{for all } \lambda, \mu \in \mathbf{D}_{p,\circ}^1(\mathbb{C}),$$

where

$$\mathbf{D}_{p,\circ}^1(\mathbb{C}) := \left\{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n \colon \sum a_\alpha |\lambda_\alpha|^2 < 1\right\} \subset \mathbf{D}_p^m(\mathbb{C}),$$

$\lambda_\alpha := \lambda_{i_1} \cdots \lambda_{i_m}$ if $\alpha = g_{i_1} \cdots g_{i_m} \in \mathbb{F}_n^+$, and $\lambda_{g_0} = 1$.

We show that the algebra $H^\infty(\mathbf{D}_{p,\circ}^1(\mathbb{C}))$ of all multipliers of the Hilbert space $H^2(\mathbf{D}_{p,\circ}^1(\mathbb{C}))$ is reflexive and coincides with the weakly closed algebra generated by the identity and the multipliers $M_{\lambda_1}, \ldots, M_{\lambda_n}$ by the coordinate functions. Moreover, the multipliers $M_{\lambda_1}, \ldots, M_{\lambda_n}$ can be identified with the operators $L_1, \ldots, L_n$, where

$$L_i := P_{F_s^2(\mathbf{D}_p^m)} W_i|_{F_s^2(\mathbf{D}_p^m)}, \quad i = 1, \ldots, n,$$

and $(W_1, \ldots, W_n)$ is the universal model associated with the noncommutative domain $\mathbf{D}_p^m$. Section 4 will play an important role in connecting the results of the present paper to analytic function theory on Reinhardt domains in $\mathbb{C}^n$, as well as, to model theory for commuting n-tuples of operators.

In Section 5, we consider noncommutative varieties $\mathcal{V}^m_{p,\mathcal{Q}}(\mathcal{H}) \subset \mathbf{D}^m_p(\mathcal{H})$ determined by sets $\mathcal{Q}$ of noncommutative polynomials. We associate with each such a variety a *universal model* $(B_1, \ldots, B_n) \in \mathcal{V}^m_{p,\mathcal{Q}}(\mathcal{N}_\mathcal{Q})$, which is the compression of $(W_1, \ldots, W_n)$ to an appropriate subspace $\mathcal{N}_\mathcal{Q}$ of the full Fock space $F^2(H_n)$. We introduce the *constrained noncommutative Berezin transform* $\mathbf{B}^c_T : B(\mathcal{N}_\mathcal{Q}) \to B(\mathcal{H})$ and use it to obtain analogues of the results of Section 2, for subvarieties. We also show that, if the constants belong to the subspace $\mathcal{N}_\mathcal{Q}$, then the C^*-algebra $C^*(B_1, \ldots, B_n)$ is irreducible and all the compacts operators in $B(\mathcal{N}_\mathcal{Q})$ are contained in the operator space $\overline{\operatorname{span}}\{B_\alpha B^*_\beta\colon \alpha, \beta \in \mathbb{F}^+_n\}$. These results are vital for the development of model theory on noncommutative varieties.

In Section 6, we obtain dilation and model theorems for the elements of the noncommutative variety $\mathcal{V}^m_{p,\mathcal{Q}}(\mathcal{H})$. First, we prove that an n-tuple of operators $T := (T_1, \ldots, T_n) \in B(\mathcal{H})^n$ is a pure element of $\mathcal{V}^m_{p,\mathcal{Q}}(\mathcal{H})$ if and only if

$$T^*_i = \big(B^*_i \otimes I_\mathcal{D}\big)\big|_\mathcal{H}, \quad i = 1, \ldots, n,$$

where $\mathcal{H}$ is an invariant subspace under each operator $B^*_i \otimes I_\mathcal{D}$, $i = 1, \ldots, n$, $\mathcal{D} := \overline{\Delta_{p,m,T}\mathcal{H}}$, and $\Delta_{p,m,T} := [(1 - p)^m(T, T^*)]^{1/2}$.

When $(T_1, \ldots, T_n) \in \mathcal{V}^m_{p,\mathcal{Q}}(\mathcal{H})$ is an n-tuple of operators (on a separable Hilbert space $\mathcal{H}$) with the radial property and $\mathcal{Q}$ is a set of homogeneous noncommutative polynomials, we show that there exists a $*$-representation $\pi : C^*(B_1, \ldots, B_n) \to B(\mathcal{K}_\pi)$ on a separable Hilbert space $\mathcal{K}_\pi$, which annihilates the compact operators and

$$p\big(\pi(B), \pi(B)^*\big) = I_{\mathcal{K}_\pi}, \quad \text{where } \pi(B) := \big(\pi(B_1), \ldots, \pi(B_n)\big),$$

such that $T^*_i = V^*_i|\mathcal{H}$ for $i = 1, \ldots, n$, where the operators

$$V_i := \begin{bmatrix} B_i \otimes I_\mathcal{D} & 0 \\ 0 & \pi(B_i) \end{bmatrix}, \quad i = 1, \ldots, n,$$

are acting on the Hilbert space $\widetilde{\mathcal{K}} := (\mathcal{N}_\mathcal{Q} \otimes \mathcal{D}) \oplus \mathcal{K}_\pi$ and $\mathcal{H}$ is identified with a $*$-cyclic co-invariant subspace of $\tilde{\mathcal{K}}$ under each operator V_i, $i = 1, \ldots, n$.

In the single variable case, when $m \geqslant 2$, $n = 1$, $p = Z$, and $\mathcal{Q} = 0$, the corresponding variety coincides with the set of all m-hypercontractions studied by Agler in [1,2], and recently by Olofsson [26,27]. When $m \geqslant 2$, $n \geqslant 2$, $p = Z_1 + \cdots + Z_n$, and $\mathcal{Q} = 0$, the elements of the corresponding domain $\mathbf{D}^m_p(\mathcal{H})$ can be seen as multivariable noncommutative analogues of Agler's m-hypercontractions.

In the particular case when $\mathcal{Q}_c$ coincides with the set of polynomials $Z_iZ_j - Z_jZ_i$, $i, j = 1, \ldots, n$, we can combine the results of Sections 4 and 6 to recover several results concerning model theory for commuting n-tuples of operators. The case $m \geqslant 2$, $n \geqslant 2$, $p = Z_1 + \cdots + Z_n$, and $\mathcal{Q} = \mathcal{Q}_c$, was studied by Athavale [9], Müller [24], Müller, Vasilescu [25], Vasilescu [52], and Curto, Vasilescu [14]. Some of these results concerning model theory were extended by S. Pott [48] to positive regular polynomials in commuting indeterminates.

We should mention that most of the results of this paper are presented in a more general setting, namely, when the polynomial p is replaced by a positive regular free holomorphic function (see Section 1 for terminology). In a future paper, we expect to use these results to obtain functional models for the elements of the noncommutative domain $\mathbf{D}^m_p(\mathcal{H})$ (respectively subvariety $\mathcal{V}^m_{p,\mathcal{Q}}(\mathcal{H})$), based on characteristic functions.

1. Noncommutative domains and universal models

In this section, we associate with each positive regular free holomorphic function f on $[B(\mathcal{H})^n]_\rho$, $\rho > 0$, and each $m, n \in \{1, 2, \ldots\}$, a noncommutative domain $\mathbf{D}_f^m(\mathcal{H}) \subset B(\mathcal{H})^n$ and a unique n-tuple $(W_1, \ldots, W_n)$ of weighted shifts. This n-tuple of operators will play the role of the *universal model* for the elements of $\mathbf{D}_f^m(\mathcal{H})$. We also introduce the n-tuple $(\Lambda_1, \ldots, \Lambda_n)$ associated with $\mathbf{D}_f^m(\mathcal{H})$, which turns out to be the universal model for the elements of the noncommutative domain $\mathbf{D}_{\tilde{f}}^m$.

Let H_n be an n-dimensional complex Hilbert space with orthonormal basis $e_1, e_2, \ldots, e_n$, where $n \in \{1, 2, \ldots\}$. We consider the full Fock space of H_n defined by

$$F^2(H_n) := \bigoplus_{k \geqslant 0} H_n^{\otimes k},$$

where $H_n^{\otimes 0} := \mathbb{C}1$ and $H_n^{\otimes k}$ is the (Hilbert) tensor product of k copies of H_n. Define the left creation operators $S_i : F^2(H_n) \to F^2(H_n)$, $i = 1, \ldots, n$, by

$$S_i \varphi := e_i \otimes \varphi, \quad \varphi \in F^2(H_n),$$

and the right creation operators $R_i : F^2(H_n) \to F^2(H_n)$, $i = 1, \ldots, n$, by $R_i \varphi := \varphi \otimes e_i$, $\varphi \in F^2(H_n)$.

The algebra F_n^∞ and its norm closed version, the noncommutative disc algebra $\mathcal{A}_n$, were introduced by the author [34] in connection with a multivariable noncommutative von Neumann inequality. F_n^∞ is the algebra of left multipliers of $F^2(H_n)$ and can be identified with the weakly closed (or w^*-closed) algebra generated by the left creation operators $S_1, \ldots, S_n$ acting on $F^2(H_n)$, and the identity. The noncommutative disc algebra $\mathcal{A}_n$ is the norm closed algebra generated by $S_1, \ldots, S_n$, and the identity. For basic properties concerning the noncommutative analytic Toeplitz algebra F_n^∞ we refer to [4,15–18,32,33,35–37,39].

Let $\mathbb{F}_n^+$ be the unital free semigroup on n generators $g_1, \ldots, g_n$ and the identity g_0. The length of $\alpha \in \mathbb{F}_n^+$ is defined by $|\alpha| := 0$ if $\alpha = g_0$ and $|\alpha| := k$ if $\alpha = g_{i_1} \cdots g_{i_k}$, where $i_1, \ldots, i_k \in \{1, \ldots, n\}$. If $X := (X_1, \ldots, X_n) \in B(\mathcal{H})^n$, where $B(\mathcal{H})$ is the algebra of all bounded linear operators on the Hilbert space $\mathcal{H}$, we denote $X_\alpha := X_{i_1} \cdots X_{i_k}$ and $X_{g_0} := I_{\mathcal{H}}$.

We say that $f = f(X_1, \ldots, X_n) := \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha X_\alpha$, $a_\alpha \in \mathbb{C}$, is a free holomorphic function on the noncommutative ball $[B(\mathcal{H})^n]_\rho$ for some $\rho > 0$, where

$$\left[B(\mathcal{H})^n\right]_\rho := \left\{(X_1, \ldots, X_n) \in B(\mathcal{H})^n \colon \left\|X_1 X_1^* + \cdots + X_n X_n^*\right\| < \rho\right\},$$

if the series $\sum_{k=0}^\infty \sum_{|\alpha|=k} a_\alpha X_\alpha$ is convergent in the operator norm topology for any $(X_1, \ldots, X_n) \in [B(\mathcal{H})^n]_\rho$. According to [44], f is a free holomorphic function on $[B(\mathcal{H})^n]_\rho$ if and only if

$$\limsup_{k \to \infty} \left(\sum_{|\alpha|=k} |a_\alpha|^2 \right)^{1/2k} \leqslant \frac{1}{\rho}.$$

Throughout this paper, we assume that $a_\alpha \geqslant 0$ for any $\alpha \in \mathbb{F}_n^+$, $a_{g_0} = 0$, and $a_{g_i} > 0$, $i = 1, \ldots, n$. A function f satisfying all these conditions on the coefficients is called a *positive regular free holomorphic function on* $[B(\mathcal{H})^n]_\rho$ for some $\rho > 0$.

Lemma 1.1. *Let f be a positive regular free holomorphic function on $[B(\mathcal{H})^n]_\rho$, $\rho > 0$, with the representation $f(X_1,\ldots,X_n) := \sum_{\alpha\in\mathbb{F}_n^+} a_\alpha X_\alpha$, $a_\alpha \in \mathbb{C}$. Then there exists $r \in (0,1)$ such that $\|f(rS_1,\ldots,rS_n)\| < 1$ and, for any $m = 1, 2, \ldots,$*

$$\bigl[1 - f(rS_1,\ldots,rS_n)\bigr]^{-m} = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} b_\alpha^{(m)} r^{|\alpha|} S_\alpha,$$

where $b_{g_0}^{(m)} = 1$ and

$$b_\alpha^{(m)} = \sum_{j=1}^{|\alpha|} \sum_{\substack{\gamma_1\cdots\gamma_j=\alpha \\ |\gamma_1|\geqslant 1,\ldots,|\gamma_j|\geqslant 1}} a_{\gamma_1}\cdots a_{\gamma_j} \binom{j+m-1}{m-1} \quad \text{if } |\alpha| \geqslant 1. \tag{1.1}$$

Proof. Due to the Schwartz type lemma for free holomorphic functions on the open unit ball $[B(\mathcal{H})^n]_1$ (see [44]), there exists $r > 0$ such that $f(rS_1,\ldots,rS_n)$ is in the noncommutative disc algebra $\mathcal{A}_n$ and $\|f(rS_1,\ldots,rS_n)\| < 1$. Therefore, the operator $I - f(rS_1,\ldots,rS_n)$ is invertible with its inverse $g(rS_1,\ldots,rS_n) := [I - f(rS_1,\ldots,rS_n)]^{-1}$ in $\mathcal{A}_n \subset F_n^\infty$. Assume that $g(rS_1,\ldots,rS_n)$ has the Fourier representation $\sum_{\alpha\in\mathbb{F}_n^+} b_\alpha^{(1)} r^{|\alpha|} S_\alpha$ for some constants $b_\alpha^{(1)} \in \mathbb{C}$. Consequently, using the fact that $r^{|\alpha|} b_\alpha^{(1)} = P_{\mathbb{C}} S_\alpha^* g(rS_1,\ldots,rS_n)(1)$, we deduce that

$$\begin{aligned} g(rS_1,\ldots,rS_n) &= I + f(rS_1,\ldots,rS_n) + f(rS_1,\ldots,rS_n)^2 + \cdots \\ &= I + \sum_{k=1}^{\infty} \sum_{|\alpha|=k} \left(\sum_{j=1}^{|\alpha|} \sum_{\substack{\gamma_1\cdots\gamma_j=\alpha \\ |\gamma_1|\geqslant 1,\ldots,|\gamma_j|\geqslant 1}} a_{\gamma_1}\cdots a_{\gamma_j} \right) r^{|\alpha|} S_\alpha. \end{aligned}$$

Due to the uniqueness of the Fourier representation of the elements in F_n^∞, we deduce relation (1.1), when $m = 1$. Now, we proceed by induction over m. Assume that relation (1.1) holds for m and let us prove it for $m + 1$. Notice that

$$\begin{aligned} &\bigl[I - f(rS_1,\ldots,rS_n)\bigr]^{-(m+1)} \\ &\quad = \bigl[I - f(rS_1,\ldots,rS_n)\bigr]^{-m} \bigl[I - f(rS_1,\ldots,rS_n)\bigr]^{-1} \\ &\quad = \left\{ I + \sum_{|\omega|\geqslant 1} \left[\sum_{j=1}^{|\omega|} \sum_{\substack{\xi_1\cdots\xi_j=\omega \\ |\xi_1|\geqslant 1,\ldots,|\xi_j|\geqslant 1}} a_{\xi_1}\cdots a_{\xi_j} \binom{j+m-1}{m-1} \right] r^{|\omega|} S_\omega \right\} \\ &\qquad \times \left\{ I + \sum_{|\sigma|\geqslant 1} \left[\sum_{k=1}^{|\sigma|} \sum_{\substack{\epsilon_1\cdots\epsilon_k=\sigma \\ |\epsilon_1|\geqslant 1,\ldots,|\epsilon_k|\geqslant 1}} a_{\epsilon_1}\cdots a_{\epsilon_k} \right] r^{|\sigma|} S_\sigma \right\} \\ &\quad = I + \sum_{|\gamma|\geqslant 1} \left[\sum_{k=1}^{|\gamma|} \sum_{\substack{\epsilon_1\cdots\epsilon_k=\gamma \\ |\epsilon_1|\geqslant 1,\ldots,|\epsilon_k|\geqslant 1}} a_{\epsilon_1}\cdots a_{\epsilon_k} + \sum_{j=1}^{|\gamma|} \sum_{\substack{\xi_1\cdots\xi_j=\gamma \\ |\xi_1|\geqslant 1,\ldots,|\xi_j|\geqslant 1}} a_{\xi_1}\cdots a_{\xi_j} \binom{j+m-1}{m-1} \right. \end{aligned}$$

$$+ \sum_{\substack{\omega\sigma=\gamma\\ |\omega|\geqslant 1, |\sigma|\geqslant 1}} \sum_{j=1}^{|\omega|} \sum_{k=1}^{|\sigma|} \sum_{\substack{\xi_1\cdots\xi_j=\gamma\\ |\xi_1|\geqslant 1,\ldots,|\xi_j|\geqslant 1}}$$

$$\times \sum_{\substack{\epsilon_1\cdots\epsilon_k=\gamma\\ |\epsilon_1|\geqslant 1,\ldots,|\epsilon_k|\geqslant 1}} \binom{j+m-1}{m-1} a_{\xi_1}\cdots a_{\xi_j} a_{\epsilon_1}\cdots a_{\epsilon_k}\Bigg] r^{|\gamma|} S_\gamma.$$

If we look closer to the sums in the brackets, we notice that each product $a_{\eta_1}\cdots a_{\eta_p}$, where $\eta_1\cdots\eta_p=\gamma$ with $\eta_1,\ldots,\eta_p\in\mathbb{F}_n^+$ and $|\eta_1|\geqslant 1,\ldots,|\eta_p|\geqslant 1$, occurs $p+1$ times. This is because

$$a_{\eta_1}\cdots a_{\eta_p} = \begin{cases} a_{\epsilon_1}\cdots a_{\epsilon_k} & \text{if } (\eta_1,\ldots,\eta_p)=(\epsilon_1,\ldots,\epsilon_k),\\ a_{\xi_1}\cdots a_{\xi_j}a_{\epsilon_1}\cdots a_{\epsilon_k} & \text{if } (\eta_1,\ldots,\eta_p)=(\xi_1,\ldots,\xi_j,\epsilon_1,\ldots,\epsilon_k) \text{ and } j=1,\ldots,p-1,\\ a_{\xi_1}\cdots a_{\xi_j} & \text{if } (\eta_1,\ldots,\eta_p)=(\xi_1,\ldots,\xi_j).\end{cases}$$

Moreover, at each occurrence, the product $a_{\eta_1}\cdots a_{\eta_p}$ has a coefficient which is equal to

$$\begin{cases} \binom{m-1}{m-1} & \text{if } (\eta_1,\ldots,\eta_p)=(\epsilon_1,\ldots,\epsilon_k),\\ \binom{j+m-1}{m-1} & \text{if } (\eta_1,\ldots,\eta_p)=(\xi_1,\ldots,\xi_j,\epsilon_1,\ldots,\epsilon_k) \text{ and } j=1,\ldots,p-1,\\ \binom{p+m-1}{m-1} & \text{if } (\eta_1,\ldots,\eta_p)=(\xi_1,\ldots,\xi_j).\end{cases}$$

Hence, we deduce that the coefficient of $a_{\eta_1}\cdots a_{\eta_p}$ is equal to

$$\sum_{j=0}^{p}\binom{j+m-1}{m-1}=\binom{p+m}{m}.$$

The latter equality can be easily deduced using the well-known relation

$$\binom{j+m}{m}=\binom{j+m-1}{m}+\binom{j+m-1}{m-1}$$

for any $j=1,\ldots,p$. Therefore, we have $[I-f(rS_1,\ldots,rS_n)]^{-(m+1)}=\sum_{|\gamma|\geqslant 1} b_\gamma^{(m+1)} r^{|\gamma|}S_\gamma$, where

$$b_\gamma^{(m+1)}=\sum_{p=1}^{|\gamma|}\sum_{\substack{\eta_1\cdots\eta_p=\gamma\\ |\eta_1|\geqslant 1,\ldots,|\eta_p|\geqslant 1}} a_{\eta_1}\cdots a_{\eta_p}\binom{p+m}{m} \quad \text{if } |\gamma|\geqslant 1.$$

This completes the induction and the proof. □

Lemma 1.2. *Let f be a positive regular free holomorphic function on $[B(\mathcal{H})^n]_\rho$, $\rho > 0$, with the representation $f(X_1, \ldots, X_n) := \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha X_\alpha$, $a_\alpha \in \mathbb{C}$, and let $g := 1 - (1 - f)^m$, $m = 1, 2, \ldots,$ have the representation $g(X_1, \ldots, X_n) := \sum_{\gamma \in \mathbb{F}_n^+} c_\gamma^{(m)} X_\gamma$, $a_\gamma \in \mathbb{C}$. Then the following relations hold*:

$$b_\beta^{(m)} = \sum_{\substack{\gamma\alpha=\beta \\ \alpha \in \mathbb{F}_n^+, |\gamma| \geqslant 1}} b_\alpha^{(m)} c_\gamma^{(m)} \quad \textit{if } |\beta| \geqslant 1 \textit{ and } m = 1, 2, \ldots, \tag{1.2}$$

and

$$b_\alpha^{(m)} = b_\alpha^{(m-1)} + \sum_{\substack{\gamma\sigma=\alpha \\ \sigma \in \mathbb{F}_n^+, |\gamma| \geqslant 1}} b_\sigma^{(m)} a_\gamma \quad \textit{if } m \geqslant 2 \textit{ and } \alpha \in \mathbb{F}_n^+. \tag{1.3}$$

Proof. Since

$$\big\{I - \big[I - f(rS_1, \ldots, rS_n)\big]^m\big\}\big[I - f(rS_1, \ldots, rS_n)\big]^{-m} = \big[I - f(rS_1, \ldots, rS_n)\big]^{-m} - I$$

and using Lemma 1.1, we have

$$\left(\sum_{k=0}^\infty \sum_{|\alpha|=k} b_\alpha^{(m)} r^{|\alpha|} S_\alpha\right)\left(\sum_{p=1}^\infty \sum_{|\gamma|=p} b_\gamma^{(m)} r^{|\gamma|} S_\gamma\right) = \sum_{q=1}^\infty \sum_{|\beta|=q} b_\beta^{(m)} r^{|\beta|} S_\beta.$$

Hence, using the uniqueness of the Fourier representation for the elements in F_n^∞, we obtain relation (1.2). To prove (1.3), assume that $m \geqslant 2$ and notice that

$$\begin{aligned}
&\big[I - f(rS_1, \ldots, rS_n)\big]^{-m} - f(rS_1, \ldots, rS_n)\big[I - f(rS_1, \ldots, rS_n)\big]^{-m} - I \\
&\quad = \big[I - f(rS_1, \ldots, rS_n)\big]^{-m+1} - I.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
\sum_{k=0}^\infty \sum_{|\alpha|=k} b_\alpha^{(m)} r^{|\alpha|} S_\alpha &= \left(\sum_{q=1}^\infty \sum_{|\gamma|=q} b_\gamma^{(m)} r^{|\gamma|} S_\gamma\right)\left(\sum_{p=0}^\infty \sum_{|\sigma|=p} b_\sigma^{(m)} r^{|\sigma|} S_\sigma\right) \\
&\quad + \sum_{k=0}^\infty \sum_{|\alpha|=k} b_\alpha^{(m-1)} r^{|\alpha|} S_\alpha.
\end{aligned}$$

Using again the uniqueness of the Fourier representation for the elements in F_n^∞, we deduce relation (1.3). This completes the proof. $\square$

According to Lemma 1.1, we have $b_\alpha^{(m)} > 0$ for any $\alpha \in \mathbb{F}_n^+$ and $m = 1, 2, \ldots.$ We define now the diagonal operators $D_i : F^2(H_n) \to F^2(H_n)$, $i = 1, \ldots, n$, by setting

$$D_i e_\alpha := \sqrt{\frac{b_\alpha^{(m)}}{b_{g_i\alpha}^{(m)}}}\, e_\alpha, \quad \alpha \in \mathbb{F}_n^+.$$

Due to Lemma 1.2, we have

$$b_{g_i\alpha}^{(m)} \geqslant \sum_{\substack{\gamma\sigma=g_i\alpha\\ \sigma\in\mathbb{F}_n^+,\ |\gamma|\geqslant 1}} b_\sigma^{(m)} a_\gamma \geqslant a_{g_i} b_\alpha^{(m)}.$$

Since $a_{g_i} > 0$ for each $i = 1, \dots, n$, we deduce that

$$\|D_i\| = \sup_{\alpha\in\mathbb{F}_n^+} \sqrt{\frac{b_\alpha^{(m)}}{b_{g_i\alpha}^{(m)}}} \leqslant \frac{1}{\sqrt{a_{g_i}}}, \quad i = 1, \dots, n.$$

Now we define the *weighted left creation operators* $W_i : F^2(H_n) \to F^2(H_n)$, $i = 1, \dots, n$, associated with the positive regular free holomorphic f by setting $W_i := S_i D_i$, where $S_1, \dots, S_n$ are the left creation operators on the full Fock space $F^2(H_n)$. Therefore, we have

$$W_i e_\alpha = \frac{\sqrt{b_\alpha^{(m)}}}{\sqrt{b_{g_i\alpha}^{(m)}}} e_{g_i\alpha}, \quad \alpha \in \mathbb{F}_n^+, \tag{1.4}$$

where the coefficients $b_\alpha^{(m)}$, $\alpha \in \mathbb{F}_n^+$, are given by relation (1.1).

Throughout this paper, we denote by *id* the identity map acting on the algebra of all bounded linear operators an a Hilbert space.

Theorem 1.3. *Let f be a positive regular free holomorphic function on $[B(\mathcal{H})^n]_\rho$, $\rho > 0$, and $m = 1, 2, \dots$. The weighted left creation operators $W_1, \dots, W_n$ associated with f and m, and defined by relation* (1.4) *have the following properties*:

(i) $\sum_{|\beta|\geqslant 1} a_\beta W_\beta W_\beta^* \leqslant I$, *where the convergence is in the strong operator topology*;
(ii) $(id - \Phi_{f,W})^m(I) = P_{\mathbb{C}}$, *where $P_{\mathbb{C}}$ is the orthogonal projection of $F^2(H_n)$ on $\mathbb{C}$, and the map $\Phi_{f,W} : B(F^2(H_n)) \to B(F^2(H_n))$ is defined by*

$$\Phi_{f,W}(X) = \sum_{|\alpha|\geqslant 1} a_\alpha W_\alpha X W_\alpha^*,$$

where the convergence is in the weak operator topology;
(iii) $\lim_{p\to\infty} \Phi_{f,W}^p(I) = 0$ *in the strong operator topology*;
(iv) $\sum_{\beta\in\mathbb{F}_n^+} b_\beta^{(m)} W_\beta[(id - \Phi_{f,W})^m(I)]W_\beta^* = I$, *where the coefficients $b_\beta^{(m)}$ are defined by* (1.1), *and the convergence is in the strong operator topology.*

Proof. Using relation (1.1), a simple calculation reveals that

$$W_\beta e_\gamma = \frac{\sqrt{b_\gamma^{(m)}}}{\sqrt{b_{\beta\gamma}^{(m)}}} e_{\beta\gamma} \quad \text{and} \quad W_\beta^* e_\alpha = \begin{cases} \frac{\sqrt{b_\gamma^{(m)}}}{\sqrt{b_\alpha^{(m)}}} e_\gamma & \text{if } \alpha = \beta\gamma, \\ 0 & \text{otherwise} \end{cases} \tag{1.5}$$

for any $\alpha, \beta \in \mathbb{F}_n^+$. Due to (1.5), we deduce that

$$W_\beta W_\beta^* e_\alpha = \begin{cases} \frac{b_\gamma^{(m)}}{b_\alpha^{(m)}} e_\alpha & \text{if } \alpha = \beta\gamma, \\ 0 & \text{otherwise.} \end{cases} \tag{1.6}$$

Since the case $m = 1$ was considered in [45], we assume that $m \geqslant 2$. Notice that

$$\left(I - \sum_{1 \leqslant |\beta| \leqslant N} a_\beta W_\beta W_\beta^*\right) e_\alpha = \frac{1}{b_\alpha^{(m)}} K_{N,\alpha} e_\alpha,$$

where $K_{N,\alpha} = b_\alpha^{(m)}$ if $\alpha = g_0$, and

$$K_{N,\alpha} = b_\alpha^{(m)} - \sum_{\beta\gamma = \alpha,\ 1 \leqslant |\beta| \leqslant N} a_\beta b_\gamma^{(m)} \quad \text{if } |\alpha| \geqslant 1.$$

Due to relation (1.3), if $1 \leqslant |\alpha| \leqslant N$, we have

$$K_{N,\alpha} = b_\alpha^{(m-1)} \leqslant b_\alpha^{(m)}.$$

On the other hand, since $a_\beta \geqslant 0$, $b_\gamma^{(m)} \geqslant 0$ for any $\alpha, \gamma \in \mathbb{F}_n^+$, we have $K_{N,\alpha} \leqslant b_\alpha^{(m)}$ if $|\alpha| \geqslant 1$. Hence, we deduce that $0 \leqslant K_{N,\alpha} \leqslant b_\alpha^{(m)}$, whenever $|\alpha| > N$. On the other hand, notice that if $1 \leqslant N_1 \leqslant N_2 \leqslant |\alpha|$, then $K_{N_2,\alpha} \leqslant K_{N_1,\alpha}$. Consequently, $\{I - \sum_{1 \leqslant |\beta| \leqslant N} a_\beta W_\beta W_\beta^*\}_{N=1}^\infty$ is a decreasing sequence of positive diagonal operators which converges in the strong operator topology. Hence, we deduce that $\sum_{|\beta| \geqslant 1} a_\beta W_\beta W_\beta^* \leqslant I$, where the convergence is in the strong operator topology.

We prove now part (ii). By (1.6), the subspaces $\mathbb{C} e_\alpha$, $\alpha \in \mathbb{F}_n^+$, are invariant under $W_\beta W_\beta^*$, $\beta \in \mathbb{F}_n^+$, and, therefore, they are also invariant under $(id - \Phi_{f,W})^m(I)$. Consequently, it is enough to show that $(id - \Phi_{f,W})^m(I)1 = 1$ and

$$\big\langle (id - \Phi_{f,W})^m(I) e_\alpha, e_\alpha \big\rangle = 0$$

for any $\alpha \in \mathbb{F}_n^+$ with $|\alpha| \geqslant 1$. The first equality is obvious due to (1.6). Using Lemma 1.2, we deduce that

$$\begin{aligned} \big\langle (id - \Phi_{f,W})^m(I) e_\alpha, e_\alpha \big\rangle &= \left\langle e_\alpha - \sum_{|\beta| \geqslant 1} c_\beta^{(m)} W_\beta W_\beta^* e_\alpha, e_\alpha \right\rangle \\ &= \frac{1}{b_\alpha^{(m)}} \left(b_\alpha^{(m)} - \sum_{\beta\gamma = \alpha,\ |\beta| \geqslant 1} c_\beta^{(m)} b_\gamma^{(m)} \right) = 0 \end{aligned}$$

if $\alpha \in \mathbb{F}_n^+$ with $|\alpha| \geqslant 1$. Therefore, $(id - \Phi_{f,W})^m(I) = P_{\mathbb{C}}$.

To prove part (iii), notice that relation (1.6) implies $\Phi_{f,W}^p(I) e_\alpha = 0$ if $p > |\alpha|$. This shows that $\lim_{p \to \infty} \Phi_{f,W}^p(I) e_\alpha = 0$ for any $\alpha \in \mathbb{F}_n^+$. By part (i), we have $\|\Phi_{f,W}^p(I)\| \leqslant 1$ for any $p \in \mathbb{N}$. Now item (iii) follows.

It remains to prove (iv). To this end, notice that

$$P_{\mathbb{C}} W_\beta^* e_\alpha = \begin{cases} \frac{1}{\sqrt{b_\beta}} & \text{if } \alpha = \beta, \\ 0 & \text{otherwise,} \end{cases} \tag{1.7}$$

and therefore $\sum_{\beta \in \mathbb{F}_n^+} b_\beta W_\beta P_{\mathbb{C}} W_\beta^* e_\alpha = e_\alpha$. Using part (ii), we complete the proof. □

We can also define the *weighted right creation operators* $\Lambda_i : F^2(H_n) \to F^2(H_n)$ by setting $\Lambda_i := R_i G_i$, $i = 1, \ldots, n$, where $R_1, \ldots, R_n$ are the right creation operators on the full Fock space $F^2(H_n)$ and each diagonal operator G_i, $i = 1, \ldots, n$, is defined by

$$G_i e_\alpha := \sqrt{\frac{b_\alpha^{(m)}}{b_{\alpha g_i}^{(m)}}} e_\alpha, \quad \alpha \in \mathbb{F}_n^+,$$

where the coefficients $b_\alpha^{(m)}$, $\alpha \in \mathbb{F}_n^+$, are given by relation (1.1). In this case, we have

$$\Lambda_\beta e_\gamma = \frac{\sqrt{b_\gamma^{(m)}}}{\sqrt{b_{\gamma\tilde{\beta}}^{(m)}}} e_{\gamma\tilde{\beta}} \quad \text{and} \quad \Lambda_\beta^* e_\alpha = \begin{cases} \frac{\sqrt{b_\gamma^{(m)}}}{\sqrt{b_\alpha^{(m)}}} e_\gamma & \text{if } \alpha = \gamma\tilde{\beta}, \\ 0 & \text{otherwise} \end{cases} \tag{1.8}$$

for any $\alpha, \beta \in \mathbb{F}_n^+$, where $\tilde{\beta}$ denotes the reverse of $\beta = g_{i_1} \cdots g_{i_k}$, i.e., $\tilde{\beta} = g_{i_k} \cdots g_{i_1}$. Using Lemma 1.2 and (1.8), we deduce that

$$\left(I - \sum_{1 \leqslant |\beta| \leqslant N} a_{\tilde{\beta}} \Lambda_\beta \Lambda_\beta^*\right) e_\alpha = \frac{1}{b_\alpha^{(m)}} \tilde{K}_{N,\alpha} e_\alpha,$$

where $\tilde{K}_{N,\alpha} = b_\alpha^{(m)}$ if $\alpha = g_0$, and

$$\tilde{K}_{N,\alpha} = b_\alpha^{(m)} - \sum_{\gamma\tilde{\beta} = \alpha,\ 1 \leqslant |\tilde{\beta}| \leqslant N} a_{\tilde{\beta}} b_\gamma^{(m)} \quad \text{if } |\alpha| \geqslant 1.$$

As in the case of weighted left creation operators, one can show that

$$\sum_{|\beta| \geqslant 1} a_{\tilde{\beta}} \Lambda_\beta \Lambda_\beta^* \leqslant I \quad \text{and} \quad (id - \Phi_{\tilde{f},\Lambda})^m (I) = P_{\mathbb{C}}, \tag{1.9}$$

where $\tilde{f}(X_1, \ldots, X_n) := \sum_{|\alpha| \geqslant 1} a_{\tilde{\alpha}} X_\alpha$, $\tilde{\alpha}$ denotes the reverse of α, and $\Phi_{\tilde{f},\Lambda}(X) := \sum_{|\alpha| \geqslant 1} a_{\tilde{\alpha}} \Lambda_\alpha X \Lambda_\alpha^*$, $X \in B(F^2(H_n))$, with the convergence is in the weak operator topology. Since

$$P_{\mathbb{C}} \Lambda_\beta^* e_\alpha = \begin{cases} \frac{1}{\sqrt{b_\alpha^{(m)}}} & \text{if } \alpha = \tilde{\beta}, \\ 0 & \text{otherwise,} \end{cases}$$

we deduce that

$$\sum_{\beta\in\mathbb{F}_n^+} b_{\tilde{\beta}}^{(m)} \Lambda_\beta\big[(id - \Phi_{\tilde{f},\Lambda})^m(I)\big]\Lambda_\beta^* = I,$$

where the convergence is in the strong operator topology. Therefore, we obtain a result similar to Theorem 1.3 for the n-tuple $(\Lambda_1, \ldots, \Lambda_n)$.

A linear map $\varphi : B(\mathcal{H}) \to B(\mathcal{H})$ is called power bounded if there exists a constant $M > 0$ such that $\|\varphi^k\| \leqslant M$ for any $k \in \mathbb{N} := \{1, 2, \ldots\}$.

Lemma 1.4. *Let $\varphi : B(\mathcal{H}) \to B(\mathcal{H})$ be a power bounded, positive linear map and let $D \in B(\mathcal{H})$ be a positive operator. If $m \in \mathbb{N}$, then*

$$(id - \varphi)^m(D) \geqslant 0 \quad \text{if and only if} \quad (id - \varphi)^k(D) \geqslant 0, \quad k = 1, 2, \ldots, m.$$

Proof. One implication is obvious. Assume that $m \geqslant 2$ and $(id-\varphi)^m(D) \geqslant 0$. Due to the identity

$$(id - \varphi)^k(D) = \sum_{p=0}^{k} (-1)^p \binom{k}{p} \varphi^p(D), \quad k \in \mathbb{N},$$

and the fact that φ is a positive linear map, we deduce that $x_j := \langle \varphi^j(id - \varphi)^{m-1}(D)h, h\rangle$ is a real number for any $h \in \mathcal{H}$ and $j = 0, 1, \ldots$. Note that, we have

$$x_j - x_{j+1} = \big\langle \varphi^j(id - \varphi)^m(D)h, h\big\rangle \geqslant 0.$$

Therefore, $\{x_j\}_{j=0}^\infty$ is a decreasing sequence of real numbers.

On the other hand, using the fact that φ is a power bounded linear map, there exists a constant $M > 0$ such that $\|\varphi^k\| \leqslant M$ for any $k \in \mathbb{N}$. Therefore, we have

$$\begin{aligned}
\left|\sum_{j=0}^{p} x_j\right| &= \left|\sum_{j=0}^{p} \big\langle\big(\varphi^j - \varphi^{j+1}\big)(id - \varphi)^{m-2}(D)h, h\big\rangle\right| \\
&= \big|\big\langle(id - \varphi)^{m-2}(D)h, h\big\rangle - \big\langle\varphi^{p+1}(id - \varphi)^{m-2}(D)h, h\big\rangle\big| \\
&\leqslant \big|\big\langle\big(id - \varphi^{p+1}\big)(id - \varphi)^{m-2}(D)h, h\big\rangle\big| \\
&\leqslant (1 + M)\big\|(id - \varphi)^{m-2}(D)\big\|\|h\|^2 < \infty
\end{aligned}$$

for any $p = 0, 1, \ldots$. Hence, we deduce that $x_j \geqslant 0$ for any $j = 0, 1, \ldots$. In particular, we have $x_0 := \langle (id - \varphi)^{m-1}(D)h, h\rangle \geqslant 0$ for any $h \in \mathcal{H}$. Therefore, $(id - \varphi)^{m-1}(D) \geqslant 0$. Iterating this process, one can show that $(id - \varphi)^k(D) \geqslant 0$ for any $k = 1, 2, \ldots, m$. The proof is complete. □

Corollary 1.5. *If φ is a positive linear map on $B(\mathcal{H})$ such that $\varphi(I) \leqslant I$ and $(id - \varphi)^m(I) \geqslant 0$ for some $m \in \mathbb{N}$, then*

$$0 \leqslant (id - \varphi)^m(I) \leqslant (id - \varphi)^{m-1}(I) \leqslant \cdots \leqslant (id - \varphi)(I) \leqslant I.$$

Given $m, n \in \{1, 2, \ldots\}$ and a positive regular free holomorphic function $f := \sum_{|\alpha| \geqslant 1} a_\alpha X_\alpha$, we define the noncommutative domain

$$\mathbf{D}_f^m(\mathcal{H}) := \left\{X := (X_1, \ldots, X_n) \in B(\mathcal{H})^n\colon\ (id - \Phi_{f,X})^k(I) \geqslant 0 \text{ for } 1 \leqslant k \leqslant m\right\},$$

where $\Phi_{f,X} : B(\mathcal{H}) \to B(\mathcal{H})$ is defined by $\Phi_{f,X}(Y) := \sum_{|\alpha| \geqslant 1} a_\alpha X_\alpha Y X_\alpha^*$, $Y \in B(\mathcal{H})$, and the convergence is in the week operator topology. For the next result, we need to denote by $(W_1^{(f)}, \ldots, W_n^{(f)})$ the weighted left creation operators $(W_1, \ldots, W_n)$ associated with $\mathbf{D}_f^{(m)}$. The notation $(\Lambda_1^{(f)}, \ldots, \Lambda_n^{(f)})$ is now clear.

Theorem 1.6. *Let $(W_1^{(f)}, \ldots, W_n^{(f)})$ (respectively $(\Lambda_1^{(f)}, \ldots, \Lambda_n^{(f)})$) be the weighted left (respectively right) creation operators associated with the noncommutative domain $\mathbf{D}_f^m$. Then the following statements hold:*

(i) $(W_1^{(f)}, \ldots, W_n^{(f)}) \in \mathbf{D}_f^m(F^2(H_n))$;
(ii) $(\Lambda_1^{(f)}, \ldots, \Lambda_n^{(f)}) \in \mathbf{D}_{\tilde{f}}^m(F^2(H_n))$;
(iii) $U^* \Lambda_i^{(f)} U = W_i^{(\tilde{f})}$, $i = 1, \ldots, n$, *where $U \in B(F^2(H_n))$ is the unitary operator defined by equation $U e_\alpha := e_{\tilde{\alpha}}$, $\alpha \in \mathbb{F}_n^+$;*
(iv) $W_i^{(f)} \Lambda_j^{(f)} = \Lambda_j^{(f)} W_i^{(f)}$ *for $i, j = 1, \ldots, n$.*

Proof. Items (i) and (ii) follow from Theorem 1.3, Lemma 1.4, and relation (1.9). Using relation (1.5) when f is replaced by $\tilde{f}$, we obtain

$$W_i^{(\tilde{f})} e_\gamma = \frac{\sqrt{b_{\tilde{\gamma}}^{(m)}}}{\sqrt{b_{\tilde{\gamma} g_i}^{(m)}}} e_{g_i \gamma}.$$

On the other hand, due to relation (1.8), we deduce that

$$U^* \Lambda_i^{(f)} U e_\gamma = U^* \left(\frac{\sqrt{b_{\tilde{\gamma}}^{(m)}}}{\sqrt{b_{\tilde{\gamma} g_i}^{(m)}}} e_{\tilde{\gamma} g_i} \right) = \frac{\sqrt{b_{\tilde{\gamma}}^{(m)}}}{\sqrt{b_{\tilde{\gamma} g_i}^{(m)}}} e_{g_i \gamma}.$$

Therefore, $U^* \Lambda_i^{(f)} U = W_i^{(\tilde{f})}$, $i = 1, \ldots, n$.

Now, using relation (1.4), (1.8), we obtain

$$\Lambda_j W_i^{(f)} e_\alpha = \frac{\sqrt{b_\alpha^{(m)}}}{\sqrt{b_{g_i \alpha}^{(m)}}} \Lambda_j^{(f)}(e_{g_i \alpha}) = \frac{\sqrt{b_\alpha^{(m)}}}{\sqrt{b_{g_i \alpha g_j}^{(m)}}} e_{g_i \alpha g_j}$$

for any $\alpha \in \mathbb{F}_n^+$ and $i, j = 1, \ldots, n$. Similar calculations reveal that $\Lambda_j^{(f)} W_i^{(f)} e_\alpha = W_i^{(f)} \Lambda_j^{(f)} e_\alpha$, which proves (iv). The proof is complete. $\square$

2. Noncommutative Berezin transforms

In this section, we introduce a *noncommutative Berezin transform* associated with each n-tuple of operators $T := (T_1, \dots, T_n)$ in the noncommutative domain $\mathbf{D}_f^m(\mathcal{H})$, and present some of its basic properties.

Let f be a positive regular free holomorphic function on a noncommutative ball $[B(\mathcal{H})^n]_\rho$, $\rho > 0$, with representation $f(X_1, \dots, X_n) := \sum_{|\alpha|\geqslant 1} a_\alpha X_\alpha$. Let $T := (T_1, \dots, T_n) \in B(\mathcal{H})^n$ be an n-tuple of operators such that the series $\sum_{|\alpha|\geqslant 1} a_\alpha T_\alpha T_\alpha^*$ is WOT-convergent, and consider the bounded linear map $\Phi_{f,T} : B(\mathcal{H}) \to B(\mathcal{H})$, given by

$$\Phi_{f,T}(X) := \sum_{|\alpha|\geqslant 1} a_\alpha T_\alpha X T_\alpha^*, \quad X \in B(\mathcal{K}), \tag{2.1}$$

where the convergence is in the week operator topology. The joint spectral radius of $T \in \mathbf{D}_f^m(\mathcal{H})$ is defined by

$$r_f(T_1, \dots, T_n) := \lim_{k\to\infty} \left\| \Phi_{f,T}^k(I) \right\|^{1/2k}.$$

We recall that the model n-tuple $(\Lambda_1, \dots, \Lambda_n)$ associated with $\mathbf{D}_f^m$ was defined in Section 1. According to the results of that section, the series $\sum_{|\alpha|\geqslant 1} a_{\tilde{\alpha}} \Lambda_\alpha \Lambda_\alpha^*$ is SOT-convergent and, therefore, so is the series $\sum_{|\alpha|\geqslant 1} a_{\tilde{\alpha}} \Lambda_\alpha \otimes T_{\tilde{\alpha}}^*$. Notice also that

$$\left\| \sum_{|\alpha|\geqslant 1} a_{\tilde{\alpha}} \Lambda_\alpha \otimes T_{\tilde{\alpha}}^* \right\| \leqslant \left\| \sum_{|\alpha|\geqslant 1} a_{\tilde{\alpha}} \Lambda_\alpha \Lambda_\alpha^* \right\| \left\| \sum_{|\alpha|\geqslant 1} a_{\tilde{\alpha}} T_{\tilde{\alpha}} T_{\tilde{\alpha}}^* \right\|$$

and

$$\left\| \left(\sum_{|\alpha|\geqslant 1} a_{\tilde{\alpha}} \Lambda_\alpha \otimes T_{\tilde{\alpha}}^* \right)^k \right\| \leqslant \left\| \Phi_{\tilde{f},\Lambda}^k(I) \right\|^{1/2} \left\| \Phi_{f,T}^k(I) \right\|^{1/2}, \quad k \in \mathbb{N}, \tag{2.2}$$

where $\tilde{f}(X_1, \dots, X_n) := \sum_{|\alpha|\geqslant 1} a_{\tilde{\alpha}} X_\alpha$ and $\Phi_{\tilde{f},\Lambda}(Y) := \sum_{|\alpha|\geqslant 1} a_{\tilde{\alpha}} \Lambda_\alpha Y \Lambda_\alpha^*$. Hence, we deduce that

$$r\left(\sum_{|\alpha|\geqslant 1} a_{\tilde{\alpha}} \Lambda_\alpha \otimes T_{\tilde{\alpha}}^* \right) \leqslant r_{\tilde{f}}(\Lambda_1, \dots, \Lambda_n) r_f(T_1, \dots, T_n),$$

where $r(A)$ denotes the usual spectral radius of an operator A. Due to the results of Section 1, we have $\|\Phi_{\tilde{f},\Lambda}(I)\| \leqslant 1$, which implies $r_{\tilde{f}}(\Lambda_1, \dots, \Lambda_n) \leqslant 1$. Consequently, we have

$$r\left(\sum_{|\alpha|\geqslant 1} a_{\tilde{\alpha}} \Lambda_\alpha \otimes T_{\tilde{\alpha}}^* \right) \leqslant r_f(T_1, \dots, T_n).$$

Therefore, if $r_f(T_1, \dots, T_n) < 1$, then the operator

$$\left(I - \sum_{|\alpha|\geqslant 1} a_{\tilde{\alpha}} \Lambda_\alpha \otimes T_{\tilde{\alpha}}^* \right)^{-1} = \sum_{k=0}^\infty \left(\sum_{|\alpha|\geqslant 1} a_{\tilde{\alpha}} \Lambda_\alpha \otimes T_{\tilde{\alpha}}^* \right)^k \tag{2.3}$$

is well defined, where the convergence is in the operator norm topology.

For each $T := (T_1, \dots, T_n) \in \mathbf{D}_f^m(\mathcal{H})$ with $r_f(T_1, \dots, T_n) < 1$, we introduce the *noncommutative Berezin transform* at T as the map $\mathbf{B}_T : B(F^2(H_n)) \to B(\mathcal{H})$ defined by

$$\big\langle \mathbf{B}_T[g]x, y\big\rangle := \left\langle \left(I - \sum_{|\alpha|\geqslant 1} \bar{a}_{\tilde{\alpha}} \Lambda_\alpha^* \otimes T_{\tilde{\alpha}}\right)^{-m} \big(g \otimes \Delta_{T,m,f}^2\big) \times \left(I - \sum_{|\alpha|\geqslant 1} a_{\tilde{\alpha}} \Lambda_\alpha \otimes T_{\tilde{\alpha}}^*\right)^{-m} (1 \otimes x), 1 \otimes y\right\rangle, \tag{2.4}$$

where $\Delta_{T,m,f} := [(id - \Phi_{f,T})^m(I)]^{1/2}$ and $x, y \in \mathcal{H}$. We remark that in the particular case when: $n = 1, m = 1, f(X) = X, \mathcal{H} = \mathbb{C}$, and $T = \lambda \in \mathbb{D}$, we recover the Berezin transform of a bounded linear operator on the Hardy space $H^2(\mathbb{D})$, i.e.,

$$\mathbf{B}_\lambda[g] = \big(1 - |\lambda|^2\big)\langle g k_\lambda, k_\lambda\rangle, \quad g \in B\big(H^2(\mathbb{D})\big),$$

where $k_\lambda(z) := (1 - \bar{\lambda}z)^{-1}$ and $z, \lambda \in \mathbb{D}$.

The noncommutative Berezin transform will play an important role in this paper. We will present some of its basic properties in this section. First, we need a few preliminary results about positive linear maps on $B(\mathcal{H})$.

Lemma 2.1. *Let $\varphi : B(\mathcal{H}) \to B(\mathcal{H})$ be a linear map and $k, q \in \mathbb{N}$. Then*

$$\sum_{p=0}^{q} \binom{p+k-1}{k-1} \varphi^p (id - \varphi)^k = id - \sum_{j=0}^{k-1} \binom{q+j}{j} \varphi^{q+1} (id - \varphi)^j. \tag{2.5}$$

Proof. Since $\sum_{p=0}^{q} \varphi^p(id - \varphi) = id - \varphi^{q+1}$, Eq. (2.5) holds for $k = 1$. We proceed now by induction over k. Assume that (2.5) holds for $k = m$. Since $\varphi(id - \varphi) = (id - \varphi)\varphi$ and

$$\binom{p+m}{m} - \binom{p+m-1}{m} = \binom{p+m-1}{m-1},$$

we have

$$\begin{aligned}
&\sum_{p=0}^{q} \binom{p+m}{m} \varphi^p (id - \varphi)^{m+1} \\
&\quad = \sum_{p=0}^{q} \binom{p+m}{m} \varphi^p (id - \varphi)^m - \sum_{p=0}^{q} \binom{p+m}{m} \varphi^{p+1} (id - \varphi)^m \\
&\quad = (id - \varphi)^m + \sum_{p=1}^{q} \left[\binom{p+m}{m} - \binom{p+m-1}{m}\right] \varphi^p (id - \varphi)^m - \binom{q+m}{m} \varphi^{q+1} (id - \varphi)^m \\
&\quad = \sum_{p=0}^{q} \binom{p+m-1}{m-1} \varphi^p (id - \varphi)^m - \binom{q+m}{m} \varphi^{q+1} (id - \varphi)^m.
\end{aligned}$$

Using the induction hypothesis, we complete the proof. □

Lemma 2.2. *Let $\varphi : B(\mathcal{H}) \to B(\mathcal{H})$ be a power bounded, positive linear map and let $D \in B(\mathcal{H})$ be a positive operator such that $(id - \varphi)^m(D) \geqslant 0$ for some $m \in \mathbb{N}$. Then the following limit exists for any $h \in \mathcal{H}$ and $k = 0, 1, \ldots, m-1$:*

$$\lim_{p\to\infty} p^k \langle \varphi^p(id-\varphi)^k(D)h, h\rangle = \begin{cases} \lim_{p\to\infty}\langle \varphi^p(D)h, h\rangle & \text{if } k = 0,\\ 0 & \text{if } k = 1, 2, \ldots, m-1. \end{cases}$$

Proof. For each $h \in \mathcal{H}$, $p = 0, 1, \ldots,$ and $r = 0, 1, \ldots, m$, denote $x_p^{(r)} := \langle \varphi^p(id-\varphi)^r(D)h, h\rangle$ and notice that, due to Lemma 1.4, $x_p^{(r)} \geqslant 0$. When $k = 0, 1, \ldots, m-1$, using the same lemma, we obtain

$$x_p^{(k)} - x_{p+1}^{(k)} = \langle \varphi^p(id-\varphi)^{k+1}(D)h, h\rangle \geqslant 0.$$

Therefore, $\{x_p^{(k)}\}_{p=0}^\infty$ is a decreasing sequence of positive numbers. In particular, when $k = 0$, we deduce that $\lim_{p\to\infty}\langle \varphi^p(D)h, h\rangle$ exists.

It remains to prove that

$$\lim_{p\to\infty} p^k \langle \varphi^p(id-\varphi)^k(D)h, h\rangle = 0 \tag{2.6}$$

for any $k = 1, \ldots, m-1$. As an intermediate step, we will also prove that

$$\sum_{p=1}^\infty p^{r-1} x_p^{(r)} < \infty \tag{2.7}$$

for $r = 1, \ldots, m$. Notice that this relation holds true if $r = 1$, due to the fact that the series

$$\sum_{p=1}^\infty \langle \varphi^p(id-\varphi)(D)h, h\rangle = \langle Dh, h\rangle - \lim_{p\to\infty}\langle \varphi^p(D)h, h\rangle$$

is convergent. We proceed now by induction over r. Assume that $1 \leqslant N \leqslant m-1$ and that relation (2.7) holds for $r = N$, i.e., $\sum_{p=1}^\infty p^{N-1}x_p^{(N)} < \infty$. We shall prove first that relation (2.6) holds for $k = N$. Due to the Cauchy criterion, we have

$$y_q := q^{N-1}x_q^{(N)} + (q+1)^{N-1}x_{q+1}^{(N)} + \cdots + (2q-1)^{N-1}x_{2q-1}^{(N)} \to 0, \quad \text{as } q \to \infty.$$

Since $\{x_q^{(N)}\}_{q=1}^\infty$ is a decreasing sequence of positive numbers, we have $q^N x_{2q-1}^{(N)} \leqslant y_q$. Now, it is clear that $(2q-1)^N x_{2q-1}^{(N)} \to 0$ as $q \to \infty$. On the other hand, since $(2q)^N x_{2q}^{(N)} \leqslant (2q)^N x_{2q-1}^{(N)}$, we have $(2q)^N x_{2q}^{(N)} \to 0$ as $q \to \infty$. Consequently, relation (2.6) holds for $k = N$.

Now, we prove that if (2.6) holds for $k = N$ (where $1 \leqslant N \leqslant m-1$) and relation (2.7) holds for $r = N$, then (2.7) holds also for $r = N+1$. Notice that

$$\begin{aligned}\sum_{p=1}^q p^N x_p^{(N+1)} &= \sum_{p=1}^q p^N \langle \varphi^p(id-\varphi)^{N+1}(D)h, h\rangle\\ &= \sum_{r=1}^q r^N x_r^{(N)} - \sum_{p=1}^q p^N x_{p+1}^{(N)}\end{aligned}$$

$$= x_1^{(N)} + \sum_{p=1}^{q}\big[(p+1)^N - p^N\big]x_{p+1}^{(N)} - (q+1)^N x_{q+1}^{(N)}$$

$$\leqslant x_1^{(N)} + N\sum_{p=1}^{q}(p+1)^{N-1}x_{p+1}^{(N)} - (q+1)^N x_{q+1}^{(N)}.$$

Using our assumptions, we conclude that (2.7) holds for $r = N+1$. This completes the proof. $\square$

Let f be a positive regular free holomorphic function on a noncommutative ball $[B(\mathcal{H})^n]_\rho$, $\rho > 0$. In what follows we introduce the noncommutative Berezin kernel associated with any n-tuple of operators $T := (T_1, \ldots, T_n)$ in the noncommutative domain $\mathbf{D}_f^m(\mathcal{H})$, and present some of its basic properties.

Lemma 2.3. *Let $T := (T_1, \ldots, T_n) \in \mathbf{D}_f^m(\mathcal{H})$ and let $K_{f,T}^{(m)} : \mathcal{H} \to F^2(H_n) \otimes \overline{\Delta_{f,m,T}(\mathcal{H})}$ be the map defined by*

$$K_{f,T}^{(m)} h := \sum_{\alpha\in\mathbb{F}_n^+} \sqrt{b_\alpha^{(m)}}\, e_\alpha \otimes \Delta_{f,m,T} T_\alpha^* h, \quad h \in \mathcal{H}, \tag{2.8}$$

where $\Delta_{f,m,T} := [(I - \Phi_{f,T})^m(I)]^{1/2}$, the positive map $\Phi_{f,T}$ is defined by (2.1) *and the coefficients $b_\alpha^{(m)}$ are given by* (1.1). *Then*

(i) $K_{f,T}^{(m)*}K_{f,T}^{(m)} = I_{\mathcal{H}} - Q_{f,T}$, *where* $Q_{f,T} := \text{SOT-}\lim_{k\to\infty} \Phi_{f,T}^k(I)$;

(ii) $K_{f,T}^{(m)}T_i^* = (W_i^* \otimes I_{\mathcal{H}})K_{f,T}^{(m)}$, $i = 1, \ldots, n$, *where $(W_1, \ldots, W_n)$ is the n-tuple of weighted left creation operators associated with the noncommutative domain $\mathbf{D}_f^m$.*

Proof. Since $\Phi_{f,T}(I) \leqslant I$ and $\Phi_{f,T}(\cdot)$ is a positive linear map, it is easy to see that $\{\Phi_{f,T}^p(I)\}_{p=1}^\infty$ is a decreasing sequence of positive operators and, consequently, $Q_{f,T} := \text{SOT-}\lim_{p\to\infty} \Phi_{f,T}^p(I)$ exists. Due to relation (1.1) and using Lemma 2.1, we deduce that

$$\left\langle \sum_{\beta\in\mathbb{F}_n^+} b_\beta^{(m)} T_\beta \Delta_{f,m,T}^2 T_\beta^* h, h\right\rangle$$

$$= \langle \Delta_{f,m,T}^2 h, h\rangle + \sum_{m=1}^{\infty}\sum_{|\beta|=m} \langle b_\beta^{(m)} T_\beta \Delta_{f,m,T}^2 T_\beta^* h, h\rangle$$

$$= \langle \Delta_{f,m,T}^2 h, h\rangle + \sum_{m=1}^{\infty}\sum_{|\beta|=m} \left\langle \left(\sum_{j=1}^{|\beta|} \binom{j+m-1}{m-1} \sum_{\substack{\gamma_1\cdots\gamma_j=\beta \\ |\gamma_1|\geqslant 1,\ldots,|\gamma_j|\geqslant 1}} a_{\gamma_1}\cdots a_{\gamma_j} \right)\right.$$

$$\left. \times\, T_{\gamma_1\cdots\gamma_j} \Delta_{f,m,T}^2 T_{\gamma_1\cdots\gamma_j}^* h, h\right\rangle$$

$$
\begin{aligned}
&= \langle \Delta_{f,m,T}^2 h, h\rangle + \sum_{p=1}^{\infty} \binom{p+m-1}{m-1} \sum_{|\alpha_1|\geqslant 1,\ldots,|\alpha_p|\geqslant 1} a_{\alpha_1}\cdots a_{\alpha_p} T_{\alpha_1}\cdots T_{\alpha_p} \Delta_{f,m,T}^2 T_{\alpha_p}^*\cdots T_{\alpha_1}^* \\
&= \lim_{k\to\infty} \sum_{p=0}^{k} \binom{p+m-1}{m-1} \big\langle \big\{ \Phi_{f,T}^p\big[(I-\Phi_{f,T})^m\big](I)\big\} h, h\big\rangle \\
&= \|h\|^2 - \lim_{k\to\infty} \sum_{j=0}^{m-1} \binom{k+j}{j} \big\langle \Phi_{f,T}^{k+1}\big[(I-\Phi_{f,T})^j\big](I) h, h\big\rangle
\end{aligned}
$$

for any $h \in \mathcal{H}$. Now, applying Lemma 2.2 to $\Phi_{f,T}$, we deduce that

$$
\sum_{\beta\in\mathbb{F}_n^+} b_\beta^{(m)} T_\beta \Delta_{f,m,T}^2 T_\beta^* = I_{\mathcal{H}} - Q_{f,T}. \tag{2.9}
$$

Due to the above calculations, we have

$$
\big\| K_{f,T}^{(m)} h\big\|^2 = \sum_{\beta\in\mathbb{F}_n^+} b_\beta^{(m)} \langle T_\beta \Delta_{f,m,T}^2 T_\beta^* h, h\rangle = \|h\|^2 - \big\| Q_{f,T}^{1/2} h\big\|^2
$$

for any $h \in \mathcal{H}$. Therefore, $K_{f,T}^{(m)}$ is a contraction and

$$
K_{f,T}^{(m)*} K_{f,T}^{(m)} = I_{\mathcal{H}} - Q_{f,T}. \tag{2.10}
$$

On the other hand, one can show that

$$
K_{f,T}^{(m)} T_i^* = \big(W_i^* \otimes I_{\mathcal{H}}\big) K_{f,T}^{(m)}, \quad i = 1,\ldots,n, \tag{2.11}
$$

where $(W_1,\ldots,W_n)$ is the n-tuple of weighted left creation operators associated with the noncommutative domain $\mathbf{D}_f^m$. Indeed, notice that, due to relation (1.5), we have

$$
W_i^* e_\alpha = \begin{cases} \dfrac{\sqrt{b_\gamma^{(m)}}}{\sqrt{b_\alpha^{(m)}}} e_\gamma & \text{if } \alpha = g_i\gamma, \\ 0 & \text{otherwise.} \end{cases}
$$

Hence, we deduce that

$$
\begin{aligned}
\big(W_i^* \otimes I_{\mathcal{H}}\big) K_{f,T}^{(m)} h &= \sum_{\alpha\in\mathbb{F}_n^+} \sqrt{b_\alpha^{(m)}}\, W_i^* e_\alpha \otimes \Delta_{f,m,T} T_\alpha^* h \\
&= \sum_{\gamma\in\mathbb{F}_n^+} \sqrt{b_{g_i\gamma}^{(m)}} \frac{\sqrt{b_\gamma^{(m)}}}{\sqrt{b_{g_i\gamma}^{(m)}}} e_\gamma \otimes \Delta_{f,m,T} T_{g_i\gamma}^* h \\
&= K_{f,T}^{(m)} T_i^* h
\end{aligned}
$$

for any $h \in \mathcal{H}$ and $i = 1,\ldots,n$, which proves our assertion. □

We can define now the *extended noncommutative Berezin transform* $\widetilde{\mathbf{B}}_T$ at any $T \in \mathbf{D}_f^m(\mathcal{H})$ by setting

$$\widetilde{\mathbf{B}}_T[g] := K_{f,T}^{(m)*}(g \otimes I_{\mathcal{H}})K_{f,T}^{(m)}, \quad g \in B\bigl(F^2(H_n)\bigr), \tag{2.12}$$

where the *noncommutative Berezin kernel* $K_{f,T}^{(m)} : \mathcal{H} \to F^2(H_n) \otimes \mathcal{H}$ is defined by

$$K_{f,T}^{(m)}h = \sum_{\alpha \in \mathbb{F}_n^+} \sqrt{b_\alpha^{(m)}} e_\alpha \otimes \Delta_{f,m,T} T_\alpha^* h, \quad h \in \mathcal{H}, \tag{2.13}$$

the defect operator $\Delta_{T,m,f} := [(id - \Phi_{f,T})^m(I)]^{1/2}$, and the coefficients $b_\alpha^{(m)}$, $\alpha \in \mathbb{F}_n^+$, are given by relation (1.1).

Proposition 2.4. *The noncommutative Berezin transforms $\widetilde{\mathbf{B}}_T$ and $\mathbf{B}_T$ coincide for any n-tuple of operators $T := (T_1, \ldots, T_n) \in \mathbf{D}_f^m(\mathcal{H})$ with joint spectral radius $r_f(T_1, \ldots, T_n) < 1$.*

Proof. Due to Lemma 1.1 and relation (2.3), the operator $(I - \sum_{|\alpha| \geqslant 1} a_{\tilde{\alpha}} \Lambda_\alpha \otimes T_{\tilde{\alpha}}^*)^{-m}$ has the Fourier representation is $\sum_{\beta \in \mathbb{F}_n^+} (\Lambda_\beta \otimes b_{\tilde{\beta}} T_{\tilde{\beta}}^*)$. Consequently, using relations (1.8) and (2.13), we obtain

$$\begin{aligned} K_{f,T}^{(m)}h &= \sum_{\alpha \in \mathbb{F}_n^+} \sqrt{b_\alpha^{(m)}} e_\alpha \otimes \Delta_{f,m,T} T_\alpha^* h \\ &= 1 \otimes \Delta_{f,m,T} h + \sum_{|\beta| \geqslant 1} b_{\tilde{\beta}}^{(m)} \Lambda_\beta(1) \otimes \Delta_{f,m,T} T_{\tilde{\beta}}^* h \\ &= (I_{F^2(H_n)} \otimes \Delta_{T,m,f}) \Bigl(I - \sum_{|\alpha| \geqslant 1} a_{\tilde{\alpha}} \Lambda_\alpha \otimes T_{\tilde{\alpha}}^*\Bigr)^{-m} (1 \otimes h) \end{aligned}$$

for any $h \in \mathcal{H}$. Taking into account relations (2.4) and (2.12), we complete the proof. □

Let us recall some definitions concerning completely bounded maps on operator spaces. We identify $M_k(B(\mathcal{H}))$, the set of $k \times k$ matrices with entries in $B(\mathcal{H})$, with $B(\mathcal{H}^{(k)})$, where $\mathcal{H}^{(k)}$ is the direct sum of k copies of $\mathcal{H}$. If $\mathcal{X}$ is an operator space, i.e., a closed subspace of $B(\mathcal{H})$, we consider $M_k(\mathcal{X})$ as a subspace of $M_k(B(\mathcal{H}))$ with the induced norm. Let $\mathcal{X}, \mathcal{Y}$ be operator spaces and $u : \mathcal{X} \to \mathcal{Y}$ be a linear map. Define the map $u_k : M_k(\mathcal{X}) \to M_k(\mathcal{Y})$ by

$$u_k\bigl([x_{ij}]_k\bigr) := \bigl[u(x_{ij})\bigr]_k.$$

We say that u is completely bounded if $\|u\|_{cb} := \sup_{k \geqslant 1} \|u_k\| < \infty$. When $\|u\|_{cb} \leqslant 1$ (respectively u_k is an isometry for any $k \geqslant 1$) then u is completely contractive (respectively isometric). We call u completely positive if u_k is positive for all $k \geqslant 1$. For more information on completely bounded maps and the classical von Neumann inequality [53], we refer to [28,29].

Let $f(X_1, \ldots, X_n) := \sum_{|\alpha| \geqslant 1} a_\alpha X_\alpha$ be a positive regular free holomorphic function on $[B(\mathcal{H})^n]_\rho$, $\rho > 0$, and let $T := (T_1, \ldots, T_n)$ be an n-tuple of operators in the noncommutative domain $\mathbf{D}_f^m(\mathcal{H})$. Recall that the positive linear map $\Phi_{f,T} : B(\mathcal{H}) \to B(\mathcal{H})$ is defined by

$\Phi_{f,T}(X) = \sum_{|\alpha|\geqslant 1} a_\alpha T_\alpha X T_\alpha^*$, where the convergence is in the weak operator topology. In the proof of Lemma 2.3, we saw that $Q_{f,T} := \text{SOT-}\lim_{k\to\infty} \Phi_{f,T}^k(I)$ exists. We call an n-tuple T *pure* (or of class $C_{\cdot 0}$) if $Q_{f,T} = 0$. We remark that if $\|\Phi_{f,T}(I)\| < 1$, then T is of class $C_{\cdot 0}$. This is due to the fact that $\|\Phi_{f,T}^k(I)\| \leqslant \|\Phi_{f,T}(I)\|^k$. Note also that, due to Theorem 1.6, the model n-tuple $W := (W_1, \dots, W_n)$ is in the noncommutative domain $\mathbf{D}_f^m(F^2(H_n))$ and, due to Theorem 1.3, it is of class $C_{\cdot 0}$.

We introduce the domain algebra $\mathcal{A}_n(\mathbf{D}_f^m)$ associated with the noncommutative domain $\mathbf{D}_f^m$ to be the norm closure of all polynomials in the weighted left creation operators $W_1, \dots, W_n$ and the identity. Using the weighted right creation operators associated with $\mathbf{D}_f^m$, one can define the corresponding domain algebra $\mathcal{R}_n(\mathbf{D}_f^m)$.

Theorem 2.5. *Let $T := (T_1, \dots, T_n)$ be a pure n-tuple of operators in the noncommutative domain $\mathbf{D}_f^m(\mathcal{H})$. Then the restriction of the noncommutative Berezin transform $\widetilde{\mathbf{B}}_T$ to $\overline{\operatorname{span}}\{W_\alpha W_\beta^*;\ \alpha, \beta \in \mathbb{F}_n^+\}$ is a unital completely contractive linear map such that*

$$\widetilde{\mathbf{B}}_T\big[W_\alpha W_\beta^*\big] = T_\alpha T_\beta^*, \quad \alpha, \beta \in \mathbb{F}_n^+.$$

In particular, the restriction of $\widetilde{\mathbf{B}}_T$ to the domain algebra $\mathcal{A}_n(\mathbf{D}_f^m)$ is a completely contractive homomorphism.

Proof. According to Lemma 2.3, $K_{f,T}^{(m)}$ is an isometry if and only if $T := (T_1, \dots, T_n) \in \mathbf{D}_f^m(\mathcal{H})$ is a pure n-tuple. Part (ii) of the same lemma and relation (2.12) imply

$$\widetilde{\mathbf{B}}_T\big[W_\alpha W_\beta^*\big] = K_{f,T}^{(m)*}\big[W_\alpha W_\beta^* \otimes I_\mathcal{H}\big) K_{f,T}^{(m)} = T_\alpha T_\beta^*, \quad \alpha, \beta \in \mathbb{F}_n^+.$$

Now, one can easily deduce that $\widetilde{\mathbf{B}}_T$ is a unital completely contractive linear map. This completes the proof. □

We say that an n-tuple of operators $X := (X_1, \dots, X_n) \in \mathbf{D}_f^m(\mathcal{H})$ has the radial property with respect to $\mathbf{D}_f^m(\mathcal{H})$ if there exists $\delta \in (0,1)$ such that $rX := (rX_1, \dots, rX_n) \in \mathbf{D}_f^m(\mathcal{H})$ for any $r \in (\delta, 1)$.

Proposition 2.6. *Any noncommutative domain $\mathbf{D}_f^m(\mathcal{H})$ contains a ball $[B(\mathcal{H})^n]_\gamma$, $\gamma > 0$, and, therefore, n-tuples of operators with the radial property.*

Proof. Since $f = \sum_{|\alpha|\geqslant 1} a_\alpha X_\alpha$ is a free holomorphic function on a certain ball $[B(\mathcal{H})^n]_\delta$, $\delta > 0$, we have $\limsup_{k\to\infty}(\sum_{|\alpha|=k} |a_\alpha|^2)^{1/2k} < \infty$. Consequently, there exists a constant $M > 0$ such that $|a_\alpha| \leqslant M^k$ for any $\alpha \in \mathbb{F}_n^+$ with $|\alpha| = k$. Let $r \in (0,1)$ be such that $Mr < 1$ and let $(X_1, \dots, X_n) \in [B(\mathcal{H})^n]_r$. Then we have

$$\big\|\Phi_{f,X}(I)\big\| \leqslant \sum_{k=1}^\infty M^k \bigg\| \sum_{|\alpha|=k} X_\alpha X_\alpha^* \bigg\| \leqslant \sum_{k=1}^\infty M^k r^{2k} = \frac{r^2 M}{1 - r^2 M},$$

which converges to zero as $r \to 0$. Since f^k, $k = 1, \dots, m$, is a free holomorphic function with $f^k(0) = 0$, a similar result holds. Therefore, there exists a ball $[B(\mathcal{H})^n]_\gamma$, $\gamma > 0$, such that

$\|\Phi_{f,X}(I)\|, \dots, \|\Phi_{f^m,X}(I)\|$ are as small as needed for any $X \in [B(\mathcal{H})^n]_\gamma$. On the other hand, we have $\Phi_{f,X}^k(I) = \Phi_{f^k,X}(I)$ and

$$(id - \Phi_{f,X})^m(I) = I - \sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} \Phi_{f^k,X}(I).$$

Now, it is clear that $(I - \Phi_{f,X})^m(I) \geqslant 0$ for any X in an appropriate ball $[B(\mathcal{H})^n]_\gamma$, $\gamma > 0$. The proof is complete. □

We remark that one can easily prove that if p is a positive regular noncommutative polynomial and $T := (T_1, \dots, T_n) \in B(\mathcal{H})^n$ is such that $(id - \Phi_{p,T})^k \geqslant cI$ for some $c > 0$ and any $1 \leqslant k \leqslant m$, then $(T_1, \dots, T_n) \in \mathbf{D}_p^m(\mathcal{H})$ has the radial property.

The next result extends Theorems 3.7 and 3.8 from [39] to our more general setting. We only sketch the proof.

Theorem 2.7. *Let $T := (T_1, \dots, T_n)$ be an n-tuple of operators with the radial property in the noncommutative domain $\mathbf{D}_f^m(\mathcal{H})$ and let $\mathcal{S} := \overline{\operatorname{span}}\{W_\alpha W_\beta^*;\ \alpha, \beta \in \mathbb{F}_n^+\}$. Then there is a unital completely contractive linear map $\Psi_{f,m,T} : \mathcal{S} \to B(\mathcal{H})$ such that*

$$\Psi_{f,m,T}(g) = \lim_{r \to 1} \mathbf{B}_{rT}[g], \quad g \in \mathcal{S}, \tag{2.14}$$

where the limit exists in the norm topology of $B(\mathcal{H})$, and

$$\Psi_{f,m,T}\big(W_\alpha W_\beta^*\big) = T_\alpha T_\beta^*, \quad \alpha, \beta \in \mathbb{F}_n^+.$$

In particular, the restriction of $\Psi_{f,m,T}$ to the domain algebra $\mathcal{A}_n(\mathbf{D}_f^m)$ is a completely contractive homomorphism. If, in addition, T is a pure n-tuple, then

$$\lim_{r \to 1} \mathbf{B}_{rT}[g] = \widetilde{\mathbf{B}}_T[g], \quad g \in \mathcal{S}.$$

Proof. Since $0 < r < 1$, $(rT_1, \dots, rT_n) \in \mathbf{D}_f^m(\mathcal{H})$ is a pure n-tuple. Indeed, it is enough to see that $\Phi_{f,rT}^k(I) \leqslant r^k \Phi_{f,T}^k(I) \leqslant r^k I$ for $k \in \mathbb{N}$. Therefore, due to relation (2.10), $K_{f,rT}$ is an isometry. Now, Lemma 2.3 implies

$$K_{f,rT}^{(m)*}\big[W_\alpha W_\beta^* \otimes I_{\mathcal{H}}\big) K_{f,rT}^{(m)} = r^{|\alpha|+|\beta|} T_\alpha T_\beta^*, \quad \alpha, \beta \in \mathbb{F}_n^+. \tag{2.15}$$

Hence, we deduce that

$$\left\| \sum_{\alpha,\beta \in \Lambda} c_{\alpha,\beta} T_\alpha T_\beta^* \right\| \leqslant \left\| \sum_{\alpha,\beta \in \Lambda} c_{\alpha,\beta} W_\alpha W_\beta^* \right\| \tag{2.16}$$

for any finite set $\Lambda \subset \mathbb{F}_n^+$ and $c_{\alpha,\beta} \in \mathbb{C}$. For each $g \in \mathcal{S}$, let $\{q_k(W_i, W_i^*)\}_{k=0}^\infty$ be a sequence of polynomials of the form $\sum_{\alpha,\beta \in \Lambda} c_{\alpha,\beta} W_\alpha W_\beta^*$ which converges to g, as $k \to \infty$. Define $\Psi_{f,m,T}(g) := \lim_{k \to \infty} q_k(T_i, T_i^*)$. The von Neumann type inequality (2.16) shows that

$\Psi_{f,m,T}(g)$ is well defined and $\|\Psi_{f,m,T}(g)\| \leqslant \|g\|$. Using the matrix version on (2.15), we deduce that $\Psi_{f,m,T}$ is a unital completely contractive linear map. To prove the second part of the theorem, one has to use the relation

$$\mathbf{B}_{rT}[g] = K_{f,rT}^{(m)*}(g \otimes I_{\mathcal{H}})K_{f,rT}^{(m)}, \quad g \in \mathcal{S},$$

and standard approximation arguments (see [39]). □

We say that a noncommutative domain $\mathbf{D}_f^m$ has the radial property if each n-tuple $X \in \mathbf{D}_f^m(\mathcal{H})$ has the radial property, where $\mathcal{H}$ is a separable infinite-dimensional Hilbert space. Notice that, if $m = 1$, then the noncommutative domain $\mathbf{D}_f^1$ has always the radial property. When $m \geqslant 1$, we have the following class of noncommutative domains with the radial property.

Example 2.8. If $p(X_1, \ldots, X_n) := a_1X_1 + \cdots + a_nX_n$, $a_i > 0$, then the noncommutative domain $\mathbf{D}_p^m(\mathcal{H})$, $m = 1, 2, \ldots$, has the radial property. Indeed, let $X := (X_1, \ldots, X_n) \in \mathbf{D}_p^m(\mathcal{H})$, $0 < r \leqslant 1$, and note that

$$\begin{aligned}(id - \Phi_{p,rX})^k(I) &= \left[(id - \Phi_{p,X}) + (1-r)\Phi_{p,X}\right]^k(I) \\ &= \sum_{j=0}^{k} \binom{k}{j}(1-r)^{k-j}\Phi_{p,X}^{k-j}(id - \Phi_{p,X})^j(I)\end{aligned}$$

for any $k = 1, \ldots, m$. By Corollary 1.5, we have $(id - \Phi_{p,X})^j(I) \geqslant 0$ for $j = 1, \ldots, m$. Now, using the fact that $\Phi_{p,X}^j$ is a positive linear map, we deduce that $(id - \Phi_{p,rX})^k(I) \geqslant 0$ for $j = 1, \ldots, m$ and $r \in (0, 1]$, which proves our assertion.

Assume that p is a regular positive noncommutative polynomial and $\mathbf{D}_p^m$ is a noncommutative domain with the radial property. Under these conditions, one can prove the following.

Corollary 2.9. *An n-tuple of operators $(T_1, \ldots, T_n) \in B(\mathcal{H})^n$ is in the noncommutative domain $\mathbf{D}_p^m(\mathcal{H})$ if and only if there exists a completely positive linear map $\Psi : C^*(W_1, \ldots, W_n) \to B(\mathcal{H})$ such that $\Psi(W_\alpha W_\beta^*) = T_\alpha T_\beta^*$, $\alpha, \beta \in \mathbb{F}_n^+$. In particular, the result holds if $p = a_1X_1 + \cdots + a_nX_n$ with $a_i > 0$.*

Proof. The direct implication is due to Theorem 2.7 and Arveson's extension theorem [5]. For the converse, use Theorem 1.6, and notice that $\Psi[(I - \Phi_{p,W})^k(I)] = (I - \Phi_{f,p})^k(I)$ for $k = 1, \ldots, m$. □

3. The Hardy algebra $F_n^\infty(\mathbf{D}_f^m)$ and a functional calculus

In this section, we introduce the Hardy algebra $F_n^\infty(\mathbf{D}_f^m)$ (respectively $R_n^\infty(\mathbf{D}_f^m)$) associated with the noncommutative domain $\mathbf{D}_f^m$ and present some basic properties. The main result is an $F_n^\infty(\mathbf{D}_f^m)$-functional calculus for completely noncoisometric n-tuples of operators in the noncommutative domain $\mathbf{D}_f^m$.

Let f be a positive regular free holomorphic function on a noncommutative ball $[B(\mathcal{H})^n]_\rho$, $\rho > 0$, with representation $f(X_1, \ldots, X_n) := \sum_{|\alpha| \geqslant 1} a_\alpha X_\alpha$. As preliminaries, we need some

inequalities concerning the coefficients $b_\alpha^{(m)}$ associated with f (see Section 1). According to Lemma 1.1, if $|\alpha| \geqslant 1$ and $|\beta| \geqslant 1$, then we have

$$b_\alpha^{(m)} b_\beta^{(m)} = \sum_{j=1}^{|\alpha|} \sum_{k=1}^{|\beta|} \binom{j+m-1}{m-1} \binom{k+m-1}{m-1} \times \Bigg[\sum_{\substack{\gamma_1 \cdots \gamma_j = \alpha \\ |\gamma_1| \geqslant 1, \ldots, |\gamma_j| \geqslant 1}} \sum_{\substack{\sigma_1 \cdots \sigma_k = \beta \\ |\sigma_1| \geqslant 1, \ldots, |\sigma_k| \geqslant 1}} a_{\gamma_1} \cdots a_{\gamma_j} a_{\sigma_1} \cdots a_{\sigma_k} \Bigg]$$

and

$$b_{\alpha\beta}^{(m)} = \sum_{p=1}^{|\alpha|+|\beta|} \binom{p+m-1}{m-1} \Bigg[\sum_{\substack{\epsilon_1 \cdots \epsilon_p = \alpha\beta \\ |\epsilon_1| \geqslant 1, \ldots, |\epsilon_p| \geqslant 1}} a_{\epsilon_1} \cdots a_{\epsilon_p} \Bigg].$$

Note that, for any $j = 1, \ldots, |\alpha|$ and $k = 1, \ldots, |\beta|$,

$$\binom{j+m-1}{m-1} \binom{k+m-1}{m-1} \leqslant M_{|\beta|,m} \binom{j+k+m-1}{m-1},$$

where $M_{|\beta|,m} := \binom{|\beta|+m-1}{m-1}$. A closer look at the above-mentioned equalities reveals that

$$b_\alpha^{(m)} b_\beta^{(m)} \leqslant M_{|\beta|,m} b_{\alpha\beta}^{(m)}, \quad \alpha \in \mathbb{F}_n^+. \tag{3.1}$$

Similarly, we obtain

$$b_\alpha^{(m)} b_\beta^{(m)} \leqslant M_{|\alpha|,m} b_{\alpha\beta}^{(m)}, \quad \beta \in \mathbb{F}_n^+.$$

Let $\varphi(W_1, \ldots, W_n) = \sum_{\beta \in \mathbb{F}_n^+} c_\beta W_\beta$ be a formal sum with the property that

$$\sum_{\beta \in \mathbb{F}_n^+} |c_\beta|^2 \frac{1}{b_\beta^{(m)}} < \infty,$$

where the coefficients b_β, $\beta \in \mathbb{F}_n^+$, are given by relation (1.1). Using relations (1.5) and (3.1), one can see that $\sum_{\beta \in \mathbb{F}_n^+} c_\beta W_\beta(p) \in F^2(H_n)$ for any $p \in \mathcal{P}$, where $\mathcal{P}$ is the set of all polynomial in $F^2(H_n)$. Indeed, for each $\gamma \in \mathbb{F}_n^+$, we have $\sum_{\beta \in \mathbb{F}_n^+} c_\beta W_\beta(e_\gamma) = \sum_{\beta \in \mathbb{F}_n^+} c_\beta \sqrt{\frac{b_\gamma^{(m)}}{b_{\beta\gamma}^{(m)}}} e_{\beta\gamma}$ and, due to inequality (3.1), we deduce that

$$\sum_{\beta \in \mathbb{F}_n^+} |c_\beta|^2 \frac{b_\gamma^{(m)}}{b_{\beta\gamma}^{(m)}} \leqslant M_{|\gamma|,m} \sum_{\beta \in \mathbb{F}_n^+} |c_\beta|^2 \frac{1}{b_\beta^{(m)}} < \infty.$$

If

$$\sup_{p\in\mathcal{P},\ \|p\|\leqslant 1}\left\|\sum_{\beta\in\mathbb{F}_n^+} c_\beta W_\beta(p)\right\| < \infty,$$

then there is a unique bounded operator acting on $F^2(H_n)$, which we denote by $\varphi(W_1,\dots,W_n)$, such that

$$\varphi(W_1,\dots,W_n)p = \sum_{\beta\in\mathbb{F}_n^+} c_\beta W_\beta(p) \quad \text{for any } p\in\mathcal{P}.$$

The set of all operators $\varphi(W_1,\dots,W_n)\in B(F^2(H_n))$ satisfying the above-mentioned properties is denoted by $F_n^\infty(\mathbf{D}_f^m)$. When $f = X_1+\cdots+X_n$ and $m=1$, $F_n^\infty(\mathbf{D}_f^m)$ coincides with the noncommutative analytic Toeplitz algebra F_n^∞, which was introduced in [34] in connection with a noncommutative multivariable von Neumann inequality. As in this particular case, one can prove that $F_n^\infty(\mathbf{D}_f^m)$ is a Banach algebra, which we call Hardy algebra associated with the noncommutative domain $\mathbf{D}_f^m$.

In a similar manner, using the weighted right creation operators $(\Lambda_1,\dots,\Lambda_n)$ associated with $\mathbf{D}_f^m$, one can define the corresponding the Hardy algebra $R_n^\infty(\mathbf{D}_f^m)$. More precisely, if $g(\Lambda_1,\dots,\Lambda_n) = \sum_{\beta\in\mathbb{F}_n^+} c_{\tilde\beta}\Lambda_\beta$ is a formal sum with the property that $\sum_{\beta\in\mathbb{F}_n^+}|c_\beta|^2\frac{1}{b_\beta^{(m)}}<\infty$, where the coefficients $b_\alpha^{(m)}$, $\alpha\in\mathbb{F}_n^+$, are given by relation (1.1), and such that

$$\sup_{p\in\mathcal{P},\ \|p\|\leqslant 1}\left\|\sum_{\beta\in\mathbb{F}_n^+} c_{\tilde\beta}\Lambda_\beta(p)\right\| < \infty,$$

then there is a unique bounded operator on $F^2(H_n)$, which we denote by $g(\Lambda_1,\dots,\Lambda_n)$, such that

$$g(\Lambda_1,\dots,\Lambda_n)p = \sum_{\beta\in\mathbb{F}_n^+} c_{\tilde\beta}\Lambda_\beta(p) \quad \text{for any } p\in\mathcal{P}.$$

The set of all operators $g(\Lambda_1,\dots,\Lambda_n)\in B(F^2(H_n))$ satisfying the above-mentioned properties is denoted by $R_n^\infty(\mathbf{D}_f^m)$.

Proposition 3.1. *Let f be a positive regular free holomorphic function on a noncommutative ball $[B(\mathcal{H})^n]_\rho$, $\rho>0$, and let $\mathbf{D}_f^m$ be the associated noncommutative domain. Then the following statements hold*:

(i) $F_n^\infty(\mathbf{D}_f^m)' = U^*(F_n^\infty(\mathbf{D}_{\tilde f}^m))U = R_n^\infty(\mathbf{D}_f^m)$, *where* $'$ *stands for the commutant and* $U\in B(F^2(H_n))$ *is the unitary operator defined by* $Ue_\alpha = e_{\tilde\alpha}$, $\alpha\in\mathbb{F}_n^+$;
(ii) $F_n^\infty(\mathbf{D}_f^m)'' = F_n^\infty(\mathbf{D}_f^m)$ *and* $R_n^\infty(\mathbf{D}_f^m)'' = R_n^\infty(\mathbf{D}_f^m)$.

Proof. Let $(W_1^{(f)},\dots,W_n^{(f)})$ (respectively $(\Lambda_1^{(f)},\dots,\Lambda_n^{(f)})$) be the weighted left (respectively right) creation operators associated with the noncommutative domain $\mathbf{D}_f^m$. Due to Theorem 1.6,

part (iii), we have $U^*(F_n^\infty(\mathbf{D}_{\tilde f}^m))U = R_n^\infty(\mathbf{D}_f^m)$. On the other hand, since $W_i^{(f)}\Lambda_j^{(f)} = \Lambda_j^{(f)}W_i^{(f)}$ for any $i, j = 1, \dots, n$, it is clear that $R_n^\infty(\mathbf{D}_f^m) \subseteq F_n^\infty(\mathbf{D}_f^m)'$. To prove the reverse inclusion, let $A \in F_n^\infty(\mathbf{D}_f^m)'$. Since $A(1) \in F^2(H_n)$, we have $A(1) = \sum_{\beta\in\mathbb{F}_n^+} c_{\tilde\beta}\frac{1}{\sqrt{b_{\tilde\beta}^{(m)}}}e_{\tilde\beta}$ for some coefficients $\{c_\beta\}_{\beta\in\mathbb{F}_n^+}$ with $\sum_{\beta\in\mathbb{F}_n^+}|c_\beta|^2\frac{1}{b_\beta^{(m)}} < \infty$. On the other hand, since $AW_i^{(f)} = W_i^{(f)}A$ for $i = 1, \dots, n$, relations (1.5) and (1.8) imply

$$\begin{aligned} Ae_\alpha &= \sqrt{b_\alpha^{(m)}}AW_\alpha(1) = \sqrt{b_\alpha^{(m)}}W_\alpha A(1) \\ &= \sum_{\beta\in\mathbb{F}_n^+} c_{\tilde\beta}\frac{\sqrt{b_\alpha^{(m)}}}{\sqrt{b_{\alpha\tilde\beta}^{(m)}}}e_{\alpha\tilde\beta} = \sum_{\beta\in\mathbb{F}_n^+} c_{\tilde\beta}\Lambda_\beta(e_\alpha) \end{aligned}$$

for any $\alpha \in \mathbb{F}_n^+$. Therefore, $A(q) = \sum_{\beta\in\mathbb{F}_n} c_{\tilde\beta}\Lambda_\beta(q)$ for any polynomial q in the full Fock space $F^2(H_n)$. Since A is a bounded operator, $g(\Lambda_1, \dots, \Lambda_n) := \sum_{\beta\in\mathbb{F}_n} c_{\tilde\beta}\Lambda_\beta$ is in $R_n^\infty(\mathbf{D}_f^m)$ and $A = g(\Lambda_1, \dots, \Lambda_n)$. Therefore, $R_n^\infty(\mathbf{D}_f^m) = F_n^\infty(\mathbf{D}_f^m)'$. The item (ii) follows easily applying part (i). This completes the proof. □

An obvious consequence of Proposition 3.1 is that $F_n^\infty(\mathbf{D}_f^m)$ is WOT-closed (respectively w^*-closed) in $B(F^2(H_n))$.

Let Q_k, $k \geqslant 0$, be the orthogonal projection of $F^2(H_n)$ on the subspace $\operatorname{span}\{e_\alpha\colon |\alpha| = k\}$. For each integer j, define the completely contractive projection $\Phi_j : B(F^2(H_n)) \to B(F^2(H_n))$ by

$$\Phi_j(A) := \sum_{k\geqslant\max\{0,-j\}} Q_kAQ_{k+j}.$$

According to Lemma 1.1 from [15], the Cesaro operators on $B(F^2(H_n))$ defined by

$$\Sigma_k(A) := \sum_{|j|<k}\left(1 - \frac{|j|}{k}\right)\Phi_j(A), \quad k \geqslant 1,$$

are completely contractive and $\Sigma_k(A)$ converges to A in the strong operator topology. Now, let $A \in F_n^\infty(\mathbf{D}_f^m)$ have the Fourier representation $\sum_{\alpha\in\mathbb{F}_n^+} a_\alpha W_\alpha$. Due to the definition of the weighted left creation operators (see (1.4)), one can check that

$$Q_{k+j}AQ_j = \left(\sum_{|\alpha|=k} a_\alpha W_\alpha\right)Q_j, \quad k \geqslant 0,\ j \geqslant 0,$$

and $Q_jAQ_{k+j} = 0$ if $k \geqslant 1$ and $j \geqslant 0$. Therefore,

$$\Sigma_k(A) = \sum_{|\alpha|\leqslant k-1}\left(1 - \frac{|\alpha|}{k}\right)a_\alpha W_\alpha$$

converges to A, as $k \to \infty$, in the strong operator topology. Therefore, we have proved the following result.

Proposition 3.2. *The algebra $F_n^\infty(\mathbf{D}_f^m)$ is the sequential SOT- (respectively WOT-, w^*-) closure of all polynomials in $W_1, \ldots, W_n$, and the identity.*

Now, we have all the ingredients to extend the corresponding results from [17,45], to our more general setting. More precisely, one can similarly prove that the following statements hold:

(i) The Hardy algebra $F_n^\infty(\mathbf{D}_f^m)$ is inverse closed.
(ii) The only normal elements in $F_n^\infty(\mathbf{D}_f^m)$ are the scalars.
(iii) Every element $A \in F_n^\infty(\mathbf{D}_f^m)$ has its spectrum $\sigma(A) \neq \{0\}$ and it is injective.
(iv) The algebra $F_n^\infty(\mathbf{D}_f^m)$ contains no non-trivial idempotents and no nonzero quasinilpotent elements.
(v) The algebra $F_n^\infty(\mathbf{D}_f^m)$ is semisimple.
(vi) If $A \in F_n^\infty(\mathbf{D}_f^m)$, $n \geqslant 2$, then $\sigma(A) = \sigma_e(A)$.

We recall that an n-tuple $(T_1, \ldots, T_n) \in \mathbf{D}_f^m(\mathcal{H})$ has the radial property with respect to $\mathbf{D}_f^m(\mathcal{H})$ if there exists a constant $\delta \in (0, 1)$ such that $(rT_1, \ldots, rT_n) \in \mathbf{D}_f^m(\mathcal{H})$ for any $r \in (\delta, 1)$.

Lemma 3.3. *Let $(T_1, \ldots, T_n)$be an n-tuple of operators with the radial property in the noncommutative domain $\mathbf{D}_f^m(\mathcal{H})$. Then*

$$g(rT_1, \ldots, rT_n)K_{f,T}^{(m)*} = K_{f,T}^{(m)*}\big(g(rW_1, \ldots, rW_n) \otimes I_{\mathcal{H}}\big) \quad \textit{for any } r \in (\delta, 1) \tag{3.2}$$

and $g(W_1, \ldots, W_n) = \sum_{\beta \in \mathbb{F}_n^+} c_\beta W_\beta$ in $F_n^\infty(\mathbf{D}_f^m)$, where

$$g(rT_1, \ldots, rT_n) := \sum_{k=0}^{\infty} \sum_{|\alpha|=k} c_\alpha r^{|\alpha|} T_\alpha,$$

with the convergence in the operator norm topology.

Proof. According to relations (1.6) and (3.1), the operators $\{W_\beta\}_{|\beta|=k}$ have orthogonal ranges and

$$\|W_\beta x\| \leqslant \frac{1}{\sqrt{b_\beta^{(m)}}} M_{|\beta|,m} \|x\|, \quad x \in F^2(H_n),$$

where $M_{|\beta|,m} := \binom{|\beta|+m-1}{m-1}$. Consequently, we deduce that

$$\left\| \sum_{|\beta|=k} b_\beta W_\beta W_\beta^* \right\| \leqslant \binom{k+m-1}{m-1} \quad \text{for any } k = 0, 1, \ldots. \tag{3.3}$$

Since $g(W_1, \ldots, W_n) \in F_n^\infty(\mathbf{D}_f^m)$, we have $\sum_{\beta \in \mathbb{F}_n^+} |c_\beta|^2 \frac{1}{b_\beta} < \infty$. Hence and using (3.3), we deduce that, for $0 < t < 1$,

$$\begin{aligned}\sum_{k=0}^{\infty} t^k \left\| \sum_{|\beta|=k} c_\beta W_\beta \right\| &\leqslant \sum_{k=0}^{\infty} t^k \left(\sum_{|\beta|=k} |c_\beta|^2 \frac{1}{b_\beta^{(m)}} \right)^{1/2} \left\| \sum_{|\beta|=k} b_\beta^{(m)} W_\beta W_\beta^* \right\|^{1/2} \\ &\leqslant \sum_{k=0}^{\infty} \left(\sum_{|\beta|=k} |c_\beta|^2 \frac{1}{b_\beta^{(m)}} \right)^{1/2} t^k \binom{k+m-1}{m-1}^{1/2} \\ &\leqslant \left(\sum_{\beta \in \mathbb{F}_n^+} |c_\beta|^2 \frac{1}{b_\beta} \right)^{1/2} \left(\sum_{k=0}^{\infty} t^{2k} \binom{k+m-1}{m-1} \right)^{1/2} < \infty,\end{aligned}$$

which proves that

$$g(tW_1, \ldots, tW_n) := \lim_{k\to\infty} \sum_{p=0}^{k} \sum_{|\alpha|=p} t^{|\alpha|} c_\alpha W_\alpha \tag{3.4}$$

is in the noncommutative domain algebra $\mathcal{A}_n(\mathbf{D}_f^m)$, where the convergence is in the operator norm. Consequently, Theorem 2.7 implies that $g(rT_1, \ldots, rT_n) := \sum_{k=0}^{\infty} \sum_{|\alpha|=k} c_\alpha r^{|\alpha|} T_\alpha$ is convergent in the operator norm topology. On the other hand, due to Lemma 2.3, we have $T_i K_{f,T}^{(m)*} = K_{f,T}^{(m)*}(W_i \otimes I_{\mathcal{H}})$ for any $i = 1, \ldots, n$. Now, one can deduce (3.2). This completes the proof. □

In what follows we show that the restriction of the noncommutative Berezin transform to the Hardy algebra $F_n^\infty(\mathbf{D}_f^m)$ provides a functional calculus associated with each pure n-tuple of operators in the noncommutative domain $\mathbf{D}_f^m(\mathcal{H})$.

Theorem 3.4. *Let $T := (T_1, \ldots, T_n)$ be a pure n-tuple of operators in the noncommutative domain $\mathbf{D}_f^m(\mathcal{H})$ and define the map*

$$\Psi_T : F_n^\infty\big(\mathbf{D}_f^m\big) \to B(\mathcal{H}) \quad \text{by} \quad \Psi_T(g) := \widetilde{\mathbf{B}}_T[g],$$

where $\widetilde{\mathbf{B}}_T$ is the noncommutative Berezin transform at $T \in \mathbf{D}_f^m(\mathcal{H})$. Then

(i) *Ψ_T is WOT-continuous (respectively SOT-continuous) on bounded sets;*
(ii) *Ψ_T is a unital completely contractive homomorphism and $\Psi_T(W_\alpha) = T_\alpha$ for $\alpha \in \mathbb{F}_n^+$.*

If, in addition, the universal model $(W_1, \ldots, W_n)$ has the radial property with respect to $\mathbf{D}_f^m(F^2(H_n))$, then

$$\Psi_T(g) = \text{SOT-}\lim_{r\to 1} g(rT_1, \ldots, rT_n)$$

for any $g := \sum_{\beta\in\mathbb{F}_n^+} c_\beta W_\beta$ in $F_n^\infty(\mathbf{D}_f^m)$, where $g(rT_1, \ldots, rT_n) := \sum_{k=0}^{\infty} \sum_{|\alpha|=k} c_\alpha r^{|\alpha|} T_\alpha$ and the convergence is in the operator norm topology.

Proof. According to Section 2 (see relation (2.11)), we have

$$\Psi_T(g) = K_{f,T}^{(m)*}(g \otimes I) K_{f,T}^{(m)}, \quad g \in F_n^\infty\big(\mathbf{D}_f^m\big), \tag{3.5}$$

where the noncommutative Berezin kernel $K_{f,T}^{(m)}$ is given by relation (2.13). Using standard facts in functional analysis, we deduce part (i).

Now, we prove part (ii). Since T is a pure n-tuple of operators, by Lemma 2.3, $K_{f,T}^{(m)}$ is an isometry. Consequently, relation (3.5) implies

$$\left\| \left[\Psi_T(g_{ij}) \right]_k \right\| \leqslant \left\| [g_{ij}]_k \right\|$$

for any operator-valued matrix $[g_{ij}]_k$ in $M_k(F_n^\infty(\mathbf{D}_f^m))$, which proves that Ψ_T is a unital completely contractive linear map. Due to Theorem 2.5, Ψ_T is a homomorphism on polynomials in $F_n^\infty(\mathbf{D}_f^m)$. By Proposition 3.2, the polynomials in $W_1, \dots, W_n$ and the identity are sequentially WOT-dense in $F_n^\infty(\mathbf{D}_f^m)$. On the other hand, due to part (i), Ψ_T is WOT-continuous on bounded sets. Now, one can use the principle of uniform boundedness to deduce that Ψ_T is also a homomorphism on $F_n^\infty(\mathbf{D}_f^m)$.

Now, we prove the last part of this theorem. Assume that the model n-tuple $(W_1, \dots, W_n)$ has the radial property with respect to $\mathbf{D}_f^m(F^2(H_n))$. First, we show that

$$g(W_1, \dots, W_n) = \text{SOT-}\lim_{t \to 1} g(tW_1, \dots, tW_n) \tag{3.6}$$

for any $g(W_1, \dots, W_n) := \sum_{\beta \in \mathbb{F}_n^+} c_\beta W_\beta \in F_n^\infty(\mathbf{D}_f)$. According to Lemma 3.3,

$$g(tW_1, \dots, tW_n) := \lim_{k \to \infty} \sum_{k=0}^{k} \sum_{|\alpha|=p} t^{|\alpha|} c_\alpha W_\alpha \tag{3.7}$$

is in the noncommutative domain algebra $\mathcal{A}_n(\mathbf{D}_f^m)$, where the convergence is in the operator norm topology. Fix now $\gamma, \sigma, \epsilon \in \mathbb{F}_n^+$ and consider the polynomial $p(W_1, \dots, W_n) := \sum_{\beta \in \mathbb{F}_n^+, |\beta| \leqslant |\gamma|} c_\beta W_\beta$. Since $W_\beta^* e_\gamma = 0$ for any $\beta \in \mathbb{F}_n^+$ with $|\beta| > |\gamma|$, we have

$$g(rW_1, \dots, rW_n)^* e_\alpha = p(rW_1, \dots, rW_n)^* e_\alpha$$

for any $\alpha \in \mathbb{F}_n^+$ with $|\alpha| \leqslant |\gamma|$ and any $r \in [0, 1]$. On the other hand, since $rW := (rW_1, \dots, rW_n) \in \mathbf{D}_f^m(F^2(H_n))$ for $r \in (\delta, 1)$, Lemma 2.3 implies

$$K_{f,rW}^{(m)} p(rW_1, \dots, rW_n)^* = \left[p(W_1, \dots, W_n)^* \otimes I_{F^2(H_n)} \right] K_{f,rW}^{(m)}$$

for any $r \in (\delta, 1)$. Using all these facts, careful calculations reveal that

$$\begin{aligned}
&\left\langle K_{f,rW}^{(m)} g(rW_1, \dots, rW_n)^* e_\gamma, e_\sigma \otimes e_\epsilon \right\rangle \\
&\quad = \left\langle K_{f,rW}^{(m)} p(rW_1, \dots, rW_n)^* e_\gamma, e_\sigma \otimes e_\epsilon \right\rangle \\
&\quad = \left\langle \left[\left(p(W_1, \dots, W_n)^* \otimes I_{F^2(H_n)} \right) \right] K_{f,rW}^{(m)} e_\gamma, e_\sigma \otimes e_\epsilon \right\rangle \\
&\quad = \sum_{\beta \in \mathbb{F}_n^+} r^{|\beta|} \sqrt{b_\beta^{(m)}} \left\langle p(W_1, \dots, W_n)^* e_\beta, e_\sigma \right\rangle \left\langle W_\beta^* e_\gamma, \Delta_{f,rW} e_\epsilon \right\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\beta\in\mathbb{F}_n^+} r^{|\beta|}\sqrt{b_\beta^{(m)}}\big\langle g(W_1,\dots,W_n)^* e_\beta, e_\sigma\big\rangle\big\langle W_\beta^* e_\gamma, \Delta_{f,rW} e_\epsilon\big\rangle \\
&= \big\langle \big[g(W_1,\dots,W_n)^*\otimes I_{F^2(H_n)}\big] K_{f,rW}^{(m)} e_\gamma, e_\sigma\otimes e_\epsilon\big\rangle
\end{aligned}
$$

for any $r\in(\delta,1)$ and $\gamma,\sigma,\epsilon\in\mathbb{F}_n^+$. Hence, since $g(rW_1,\dots,rW_n)$ and $g(W_1,\dots,W_n)$ are bounded operators, we deduce that

$$
K_{f,rW}^{(m)} g(rW_1,\dots,rW_n)^* = \big[g(W_1,\dots,W_n)^*\otimes I_{F^2(H_n)}\big] K_{f,rW}^{(m)}.
$$

Since the n-tuple $rW := (rW_1,\dots,rW_n)\in \mathbf{D}_f^{(m)}(F^2(H_n))$ is pure, the Berezin kernel $K_{f,rW}^{(m)}$ is an isometry and, therefore, the equality above implies

$$
\big\|g(rW_1,\dots,rW_n)\big\| \leqslant \big\|g(W_1,\dots,W_n)\big\| \quad \text{for any } r\in(\gamma,1). \tag{3.8}
$$

Hence, and due to the fact that $g(W_1,\dots,W_n)e_\alpha = \lim_{r\to 1} g(rW_1,\dots,rW_n)e_\alpha$ for any $\alpha\in\mathbb{F}_n^+$, an approximation argument implies relation (3.6).

According to Lemma 3.3, we have

$$
g(rT_1,\dots,rT_n)K_{f,T}^{(m)*} = K_{f,T}^{(m)*}\big(g(rW_1,\dots,rW_n)\otimes I_{\mathcal{H}}\big) \quad \text{for any } r\in(\delta,1). \tag{3.9}
$$

On the other hand, since the map $Y\mapsto Y\otimes I_{\mathcal{H}}$ is SOT-continuous on bounded sets, relations (3.6) and (3.8) imply that

$$
\text{SOT-}\lim_{r\to 1}\big[g(rW_1,\dots,rW_n)\otimes I_{\mathcal{H}}\big] = g(W_1,\dots,W_n)\otimes I_{\mathcal{H}}. \tag{3.10}
$$

Hence, using relation (3.9) and that $K_{f,T}^{(m)}$ is an isometry, we deduce that

$$
\text{SOT-}\lim_{r\to 1} g(rT_1,\dots,rT_n) = K_{f,T}^{(m)*}\big[g(W_1,\dots,W_n)\otimes I_{\mathcal{H}}\big]K_{f,T}^{(m)} = \widetilde{\mathbf{B}}_T[g]. \tag{3.11}
$$

This completes the proof. □

We need now the following technical result concerning the Berezin transform and the radial property.

Lemma 3.5. *If $T := (T_1,\dots,T_n)\in\mathbf{D}_f^m(\mathcal{H})$ and the universal model $(W_1,\dots,W_n)$ have the radial property, then there exists $\delta\in(0,1)$ such that the noncommutative Berezin kernel satisfies the relation*

$$
K_{f,rT}^{(m)*}\big(g(W_1,\dots,W_n)\otimes I_{\mathcal{H}}\big) = g(rT_1,\dots,rT_n)K_{f,rT}^{(m)*} \tag{3.12}
$$

for any $g(W_1,\dots,W_n)\in F_n^\infty(\mathbf{D}_f^m)$ and $r\in(\delta,1)$.

If, in addition, $T := (T_1,\dots,T_n)\in\mathbf{D}_f^m(\mathcal{H})$ is a pure n-tuple of operators, then

$$
\mathbf{B}_{rT}[g] = \widetilde{\mathbf{B}}_T[g_r], \quad r\in(\delta,1),
$$

where $g_r(W_1,\dots,W_n) := g(rW_1,\dots,rW_n)$.

Proof. First, notice that Lemma 2.3 implies

$$K_{f,rT}^{(m)*}\big[p(W_1,\dots,W_n)\otimes I_{\mathcal{H}}\big]=p(rT_1,\dots,rT_n)K_{f,rT}^{(m)*} \tag{3.13}$$

for any polynomial $p(W_1,\dots,W_n)$ and $r\in(\delta,1)$. Since $rT:=(rT_1,\dots,rT_n)\in \mathbf{D}_f^{(m)}(\mathcal{H})$, relation (3.7) and Theorem 2.7 imply

$$\lim_{k\to\infty}\sum_{|\alpha|\leqslant k} t^{|\alpha|}r^{|\alpha|}c_\alpha T_\alpha = g_t(rT_1,\dots,rT_n)\quad \text{for any } t\in[0,1),\ r\in(\delta,1),$$

where the convergence is in the operator norm topology. Using relation (3.13), when

$$p(W_1,\dots,W_n):=\sum_{q=0}^{k}\sum_{|\alpha|=q} t^{|\alpha|}c_\alpha W_\alpha,$$

and taking the limit as $k\to\infty$, we get

$$K_{f,rT}^{(m)*}\big[g_t(W_1,\dots,W_n)\otimes I_{\mathcal{H}}\big]=g_t(rT_1,\dots,rT_n)K_{f,rT}^{(m)*}\quad \text{for } r\in(\delta,1). \tag{3.14}$$

On the other hand, let us prove that

$$\lim_{t\to 1} g_t(rT_1,\dots,rT_n)=g(rT_1,\dots,rT_n), \tag{3.15}$$

where the convergence is in the operator norm topology. Notice that, if $\epsilon>0$, there is $m_0\in\mathbb{N}$ such that

$$\sum_{k=m_0}^{\infty} r^k\binom{k+m-1}{m-1}<\frac{\epsilon}{4M},\qquad \text{where } M:=\big\|g(W_1,\dots,W_n)(1)\big\|.$$

Since $(T_1,\dots,T_n)\in\mathbf{D}_f^{(m)}(\mathcal{H})$, Theorem 2.7 and relation (3.3) imply

$$\Big\|\sum_{|\beta|=k} b_\alpha T_\beta T_\beta^*\Big\|\leqslant\Big\|\sum_{|\beta|=k} b_\alpha W_\beta W_\beta^*\Big\|\leqslant\binom{k+m-1}{m-1}.$$

Now, we can deduce that

$$\begin{aligned}\sum_{k=m_0}^{\infty} r^k\Big\|\sum_{|\beta|=k} c_\beta T_\beta\Big\| &\leqslant \sum_{k=m_0}^{\infty} r^k\Big(\sum_{|\beta|=k}|c_\alpha|^2\frac{1}{b_\beta}\Big)^{1/2}\Big\|\sum_{|\beta|=k} b_\beta T_\beta T_\beta^*\Big\|^{1/2}\\ &\leqslant M\sum_{k=m_0}^{\infty} r^k\binom{k+m-1}{m-1}<\frac{\epsilon}{4}.\end{aligned}$$

Consequently, there exists $0<d<1$ such that

$$\begin{aligned}
&\left\| \sum_{k=0}^{\infty} \sum_{|\alpha|=k} t^{|\alpha|} r^{|\alpha|} c_\alpha T_\alpha - \sum_{k=0}^{\infty} \sum_{|\alpha|=k} r^{|\alpha|} c_\alpha T_\alpha \right\| \\
&\quad \leqslant \frac{\epsilon}{2} + \left\| \sum_{k=1}^{m_0-1} r^k \bigl(t^k - 1\bigr) \sum_{|\beta|=k} c_\beta T_\beta \right\| \\
&\quad \leqslant \frac{\epsilon}{2} + M \sum_{k=1}^{m_0-1} r^k \bigl(t^k - 1\bigr) \binom{k+m-1}{m-1} < \epsilon
\end{aligned}$$

for any $t \in (d, 1)$. Hence, we deduce (3.15). Using relations (3.10), (3.15), and taking the limit in (3.14), as $t \to 1$, we obtain (3.12). Now, assume that T is a pure n-tuple. Based on Proposition 2.4 and relations (2.12), (3.2), and (3.12), we deduce that $\mathbf{B}_{rT}[g] = \mathbf{B}_T[g_r]$ for $r \in (\delta, 1)$. The proof is complete. $\square$

Using Theorem 3.4 and Lemma 3.5, we can deduce the following Fatou type result.

Corollary 3.6. *Let $T := (T_1, \ldots, T_n) \in \mathbf{D}_f^m(\mathcal{H})$ be a pure n-tuple of operators and assume that $(W_1, \ldots, W_n)$ has the radial property. Then*

$$\text{SOT-}\lim_{r\to 1} \mathbf{B}_{rT}[g] = \widetilde{\mathbf{B}}_T[g] \quad \textit{for any } g \in F_n^\infty\bigl(\mathbf{D}_f^m\bigr).$$

Proof. Recall that $\Phi_{f,T}(X) := \sum_{|\alpha|\geqslant 1} a_\alpha T_\alpha X T_\alpha^*$, where the series is WOT-convergent. Since the sequence $\sum_{1\leqslant|\alpha|\leqslant k} a_\alpha r^{2|\alpha|} W_\alpha W_\alpha^*$ is bounded and SOT-convergent to $\Phi_{f,rW}(I)$, as $k \to \infty$, the proof of Theorem 2.5 implies

$$\Phi_{f,rT}(I) = K_{f,T}^{(m)*}\bigl[\Phi_{f,rW}(I) \otimes I_{\mathcal{H}}\bigr] K_{f,T}^{(m)}$$

and, consequently,

$$(I - \Phi_{f,rT})^m(I) = K_{f,T}^{(m)*}\bigl[(I - \Phi_{f,rW})^m(I) \otimes I_{\mathcal{H}}\bigr] K_{f,T}^{(m)}.$$

Since $(W_1, \ldots, W_n)$ has the radial property, so does $(T_1, \ldots, T_n)$. Using now Theorem 3.4 and Lemma 3.5, we can complete the proof. $\square$

An n-tuple $T := (T_1, \ldots, T_n) \in \mathbf{D}_f^m(\mathcal{H})$ is called completely non-coisometric (c.n.c) with respect to the noncommutative domain $\mathbf{D}_f^m(\mathcal{H})$ if there is no vector $h \in \mathcal{H}$, $h \neq 0$, such that $\langle \Phi_{f,T}^k(I)h, h\rangle = \|h\|^2$ for any $k = 1, 2, \ldots$. Due to relation (2.10), we have

$$\bigl\| K_{f,T}^{(m)} h \bigr\|^2 = \|h\|^2 - \bigl\| Q_{f,T}^{1/2} h \bigr\|^2, \quad h \in \mathcal{H},$$

where $Q_{f,T} := \text{SOT-}\lim_{k\to\infty} \Phi_{f,T}^k(I)$. Notice that T is c.n.c. if and only if the noncommutative Berezin kernel $K_{f,T}^{(m)}$ is one-to-one.

Now, we can present an $F_n^\infty(\mathbf{D}_f^m)$-functional calculus for c.n.c. n-tuples of operators in the noncommutative domain $\mathbf{D}_f^m(\mathcal{H})$.

Theorem 3.7. *Let $\mathbf{D}_f^m$ be a noncommutative domain such that the universal model $(W_1, \ldots, W_n)$ has the radial property. If $T := (T_1, \ldots, T_n) \in \mathbf{D}_f^m(\mathcal{H})$ is a completely non-coisometric n-tuple of operators with the radial property, then*

$$\Phi(g) := \text{SOT-}\lim_{r\to 1} g_r(T_1, \ldots, T_n), \quad g = g(W_1, \ldots, W_n) \in F_n^\infty\big(\mathbf{D}_f^m\big),$$

exists in the strong operator topology and defines a map $\Phi : F_n^\infty(\mathbf{D}_f^m) \to B(\mathcal{H})$ with the following properties:

(i) $\Phi(g) = \text{SOT-}\lim_{r\to 1} \mathbf{B}_{rT}[g]$, *where $\mathbf{B}_{rT}$ is the noncommutative Berezin transform at $rT \in \mathbf{D}_f^m(\mathcal{H})$*;
(ii) *Φ is WOT-continuous* (*respectively SOT-continuous*) *on bounded sets*;
(iii) *Φ is a unital completely contractive homomorphism.*

Proof. Let $\delta \in (0,1)$ be such that $(rT_1, \ldots, rT_n) \in \mathbf{D}_f^m(\mathcal{H})$ and $(rW_1, \ldots, rW_n) \in \mathbf{D}_f^m(F^2(H_n))$ for any $r \in (\delta, 1)$. Due to (3.8) and taking the limit in relation (3.2), as $r \to 1$, we deduce that the map $G : \text{range}\, K_{f,T}^{(m)*} \to \mathcal{H}$ given by $Gy := \lim_{r\to 1} g_r(T_1, \ldots, T_n)y$, $y \in \text{range}\, K_{f,T}^{(m)*}$, is well defined, linear, and

$$\big\|GK_{f,T}^{(m)*}\varphi\big\| \leqslant \limsup_{r\to 1}\big\|g_r(W_1, \ldots, W_n)\big\|\,\big\|K_{f,T}^{(m)*}\varphi\big\| \leqslant \big\|g(W_1, \ldots, W_n)\big\|\,\big\|K_{f,T}^{(m)*}\varphi\big\|$$

for any $\varphi \in F^2(H_n) \otimes \mathcal{H}$.

Now, assume that $T = (T_1, \ldots, T_n) \in \mathcal{D}_f(\mathcal{H})$ is c.n.c. Since the Berezin kernel $K_{f,T}^{(m)}$ is one-to-one, its range is dense in $\mathcal{H}$. Consequently, the map G has a unique extension to a bounded linear operator on $\mathcal{H}$, denoted also by G, with $\|G\| \leqslant \|g(W_1, \ldots, W_n)\|$. Let us show that

$$\lim_{r\to 1} g_r(T_1, \ldots, T_n)h = Gh \quad \text{for any } h \in \mathcal{H}. \tag{3.16}$$

Let $\{y_k\}_{k=1}^\infty$ be a sequence of vectors in the range of $K_{f,T}^*$, which converges to y. According to Theorem 2.7 and relations (3.7), (3.8), we have

$$\big\|g_r(T_1, \ldots, T_n)\big\| \leqslant \big\|g_r(W_1, \ldots, W_n)\big\| \leqslant \big\|g(W_1, \ldots, W_n)\big\|$$

for any $r \in (\delta, 1)$. Let $\{y_k\}_{k=1}^\infty$ be a sequence of vectors in the range of $K_{f,T}^{(m)*}$, which converges to y, and notice that

$$\begin{aligned}\big\|Gh - g_r(T_1, \ldots, T_n)h\big\| &\leqslant \|Gh - Gy_k\| + \big\|Gy_k - g_r(T_1, \ldots, T_n)y_k\big\| \\ &\quad + \big\|g_r(T_1, \ldots, T_n)y_k - g_r(T_1, \ldots, T_n)h\big\| \\ &\leqslant 2\big\|g(W_1, \ldots, W_n)\big\|\,\|h - y_k\| + \big\|Gy_k - g_r(T_1, \ldots, T_n)y_k\big\|.\end{aligned}$$

Since $\lim_{r\to 1} g_r(T_1, \ldots, T_n)y_k = Gy_k$, relation (3.16) follows. Due to Lemma 3.5, we have

$$g_r(T_1, \ldots, T_n) = K_{f,rT}^{(m)*}\big[g(W_1, \ldots, W_n) \otimes I_{\mathcal{H}}\big]K_{f,rT}^{(m)}, \tag{3.17}$$

which together with (3.16) imply part (i) of the theorem.

Now let us prove part (ii). Due to relation (3.17), we have

$$\big\|g_r(T_1, \ldots, T_n)\big\| \leqslant \big\|g(W_1, \ldots, W_n)\big\|$$

and, therefore, $\|\Phi(g)\| \leqslant \|g\|$ for $g \in F_n^\infty(\mathbf{D}_f^m)$. Taking $r \to 1$ in relation (3.2) of Lemma 3.3 and using part (i), we obtain

$$\Phi(g) K_{f,T}^{(m)*} = K_{f,T}^{(m)*}(g \otimes I), \quad g \in F_n^\infty\big(\mathbf{D}_f^m\big). \tag{3.18}$$

Let $\{g_i\}$ be a bounded net in $F_n^\infty(\mathbf{D}_f^m)$ such that $g_i \to g \in F_n^\infty(\mathbf{D}_f^m)$ in the weak (respectively strong) operator topology. Then $g_i \otimes I$ converges to $g \otimes I$ in the same topologies. By (3.18), we have $\Phi(g_i) K_{f,T}^{(m)*} = K_{f,T}^{(m)*}(g_i \otimes I)$. Since the range of $K_{f,T}^{(m)*}$ is dense in $\mathcal{H}$ and $\{\Phi(g_i)\}$ is bounded, an approximation argument shows that $\Phi(g_i) \to \Phi(g)$ in the weak (respectively strong) operator topology.

To prove (iii), note that (3.17) and the fact that $K_{f,rT}$ is an isometry for $r \in (\delta, 1)$ imply

$$\big\|\big[g_{ij}(rT_1, \ldots, rT_n)\big]_k\big\| \leqslant \big\|[g_{ij}]_k\big\|$$

for any operator-valued matrix $[g_{ij}]_k \in M_k(F_n^\infty(\mathbf{D}_f^m))$ and $r \in (\delta, 1)$. Hence, and due to the fact that $\Phi(g_{ij}) = \text{SOT-}\lim_{r\to 1} g_{ij}(rT_1, \ldots, rT_n)$, we deduce that Φ is completely contractive map. On the other hand, due to Theorem 2.7, Φ is a homomorphism on polynomials in $W_1, \ldots, W_n$ and the identity. Since these polynomials are sequentially WOT-dense in $F_n^\infty(\mathbf{D}_f^m)$ (see Proposition 3.2) and Φ is WOT-continuous on bounded sets, we deduce part (iii). The proof is complete. □

Consider the particular case when the domain $\mathbf{D}_p^m$, $m \geqslant 1$, is determined by the noncommutative polynomial $p = a_1 Z_1 + \cdots + a_n Z_n$, $a_i > 0$. Due to Example 2.8, $\mathbf{D}_p^m$ has the radial property. Therefore, according to Theorem 3.7, there is an $F_n^\infty(\mathbf{D}_p^m)$-functional calculus for any c.n.c. n-tuple of operators in $\mathbf{D}_p^m(\mathcal{H})$. When $m \geqslant 2$, $n = 1$, and $p = Z$, we obtain a functional calculus for Agler's m-hypercontractions. On the other hand, if $m = 1$, $n = 1$, and $p = Z_1 + \cdots + Z_n$, we obtain the F_n^∞-functional calculus for row contractions [35]. Moreover, if $m = 1$, $n = 1$, and $p = Z$, we obtain the Nagy–Foias H^∞-functional calculus for c.n.c. contractions. We remark that the H^∞-functional calculus works for a larger class of contractions (see [51]).

4. Weighted shifts, symmetric weighted Fock spaces, and multipliers

In this section, we find all the eigenvectors for $W_1^*, \ldots, W_n^*$, where $(W_1, \ldots, W_n)$ is the universal model associated with the noncommutative domain $\mathbf{D}_f^m$. As consequences, we identify the w^*-continuous multiplicative linear functional on the Hardy algebra $F_n^\infty(\mathbf{D}_f^m)$ and find the joint right spectrum of $(W_1, \ldots, W_n)$. We introduce the symmetric weighted Fock space $F_s^2(\mathbf{D}_f^m)$ and identify it with a reproducing kernel Hilbert space $H^2(\mathbf{D}_{f,\circ}^1(\mathbb{C}))$. We also show that the algebra of all its multipliers is reflexive. This section plays an important role in connecting the results of the present paper to analytic function theory on Reinhardt domains in $\mathbb{C}^n$, as well as, to model theory for commuting n-tuples of operators.

Let $f = \sum_{|\alpha| \geqslant 1} a_\alpha X_\alpha$ be a positive regular free holomorphic function on $[B(\mathcal{H})^n]$, $\rho > 0$, and define

$$\mathbf{D}_{f,\circ}^1(\mathbb{C}) := \left\{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n\colon \sum_{|\alpha| \geqslant 1} a_\alpha |\lambda_\alpha|^2 < 1\right\} \subset \mathbf{D}_f^m(\mathbb{C}),$$

where $\lambda_\alpha := \lambda_{i_1} \cdots \lambda_{i_m}$ if $\alpha = g_{i_1} \cdots g_{i_m} \in \mathbb{F}_n^+$, and $\lambda_{g_0} = 1$.

Theorem 4.1. *Let* $(W_1, \dots, W_n)$ (*respectively* $(\Lambda_1, \dots, \Lambda_n)$) *be the weighted left* (*respectively right*) *creation operators associated with the noncommutative domain* $\mathbf{D}_f^m$. *The eigenvectors for* $W_1^*, \dots, W_n^*$ (*respectively* $\Lambda_1^*, \dots, \Lambda_n^*$) *are precisely the vectors*

$$z_\lambda := \left(I - \sum_{|\alpha| \geqslant 1} a_{\tilde{\alpha}} \bar{\lambda}_\alpha \Lambda_\alpha\right)^{-m}(1) \in F^2(H_n) \quad \text{for } \lambda = (\lambda_1, \dots, \lambda_n) \in \mathbf{D}_{f,\circ}^1(\mathbb{C}),$$

where $\tilde{\alpha}$ *denotes the reverse of* α. *They satisfy the equations*

$$W_i^* z_\lambda = \bar{\lambda}_i z_\lambda, \qquad \Lambda_i^* z_\lambda = \bar{\lambda}_i z_\lambda \quad \text{for } i = 1, \dots, n,$$

and each vector z_λ *is cyclic for* $R_n^\infty(\mathbf{D}_f^m)$.

If $\lambda := (\lambda_1, \dots, \lambda_n) \in \mathbf{D}_{f,\circ}^1(\mathbb{C})$ *and* $\varphi(W_1, \dots, W_n) := \sum_{\beta \in \mathbb{F}_n^+} c_\beta W_\beta$ *is in the Hardy algebra* $F_n^\infty(\mathbf{D}_f^m)$, *then* $\sum_{\beta \in \mathbb{F}_n^+} |c_\beta||\lambda_\beta| < \infty$ *and the map*

$$\Phi_\lambda : F_n^\infty(\mathbf{D}_f) \to \mathbb{C}, \quad \Phi_\lambda\big(\varphi(W_1, \dots, W_n)\big) := \varphi(\lambda),$$

is w^**-continuous and multiplicative. Moreover,* $\varphi(W_1, \dots, W_n)^* z_\lambda = \overline{\varphi(\lambda)} z_\lambda$ *and*

$$\varphi(\lambda) = \big\langle \varphi(W_1, \dots, W_n)1, z_\lambda \big\rangle = \big\langle \varphi(W_1, \dots, W_n) u_\lambda, u_\lambda \big\rangle,$$

where $u_\lambda := \frac{z_\lambda}{\|z_\lambda\|}$.

Proof. Since $\sum_{|\alpha| \geqslant 1} a_{\tilde{\alpha}} \Lambda_\alpha \Lambda_\alpha^*$ is SOT-convergent and, for any $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbf{D}_{f,\circ}^1(\mathbb{C})$,

$$\begin{aligned} \left\| \sum_{|\alpha| \geqslant 1} a_{\tilde{\alpha}} \bar{\lambda}_{\tilde{\alpha}} \Lambda_\alpha \right\| &\leqslant \left\| \sum_{|\alpha| \geqslant 1} a_{\tilde{\alpha}} \Lambda_\alpha \Lambda_\alpha^* \right\| \left(\sum_{|\alpha| \geqslant 1} a_\alpha |\lambda_\alpha|^2 \right)^{1/2} \\ &\leqslant \left(\sum_{|\alpha| \geqslant 1} a_\alpha |\lambda_\alpha|^2 \right)^{1/2} < 1, \end{aligned}$$

the operator $(I - \sum_{|\alpha| \geqslant 1} a_{\tilde{\alpha}} \bar{\lambda}_{\tilde{\alpha}} \Lambda_\alpha)^{-m}$ is well defined. Due to the results of Section 1 (see Lemma 1.1), we have

$$\left(I - \sum_{|\alpha| \geqslant 1} a_{\tilde{\alpha}} \bar{\lambda}_{\tilde{\alpha}} \Lambda_\alpha\right)^{-m} = \sum_{\beta \in \mathbb{F}_n^+} b_{\tilde{\beta}}^{(m)} \bar{\lambda}_{\tilde{\beta}} \Lambda_\beta,$$

where the coefficients b_β, $\beta \in \mathbb{F}_n^+$, are defined by relation (1.1). Hence, and using relation (1.8), we obtain

$$z_\lambda = \left(I - \sum_{|\alpha| \geqslant 1} a_{\tilde{\alpha}} \bar{\lambda}_\alpha \Lambda_\alpha\right)^{-m}(1) = \sum_{\beta \in \mathbb{F}_n^+} b_{\tilde{\beta}}^{(m)} \bar{\lambda}_{\tilde{\beta}} \Lambda_\beta(1) = \sum_{\beta \in \mathbb{F}_n^+} \sqrt{b_\beta^{(m)}} \bar{\lambda}_\beta e_\beta. \tag{4.1}$$

The fact that $z_\lambda \in F^2(H_n)$ is a cyclic vector for $R_n^\infty(\mathcal{D}_f)$ is obvious.

Now, notice that if $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbf{D}^1_{f,\circ}(\mathbb{C})$, then λ is of class $C_{\cdot 0}$ with respect to $\mathbf{D}^1_{f,\circ}(\mathbb{C})$. Using relation (2.9) in our particular case, we get

$$\left(1 - \sum_{|\alpha|\geqslant 1} a_\alpha |\lambda_\alpha|^2\right)^m \left(\sum_{\beta\in\mathbb{F}_n^+} b_\beta^{(m)} |\lambda_\beta|^2\right) = 1.$$

Consequently, we have

$$\|z_\lambda\| = \frac{1}{\sqrt{(1 - \sum_{|\alpha|\geqslant 1} a_\alpha |\lambda_\alpha|^2)^m}}. \tag{4.2}$$

Due to relation (1.5), we have

$$W_i^* e_\alpha = \begin{cases} \frac{\sqrt{b_\gamma^{(m)}}}{\sqrt{b_\alpha^{(m)}}} e_\gamma & \text{if } \alpha = g_i\gamma, \\ 0 & \text{otherwise.} \end{cases}$$

A simple computation shows that $W_i^* z_\lambda = \bar{\lambda}_i z_\lambda$ for $i = 1, \dots, n$. Similarly, one can use relation (1.8) to prove that $\Lambda_i^* z_\lambda = \bar{\lambda}_i z_\lambda$ for $i = 1, \dots, n$.

Conversely, let $z = \sum_{\beta\in\mathbb{F}_n^+} c_\beta e_\beta \in F^2(H_n)$ and assume that $W_i^* z = \bar{\lambda}_i z$, $i = 1, \dots, n$, for some n-tuple $(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$. Using the definition of the weighted left creation operators $W_1, \dots, W_n$, we deduce that

$$\begin{aligned} c_\alpha &= \langle z, e_\alpha\rangle = \left\langle z, \sqrt{b_\alpha^{(m)}} W_\alpha(1)\right\rangle \\ &= \sqrt{b_\alpha^{(m)}}\left\langle W_\alpha^* z, 1\right\rangle = \sqrt{b_\alpha^{(m)}}\bar{\lambda}_\alpha \langle z, 1\rangle \\ &= c_0\sqrt{b_\alpha^{(m)}}\bar{\lambda}_\alpha \end{aligned}$$

for any $\alpha \in \mathbb{F}_n^+$, whence $z = a_0 \sum_{\beta\in\mathbb{F}_n^+} \sqrt{b_\beta^{(m)}}\bar{\lambda}_\beta e_\beta$. Since $z \in F^2(H_n)$, we must have

$$\sum_{\beta\in\mathbb{F}_n^+} b_\beta^{(m)} |\lambda_\beta|^2 < \infty.$$

On the other hand, relation (1.1) implies

$$\left(\sum_{j=0}^k \left(\sum_{|\alpha|\geqslant 1} a_\alpha |\lambda_\alpha|^2\right)^j\right)^m \leqslant \sum_{\beta\in\mathbb{F}_n^+} b_\beta^{(m)} |\lambda_\beta|^2 < \infty$$

for any $k \in \mathbb{N}$. Letting $k \to \infty$ in the relation above, we must have $\sum_{|\alpha|\geqslant 1} a_\alpha |\lambda_\alpha|^2 < 1$, whence $(\lambda_1, \dots, \lambda_n) \in \mathbf{D}^1_{f,\circ}(\mathbb{C})$. A similar result can be proved for the weighted right creation operators $\Lambda_1, \dots, \Lambda_n$ if one uses relation (1.8).

Now, let us prove the last part of the theorem. Since $\varphi(W_1, \ldots, W_n) = \sum_{\beta\in\mathbb{F}_n^+} c_\beta W_\beta$ is in the Hardy algebra $F_n^\infty(\mathbf{D}_f^m)$, we have $\sum_{\beta\in\mathbb{F}_n^+} |c_\beta|^2 \frac{1}{b_\beta^{(m)}} < \infty$ (see Section 3). As shown above, if $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbf{D}_{f,\circ}^1(\mathbb{C})$, then $\sum_{\beta\in\mathbb{F}_n^+} b_\beta^{(m)} |\lambda_\beta|^2 < \infty$. Applying Cauchy's inequality, we have

$$\sum_{\beta\in\mathbb{F}_n^+} |c_\beta||\lambda_\beta| \leqslant \left(\sum_{\beta\in\mathbb{F}_n^+} |c_\beta|^2 \frac{1}{b_\beta^{(m)}}\right)^{1/2} \left(\sum_{\beta\in\mathbb{F}_n^+} b_\beta^{(m)} |\lambda_\beta|^2\right)^{1/2} < \infty.$$

Note also that

$$\begin{aligned}\big\langle \varphi(W_1, \ldots, W_n)1, z_\lambda\big\rangle &= \left\langle \sum_{\beta\in\mathbb{F}_n^+} c_\beta \frac{1}{\sqrt{b_\beta^{(m)}}} e_\beta, \sum_{\beta\in\mathbb{F}_n^+} \sqrt{b_\beta^{(m)}}\, \bar{\lambda}_\beta e_\beta \right\rangle \\ &= \sum_{\beta\in\mathbb{F}_n^+} c_\beta \lambda_\beta = \varphi(\lambda_1, \ldots, \lambda_n).\end{aligned}$$

Now, for each $\beta \in \mathbb{F}_n^+$, we have

$$\begin{aligned}\left\langle \varphi(W_1, \ldots, W_n)^* z_\lambda, \frac{1}{\sqrt{b_\alpha^{(m)}}} e_\beta \right\rangle &= \big\langle z_\lambda, \varphi(W_1, \ldots, W_n) W_\beta(1)\big\rangle \\ &= \overline{\lambda_\beta \varphi(\lambda)} = \left\langle \overline{\varphi(\lambda)} z_\lambda, \frac{1}{\sqrt{b_\alpha^{(m)}}} e_\beta \right\rangle.\end{aligned}$$

Hence, we deduce that

$$\varphi(W_1, \ldots, W_n)^* z_\lambda = \overline{\varphi(\lambda)} z_\lambda. \tag{4.3}$$

One can easily see that

$$\begin{aligned}\big\langle \varphi(W_1, \ldots, W_n) u_\lambda, u_\lambda\big\rangle &= \frac{1}{\|z_\lambda\|^2} \big\langle z_\lambda, \varphi(W_1, \ldots, W_n)^* z_\lambda\big\rangle \\ &= \frac{1}{\|z_\lambda\|^2} \big\langle z_\lambda, \overline{\varphi(\lambda)} z_\lambda\big\rangle = \varphi(\lambda).\end{aligned}$$

The fact that the map Φ_λ is multiplicative and w^*-continuous is now obvious. This completes the proof. □

As in [17], in the particular case when $m = 1$ and $f = X_1 + \cdots + X_n$, one can similarly prove (using Theorem 4.1) the following.

Proposition 4.2. *A map $\varphi : F_n^\infty(\mathbf{D}_f^m) \to \mathbb{C}$ is a w^*-continuous multiplicative linear functional if and only if there exists $\lambda \in \mathbf{D}_{f,\circ}^1(\mathbb{C})$ such that*

$$\varphi(A) = \varphi_\lambda(A) := \langle A u_\lambda, u_\lambda\rangle, \quad A \in F_n^\infty(\mathcal{D}_f),$$

where $u_\lambda := \frac{z_\lambda}{\|z_\lambda\|}$.

We recall that the joint right spectrum $\sigma_r(T_1,\dots,T_n)$ of an n-tuple $(T_1,\dots,T_n)$ of operators in $B(\mathcal{H})$ is the set of all n-tuples $(\lambda_1,\dots,\lambda_n)$ of complex numbers such that the right ideal of $B(\mathcal{H})$ generated by the operators $\lambda_1 I - T_1,\dots,\lambda_n I - T_n$ does not contain the identity operator. We recall [47] that $(\lambda_1,\dots,\lambda_n)\notin\sigma_r(T_1,\dots,T_n)$ if and only if there exists $\delta>0$ such that $\sum_{i=1}^n(\lambda_i I - T_i)(\bar{\lambda}_i I - T_i^*)\geqslant \delta I$.

Theorem 4.1 implies the following result. Since the proof is similar to the proof of [37, Theorem 5.1], we shall omit it.

Proposition 4.3. *If* $(W_1,\dots,W_n)$ *are the weighted left creation operators associated with the noncommutative domain* $\mathbf{D}_f^m$*, then the right joint spectrum* $\sigma_r(W_1,\dots,W_n)$ *coincide with* $\mathbf{D}_f^1(\mathbb{C})$.

Now, we define the symmetric weighted Fock space associated with the noncommutative domain $\mathbf{D}_f^m$. We need a few definitions. For each $\lambda=(\lambda_1,\dots,\lambda_n)\in\mathbb{C}^n$ and each n-tuple $\mathbf{k}:=(k_1,\dots,k_n)\in\mathbb{N}_0^n$, where $\mathbb{N}_0:=\{0,1,\dots\}$, let $\lambda^{\mathbf{k}}:=\lambda_1^{k_1}\cdots\lambda_n^{k_n}$. For each $\mathbf{k}\in\mathbb{N}_0$, we denote

$$\Lambda_{\mathbf{k}}:=\left\{\alpha\in\mathbb{F}_n^+ :\ \lambda_\alpha=\lambda^{\mathbf{k}} \text{ for all } \lambda\in\mathbb{C}^n\right\}.$$

For each $\mathbf{k}\in\mathbb{N}_0^n$, define the vector

$$w^{\mathbf{k}}:=\frac{1}{\gamma_{\mathbf{k}}^{(m)}}\sum_{\alpha\in\Lambda_{\mathbf{k}}}\sqrt{b_\alpha^{(m)}}e_\alpha\in F^2(H_n),\quad \text{where } \gamma_{\mathbf{k}}^{(m)}:=\sum_{\alpha\in\Lambda_{\mathbf{k}}}b_\alpha^{(m)}$$

and the coefficients $b_\alpha^{(m)}$, $\alpha\in\mathbb{F}_n^+$, are defined by relation (1.1). Note that the set $\{w^{\mathbf{k}}:\ \mathbf{k}\in\mathbb{N}_0^n\}$ consists of orthogonal vectors in $F^2(H_n)$ and $\|w^{\mathbf{k}}\|=\frac{1}{\sqrt{\gamma_{\mathbf{k}}^{(m)}}}$. We denote by $F_s^2(\mathbf{D}_f^m)$ the closed span of these vectors, and call it the symmetric weighted Fock space associated with the noncommutative domain $\mathbf{D}_f^m$.

If $\mathcal{Q}$ is a set of noncommutative polynomials, we define the subspace $\mathcal{M}_{\mathcal{Q}}$ of $F^2(H_n)$ by setting

$$\mathcal{M}_{\mathcal{Q}}:=\overline{\operatorname{span}}\left\{W_\alpha q(W_1,\dots,W_n)W_\beta(1):\ q\in\mathcal{Q},\ \alpha,\beta\in\mathbb{F}_n^+\right\}.$$

Theorem 4.4. *Let* $f=\sum_{|\alpha|\geqslant 1}a_\alpha X_\alpha$ *be a positive regular free holomorphic function on* $[B(\mathcal{H})^n]_\rho$, $\rho>0$, *and let* $\mathcal{Q}_c$ *be the set of all polynomials of the form*

$$Z_iZ_j-Z_jZ_i,\quad i,j=1,\dots,n.$$

Then the following statements hold:

(i) $F_s^2(\mathbf{D}_f^m)=\overline{\operatorname{span}}\{z_\lambda:\ \lambda\in\mathbf{D}_{f,\circ}^1(\mathbb{C})\}=\mathcal{N}_{\mathcal{Q}_c}:=F^2(H_n)\ominus\mathcal{M}_{\mathcal{Q}_c}$.

(ii) *The symmetric weighted Fock space* $F_s^2(\mathbf{D}_f^m)$ *can be identified with the Hilbert space* $H^2(\mathbf{D}_{f,\circ}^1(\mathbb{C}))$ *of all functions* $\varphi:\mathbf{D}_{f,\circ}^1(\mathbb{C})\to\mathbb{C}$ *which admit a power series representation* $\varphi(\lambda)=\sum_{\mathbf{k}\in\mathbb{N}_0}c_{\mathbf{k}}\lambda^{\mathbf{k}}$ *with*

$$\|\varphi\|_2=\sum_{\mathbf{k}\in\mathbb{N}_0}|c_{\mathbf{k}}|^2\frac{1}{\gamma_{\mathbf{k}}^{(m)}}<\infty.$$

More precisely, every element $\varphi = \sum_{\mathbf{k}\in\mathbb{N}_0} c_{\mathbf{k}} w^{\mathbf{k}}$ *in* $F_s^2(\mathbf{D}_f^m)$ *has a functional representation on* $\mathbf{D}^1_{f,\circ}(\mathbb{C})$ *given by*

$$\varphi(\lambda) := \langle \varphi, z_\lambda\rangle = \sum_{\mathbf{k}\in\mathbb{N}_0} c_{\mathbf{k}}\lambda^{\mathbf{k}}, \quad \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbf{D}^1_{f,\circ}(\mathbb{C}),$$

and

$$\left|\varphi(\lambda)\right| \leqslant \frac{\|\varphi\|_2}{\sqrt{(1-\sum_{|\alpha|\geqslant 1} a_\alpha |\lambda_\alpha|^2)^m}}, \quad \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbf{D}^1_{f,\circ}(\mathbb{C}).$$

(iii) *The mapping* $K_f : \mathbf{D}^1_{f,\circ}(\mathbb{C}) \times \mathbf{D}^1_{f,\circ}(\mathbb{C}) \to \mathbb{C}$ *defined by*

$$K_f(\mu, \lambda) := \frac{1}{(1-\sum_{|\alpha|\geqslant 1} a_\alpha \mu_\alpha \bar{\lambda}_\alpha)^m} \quad \textit{for all } \lambda, \mu \in \mathbf{D}^1_{f,\circ}(\mathbb{C})$$

is positive definite, and $K_f(\mu, \lambda) = \langle z_\lambda, z_\mu\rangle$.

Proof. First, we prove that

$$\overline{\operatorname{span}}\{z_\lambda\colon \lambda \in \mathbf{D}^1_{f,\circ}(\mathbb{C})\} \subseteq F_s^2(\mathbf{D}_f^m) \subseteq \mathcal{N}_{\mathcal{Q}_c}.$$

Notice that the first inclusion is due to that fact that $z_\lambda = \sum_{\mathbf{k}\in\mathbb{N}_0^n} \bar{\lambda}^{\mathbf{k}} \gamma_{\mathbf{k}} w^{\mathbf{k}}$ for $\lambda \in \mathbf{D}^1_{f,\circ}(\mathbb{C})$. To prove the second inclusion, note that, due to relation (1.5), we have

$$\begin{aligned}\left\langle w^{\mathbf{k}}, W_\gamma(W_jW_i - W_iW_j)W_\beta(1)\right\rangle &= \frac{1}{\gamma_{\mathbf{k}}}\left\langle \sum_{\alpha\in\Lambda_{\mathbf{k}}} \sqrt{b_\alpha^{(m)}} e_\alpha, \frac{1}{\sqrt{b^{(m)}_{\gamma g_j g_i \beta}}} e_{\gamma g_j g_i \beta} - \frac{1}{\sqrt{b^{(m)}_{\gamma g_i g_j \beta}}} e_{\gamma g_i g_j \beta}\right\rangle \\ &= 0\end{aligned}$$

for any $\mathbf{k} \in \mathbb{N}_0^n$, $\alpha, \beta \in \mathbb{F}_n^+$, $i, j = 1, \ldots, n$. This shows that $w^{\mathbf{k}} \in \mathcal{N}_{\mathcal{Q}_c}$ and proves our assertion. To complete the proof of part (i), it is enough to show that

$$\mathcal{N}_{\mathcal{Q}_c} \subseteq \overline{\operatorname{span}}\{z_\lambda\colon \lambda \in \mathbf{D}^1_{f,\circ}(\mathbb{C})\}.$$

To this end, assume that there is a vector $x := \sum_{\beta\in\mathbb{F}_n^+} c_\beta e_\beta \in \mathcal{N}_{\mathcal{Q}_c}$ and $x \perp z_\lambda$ for all $\lambda \in \mathbf{D}^1_{f,\circ}(\mathbb{C})$. Then, using (4.1), we obtain

$$\left\langle \sum_{\beta\in\mathbb{F}_n^+} c_\beta e_\beta, z_\lambda\right\rangle = \sum_{\mathbf{k}\in\mathbb{N}_0^n}\left(\sum_{\beta\in\Lambda_{\mathbf{k}}} c_\beta\sqrt{b_\beta^{(m)}}\right)\lambda^{\mathbf{k}} = 0$$

for any $\lambda \in \mathbf{D}^1_{f,\circ}(\mathbb{C})$. Since $\mathbf{D}^1_{f,\circ}(\mathbb{C})$ contains an open ball in $\mathbb{C}^n$, we deduce that

$$\sum_{\beta\in\Lambda_{\mathbf{k}}} c_\beta\sqrt{b_\beta^{(m)}} = 0 \quad \text{for all } \mathbf{k} \in \mathbb{N}_0^n. \tag{4.4}$$

Fix $\beta_0 \in \Lambda_{\mathbf{k}}$ and let $\beta \in \Lambda_{\mathbf{k}}$ be such that β is obtained from β_0 by transposing just two generators. So we can assume that $\beta_0 = \gamma g_j g_i \omega$ and $\beta = \gamma g_i g_j \omega$ for some $\gamma, \omega \in \mathbb{F}_n^+$ and $i \neq j$, $i, j = 1, \dots, n$. Since $x \in \mathcal{N}_{\mathcal{Q}_c} = F^2(H_n) \ominus \mathcal{M}_{\mathcal{Q}_c}$, we must have

$$\langle x, W_\gamma (W_j W_i - W_i W_j) W_\omega(1) \rangle = 0,$$

which implies

$$\frac{c_{\beta_0}}{\sqrt{b_{\beta_0}^{(m)}}} = \frac{c_\beta}{\sqrt{b_\beta^{(m)}}}.$$

Since any element $\gamma \in \Lambda_{\mathbf{k}}$ can be obtained from β_0 by successive transpositions, repeating the above argument, we deduce that

$$\frac{c_{\beta_0}}{\sqrt{b_{\beta_0}^{(m)}}} = \frac{c_\gamma}{\sqrt{b_\gamma^{(m)}}} \quad \text{for all } \gamma \in \Lambda_{\mathbf{k}}.$$

Setting $t := c_{\beta_0} / \sqrt{b_{\beta_0}^{(m)}}$, we have $c_\gamma = t \sqrt{b_\gamma^{(m)}}$, $\gamma \in \Lambda_{\mathbf{k}}$, and relation (4.4) implies $t = 0$ (remember that $b_\beta > 0$). Therefore, $c_\gamma = 0$ for any $\gamma \in \Lambda_{\mathbf{k}}$ and $\mathbf{k} \in \mathbb{N}_0^n$, so $x = 0$. Consequently, we have $\overline{\operatorname{span}}\{z_\lambda \colon \lambda \in \mathbf{D}_{f,\circ}^1(\mathbb{C})\} = \mathcal{N}_{\mathcal{Q}_c}$.

Now, let us prove part (ii) of the theorem. Since the set $\{w^{\mathbf{k}} \colon \mathbf{k} \in \mathbb{N}_0^n\}$ consists of orthogonal vectors in $F^2(H_n)$ with $\|w^{\mathbf{k}}\| = 1/\sqrt{\gamma_{\mathbf{k}}^{(m)}}$, and $F_s^2(\mathbf{D}_f^m)$ the closed span of these vectors, any $\varphi \in F_s^2(\mathbf{D}_f^m)$ has a unique representation $\varphi = \sum_{\mathbf{k} \in \mathbb{N}_0} c_{\mathbf{k}} w^{\mathbf{k}}$ with $\|\varphi\|_2 = \sum_{\mathbf{k} \in \mathbb{N}_0} |c_{\mathbf{k}}|^2 \frac{1}{\gamma_{\mathbf{k}}^{(m)}} < \infty$. Note that

$$\langle w^{\mathbf{k}}, z_\lambda \rangle = \frac{1}{\gamma_{\mathbf{k}}} \left\langle \sum_{\beta \in \Lambda_{\mathbf{k}}} \sqrt{b_\beta^{(m)}} e_\beta, z_\lambda \right\rangle = \frac{1}{\gamma_{\mathbf{k}}} \sum_{\beta \in \Lambda_{\mathbf{k}}} b_\beta^{(m)} \lambda_\beta = \lambda^{\mathbf{k}}$$

for any $\lambda \in \mathbf{D}_{f,\circ}^1(\mathbb{C})$ and $\mathbf{k} \in \mathbb{N}_0^n$. Hence, every element $\varphi = \sum_{\mathbf{k} \in \mathbb{N}_0} c_{\mathbf{k}} w^{\mathbf{k}}$ in $F_s^2(\mathbf{D}_f^m)$ has a functional representation on $\mathbf{D}_{f,\circ}^1(\mathbb{C})$ given by

$$\varphi(\lambda) := \langle \varphi, z_\lambda \rangle = \sum_{\mathbf{k} \in \mathbb{N}_0} c_{\mathbf{k}} \lambda^{\mathbf{k}}, \quad \lambda = (\lambda_1, \dots, \lambda_n) \in \mathbf{D}_{f,\circ}^1(\mathbb{C}),$$

and, due to (4.2),

$$|\varphi(\lambda)| \leqslant \|\varphi\|_2 \|z_\lambda\| = \frac{\|\varphi\|_2}{\sqrt{(1 - \sum_{|\alpha| \geqslant 1} a_\alpha |\lambda_\alpha|^2)^m}}.$$

The identification of $F_s^2(\mathbf{D}_f^m)$ with $H^2(\mathbf{D}_{f,\circ}^1(\mathbb{C}))$ is now clear.

As in the proof of Theorem 4.1, we deduce that

$$\left(I - \sum_{|\alpha| \geqslant 1} a_{\tilde{\alpha}} \bar{\lambda}_{\tilde{\alpha}} \Lambda_\alpha \right)^{-m} = \sum_{\beta \in \mathbb{F}_n^+} b_{\tilde{\beta}}^{(m)} \bar{\lambda}_{\tilde{\beta}} \Lambda_\beta$$

if $(\lambda_1, \ldots, \lambda_n) \in \mathbf{D}^1_{f,\circ}(\mathbb{C})$. Similarly, if $(\mu_1, \ldots, \mu_n) \in \mathbf{D}^1_{f,\circ}(\mathbb{C}) = \mathbf{D}^1_{f,\circ}(\mathbb{C}) \cap \mathbf{D}^1_{\tilde{f},\circ}(\mathbb{C})$, we deduce that

$$\sum_{\beta \in \mathbb{F}_n^+} b_\beta \mu_\beta \bar{\lambda}_\beta = \Big(I - \sum_{|\alpha| \geqslant 1} a_\alpha \mu_\alpha \bar{\lambda}_\alpha \Big)^{-m}.$$

Since

$$K_f(\mu, \lambda) = \sum_{\beta \in \mathbb{F}_n^+} b_\beta^{(m)} \mu_\beta \bar{\lambda}_\beta = \langle z_\lambda, z_\mu \rangle,$$

the result in part (iii) follows. The proof is complete. □

Let J_c be the w^*-closed two-sided ideal of the Hardy algebra $F_n^\infty(\mathbf{D}_f^m)$ generated by the commutators

$$W_i W_j - W_j W_i, \quad i, j = 1, \ldots, n.$$

Since $W_i W_j - W_j W_i \in J_c$ and every permutation of k objects is a product of transpositions, it is clear that $W_\alpha W_\beta - W_\beta W_\alpha \in J_c$ for any $\alpha, \beta \in \mathbb{F}_n^+$. Consequently, $W_\gamma (W_\alpha W_\beta - W_\beta W_\alpha) W_\omega \in J_c$ for any $\alpha, \beta, \gamma, \omega \in \mathbb{F}_n^+$. Since the polynomials in $W_1, \ldots, W_n$ are w^* dense in $F_n^\infty(\mathbf{D}_f^m)$, we deduce that J_c coincides with the w^*-closure of the commutator ideal of $F_n^\infty(\mathbf{D}_f^m)$.

Define the operators on $F_s^2(\mathbf{D}_f^m)$ by

$$L_i := P_{F_s^2(\mathcal{D}_f)} W_i|_{F_s^2(\mathcal{D}_f)}, \quad i = 1, \ldots, n,$$

where $W_1, \ldots, W_n$ are the weighted left creation operators associated with $\mathbf{D}_f^m$. Let

$$\varphi(W_1, \ldots, W_n) \in F_n^\infty\big(\mathbf{D}_f^m\big)$$

and denote $M_\varphi := P_{F_s^2(\mathbf{D}_f^m)} \varphi(W_1, \ldots, W_n)|_{F_s^2(\mathbf{D}_f^m)}$. According to Theorems 4.1 and 4.4, the vector z_λ is in $F_s^2(\mathbf{D}_f^m)$ for $\lambda \in \mathbf{D}^1_{f,\circ}(\mathbb{C})$, and $\varphi(W_1, \ldots, W_n)^* z_\lambda = \overline{\varphi(\lambda)} z_\lambda$. Consequently, we have

$$\begin{aligned}
[M_\varphi \psi](\lambda) &= \langle M_\varphi \psi, z_\lambda \rangle \\
&= \big\langle \varphi(W_1, \ldots, W_n)\psi, z_\lambda \big\rangle \\
&= \big\langle \psi, \varphi(W_1, \ldots, W_n)^* z_\lambda \big\rangle \\
&= \big\langle \psi, \overline{\varphi(\lambda)} z_\lambda \big\rangle = \varphi(\lambda)\psi(\lambda)
\end{aligned}$$

for any $\psi \in F_s^2(\mathbf{D}_f^m)$ and $\lambda \in \mathbf{D}^1_{f,\circ}(\mathbb{C})$. Therefore, the operators in $P_{F_s^2(\mathbf{D}_f^m)} F_n^\infty(\mathbf{D}_f^m)|_{F_s^2(\mathbf{D}_f^m)}$ are "analytic" multipliers of $F_s^2(\mathbf{D}_f^m)$. Moreover,

$$\|M_\varphi\| = \sup\big\{\|\varphi f\|_2\colon\ f \in F_s^2\big(\mathbf{D}_f^m\big),\ \|f\| \leqslant 1\big\}.$$

In particular, for each $i = 1, \ldots, n$, L_i is the multiplier M_{λ_i} by the coordinate function. Let $H^\infty(\mathbf{D}^1_{f,\circ}(\mathbb{C}))$ be the algebra of all multipliers of the Hilbert space $H^2(\mathbf{D}^1_{f,\circ}(\mathbb{C}))$. In what follows, we show that the algebra $H^\infty(\mathbf{D}^1_{f,\circ}(\mathbb{C}))$ is reflexive.

First, we need to recall some definitions. If $A \in B(\mathcal{H})$ then the set of all invariant subspaces of A is denoted by $\operatorname{Lat} A$. For any $\mathcal{U} \subset B(\mathcal{H})$ we define

$$\operatorname{Lat} \mathcal{U} = \bigcap_{A \in \mathcal{U}} \operatorname{Lat} A.$$

If $\mathcal{S}$ is any collection of subspaces of $\mathcal{H}$, then we define $\operatorname{Alg} \mathcal{S}$ by setting

$$\operatorname{Alg} \mathcal{S} := \{A \in B(\mathcal{H}) \colon \mathcal{S} \subset \operatorname{Lat} A\}.$$

We recall that the algebra $\mathcal{U} \subset B(\mathcal{H})$ is reflexive if $\mathcal{U} = \operatorname{Alg} \operatorname{Lat} \mathcal{U}$.

Theorem 4.5. *The algebra $H^\infty(\mathbf{D}^1_{f,\circ}(\mathbb{C}))$ is reflexive and coincides with the weakly closed algebra generated by the operators $L_1, \ldots, L_n$ and the identity.*

Proof. First we show that $H^\infty(\mathbf{D}^1_{f,\circ}(\mathbb{C}))$ is included in the weakly closed algebra generated by the operators $L_1, \ldots, L_n$ and the identity. Suppose that $g = \sum_{\mathbf{k} \in \mathbb{N}_0} c_{\mathbf{k}} w^{\mathbf{k}}$ is a bounded multiplier, i.e., $M_g \in B(F^2_s(\mathcal{D}_f))$. As in Section 3, using Cesaro means, one can find a sequence of polynomials $p_m = \sum c^{(m)}_{\mathbf{k}} w^{\mathbf{k}}$ such that M_{p_m} converges to M_g in the strong operator topology and, consequently, in the WOT-topology. Since M_{p_m} is a polynomial in $L_1, \ldots, L_n$ and the identity, our assertion follows.

Now, let $X \in B(F^2_s(\mathbf{D}^m_f))$ be an operator that leaves invariant all the invariant subspaces under each operator $L_1, \ldots, L_n$. Due to Theorem 4.1, we have $L_i^* z_\lambda = \bar{\lambda}_i z_\lambda$ for any $\lambda \in \mathbf{D}^1_{f,\circ}(\mathbb{C})$ and $i = 1, \ldots, n$. Since X^* leaves invariant all the invariant subspaces under $L_1^*, \ldots, L_n^*$, the vector z_λ must be an eigenvector for X^*. Consequently, there is a function $\varphi : \mathbf{D}^1_{f,\circ}(\mathbb{C}) \to \mathbb{C}$ such that $X^* z_\lambda = \overline{\varphi(\lambda)} z_\lambda$ for any $\lambda \in \mathbf{D}^1_{f,\circ}(\mathbb{C})$. Notice that, if $f \in F^2_s(\mathbf{D}^m_f)$, then, due to Theorem 4.4, Xf has the functional representation

$$(Xf)(\lambda) = \langle Xf, z_\lambda \rangle = \langle f, X^* z_\lambda \rangle = \varphi(\lambda) f(\lambda) \quad \text{for all } \lambda \in \mathbf{D}^1_{f,\circ}(\mathbb{C}).$$

In particular, if $f = 1$, then the functional representation of $X(1)$ coincide with φ. Consequently, φ admits a power series representation on $\mathbf{D}^1_{f,\circ}(\mathbb{C})$ and can be identified with $X(1) \in F^2_s(\mathbf{D}^m_f)$. Moreover, the equality above shows that $\varphi f \in H^2(\mathbf{D}^1_{f,\circ}(\mathbb{C}))$ for any $f \in F^2_s(\mathbf{D}^m_f)$. This shows that φ is in $H^\infty(\mathbf{D}^1_{f,\circ}(\mathbb{C}))$ and completes the proof of reflexivity. Hence, $H^\infty(\mathbf{D}^1_{f,\circ}(\mathbb{C}))$ is a WOT-closed algebra containing $L_1, \ldots, L_n$ and the identity. This implies the second part of the theorem. □

5. Noncommutative varieties, Berezin transforms, and universal models

In this section, we consider noncommutative varieties $\mathcal{V}^m_{f,\mathcal{Q}}(\mathcal{H}) \subset \mathbf{D}^m_f(\mathcal{H})$ determined by sets $\mathcal{Q}$ of noncommutative polynomials, and associate with each such a variety a universal model $(B_1, \ldots, B_n) \in \mathcal{V}^m_{f,\mathcal{Q}}(\mathcal{N}_\mathcal{Q})$, where $\mathcal{N}_\mathcal{Q}$ is an appropriate subspace of the full Fock space.

We introduce a *constrained noncommutative Berezin transform* and use it to obtain analogues of the results of Section 2, for subvarieties. We also show that, under a natural condition, the C^*-algebra $C^*(B_1,\dots,B_n)$ is irreducible and all the compacts operators in $B(\mathcal{N}_\mathcal{Q})$ are contained in the operator space $\overline{\operatorname{span}}\{B_\alpha B_\beta^*\colon \alpha,\beta\in\mathbb{F}_n^+\}$. These results are vital for the development of a model theory on noncommutative varieties.

Let $f:=\sum_{|\alpha|\geqslant 1} a_\alpha X_\alpha$ be a positive regular free holomorphic function on $[B(\mathcal{H})^n]_\rho$, $\rho>0$, and let $W_1,\dots,W_n$ be the weighted left creation operators associated with the noncommutative domain $\mathbf{D}_f^m$. Let $\mathcal{Q}$ be a family of noncommutative polynomials and define the noncommutative variety

$$\mathcal{V}_{f,\mathcal{Q}}^m(\mathcal{H}):=\bigl\{(X_1,\dots,X_n)\in\mathbf{D}_f^m(\mathcal{H})\colon\ q(X_1,\dots,X_n)=0 \text{ for any } q\in\mathcal{Q}\bigr\}.$$

We associate with $\mathcal{V}_{f,\mathcal{Q}}^m$ the operators $B_1,\dots,B_n$ defined as follows. Consider the subspaces

$$\mathcal{M}_\mathcal{Q}:=\overline{\operatorname{span}}\bigl\{W_\alpha q(W_1,\dots,W_n)W_\beta(1)\colon\ q\in\mathcal{Q},\ \alpha,\beta\in\mathbb{F}_n^+\bigr\} \tag{5.1}$$

and $\mathcal{N}_\mathcal{Q}:=F^2(H_n)\ominus\mathcal{M}_\mathcal{Q}$. We assume that $\mathcal{N}_\mathcal{Q}\neq\{0\}$. It is easy to see that $\mathcal{N}_\mathcal{Q}$ is invariant under each operator $W_1^*,\dots,W_n^*$ and $\Lambda_1^*,\dots,\Lambda_n^*$. Define $B_i:=P_{\mathcal{N}_\mathcal{Q}}W_i|_{\mathcal{N}_\mathcal{Q}}$ and $C_i:=P_{\mathcal{N}_\mathcal{Q}}\Lambda_i|\mathcal{N}_\mathcal{Q}$ for $i=1,\dots,n$, where $P_{\mathcal{N}_\mathcal{Q}}$ is the orthogonal projection of $F^2(H_n)$ onto $\mathcal{N}_\mathcal{Q}$. Notice that $q(B_1,\dots,B_n)=0$ for any $q\in\mathcal{Q}$. By taking the compression to the subspace $\mathcal{N}_\mathcal{Q}$, in Theorem 1.3, we obtain similar results, where the universal model $(W_1,\dots,W_n)$ is replaced by the n-tuple $(B_1,\dots,B_n)$. In particular, we deduce that $(B_1,\dots,B_n)\in\mathcal{V}_{f,\mathcal{Q}}^m(\mathcal{N}_\mathcal{Q})$ is a pure n-tuple of operators which will play the role of universal model for the noncommutative variety $\mathcal{V}_{f,\mathcal{Q}}^m$.

For each n-tuple $T:=(T_1,\dots,T_n)\in\mathcal{V}_{f,\mathcal{Q}}^m(\mathcal{H})$ with $r_f(T_1,\dots,T_n)<1$, we introduce the *constrained noncommutative Berezin transform* at T as the map $\mathbf{B}_T^c: B(\mathcal{N}_\mathcal{Q})\to B(\mathcal{H})$ defined by

$$\begin{aligned}\bigl\langle\mathbf{B}_T^c[g]x,y\bigr\rangle:=\Biggl\langle&\Biggl(I-\sum_{|\alpha|\geqslant 1}\bar a_{\tilde\alpha}C_\alpha^*\otimes T_{\tilde\alpha}\Biggr)^{-m}\bigl(g\otimes\Delta_{T,m,f}^2\bigr)\\&\times\Biggl(I-\sum_{|\alpha|\geqslant 1}a_{\tilde\alpha}C_\alpha\otimes T_{\tilde\alpha}^*\Biggr)^{-m}(1\otimes x),1\otimes y\Biggr\rangle,\end{aligned} \tag{5.2}$$

where $\Delta_{T,m,f}:=[(id-\Phi_{f,T})^m(I)]^{1/2}$ and $x,y\in\mathcal{H}$. We define the *extended constrained noncommutative Berezin transform* $\widetilde{\mathbf{B}}_T^c$ at any $T\in\mathcal{V}_{f,\mathcal{Q}}^m(\mathcal{H})$ by setting

$$\widetilde{\mathbf{B}}_T^c[g]:=K_{f,T,\mathcal{Q}}^{(m)*}(g\otimes I_\mathcal{H})K_{f,T,\mathcal{Q}}^{(m)},\quad g\in B(\mathcal{N}_\mathcal{Q}), \tag{5.3}$$

where the *constrained noncommutative Berezin kernel* associated with the n-tuple $T\in\mathcal{V}_{f,\mathcal{Q}}^m(\mathcal{H})$ is the bounded operator $K_{f,T,\mathcal{Q}}^{(m)}:\mathcal{H}\to\mathcal{N}_\mathcal{Q}\otimes\overline{\Delta_{f,m,T}\mathcal{H}}$ defined by

$$K_{f,T,\mathcal{Q}}^{(m)}:=(P_{\mathcal{N}_\mathcal{Q}}\otimes I_{\overline{\Delta_{f,m,T}\mathcal{H}}})K_{f,T}^{(m)},$$

where $K_{f,T}^{(m)}$ is the Berezin kernel associated with $T\in\mathbf{D}_f^m(\mathcal{H})$.

Using the results from Section 2 (see Proposition 2.4), one can show that the constrained noncommutative Berezin transforms $\widetilde{\mathbf{B}}^c_T$ and $\mathbf{B}^c_T$ coincide for any n-tuple of operators $T := (T_1, \ldots, T_n) \in \mathcal{V}^m_{f,\mathcal{Q}}(\mathcal{H})$ with joint spectral radius $r_f(T_1, \ldots, T_n) < 1$.

Theorem 5.1. *Let f be a positive regular free holomorphic function on $[B(\mathcal{H})^n]_\rho$, $\rho > 0$, and let $\mathcal{Q}$ be a family of noncommutative polynomials such that $\mathcal{N}_\mathcal{Q} \neq \{0\}$. If $T := (T_1, \ldots, T_n)$ is a pure n-tuple of operators in the noncommutative variety $\mathcal{V}^m_{f,\mathcal{Q}}(\mathcal{H})$, then the restriction of the constrained noncommutative Berezin transform $\widetilde{\mathbf{B}}^c_T$ to $\overline{\operatorname{span}}\{B_\alpha B^*_\beta\colon \alpha, \beta \in \mathbb{F}^+_n\}$ is a unital completely contractive linear map such that*

$$\widetilde{\mathbf{B}}^c_T\big(B_\alpha B^*_\beta\big) = T_\alpha T^*_\beta, \quad \alpha, \beta \in \mathbb{F}^+_n.$$

Proof. Using Lemma 2.3, we have

$$\big\langle K^{(m)}_{f,T} x, W_\alpha q(W_1, \ldots, W_n) W_\beta(1) \otimes y\big\rangle = \big\langle x, T_\alpha q(T_1, \ldots, T_n) T_\beta K^{(m)*}_{f,T}(1 \otimes y\big\rangle = 0$$

for any $x \in \mathcal{H}$, $y \in \overline{\Delta_{f,m,T}\mathcal{H}}$, and $q \in \mathcal{Q}$. Hence, we deduce that

$$\operatorname{range} K^{(m)}_{f,T} \subseteq \mathcal{N}_\mathcal{Q} \otimes \overline{\Delta_{f,m,T}\mathcal{H}}. \tag{5.4}$$

Due to the definition of the constrained Berezin kernel associated with the n-tuple $T \in \mathcal{V}^m_{f,\mathcal{Q}}(\mathcal{H})$, and using Lemma 2.3 and relation (5.4), we obtain

$$K^{(m)}_{f,T,\mathcal{Q}} T^*_\alpha = \big(B^*_\alpha \otimes I_\mathcal{H}\big) K^{(m)}_{f,T,\mathcal{Q}}, \quad \alpha \in \mathbb{F}^+_n. \tag{5.5}$$

Since (5.4) holds and $K^{(m)}_{f,T}$ is an isometry, so is $K^{(m)}_{f,T,\mathcal{Q}}$. Consequently, using relation (5.5), we deduce that

$$\widetilde{\mathbf{B}}^c_T\big(B_\alpha B^*_\beta\big) = K^{(m)*}_{f,T,\mathcal{Q}}\big(B_\alpha B^*_\beta \otimes I_\mathcal{H}\big) K^{(m)}_{f,T,\mathcal{Q}} = T_\alpha T^*_\beta, \quad \alpha, \beta \in \mathbb{F}^+_n.$$

Now, one can easily deduce that $\widetilde{\mathbf{B}}^c_T$ is a unital completely contractive linear map on $\overline{\operatorname{span}}\{B_\alpha B^*_\beta\colon \alpha, \beta \in \mathbb{F}^+_n\}$. The proof is complete. □

We recall that an n-tuple of operators $T := (T_1, \ldots, T_n) \in \mathcal{V}^m_{f,\mathcal{Q}}(\mathcal{H})$ has the radial property with respect to the noncommutative variety $\mathcal{V}^m_{f,\mathcal{Q}}(\mathcal{H})$ if there is $\delta \in (0, 1)$ such that $rT := (rT_1, \ldots, rT_n) \in \mathcal{V}^m_{f,\mathcal{Q}}(\mathcal{H})$ for any $r \in (\delta, 1)$.

Theorem 5.2. *Let f be a positive regular free holomorphic function on $[B(\mathcal{H})^n]_\rho$, $\rho > 0$, and let $\mathcal{Q}$ be a set of homogeneous polynomials. Let $T := (T_1, \ldots, T_n)$ be an n-tuple of operators with the radial property in the noncommutative variety $\mathcal{V}^m_{f,\mathcal{Q}}(\mathcal{H})$ and let $\mathcal{S} := \overline{\operatorname{span}}\{B_\alpha B^*_\beta;\ \alpha, \beta \in \mathbb{F}^+_n\}$. Then there is a unital completely contractive linear map $\Psi_{f,T,\mathcal{Q}} : \mathcal{S} \to B(\mathcal{H})$ such that*

$$\Psi_{f,T,\mathcal{Q}}(g) = \lim_{r \to 1} \mathbf{B}^c_{rT}[g], \quad g \in \mathcal{S}, \tag{5.6}$$

where the limit exists in the norm topology of $B(\mathcal{H})$, and $\Psi_{f,T,\mathcal{Q}}(B_\alpha B_\beta^*) = T_\alpha T_\beta^*$, $\alpha, \beta \in \mathbb{F}_n^+$. If, in addition, T is a pure n-tuple of operators, then

$$\lim_{r\to 1} \mathbf{B}_{rT}^{c}[g] = \widetilde{\mathbf{B}}_T^{c}[g], \quad g \in \mathcal{S},$$

where the limit exists in the norm topology of $B(\mathcal{H})$.

Proof. Let $\delta \in (0,1)$ be such that $rT := (rT_1, \dots, rT_n) \in \mathbf{D}_f^m(\mathcal{H})$ for any $r \in (\delta, 1)$. Since $\mathcal{Q}$ consists of homogeneous polynomials we also have $rT \in \mathcal{V}_{f,\mathcal{Q}}^m(\mathcal{H})$. Moreover, we can show, as in the proof of Theorem 5.1, that range $K_{f,rT}^{(m)} \subseteq \mathcal{N}_\mathcal{Q} \otimes \mathcal{H}$ for any $r \in (\delta, 1)$, where $K_{f,rT}^{(m)}$ is the Berezin kernel associated with $rT \in \mathbf{D}_f^m(\mathcal{H})$. Moreover,

$$K_{f,rT,\mathcal{Q}}^{(m)} r^{|\alpha|} T_\alpha^* = \left(B_\alpha^* \otimes I_\mathcal{H}\right) K_{f,rT,\mathcal{Q}}^{(m)}, \quad \alpha \in \mathbb{F}_n^+,$$

where $K_{f,rT,\mathcal{Q}}^{(m)} := (P_{\mathcal{N}_\mathcal{Q}} \otimes I_\mathcal{H}) K_{f,rT}^{(m)}$ is the constrained Berezin kernel and $B_i := P_{\mathcal{N}_\mathcal{Q}} W_i|_{\mathcal{N}_\mathcal{Q}}$, $i = 1, \dots, n$. Since rT is pure, $K_{f,rT,\mathcal{Q}}^{(m)}$ is an isometry. Consequently, as in the proof of Theorem 2.7, we deduce that there is a unique unital completely contractive linear map $\Psi_{p,T,\mathcal{Q}} : \mathcal{S} \to B(\mathcal{H})$ such that $\Psi_{p,T,\mathcal{Q}}(B_\alpha B_\beta^*) = T_\alpha T_\beta^*$, $\alpha, \beta \in \mathbb{F}_n^+$. The rest of the proof is similar to that of Theorem 2.7. We shall omit it. $\square$

Assume now that p is a positive regular noncommutative polynomial and let $\mathbf{D}_p^m$ be the noncommutative domain it generates. The next result will play an important role in Section 6, where we develop a model theory on noncommutative subvarieties of $\mathbf{D}_p^m$.

Theorem 5.3. *Let $\mathcal{Q}$ be a set of noncommutative polynomials such that $1 \in \mathcal{N}_\mathcal{Q}$, and let $(B_1, \dots, B_n)$ be the universal model associated with the noncommutative variety $\mathcal{V}_{p,\mathcal{Q}}^m$. Then all the compact operators in $B(\mathcal{N}_\mathcal{Q})$ are contained in the operator space*

$$\overline{\operatorname{span}}\left\{B_\alpha B_\beta^*\colon \alpha, \beta \in \mathbb{F}_n^+\right\}.$$

Moreover, the C^*-algebra $C^*(B_1, \dots, B_n)$ is irreducible.

Proof. Since $1 \in \mathcal{N}_\mathcal{Q}$ and $\mathcal{N}_\mathcal{Q}$ is an invariant subspace W_i^*, $i = 1, \dots, n$, we use Theorem 1.3 to obtain

$$(id - \Phi_{p,B})^m (I_{\mathcal{N}_\mathcal{Q}}) = P_{\mathcal{N}_\mathcal{Q}}\left[(id - \Phi_{p,W})^m (I_{F^2(H_n)})\right]\big|_{\mathcal{N}_\mathcal{Q}} = P_{\mathcal{N}_\mathcal{Q}} P_\mathbb{C}|_{\mathcal{N}_\mathcal{Q}} = P_\mathbb{C}^{\mathcal{N}_\mathcal{Q}},$$

where $P_\mathbb{C}^{\mathcal{N}_\mathcal{Q}}$ is the orthogonal projection of $\mathcal{N}_\mathcal{Q}$ onto $\mathbb{C}$. Fix

$$g(W_1, \dots, W_n) := \sum_{|\alpha| \leqslant m} d_\alpha W_\alpha \quad \text{and} \quad \xi := \sum_{\beta \in \mathbb{F}_n^+} c_\beta e_\beta \in \mathcal{N}_J \subset F^2(H_n),$$

and note that

$$P_\mathbb{C}^{\mathcal{N}_\mathcal{Q}} g(B_1, \dots, B_n)^* \xi = \left\langle \xi, g(B_1, \dots, B_n)(1)\right\rangle.$$

Consequently, we have

$$q(B_1,\dots,B_n)P_{\mathbb{C}}^{\mathcal{N}_J}g(B_1,\dots,B_n)^*\xi=\big\langle\xi,g(B_1,\dots,B_n)(1)\big\rangle q(B_1,\dots,B_n)(1) \tag{5.7}$$

for any polynomial $q(B_1,\dots,B_n)$. Hence, we deduce that the operator $q(B_1,\dots,B_n)P_{\mathbb{C}}^{\mathcal{N}_J}g(B_1,\dots,B_n)^*$ has rank one and, since $P_{\mathbb{C}}^{\mathcal{N}_\mathcal{Q}}=(id-\Phi_{p,B})^m(I_{\mathcal{N}_\mathcal{Q}})$, it is in the operator space $\overline{\operatorname{span}}\{B_\alpha B_\beta^*\colon \alpha,\beta\in\mathbb{F}_n^+\}$. On the other hand, due to the fact that the set of all vectors of the form $\sum_{|\alpha|\leqslant m}d_\alpha B_\alpha(1)$ with $m\in\mathbb{N}$, $d_\alpha\in\mathbb{C}$, is dense in $\mathcal{N}_\mathcal{Q}$, relation (5.7) implies that all compact operators in $B(\mathcal{N}_\mathcal{Q})$ are included in the operator space $\overline{\operatorname{span}}\{B_\alpha B_\beta^*\colon \alpha,\beta\in\mathbb{F}_n^+\}$.

To prove the last part of this theorem, let $\mathcal{M}\neq\{0\}$ be a subspace of $\mathcal{N}_\mathcal{Q}\subseteq F^2(H_n)$, which is jointly reducing for each operator B_i, $i=1,\dots,n$. Let $\varphi\in\mathcal{M}$, $\varphi\neq 0$, and assume that $\varphi=c_0+\sum_{|\alpha|\geqslant 1}c_\alpha e_\alpha$. If c_β is a nonzero coefficient of φ, then $P_\mathbb{C}B_\beta^*\varphi=\frac{1}{\sqrt{b_\beta^{(m)}}}c_\beta$. Indeed, since $1\in\mathcal{N}_\mathcal{Q}$, one can use relation (1.5) to deduce that

$$\big\langle P_\mathbb{C}B_\beta^*\varphi,1\big\rangle=\big\langle P_{\mathcal{N}_J}W_\beta^*\varphi,1\big\rangle=\big\langle W_\beta^*\varphi,1\big\rangle=\frac{1}{\sqrt{b_\beta^{(m)}}}c_\beta.$$

Since $\langle P_\mathbb{C}B_\beta^*\varphi,e_\gamma\rangle=0$ for any $\gamma\in\mathbb{F}_n^+$ with $|\gamma|\geqslant 1$, our assertion follows. On the other hand, since $P_{\mathbb{C}}^{\mathcal{N}_\mathcal{Q}}=(id-\Phi_{p,B})^m(I_{\mathcal{N}_\mathcal{Q}})$ and $\mathcal{M}$ is reducing for $B_1,\dots,B_n$, we deduce that $c_\beta\in\mathcal{M}$, so $1\in\mathcal{M}$. Using once again that $\mathcal{M}$ is invariant under the operators $B_1,\dots,B_n$, we have $\mathcal{E}\subseteq\mathcal{M}$. On the other hand, since $\mathcal{E}$ is dense in $\mathcal{N}_\mathcal{Q}$, we deduce that $\mathcal{N}_\mathcal{Q}\subset\mathcal{M}$. Therefore $\mathcal{N}_\mathcal{Q}=\mathcal{M}$. This completes the proof. □

We say that two n-tuples of operators $(T_1,\dots,T_n)$, $T_i\in B(\mathcal{H})$, and $(T_1',\dots,T_n')$, $T_i'\in B(\mathcal{H}')$, are unitarily equivalent if there exists a unitary operator $U:\mathcal{H}\to\mathcal{H}'$ such that

$$T_i=U^*T_i'U \quad \text{for any } i=1,\dots,n.$$

If $(B_1,\dots,B_n)$ is the universal model associated with the noncommutative variety $\mathcal{V}_{p,\mathcal{Q}}^m$, then the n-tuple $(B_1\otimes I_\mathcal{H},\dots,B_n\otimes I_\mathcal{H})$ is called constrained weighted shift with multiplicity $\dim\mathcal{H}$. Using Theorem 5.3, one can easily prove that two constrained weighted shifts associated with the noncommutative variety $\mathcal{V}_{p,\mathcal{Q}}^m$ are unitarily equivalent if and only if their multiplicities are equal.

We remark that all the results of this section are true in the commutative case, i.e., when

$$\mathcal{Q}_c:=\{Z_iZ_j-Z_jZ_i\colon i,j=1,\dots,n\}.$$

According to the results of Section 4 (see Theorem 4.4 and the remarks preceding Theorem 4.5), the space $\mathcal{N}_{\mathcal{Q}_c}$ coincides with the symmetric weighted Fock space $F_s^2(\mathbf{D}_f^m)$, which can be identified with the Hilbert space $H^2(\mathbf{D}_{f,\circ}^1(\mathbb{C}))$. Moreover, under this identification, the operators B_i, $i=1,\dots,n$, become the multipliers M_{λ_i} by the coordinate functions on the Hilbert space $H^2(\mathbf{D}_{f,\circ}^1(\mathbb{C}))$.

6. Model theory on noncommutative varieties

In this section, we obtain dilation and model theorems for the elements of the noncommutative variety $\mathcal{V}^m_{f,\mathcal{Q}}(\mathcal{H}) \subset \mathbf{D}^m_f(\mathcal{H})$ generated by a set $\mathcal{Q}$ of noncommutative polynomials.

We recall that $\mathcal{N}_\mathcal{Q} := F^2(H_n) \ominus \mathcal{M}_\mathcal{Q}$, where the subspace $\mathcal{M}_\mathcal{Q}$ is defined by (5.1). We keep the notations of the previous sections. Our first dilation result on noncommutative varieties is the following.

Theorem 6.1. *Let f be a positive regular free holomorphic function on $[B(\mathcal{H})^n]_\rho$, $\rho > 0$, and let $\mathcal{Q}$ be a family of noncommutative polynomials such that $\mathcal{N}_\mathcal{Q} \neq \{0\}$. If $T := (T_1, \dots, T_n)$ is an n-tuple of operators in the noncommutative variety $\mathcal{V}^m_{f,\mathcal{Q}}(\mathcal{H})$, then there exists a Hilbert space $\mathcal{K}$ and n-tuple $(U_1 \dots, U_n) \in \mathcal{V}^m_{f,\mathcal{Q}}(\mathcal{K})$ with $\Phi_{f,U}(I_\mathcal{K}) = I_\mathcal{K}$ and such that*

(i) *$\mathcal{H}$ can be identified with a co-invariant subspace of $\widetilde{\mathcal{K}} := (\mathcal{N}_\mathcal{Q} \otimes \overline{\Delta_{f,m,T}\mathcal{H}}) \oplus \mathcal{K}$ under the operators*

$$V_i := \begin{bmatrix} B_i \otimes I_{\overline{\Delta_{f,m,T}\mathcal{H}}} & 0 \\ 0 & U_i \end{bmatrix}, \quad i = 1, \dots, n,$$

where $\Delta_{f,m,T} := [(id - \Phi_{f,T})^m(I)]^{1/2}$;

(ii) *$T_i^* = V_i^*|\mathcal{H}$ for $i = 1, \dots, n$.*

Moreover, $\mathcal{K} = \{0\}$ if and only if $(T_1, \dots, T_n)$ is pure n-tuple of operators in $\mathcal{V}^m_{f,\mathcal{Q}}(\mathcal{H})$, i.e., $\Phi^k_{f,T}(I) \to 0$ strongly, as $k \to 0$.

Proof. We recall that the operator $Q_{f,T} := \text{SOT-}\lim_{k\to\infty} \Phi^k_{f,T}(I)$ is well defined. We use it to define

$$Y : \mathcal{H} \to \mathcal{K} := \overline{Q^{1/2}_{f,T}\mathcal{H}} \quad \text{by } Yh := Q^{1/2}_{f,T}h,\ h \in \mathcal{H}.$$

For each $i = 1, \dots, n$, let $L_i : Q^{1/2}_{f,T}\mathcal{H} \to \mathcal{K}$ be given by

$$L_i Yh := YT_i^*h, \quad h \in \mathcal{H}. \tag{6.1}$$

Note that L_i, $i = 1, \dots, n$, are well defined due to the fact that

$$\begin{aligned} \|L_i Yh\|^2 &= \langle T_i Q_{f,T} T_i^* h, h\rangle \leqslant \frac{1}{a_{g_i}} \langle \Phi_{f,T}(Q_{f,T})h, h\rangle \\ &= \frac{1}{a_{g_i}} \big\| Q^{1/2}_{f,T} h \big\|^2 = \frac{1}{a_{g_i}} \|Yh\|^2. \end{aligned}$$

Since f is positive regular free holomorphic function, we have $a_{g_i} \neq 0$ for any $i = 1, \dots, n$. Consequently, L_i can be extended to a bounded operator on $\mathcal{K}$, which will also be denoted by L_i. Now, setting $U_i := L_i^*$, $i = 1, \dots, n$, relation (6.1) implies

$$Y^* U_i = T_i Y^*, \quad i = 1, \dots, n. \tag{6.2}$$

Using this relation and the fact that $\Phi_{f,T}(Q_{f,T}) = Q_{f,T}$, we deduce that

$$Y^*\Phi_{f,U}(I_{\mathcal{K}})Y = \Phi_{f,T}\big(YY^*\big) = YY^*.$$

Hence,

$$\big\langle \Phi_{f,U}(I_{\mathcal{K}})Yh, Yh\big\rangle = \langle Yh, Yh\rangle, \quad h \in \mathcal{H},$$

which implies $\Phi_{f,U}(I_{\mathcal{K}}) = I_{\mathcal{K}}$. Now, using relation (6.2), we obtain

$$Y^*q(U_1, \ldots, U_n) = q(T_1, \ldots, T_n)Y^* = 0, \quad q \in \mathcal{Q}.$$

Since Y^* is injective on $\mathcal{K} = \overline{Y\mathcal{H}}$, we have $q(U_1, \ldots, U_n) = 0$ for any $q \in \mathcal{Q}$. Let $V : \mathcal{H} \to [\mathcal{N}_{\mathcal{Q}} \otimes \mathcal{H}] \oplus \mathcal{K}$ be defined by

$$V := \begin{bmatrix} K^{(m)}_{f,T,\mathcal{Q}} \\ Y \end{bmatrix}.$$

Notice that V is an isometry. Indeed, due to relations (2.10) and (5.4), we have

$$\begin{aligned}\|Vh\|^2 &= \big\|K^{(m)}_{f,T,\mathcal{Q}}h\big\|^2 + \|Yh\|^2 \\ &= \|h\|^2 - \text{SOT-}\lim_{k\to\infty}\big\langle \Phi^k_{f,T}(I)h, h\big\rangle + \|Yh\|^2 = \|h\|^2\end{aligned}$$

for any $h \in \mathcal{H}$. Now, using relations (5.5), (6.1), and (6.2), we obtain

$$\begin{aligned}VT_i^*h &= K^{(m)}_{f,T,\mathcal{Q}}T_i^*h \oplus YT_i^*h = \big(B_i^* \otimes I_{\mathcal{H}}\big)K^{(m)}_{f,T,\mathcal{Q}}h \oplus U_i^*Yh \\ &= \begin{bmatrix} B_i^* \otimes I_{\overline{\Delta_{f,m,T}\mathcal{H}}} & 0 \\ 0 & U_i^* \end{bmatrix} Vh\end{aligned}$$

for any $h \in \mathcal{H}$ and $i = 1, \ldots, n$. Identifying $\mathcal{H}$ with $V\mathcal{H}$ we complete the proof of (i) and (ii). The last part of the theorem is obvious. $\square$

We need the following result concerning power bounded positive linear maps on $B(\mathcal{H})$.

Lemma 6.2. *Let $\varphi : B(\mathcal{H}) \to B(\mathcal{H})$ be a power bounded positive linear map and let $D \in B(\mathcal{H})$ be a positive operator such that $\varphi(D) \leqslant D$. If $m \geqslant 1$, then*

$$(id - \varphi)^m(D) = 0 \quad \textit{if and only if} \quad \varphi(D) = D.$$

In particular, if φ is a positive linear map such that $\varphi(I) \leqslant I$ and $(id - \varphi)^m(I) = 0$, then $\varphi(I) = I$.

Proof. According to Lemma 2.1, we have

$$\sum_{p=0}^{q} \binom{p+m-1}{m-1} \varphi^p(id - \varphi)^m(D) = D - \sum_{j=0}^{m-1} \binom{q+j}{j} \varphi^{q+1}(id - \varphi)^j(D)$$

for any $q \in \mathbb{N}$. Consequently, if $(id - \varphi)^m(D) = 0$, then

$$D = \lim_{q\to\infty} \sum_{j=0}^{m-1} \binom{q+j}{j} \varphi^{q+1}(id - \varphi)^j(D).$$

Using Lemma 2.2, we deduce that $D = \lim_{q\to\infty} \varphi^q(D)$. Since φ is a positive linear map and $\varphi(D) \leqslant D$, we have

$$D = \lim_{q\to\infty} \varphi^q(D) \leqslant \cdots \leqslant \varphi^2(D) \leqslant \varphi(D) \leqslant D.$$

Hence, we deduce that $\varphi(D) = D$. The converse is obvious. $\square$

Let $C^*(\Gamma)$ be the C^*-algebra generated by a set of operators $\Gamma \subset B(\mathcal{K})$ and the identity. A subspace $\mathcal{H} \subset \mathcal{K}$ is called $*$-cyclic for Γ if $\mathcal{K} = \overline{\operatorname{span}}\{Xh, X \in C^*(\Gamma), h \in \mathcal{H}\}$. The main result of this section is the following model theorem for the elements of a noncommutative variety $\mathcal{V}^m_{p,\mathcal{Q}}(\mathcal{H})$.

Theorem 6.3. *Let p be a positive regular noncommutative polynomial and let $\mathcal{Q}$ be a set of homogeneous polynomials. Let $\mathcal{H}$ be a separable Hilbert space, and $T := (T_1, \ldots, T_n)$ be an n-tuple of operators in the noncommutative variety $\mathcal{V}^m_{p,\mathcal{Q}}(\mathcal{H})$ with the radial property, i.e.,*

$$rT := (rT_1, \ldots, rT_n) \in \mathcal{V}^m_{p,\mathcal{Q}}(\mathcal{H}) \quad \textit{for any } r \in (\delta, 1)$$

and some $\delta \in (0, 1)$.

Then there exists a $*$-representation $\pi : C^*(B_1, \ldots, B_n) \to B(\mathcal{K}_\pi)$ on a separable Hilbert space $\mathcal{K}_\pi$, which annihilates the compact operators and

$$\Phi_{p,\pi(B)}(I_{\mathcal{K}_\pi}) = I_{\mathcal{K}_\pi},$$

such that:

(i) *$\mathcal{H}$ can be identified with a $*$-cyclic co-invariant subspace of $\widetilde{\mathcal{K}} := (\mathcal{N}_\mathcal{Q} \otimes \overline{\Delta_{p,m,T}\mathcal{H}}) \oplus \mathcal{K}_\pi$ under each operator*

$$V_i := \begin{bmatrix} B_i \otimes I_{\overline{\Delta_{p,m,T}\mathcal{H}}} & 0 \\ 0 & \pi(B_i) \end{bmatrix}, \quad i = 1, \ldots, n,$$

where $\Delta_{p,m,T} := [(id - \Phi_{p,T})^m(I)]^{1/2}$;

(ii) *$T_i^* = V_i^*|\mathcal{H}$ for $i = 1, \ldots, n$.*

Proof. Applying Arveson extension theorem [5] to the map $\Psi_{p,T,\mathcal{Q}}$ of Theorem 5.2, we find a unital completely positive linear map $\Psi_{p,T,\mathcal{Q}} : C^*(B_1, \ldots, B_n) \to B(\mathcal{H})$ such that $\Psi_{p,T,\mathcal{Q}}(B_\alpha B_\beta^*) = T_\alpha T_\beta^*$ for $\alpha, \beta \in \mathbb{F}_n^+$. Let $\tilde{\pi} : C^*(B_1, \ldots, B_n) \to B(\widetilde{\mathcal{K}})$ be a minimal Stinespring dilation [49] of $\Psi_{p,T,\mathcal{Q}}$. Then

$$\Psi_{p,T,\mathcal{Q}}(X) = P_\mathcal{H}\tilde{\pi}(X)|\mathcal{H}, \quad X \in C^*(B_1, \ldots, B_n),$$

and $\widetilde{\mathcal{K}}=\overline{\operatorname{span}}\{\tilde{\pi}(X)h\colon h\in\mathcal{H}\}$. Now, one can easily see that $P_{\mathcal{H}}\tilde{\pi}(B_i)|_{\mathcal{H}^\perp}=0$, $i=1,\dots,n$. Consequently, $\mathcal{H}$ is an invariant subspace under each $\tilde{\pi}(B_i)^*$, $i=1,\dots,n$, and

$$\tilde{\pi}(B_i)^*|\mathcal{H}=\Psi_{p,T,\mathcal{Q}}\big(B_i^*\big)=T_i^*,\quad i=1,\dots,n. \tag{6.3}$$

Since $1\in\mathcal{N}_{\mathcal{Q}}$, Theorem 5.3 implies that all the compact operators $\mathcal{C}(\mathcal{N}_{\mathcal{Q}})$ in $B(\mathcal{N}_{\mathcal{Q}})$ are contained in the C^*-algebra $C^*(B_1,\dots,B_n)$. Due to standard theory of representations of C^*-algebras [6], representation $\tilde{\pi}$ decomposes into a direct sum $\tilde{\pi}=\pi_0\oplus\pi$ on $\widetilde{\mathcal{K}}=\mathcal{K}_0\oplus\mathcal{K}_\pi$, where π_0,π are disjoint representations of $C^*(B_1,\dots,B_n)$ on the Hilbert spaces

$$\mathcal{K}_0:=\overline{\operatorname{span}}\big\{\tilde{\pi}(X)\widetilde{\mathcal{K}}\colon X\in\mathcal{C}(\mathcal{N}_{\mathcal{Q}})\big\}\quad\text{and}\quad\mathcal{K}_\pi:=\mathcal{K}_0^\perp,$$

respectively, such that π annihilates the compact operators in $B(\mathcal{N}_{\mathcal{Q}})$, and π_0 is uniquely determined by the action of $\tilde{\pi}$ on the ideal $\mathcal{C}(\mathcal{N}_{\mathcal{Q}})$ of compact operators. Since every representation of $\mathcal{C}(\mathcal{N}_{\mathcal{Q}})$ is equivalent to a multiple of the identity representation, we deduce that

$$\mathcal{K}_0\simeq\mathcal{N}_{\mathcal{Q}}\otimes\mathcal{G},\qquad\pi_0(X)=X\otimes I_{\mathcal{G}},\quad X\in C^*(B_1,\dots,B_n), \tag{6.4}$$

for some Hilbert space $\mathcal{G}$. Using Theorem 5.3 and its proof, one can easily see that

$$\begin{aligned}\mathcal{K}_0&:=\overline{\operatorname{span}}\big\{\tilde{\pi}(X)\mathcal{K}\colon X\in\mathcal{C}(\mathcal{N}_{\mathcal{Q}})\big\}\\&=\overline{\operatorname{span}}\big\{\tilde{\pi}\big(B_\beta P_{\mathbb{C}}^{\mathcal{N}_{\mathcal{Q}}}B_\alpha^*\big)\mathcal{K}\colon \alpha,\beta\in\mathbb{F}_n^+\big\}\\&=\overline{\operatorname{span}}\big\{\tilde{\pi}(B_\beta)\big[(id-\Phi_{p,\tilde{\pi}(B)})^m(I_{\mathcal{K}})\big]\mathcal{K}\colon \beta\in\mathbb{F}_n^+\big\}.\end{aligned}$$

According to Theorem 5.3, the operator $(id-\Phi_{p,B})^m(I_{\mathcal{N}_{\mathcal{Q}}})=P_{\mathbb{C}}^{\mathcal{N}_{\mathcal{Q}}}$ is a projection of rank one in $C^*(B_1,\dots,B_n)$. Hence, we deduce that $(id-\Phi_{p,\pi(B)})^m(I_{\mathcal{K}_\pi})=0$ and

$$\dim\mathcal{G}=\dim\big[\operatorname{range}\pi\big(P_{\mathbb{C}}^{\mathcal{N}_{\mathcal{Q}}}\big)\big].$$

Since the Stinespring representation $\tilde{\pi}$ is minimal, we can use the proof of Theorem 5.3 to deduce that

$$\operatorname{range}\tilde{\pi}\big(P_{\mathbb{C}}^{\mathcal{N}_{\mathcal{Q}}}\big)=\overline{\operatorname{span}}\big\{\tilde{\pi}\big(P_{\mathbb{C}}^{\mathcal{N}_{\mathcal{Q}}}\big)\tilde{\pi}\big(B_\beta^*\big)h\colon \beta\in\mathbb{F}_n^+,h\in\mathcal{H}\big\}.$$

On the other hand, it is easy to see that

$$\begin{aligned}&\big\langle\tilde{\pi}\big(P_{\mathbb{C}}^{\mathcal{N}_{\mathcal{Q}}}\big)\tilde{\pi}\big(B_\alpha^*\big)h,\tilde{\pi}\big(P_{\mathbb{C}}^{\mathcal{N}_{\mathcal{Q}}}\big)\tilde{\pi}\big(B_\beta^*\big)k\big\rangle\\&\quad=\big\langle h,T_\alpha\big[(id-\Phi_{p,T})^m(I_{\mathcal{H}})\big]T_\beta^*h\big\rangle=\big\langle\Delta_{p,m,T}T_\alpha^*h,\Delta_{p,m,T}T_\beta^*k\big\rangle\end{aligned}$$

for any $h,k\in\mathcal{H}$ and $\alpha,\beta\in\mathbb{F}_n^+$. This implies the existence of a unitary operator Λ: $\operatorname{range}\tilde{\pi}(P_{\mathbb{C}}^{\mathcal{N}_{\mathcal{Q}}})\to\overline{\Delta_{p,m,T}\mathcal{H}}$ defined by

$$\Lambda\big[\tilde{\pi}\big(P_{\mathbb{C}}^{\mathcal{N}_{\mathcal{Q}}}\big)\tilde{\pi}\big(B_\alpha^*\big)h\big]:=\Delta_{p,m,T}T_\alpha^*h,\quad h\in\mathcal{H},\ \alpha\in\mathbb{F}_n^+.$$

This shows that

$$\dim\big[\operatorname{range}\pi\big(P_{\mathbb{C}}^{\mathcal{N}_{\mathcal{Q}}}\big)\big]=\dim\overline{\Delta_{p,m,T}\mathcal{H}}=\dim\mathcal{G}.$$

Using relations (6.3) and (6.4), and identifying $\mathcal{G}$ with $\overline{\Delta_{p,m,T}\mathcal{H}}$, we obtain the required dilation. On the other hand, due to the fact that $(id-\Phi_{p,\pi(B)})^m(I_{\mathcal{K}_\pi})=0$, we can use Lemma 6.2 to deduce that $\Phi_{p,\pi(B)}(I_{\mathcal{K}_\pi})=I_{\mathcal{K}_\pi}$. The proof is complete. □

A few remarks are needed. A closer look at Theorem 6.3 reveals that one can replace the polynomial p with a positive regular free holomorphic function f and obtain a model theorem for any n-tuple $(T_1,\dots,T_n)\in\mathcal{V}_{f,\mathcal{Q}}^m(\mathcal{H})$ with the radial property. More precisely, one can show that there is a $*$-representation $\tilde{\pi}:C^*(B_1,\dots,B_n)\to B(\mathcal{K}_\pi)$ such that $\mathcal{H}$ is an invariant subspace under each operator $\tilde{\pi}(B_i)^*$ and $T_i^*=\tilde{\pi}(B_i)^*|_{\mathcal{H}}$ for $i=1,\dots,n$.

On the other hand, notice that using the proof of Theorem 6.3 and due to the standard theory of representations of C^*-algebras, one can deduce the following Wold type decomposition for non-degenerate $*$-representations of the C^*-algebra $C^*(B_1,\dots,B_n)$, generated by the constrained weighted shifts associated with $\mathcal{V}_{p,\mathcal{Q}}^m$, and the identity.

Corollary 6.4. *Let p be a positive regular noncommutative polynomial and let $\mathcal{Q}$ be a set of noncommutative polynomials such that $1\in\mathcal{N}_{\mathcal{Q}}$. Let $(B_1,\dots,B_n)$ be the universal model associated with the noncommutative variety $\mathcal{V}_{p,\mathcal{Q}}^{(m)}$. If $\pi:C^*(B_1,\dots,B_n)\to B(\mathcal{K})$ is a nondegenerate $*$-representation of $C^*(B_1,\dots,B_n)$ on a separable Hilbert space $\mathcal{K}$, then π decomposes into a direct sum*

$$\pi=\pi_0\oplus\pi_1\quad \text{on } \mathcal{K}=\mathcal{K}_0\oplus\mathcal{K}_1,$$

where π_0 and π_1 are disjoint representations of $C^*(B_1,\dots,B_n)$ on the Hilbert spaces

$$\mathcal{K}_0:=\overline{\operatorname{span}}\big\{\pi(B_\beta)\big[(id-\Phi_{p,\pi(B)})^m(I_{\mathcal{K}})\big]\mathcal{K}\colon\ \beta\in\mathbb{F}_n^+\big\}\quad \text{and}\quad \mathcal{K}_1:=\mathcal{K}_0^\perp,$$

respectively, where $\pi(B):=(\pi(B_1),\dots,\pi(B_n))$. Moreover, up to an isomorphism,

$$\mathcal{K}_0\simeq\mathcal{N}_{\mathcal{Q}}\otimes\mathcal{G},\qquad \pi_0(X)=X\otimes I_{\mathcal{G}}\quad \text{for } X\in C^*(B_1,\dots,B_n),$$

where $\mathcal{G}$ is a Hilbert space with $\dim\mathcal{G}=\dim\{\operatorname{range}[(id-\Phi_{p,\pi(B)})^m(I_{\mathcal{K}})]\}$, and π_1 is a $*$-representation which annihilates the compact operators and

$$\Phi_{p,\pi_1(B)}(I_{\mathcal{K}_1})=I_{\mathcal{K}_1}.$$

If π' is another nondegenerate $*$-representation of $C^*(B_1,\dots,B_n)$ on a separable Hilbert space $\mathcal{K}'$, then π is unitarily equivalent to π' if and only if $\dim\mathcal{G}=\dim\mathcal{G}'$ and π_1 is unitarily equivalent to π_1'.

We remark that under the hypotheses and notations of Corollary 6.4, and setting $V_i:=\pi(B_i)$, $i=1,\dots,n$, the following statements are equivalent:

(i) $V:=(V_1,\dots,V_n)$ is a constrained weighted shift in the noncommutative variety $\mathcal{V}_{p,\mathcal{Q}}^m(\mathcal{K})$;

(ii) SOT-$\lim_{k\to\infty} \Phi^k_{p,V}(I) = 0$;
(iii) $\mathcal{K} = \overline{\operatorname{span}}\{V_\beta[(id - \Phi_{p,V})^m(I)]\mathcal{K}\colon \beta \in \mathbb{F}_n^+\}$;
(iv) $\sum_{\beta\in\mathbb{F}_n^+} b_\beta^{(m)} V_\beta[(id - \Phi_{p,V})^m(I)]V_\beta^* = I_{\mathcal{K}}$, where $b_\beta^{(m)}$ are the coefficients defined by (1.1).

We mention that, under the additional condition that

$$\overline{\operatorname{span}}\{B_\alpha B_\beta^*\colon \alpha, \beta \in \mathbb{F}_n^+\} = C^*(B_1, \dots, B_n),$$

the map $\Psi_{p,T,\mathcal{Q}}$ in the proof of Theorem 6.3 is unique. The uniqueness of the minimal Stinespring representation [49] and the above-mentioned Wold type decomposition imply the uniqueness of the minimal dilation of Theorem 6.3.

Corollary 6.5. *Let $V := (V_1, \dots, V_n) \in \mathcal{V}^m_{p,\mathcal{Q}}(\mathcal{K})$ be the dilation of $T := (T_1, \dots, T_n) \in \mathcal{V}^m_{p,\mathcal{Q}}(\mathcal{H})$, given by Theorem* 6.3. *Then,*

(i) *V is a constrained weighted shift if and only if T is a pure n-tuple of operators*;
(ii) *$\Phi_{p,V}(I_{\widetilde{\mathcal{K}}}) = I_{\widetilde{\mathcal{K}}}$ if and only if $\Phi_{p,T}(I_{\mathcal{H}}) = I_{\mathcal{H}}$.*

Proof. According to Theorem 6.3, we have

$$\Phi^k_{p,T}(I_{\mathcal{H}}) = P_{\mathcal{H}} \begin{bmatrix} \Phi^k_{p,B}(I_{\mathcal{N}_\mathcal{Q}}) \otimes I_{\overline{\Delta_{p,m,T}\mathcal{H}}} & 0 \\ 0 & I_{\mathcal{K}_\pi} \end{bmatrix} \Bigg| \mathcal{H} \quad \text{for } k = 1, 2, \dots,$$

which implies

$$\text{SOT-}\lim_{k\to\infty} \Phi^k_{p,T}(I_{\mathcal{H}}) = P_{\mathcal{H}} \begin{bmatrix} 0 & 0 \\ 0 & I_{\mathcal{K}_\pi} \end{bmatrix} \Bigg| \mathcal{H}.$$

Consequently, T is pure if and only if $P_{\mathcal{H}} P_{\mathcal{K}_\pi}|\mathcal{H} = 0$. The latter condition is equivalent to $\mathcal{H} \perp (0 \oplus \mathcal{K}_\pi)$, which, according to Theorem 6.3, is equivalent to $\mathcal{H} \subset \mathcal{N}_\mathcal{Q} \otimes \overline{\Delta_{p,m,T}\mathcal{H}}$. On the other hand, since $\mathcal{N}_\mathcal{Q} \otimes \overline{\Delta_{p,m,T}\mathcal{H}}$ is reducing for $V_1, \dots, V_n$, and $\widetilde{\mathcal{K}}$ is the smallest reducing subspace for $V_1, \dots, V_n$, which contains $\mathcal{H}$, we must have $\widetilde{\mathcal{K}} = \mathcal{N}_\mathcal{Q} \otimes \overline{\Delta_{p,m,T}\mathcal{H}}$. Therefore, item (i) holds.

To prove part (ii), note that

$$(id - \Phi_{p,V})^m(I_{\widetilde{\mathcal{K}}}) = \begin{bmatrix} [(id - \Phi_{p,B})^m(I_{\mathcal{N}_\mathcal{Q}})] \otimes I_{\overline{\Delta_{p,m,T}\mathcal{H}}} & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence, we deduce that $(id - \Phi_{p,V})^m(I_{\widetilde{\mathcal{K}}}) = 0$ if and only if

$$\left[(id - \Phi_{p,B})^m(I_{\mathcal{N}_\mathcal{Q}})\right] \otimes I_{\overline{\Delta_{p,m,T}\mathcal{H}}} = 0.$$

On the other hand, we know that $(id - \Phi_{p,B})^m(I_{\mathcal{N}_\mathcal{Q}}) = P_{\mathbb{C}}^{\mathcal{N}_\mathcal{Q}}$. Consequently,

$$(id - \Phi_{p,V})^m(I_{\widetilde{\mathcal{K}}}) = 0$$

if and only if $\Delta_{p,m,T} = 0$. Now, using Lemma 6.2, we obtain the equivalence in part (ii). The proof is complete. $\square$

We mention now a few remarkable particular cases, when Theorem 6.3 applies.

Remark 6.6.

(i) In the particular case when $m = 1$, $n = 1$, $p = X$, and $\mathcal{Q} = 0$, we obtain the classical isometric dilation theorem for contractions obtained by Sz.-Nagy (see [50,51]).
(ii) When $m = 1$, $n \geqslant 2$, $p = X_1 + \cdots + X_n$, and $\mathcal{Q} = 0$ we obtain the noncommutative dilation theorem for row contractions (see [13,23,31]).
(iii) In the single variable case, when $m \geqslant 2$, $n = 1$, $p = X$, and $\mathcal{Q} = 0$, the corresponding domain coincides with the set of all m-hypercontractions studied by Agler in [1,2], and recently by Olofsson [26,27].
(iv) When $m \geqslant 2$, $n \geqslant 2$, $p = X_1 + \cdots + X_n$, and $\mathcal{Q} = 0$, the elements of the corresponding domain $\mathbf{D}_p^m(\mathcal{H})$ can be seen as multivariable noncommutative analogues of Agler's m-hypercontractions.
(v) In the particular case when $\mathcal{Q}_c$ consists of the polynomials $Z_i Z_j - Z_j Z_i$, $i, j = 1, \ldots, n$, we recover several results concerning model theory for commuting n-tuples of operators. The case $n \geqslant 2$, $m \geqslant 2$, $p = X_1 + \cdots + X_n$, and $\mathcal{Q} = \mathcal{Q}_c$, was studied by Athavale [9], Müller [24], Müller, Vasilescu [25], Vasilescu [52], and Curto, Vasilescu [14].
(vi) When p is a positive regular noncommutative polynomial and $\mathcal{Q}$ consists of the polynomials

$$W_i W_j - W_j W_i, \quad i, j = 1, \ldots, n,$$

we obtain the dilation theorem of S. Pott [48].
(vii) When $m = 1$, $n \geqslant 1$, and p is any positive regular noncommutative polynomial we find the dilation theorem obtained in [45] (see also [3]).

We expect to use the results of the present paper to obtain functional models for the elements of the noncommutative domain $\mathbf{D}_f^m(\mathcal{H})$ (respectively subvariety $\mathcal{V}_{f,\mathcal{Q}}^m(\mathcal{H})$), based on characteristic functions.

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Properties of positive solutions to an elliptic equation with negative exponent ☆

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Abstract

In this paper, we study some quantitative properties of positive solutions to a singular elliptic equation with negative power on the bounded smooth domain or in the whole Euclidean space. Our model arises in the study of the steady states of thin films and other applied physics as well as differential geometry. We can get some useful local gradient estimate and L^1 lower bound for positive solutions of the elliptic equation. A uniform positive lower bound for convex positive solutions is also obtained. We show that in lower dimensions, there is no stable positive solutions in the whole space. In the whole space of dimension two, we can show that there is no positive smooth solution with finite Morse index. Symmetry properties of related integral equations are also given.

Keywords: Positive solutions; Negative power; Lower bound; Gradient estimate

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1. Introduction

In this paper, we study some properties of positive solutions of the following elliptic equation:

$$\Delta u = u^{\tau}, \quad \text{in } \Omega, \tag{1}$$

where $\tau < 0$ and $\Omega \subset \mathbf{R}^n$ is a regular domain.

Problem (1) arises in many branches of applied sciences. For example, it can be considered as steady states of thin films. Equations of the type

$$u_t = -\nabla \cdot \big(f(u)\nabla\Delta u\big) - \nabla \cdot \big(g(u)\nabla u\big) \tag{2}$$

have been used to model the dynamics of thin films of viscous fluids, where $z = u(x,t)$ is the height of the air/liquid interface. The zero set $\Sigma_u = \{u = 0\}$ is the liquid/solid interface and is sometimes called set of *ruptures*. Ruptures play a very important role in the study of thin films. The coefficient $f(u)$ reflects surface tension effects—a typical choice is $f(u) = u^3$. The coefficient of the second-order term can reflect additional forces such as gravity $g(u) = u^3$, van der Waals interactions $g(u) = u^m$, $m < 0$. For backgrounds of (2), we refer to [2,3,25–27], and the references therein.

In general, let us assume that $f(u) = u^p$, $g(u) = u^m$, where $p, m \in R$. Then if we consider the steady-state of (2), we see that u satisfying

$$u^p \nabla\Delta u + u^m \nabla u = \mathcal{C}$$

is a steady state of (2). Here $\mathcal{C} = (C_1, C_2, \ldots, C_n)$ is some constant vector. By assuming $\mathcal{C} = \mathbf{0}$ (which prevents linear term on x), we obtain

$$\Delta u + \frac{u^{\tau}}{\tau} - C = 0 \quad \text{in } \Omega, \tag{3}$$

where $\tau = m - p + 1$ and C is some constant. (Here we have assumed that $\tau \neq 0$. If $\tau = 0$, we have to replace $\frac{u^{\tau}}{\tau}$ by $\log u$.) Note that solutions to (1) are steady-states of (3) but the reverse is not true.

When $\Omega = \mathbf{R}^n$, the radially symmetric solutions to (3) has been studied in [24].

When $\tau = -2$, problem (1) also arises in the study of MEMS. We refer to [11,15–17,19,20] and the references therein.

When $\tau = -1$, Eq. (1) is related to the study of singular minimal hypersurfaces with symmetry. See [31,35] and the references therein.

In this paper, we study *quantitative* properties of solutions to (1), including the gradient estimates, L^1-estimates, global upper bounds, Liouville properties, classification of stable and finite Morse index solutions, and symmetry properties.

Here we state the main results.

The first eight theorems concerns solutions to (1).

Theorem 1 *(Gradient estimates). Let $u \in C^2(\Omega)$ be a positive solution to Eq.* (1) *in Ω. Then for any $R > 0$, $x_0 \in \Omega$ with $B_R(x_0) \subset \Omega$ we have absolute constant $C = C(R)$ such that*

$$\left|\nabla u(x)\right|^2 \leqslant C u(x)^2 + u(x)^{1+\tau} \tag{4}$$

for all $x \in B_R(x_0)$.

Theorem 2 *(L^1-estimates). Let $u \in C^2(\Omega)$ be a positive solution to Eq.* (1) *in Ω. Then for any $R > 0$ and $x_0 \in \Omega$ (with $B_R(x_0) \subset \Omega$), we have absolute constant $C(n,\tau)$ such that*

$$\int_{B_R(x_0)} u \geqslant C(n,\alpha) R^{n+\frac{2}{1-\tau}}. \tag{5}$$

Theorem 3 *(Global upper bound). Let $1 \leqslant u \in C^2(\mathbf{R}^n)$ be a positive solution to Eq.* (1) *in $\mathbf{R}^n$. Then we have absolute constant $C(n)$ such that*

$$u(x) \leqslant C(n)\big(|x|^2 + 1\big), \tag{6}$$

and

$$\big|\nabla u(x)\big| \leqslant C(n)\big(|x| + 1\big), \tag{7}$$

for all $x \in \mathbf{R}^n$.

Theorem 4 *(Liouville properties). Assume that $n > 2$ and $\tau \leqslant -1 - \frac{2n}{n-2}$. We have the* Liouville property *that there is no positive convex solution $u(x)$ to* (1) *both on the whole space $\mathbf{R}^n$ and on the half space $\mathbf{R}^n_+$ with $u(x) \geqslant 1$ everywhere.*

Theorem 5 *(Compactness). Assume that $n > 2$ and $\tau \leqslant -1 - \frac{2n}{n-2}$. Let Ω be a bounded or unbounded smooth convex domain. Let u be a positive convex solution to* (1) *on Ω. Then, we have a uniform constant $C = C(\Omega)$ such that*

$$u(x) \geqslant C, \quad x \in \Omega.$$

Theorem 6 *(Stable solutions). If $2 \leqslant n < 2 + \frac{4}{1-\tau}(-\tau + \sqrt{\tau^2 - \tau})$, then there are no* stable *positive solutions to* (1) *in $\mathbf{R}^n$.*

Theorem 7 *(Finite Morse index solutions). There is no finite Morse index positive solutions to* (1) *in $\mathbf{R}^2$.*

Theorem 8 *(Existence). Assume $\tau \leqslant 0$, $\tau \neq -1$, and let $\Omega \subset \mathbf{R}^n$ be a bounded smooth domain. Given any smooth positive boundary data ϕ. Assume that $\underline{u}$ is a sub-solution to* (1) *with $\underline{u} \leqslant \phi$ on $\partial\Omega$. Then there is smooth positive solution to* (1) *with boundary data ϕ.*

We remark that in the special case when $\tau = -1$ Theorem 6 has been obtained in [31].

In the next two theorems, we will study a related integral equation

$$u(x) = h(x) - \int_{\mathbf{R}^n} |x-y|^{\mu-n} u(y)^\tau \, dy, \tag{8}$$

with $n \geqslant 2$, $0 < \mu < n$, $h(x)$ is a positive smooth function, and $\tau < 0$.

We prove the following two symmetry properties.

Theorem 9. *Given some $\beta > 1$ and $q > 1$. Let*

$$u^{\tau-1} \in L^{\beta}(\mathbf{R}^n), \tag{9}$$

be a positive solution of Eq. (8) *with*

$$\tau < 0 \quad \text{and} \quad \beta = \frac{\tau - 1}{\frac{n-\mu}{n}\tau - 1} > \frac{2n}{n-\mu}.$$

Assume that for some plane π, we have $h(x) = h(\pi(x))$. Then $u(x)$ is symmetric to the plane π.

Theorem 10. *Assume that $h(x) = h$ is a constant function. Given some $\beta > 1$ and $q > 1$. Assume that*

$$\tau < 0 \quad \text{and} \quad \beta = \frac{\tau - 1}{\frac{n-\mu}{n}\tau - 1} > \frac{2n}{n-\mu}.$$

Then, for any positive solution to (8) *with*

$$u^{\tau-1} \in L^{\beta}(\mathbf{R}^n), \tag{10}$$

and

$$|x|^{\mu-n} u\left(\frac{x}{|x|^2}\right) \in L^q(\mathbf{R}^n), \tag{11}$$

u is radial symmetric at zero.

The plan of our paper is follows. In last two sections, we discuss symmetry properties of related integral equations (8). We prove Theorem 1 in Section 2 and prove Theorem 2 in Section 3. In Section 5, we prove Theorem 3. Theorem 5 is proved in Section 7. The Liouville property is proved in Section 6. The proof of Theorems 6 and 7 is given in Section 8. As we mentioned before, we shall discuss the existence theory of positive solutions to (1) in Section 9. Many consequences of Theorem 2 will be discussed in Section 4.

In the following, we shall use C to denote different constants which depend only on n, τ, μ, and the solution u in varying places.

2. Proof of Theorem 1

In this section, we use the maximum principle trick (see [12,30,40]) to obtain the gradient estimate for positive solutions of the elliptic equation (1), which is Theorem 1.

Proof of Theorem 1. Recall the following *basic formula*. Let v be any smooth function in $\mathbb{R}^n$. Then, we have

$$\frac{1}{2}\Delta|\nabla v|^2 = (\nabla\Delta v, \nabla v) + \left|D^2 v\right|^2,$$

which can be proved by an elementary calculation.

Let us begin with the local gradient estimate for positive solution u to the following general elliptic equation:

$$\Delta u = f(u), \quad \text{in } \mathbf{R}^n.$$

In our case we shall set $f(u) = u^\tau$. Set

$$w = \log u.$$

Then we have

$$\nabla w = u^{-1}\nabla u,$$

and we can get

$$\Delta w = -|\nabla w|^2 + F(w),$$

where

$$F(w) = u^{-1} f(u) = e^{-w} f\big(e^w\big).$$

In particular, $F(w) = e^{(-1+\tau)w}$ and $F'(w) = (-1+\tau)F(w)$ for our case.

Assume $R_2 > R > 0$. Let ϕ be a cut-off function in $B_{R_2}(0)$ with $\phi = 1$ on $B_R(0)$. Define

$$P = \phi|\nabla w|^2,$$

which is usually called the *Harnack quantity* for the solution u.

At the maximum point of P, we have the first order condition

$$\nabla P = 0,$$

which implies that

$$\nabla|\nabla w|^2 = -\phi^{-2}\nabla\phi P$$

and the second order condition:

$$0 \geqslant \Delta P = P_0(\phi)P + \phi\Delta|\nabla w|^2, \tag{12}$$

where

$$P_0(\phi) = \Delta\phi - 2|\nabla\phi|^2\phi^{-2}.$$

Using the *basic formula*, we have

$$\phi\Delta|\nabla w|^2 = 2\phi\big|D^2 w\big|^2 + 2\phi(\nabla\Delta w, \nabla w).$$

Note that

$$\phi\left|D^2 w\right|^2 \geqslant \frac{2\phi}{n}|\Delta w|^2 = \frac{2}{n\phi}\bigl(-P + \phi F(w)\bigr)^2,$$

and

$$2\phi(\nabla\Delta w, \nabla w) \geqslant 2F'P - 2\phi\bigl(\nabla|\nabla w|^2, \nabla w\bigr) = 2F'P - 2\phi^{-1}(\nabla\phi, \nabla w)P,$$

and then, for any $\mu > 0$,

$$2\phi(\nabla\Delta w, \nabla w) \geqslant 2F'P - 2\mu^{-1}\phi^{-2}|\nabla\phi|^2 P - \mu\phi^{-1}P^2.$$

Choose $\mu = \frac{1}{4n}$. Then

$$2\phi(\nabla\Delta w, \nabla w) \geqslant 2F'P - 4n\phi^{-2}|\nabla\phi|^2 P - \frac{1}{4n}\phi^{-1}P^2.$$

Hence,

$$\phi\Delta|\nabla w|^2 \geqslant \frac{2}{n\phi}\bigl(-P + \phi F(w)\bigr)^2 + 2F'P - 4n\phi^{-2}|\nabla\phi|^2 P - \frac{1}{4n}\phi^{-1}P^2.$$

Then from (12) we have

$$A(\phi, F')P \geqslant \frac{2}{n}(-P + \phi F)^2 - \frac{1}{4n}P^2.$$

Here

$$A(\phi, F') = 4n\phi^{-1}|\nabla\phi|^2 - 2\phi F' - \phi P_0(\phi).$$

If $P \leqslant 2\phi F$, then we have

$$|\nabla w|^2 \leqslant 2F.$$

We remark that in this case, we have

$$|\nabla u|^2 \leqslant 2u^2 F = 2uf(u).$$

Otherwise, we have

$$-P + \phi F \leqslant -P/2 \leqslant 0$$

and

$$\frac{2}{n}(-P + \phi F)^2 - \frac{1}{4n}P^2 \geqslant \frac{1}{4n}P^2.$$

Hence, we have

$$P \leqslant 4nA(\phi, F').$$

In conclusion, we have on $B_R(0)$,

$$|\nabla w|^2 \leqslant \max\bigl(4nA(\phi, F'), 2F\bigr),$$

which implies the conclusion of Theorem 1. □

We remark that our local gradient estimate can be extended to other elliptic equation like

$$-\Delta u = u^\tau.$$

As a consequence of the local gradient estimate in Theorem 1, we have the following improvement of [31, Theorem 7.1].

Corollary 11. *For every compact sub-domain K of Ω, there is a constant $C = C(n, K, \Omega)$ such that for any sequence (u_j) of positive solutions to Eq.* (1) *with $\tau = -1$ and with boundary data $\phi_j \leqslant M$, we have*

$$|\nabla u_j| \leqslant (M+1)C, \quad \text{on } K. \tag{13}$$

Hence, the limit u of any convergent subsequence of (u_j) is a Lipschitz continuous weak solution to a free boundary problem of the equation

$$\Delta u = u^\tau \chi_{\{u>0\}}. \tag{14}$$

For the proof of Corollary 11, we just note that the solution u is a subharmonic function and it attains its maximum only on the boundary $\partial\Omega$. Then we use the gradient bound to get the conclusion. We remark that the result is optimal in the sense that u is not differentiable at its zero point as noticed in [31] (see also [23]). It is unclear how large the zero level set $\Sigma_u(u) = \{u = 0\}$ is. However, a general study was made in the paper [23] and a partial result was obtained in [19].

Note that the gradient estimate implies the Harnack inequality for positive solutions with $u(x) \geqslant 1$ for all $x \in B_R(x_0)$. One can use this fact to derive a compactness result. The application of such compactness result on the existence theory of positive solutions to (1) will be discussed in Section 9.

3. Proof of Theorem 2

Our aim in this section is to obtain an L^1-estimates for solutions of (1). We shall prove a more general version of Theorem 2.

Theorem 12. *Assume $\tau \leqslant 0$ and assume that $\Omega \subset \mathbf{R}^n$ is an open subset in $\mathbf{R}^n$. Let $f : \mathbf{R} \to \mathbf{R}_+$ be a positive function such that*

$$s^{\frac{\tau}{\tau-1}} f(s)^{\frac{1}{1-\tau}} \geqslant C_0, \quad \text{for } s > 0,$$

for some constant C_0. Let $u \in C^0(\Omega)$ be a positive weak solution to the equation

$$\Delta u = f(u) \tag{15}$$

in Ω. Then for any $R > 0$ and $x_0 \in \Omega$ (with $B_R(x_0) \subset \Omega$), we have absolute constant $C(n,\tau)$ such that

$$\int_{B_R(x_0)} u \geqslant C(n,\alpha) R^{n+\frac{2}{1-\tau}}. \tag{16}$$

Proof. Without loss of generality, we take $x_0 = 0$. Let $R_2 = 2R_1 > 0$ and let $\xi(|x|)$ be a cut-off function with its support in the ball $B_{R_2}(0)$, $\xi(|x|) = 1$ on $B_{R_1}(0)$, and

$$|\nabla \xi| \leqslant 4/R_1, \qquad |\Delta \xi| \leqslant 100/R_1^2.$$

Multiplying both sides of (1) by $|x|^2\xi$ and integrating over the ball $B_{R_2}(0)$, we then get

$$\int u \Delta\big(|x|^2\xi\big) = \int f(u)|x|^2\xi.$$

Note that the right-hand side is bigger than

$$\int_{B_{R_1}(0)} f(u)|x|^2;$$

and the left-hand side is less than

$$C \int_{B_{R_2}(0)} u,$$

where $C > 0$ is an absolute constant depending only on the dimension n. That implies that

$$\int_{B_{R_1}(0)} f(u)|x|^2 \leqslant C \int_{B_{R_2}(0)} u.$$

Let $p > 1$. Then we have

$$\Bigg(\int_{B_{R_1}(0)} f(u)|x|^2\Bigg)^{1/p} \Bigg(\int_{B_{R_1}(0)} u\Bigg)^{(p-1)/p} \leqslant C^{1/p} \int_{B_{R_2}(0)} u.$$

Using Hölder's inequality to the left-hand side, we get

$$\int_{B_{R_1}(0)} |x|^{2/p} f(u)^{1/p} u^{(p-1)/p} \leqslant C^{1/p} \int_{B_{R_2}(0)} u.$$

Choose $p=-\tau+1$. Then by our assumption on f, we have

$$\int_{B_{R_1}(0)} |x|^{2/p} \leqslant C^{1/p} \int_{B_{R_2}(0)} u.$$

It is elementary to compute that

$$\int_{B_{R_1}(0)} |x|^{2/p} = \frac{p\omega_n}{2+p(n-1)} R_1^{n+\frac{2}{p}},$$

ω_n is the volume of the unit ball $B_1(0)$. This implies the conclusion of Theorem 12. □

We remark that our argument above can also be used to smooth positive solutions to the following equation:

$$-\Delta u = u^\tau, \quad \text{in } \Omega,$$

with $\tau \leqslant 0$.

Note that Theorem 2 follows from Theorem 12 when $f(u)=u^\tau$. Another proof of Theorem 2 is to use the convexity of $f(u)$, the spherical average method, and an ODE comparison lemma to get the L^1 lower bound. However, our proof above is more general and works also on manifolds.

There are many consequences of Theorem 2, and they will be discussed in Section 4.

4. Consequences of Theorem 2

As an easy consequence of Theorem 2, we have

Corollary 13. *Assume $\tau \leqslant 0$ and $\Omega \subset \mathbf{R}^n$. Let f be as in Theorem 2 above. Let $u \in C^0(\Omega)$ be a positive weak solution to the equation to (15) in Ω. Then for any $R>0$ and $x_0 \in \Omega$ (with $B_R(x_0) \subset \mathbf{R}^n$), we have absolute constant $C(n,\tau)$ such that*

$$\max_{\partial B_R(x_0)} u = \sup_{B_R(x_0)} u \geqslant C(n,\alpha) R^{\frac{2}{1-\tau}}. \tag{17}$$

The proof of this is direct by using the L^1 lower bound (5) since u is subharmonic and the maximum occurs only at boundary point.

Corollary 13 immediately implies the following.

Corollary 14. *Assume $\tau \leqslant 0$ and assume that $\Omega \subset \mathbf{R}^n$ is an open subset in $\mathbf{R}^n$. Let $f:\mathbf{R} \to \mathbf{R}_+$ be a positive function such that*

$$s^{\frac{\tau}{\tau-1}} f(s)^{\frac{1}{1-\tau}} \geqslant C_0, \quad \textit{for } s>0,$$

for some constant C_0. Then there is a positive constant $C=C(\Omega)$ such that if the positive boundary data $\phi \leqslant C$ on $\partial\Omega$, the Dirichlet problem to Eq. (15) *in Ω with $u=\phi$ on the boundary $\partial\Omega$ has no nontrivial positive weak C^0-solution.*

In fact, we take a ball $B_R(x_0)$ in the domain Ω and let $C = C(n,\alpha)R^{\frac{2}{1-\tau}}$. Then we have $\sup_\Omega u = \sup_{\partial\Omega}\phi > C$.

Using Theorem 2, we can easily derive the following.

Proposition 15. *There is no positive solution to* (1) *in a cone-like unbounded domain Ω with the bound*

$$R^{-n-\frac{2}{1-\tau}+\sigma}\int\limits_{B_R} u\,dx \leqslant K$$

for every ball $B_R \subset \Omega$ with $R \geqslant 1$, for some constant $K > 0$ and $\sigma > 0$.

Proof. Assume we have a positive solution u. Note that we can choose an arbitrary large ball B_R in the domain Ω. Then, using our Theorem 2, we get

$$R^\sigma \leqslant K,$$

which is not true by sending $R \to +\infty$. $\square$

We point out that for solution u to (1), the quantity

$$R^{-n-\frac{2p}{1-\tau}}\int\limits_{B_R(0)} u^p$$

is dimensionless. By this, we mean that if u is a solution to (1) in the ball $B_R(0)$, then the function

$$v(x) = R^{-\frac{2}{1-\tau}}u(Rx)$$

is a solution to (1) in $B_1(0)$ with

$$R^{-n-\frac{2}{1-\tau}p}\int\limits_{B_R} u^p\,dx = \int\limits_{B_1(0)} v^p.$$

Hence, in this sense, the L^1 lower bound estimate in Theorem 2 is the best one.

5. Proof of Theorem 3

In this section, we derive a *global upper bound* for positive solutions to (1) in $\mathbf{R}^n$.

Assume that $u(x) \geqslant 1$ on $B_R(0)$ satisfies

$$\Delta u = u^\tau.$$

Then

$$0 < \Delta u \leqslant 1.$$

Using the gradient estimate in Theorem 1, we can easily get that

$$u(x) \leqslant u(0)e^{CR}$$

for all $x \in B_R(0)$. Here C is a uniform constant. However, this estimate is too rough.

We now use the mean value property to do better. Fix $0 < r = |x| \leqslant R/4$. Since $u > 0$ is subharmonic, on one hand, we have

$$u(x) \leqslant \frac{1}{|B_r(x)|} \int\limits_{B_r(x)} u.$$

On the other hand, let

$$w(y) = u(y) - \frac{|y|^2}{2n}.$$

Then

$$\Delta w = \Delta u - 1 \leqslant 0.$$

Hence w is superharmonic, and for $R = 2r$,we have

$$1 = w(0) \geqslant \frac{1}{|B_R(0)|} \int\limits_{B_R(0)} u - \frac{2|x|^2}{n(n+2)}. \tag{18}$$

Note that $B_r(x) \subset B_R(0)$ and

$$\frac{1}{|B_R(0)|} \int\limits_{B_R(0)} u \geqslant \frac{|B_r(x)|}{|B_R(0)|} \frac{1}{|B_r(x)|} \int\limits_{B_r(x)} u \geqslant \frac{1}{2^n} \frac{1}{|B_r(x)|} \int\limits_{B_r(x)} u \geqslant \frac{u(x)}{2^n}.$$

Hence, using (18), we have (6):

$$u(x) \leqslant C_n\big(|x|^2 + 1\big).$$

Using the standard interpolation argument and $0 < \Delta \leqslant 1$, we get the gradient growth:

$$\big|\nabla u(x)\big| \leqslant C_n\big(|x| + 1\big). \qquad \square$$

The important consequence of Theorem 3 is the following well-known result.

Corollary 16. *Assume that $u > 0$ is a positive solution to the equation*

$$\Delta u = 1 \quad \textit{in } \mathbf{R}^n.$$

Then u is a polynomial of the form

$$a_0 + \sum_{j=1}^{n} a_j x_j^2,$$

where $a_0 > 0$, $a_j \geqslant 0$ *for* $j = 1, \ldots, n$, *and* $2\sum_j a_j = 1$.

Proof. We may assume $u(x) \geqslant 1$ by considering $u(x) + 1$ if necessary. Using our upper bound estimate in Theorem 3, we know u has at most quadratic growth. Consider $w(x) = u(x) - \frac{|x|^2}{2n}$. Then w is a harmonic function with quadratic growth. Hence, w is a quadratic polynomial, and so is u, which gives the conclusion. □

6. Liouville property for *convex* solutions

We prove Theorem 4 in this section. Choose any positive number $k > 1$ and let $\Omega = \{x \in \mathbf{R}^n;\ u(x) \leqslant k\}$. By our assumption ($u$ is *convex*), Ω is a bounded convex domain. Recall the Pohozaev identity formally. Let $g(u) = -u^\tau$ and let

$$G(u) = \frac{1}{1+\tau}\left[k^{1+\tau} - u^{1+\tau}\right].$$

Multiplying by $x\nabla u$ the equation

$$-\Delta u = g(u),$$

we have

$$0 = \operatorname{div}\left(\nabla u(x\nabla u) - x\frac{|\nabla u|^2}{2} + xG(u)\right) + \frac{n-2}{2}|\nabla u|^2 - nG(u).$$

Note that $1 + \tau < 0$. By integrating the equation above over Ω, we get

$$\int_\Omega \frac{n-2}{2}|\nabla u|^2 + \frac{n}{1+\tau}\left[u^{1+\tau} - k^{1+\tau}\right] + \frac{1}{2}\int_{\partial\Omega} \partial_\nu u(x \cdot \nabla u) = 0.$$

Note that by multiplying by u the equation, we have

$$\int_\Omega |\nabla u|^2 - \int_{\partial\Omega} u\partial_\nu u = \int_\Omega u^{1+\tau}.$$

Hence, using $\partial_\nu u > 0$ on the boundary $\partial\Omega$, we have

$$\left[\frac{n-2}{2} + \frac{n}{1+\tau}\right]\int_\Omega u^{1+\tau} < \frac{nk^{1+\tau}|\Omega|}{1+\tau} < 0.$$

By this we get a contradiction, and then Theorem 4 is true. So we are done.

7. Compactness result

In this section, we study the point-wise lower bound of positive solutions to Eq. (1), and prove Theorem 5.

Proof of Theorem 5. We prove it by contradiction. For otherwise, we have a sequence of positive *convex* solutions $\{u_j\}$ and a sequence of points $\{x_j\} \subset \Omega$ such that

$$u_j(x_j) = \min_{\Omega} u_j(x) \to 0.$$

Choose

$$\lambda_j = u_j(x_j)^{\frac{1-\tau}{2}} \to 0.$$

Set

$$v_j(x) = \lambda_j^{-\frac{2}{1-\tau}} u_j(x_j + \lambda_j x),$$
$$\Omega_j := \left\{ x \in \mathbf{R}^n;\ x_j + \lambda_j x \in \Omega \right\},$$

and

$$B_j = B_{R\lambda_j^{-1}}(0).$$

Then it is elementary to see that

$$\Delta v_j = v_j^{\tau}, \quad \text{in } \Omega_j,$$

and

$$v_j(x) \geqslant v_j(0) = 1.$$

Let

$$\widehat{\Omega} = \lim_j \Omega_j.$$

Assume that $\lambda_j d(x_j, \partial\Omega) \to \infty$. Then $\widehat{\Omega} = \mathbf{R}^n$ and by our Harnack gradient estimate and the standard L^p theory, we can extract a convergent subsequence in $C^2(B_r(0))$ for any $r > 0$, still denoted by $\{v_j\}$, with its limit $\bar{v}$ being a positive convex function satisfying

$$\Delta \bar{v} = \bar{v}^{\tau}, \quad \text{in } \mathbf{R}^n, \qquad \bar{v}(x) \geqslant 1 = v(0).$$

If $\lambda_j d(x_j, \partial\Omega) \leqslant C$ for some constant C, then we have $\widehat{\Omega} = \mathbf{R}^n_+$ and we can get a positive solution $v \in C^2(\mathbf{R}^n_+)$, i.e.,

$$\Delta \bar{v} = \bar{v}^{\tau}, \quad \text{in } \mathbf{R}^n_+,$$

and

$$\bar{v}(x) \geqslant 1 = \bar{v}(0).$$

However, both cases give us a contradiction by our Theorem 4. Then, we have proved Theorem 5. □

Note that the convexity property in Theorem 5 cannot be removed since $u(x) = A|x|^{\frac{2}{1-\tau}}$ is a non-negative solution to (1) with

$$A = \left[\frac{(1-\tau)^2}{2(n+1) - 2(n-1)\tau}\right]^{\frac{1}{1-\tau}}.$$

8. Finite Morse index solutions

From the variational point of view, it is also very interesting to discuss positive solutions with finite Morse index to the following equation:

$$\Delta u = u^{\tau}, \quad \text{in } \mathbf{R}^n, \tag{19}$$

where $\tau < 0$, with finite Morse index. Assume that $u \in C^2$ is a positive solution to (19). Define

$$E(\phi) = \int_{\mathbf{R}^n} \left(|\nabla \phi|^2 + \tau u^{\tau-1} \phi^2\right),$$

where $\phi \in C_0^2(\mathbf{R}^n)$. By definition, we say the positive solution u to (19) with *finite Morse index k* if there exist L^2 orthogonal nontrivial functions $\{\phi_j\}_{j=1}^k \subset C_0^2(\mathbf{R}^n)$ such that we have $E(\phi) < 0$ for $\phi \in \mathbf{W} := \operatorname{span}\{\phi_j\} - \{0\}$, and $E(\phi) \geqslant 0$ for $\phi \perp \mathbf{W}$. If $k = 0$, we say that the solution u is *stable*.

Assume that u is the positive solution to (19) with finite Morse index k. Choose a large ball $B_R(0)$ which contains the supports of all ϕ_j's. Let

$$T_r = B_{R+1+r}(0) - B_{R+1}(0).$$

Then we have

$$E(\phi) \geqslant 0 \tag{20}$$

for all $\phi \in C_0^\infty(T_r)$. Let ξ be a smooth cut-off function with compact support in T_r. Let $\phi = u^{-q}\xi$. Then we have the following stability condition for any $\epsilon > 0$,

$$\begin{aligned}
&(-\tau) \int u^{-2q-1+\tau} \xi^2 \\
&\quad \leqslant \int \left|u^{-1} D\xi - q u^{-q-1} \xi Du\right|^2 \\
&\quad \leqslant \left(1 + \frac{|q|}{2\epsilon}\right) \int u^{-2q} |D\xi|^2 + \left(q^2 + 2|q|\epsilon\right) \int u^{-2q-2} \xi^2 |Du|^2.
\end{aligned}$$

Using the weak form of Eq. (19) with the testing function $\xi^2 u^{-\beta}$, $\beta = 2q + 1 > 0$, we have

$$\beta \int u^{-\beta-1}\xi^2 |Du|^2 \leqslant \int u^{-\beta+\tau}\xi^2 + 2\int u^{-\beta}\xi |Du||D\xi|,$$

and then we have, using the Cauchy–Schwarz inequality, for any $\delta > 0$,

$$(\beta - 2\delta)\int u^{-\beta-1}\xi^2|Du|^2 \leqslant \int u^{-\beta+\tau}\xi^2 + \frac{1}{2\delta}\int u^{-\beta+1}|D\xi|^2.$$

Inserting this into the stability condition we get

$$(-\tau)\int u^{-2q-1+\tau}\xi^2 \leqslant C(\epsilon,\delta,q)\int u^{-2q}|D\xi|^2 + \left(\frac{q^2+2|q|\epsilon}{2q+1-2\delta}\right)\int u^{-2q-1+\tau}\xi^2.$$

Choose $-\frac{1}{2} < q < -\tau + \sqrt{\tau^2-\tau}$ and ϵ, δ small enough depending on q, we can have

$$\frac{q^2+2|q|\epsilon}{2q+1-2\delta} < -\tau.$$

Hence, for some constant $C(\tau,q)$, we have

$$\int u^{-2q-1+\tau}\xi^2 \leqslant C(\tau,q)\int u^{-2q}|D\xi|^2.$$

Take $q > 0$ and replace ξ by $\xi^{q+\frac{1-\tau}{2}}$ to get

$$\int\left(\frac{\xi}{u}\right)^{2q+1-\tau} \leqslant C(\tau,q)\int\left(\frac{\xi}{u}\right)^{2q}\xi^{-1-\tau}|D\xi|^2.$$

Here $C(\tau,q)$ is another constant. Using the Young inequality

$$ab \leqslant \frac{(\epsilon a)^\alpha}{\alpha} + \frac{\alpha-1}{\alpha}\left(\frac{b}{\epsilon}\right)^{\frac{\alpha}{\alpha-1}}$$

with $\alpha = \frac{q+\frac{1-\tau}{2}}{q}$ ($\frac{\alpha}{\alpha-1} = \frac{2q}{1-\tau}+1$), $a = (\frac{\xi}{u})^2$, and $b = (\xi^{-\frac{1+\tau}{2}}|D\xi|)^2$, and choosing ϵ small, we get that

$$\int\left(\frac{\xi}{u}\right)^{2q+1-\tau} \leqslant C(\tau,q)\int\left(\xi^{-\frac{1+\tau}{2}}|D\xi|\right)^{\frac{4q}{1-\tau}+2}. \tag{21}$$

Take $0 < q < -\tau + \sqrt{\tau^2-\tau}$ such that

$$n \leqslant \frac{4q}{1-\tau} + 2$$

and then we can find that

$$\int_{T_r} \left(\frac{1}{u}\right)^{2q+1-\tau} \leqslant C(R,\tau,q),$$

for all $r > 0$. Note that the restriction of q is

$$\frac{(n-2)(1-\tau)}{4} \leqslant q < -\tau + \sqrt{\tau^2 - \tau},$$

which implies that

$$n < 2 + \frac{4}{1-\tau}\bigl(-\tau + \sqrt{\tau^2 - \tau}\bigr).$$

Set q such that $n = \frac{4q}{1-\tau} + 2$ and $p = 2q + 1 - \tau$. Then $p > n$, and we use the lower bound of u to get that

$$\int_{B_r(0)} \left(\frac{1}{u}\right)^{p} \leqslant C(R,\tau,q),$$

for any $r > 0$. Note that $1 - \frac{n}{p} = 1 - \frac{2}{1-\tau} = -\frac{\tau+1}{1-\tau}$. Using our equation we find that

$$|\Delta u| \in L^p\bigl(\mathbf{R}^n\bigr).$$

Using the standard L^p estimate we find that

Theorem 17. *Let u is a positive solution with finite Morse index on $\mathbf{R}^n$. We now assume that $u(x) \geqslant u(0) = 1$. Then,*

$$\bigl|Du(x)\bigr| \leqslant C|x|^{1-\frac{n}{p}} = C|x|^{-\frac{\tau+1}{1-\tau}},$$

and then we have the growth estimate

$$u(x) \leqslant C\bigl(1 + |x|^{-\frac{2\tau}{1-\tau}}\bigr).$$

In the case when the solution u is stable, we can take $T_r = B_r(0)$. In this case, let ξ be a cut-off function such that $\xi = 1$ on the ball $B_R(0)$,

$$\xi(x) = 2 - \frac{\log|x|}{\log R},$$

for $x \in B_{R^2}(0) - B_R(0)$, and $\xi = 0$ outside $B_{R^2}(0)$. Then we get from the estimate (21) that

$$\int_{B_R(0)} \frac{1}{u^n} \leqslant \frac{C}{(\log R)^{n-1}} \to 0,$$

which is impossible.

This proves Theorem 6.

We now prove Theorem 7.

Proof. Using the test function $\phi = \xi$ in (20), we obtain that

$$\int_{B_R(0)} u^{\tau-1} \leqslant C \tag{22}$$

where C is independent of $R > 1$.

We now perform the following scaling:

$$u(r,\theta) = A_\tau r^{\frac{2}{1-\tau}} v(t,\theta), \qquad t = \log r, \quad r = |x|, \tag{23}$$

where $A_\tau = (\frac{1-\tau}{2})^{\frac{2}{1-\tau}}$.

Thus we obtain that $v(t,\theta)$ satisfies

$$v_{tt} + \frac{4}{1-\tau} v_t + v_{\theta\theta} + \frac{4}{(1-\tau)^2} v - \frac{4v^\tau}{(1-\tau)^2} = 0, \quad t \in (-\infty, +\infty),\ \theta \in S^1. \tag{24}$$

We first claim

$$v(t,\theta) \geqslant C \quad \text{for } t > 2. \tag{25}$$

In fact, from (22) and (24), we obtain that

$$\int_t^{t+1} v^{\tau-1}(t,\theta)\, ds\, d\theta \to 0, \quad \text{as } t \to +\infty. \tag{26}$$

Let $m = v^{\tau-1}$. Then it is easy to see that m satisfies

$$m_{tt} + \frac{4}{1-\tau} m_t + m_{\theta\theta} + C_1 m^2 \geqslant 0. \tag{27}$$

Let us fix a point $\mathbf{x}_0 = (t_0, \theta_0) \in (1, \infty) \times S^1$. Set $\hat{m} = e^{\frac{2}{1-\tau}(t_0 - t)} m$. Then $\hat{m}$ satisfies

$$\hat{m}_{tt} + \hat{m}_{\theta\theta} + C_2 \hat{m}^2 \geqslant 0 \tag{28}$$

for $(t,\theta) \in [t_0 - 1, t_0 + 1] \times S^1$.

By Lemma 2.2 of [22] (see also [1, Theorem 1.7]), we see that there exists $\eta_0 > 0$ such that for any $r > 0$ if $\int_{B_r(\mathbf{x}_0)} \hat{m}\, dx \leqslant \eta_0$, then

$$\hat{m}(t,\theta) \leqslant \frac{C}{r^2} \int_{B_r(\mathbf{x}_0)} \hat{m}(x)\, dx \quad \text{for } (t,\theta) \in B_{r/2}(\mathbf{x}_0).$$

Choosing $t_0 > 8$ large enough so that

$$\int_t^{t+1} v^{\tau-1}(t,\theta)\,d\theta < e^{-8}\eta_0, \quad \text{for } t > \frac{t_0}{2}. \tag{29}$$

Then

$$\int_t^{t+1} \hat{m}(t,\theta)\,ds\,d\theta < \frac{1}{2}\eta_0, \quad \text{for } t > \frac{t_0}{2}. \tag{30}$$

Thus

$$\hat{m}(t,\theta) \leqslant C$$

for $(t,\theta) \in B_{\frac{1}{2}}((t_0,\theta_0))$, which is equivalent to that $v(t,\theta) \geqslant C$.

(25) implies that $v(t,\theta) \geqslant C$. On the other hand, it is easy to see that by the Harnack inequality, $v(t,\theta) \leqslant C$. By the results of L. Simon [34], $v(t,\theta) \to v(\theta)$, where $v(\theta)$ satisfies

$$v_{\theta\theta} + \frac{4}{(1-\tau)^2}v - v^{\tau} = 0, \quad v \text{ is } 2\pi\text{-periodic}. \tag{31}$$

By Theorem 2.1 of [8], $v(\theta) \equiv$ *constant* if $\tau \neq -3$, and $v(\theta) = (\lambda\cos^2\theta + \lambda^{-1}\sin^2\theta)^{1/2}$ for $\tau = -3$. This implies

$$\lim_{r\to+\infty} |x|^{-\frac{2}{1-\tau}} u(x) \geqslant \frac{2\mu}{(-\tau)} \tag{32}$$

for some $\mu > 0$.

Next, by explicitly solving the equation (it is an Euler equation), one finds that any nontrivial solution of

$$-k'' - \frac{1}{r}k' - \big(\mu/r^2\big)k = 0 \tag{33}$$

has infinitely many (and unbounded) positive zeros if $\mu > 0$. (Note that under the changes: $r = e^s$ and $\tilde{k}(s) = k(r)$, we see that $\tilde{k}(s)$ satisfies the equation

$$\tilde{k}''(s) + \mu\tilde{k}(s) = 0.$$

It is easily seen that $\tilde{k}(s)$ has infinitely many positive zeroes for any $\mu > 0$.) Thus, we can easily deduce that q has infinitely many positive zeros. Our claim holds.

We denote the zeroes of k as $0 < r_1 < r_2 < \cdots < r_k < \cdots$ where $r_k \to +\infty$ as $k \to +\infty$. Let k_0 be such that

$$\frac{(-\tau)}{u^{1-\tau}} \geqslant \frac{2\mu}{r^2}, \quad r > r_k,\ k \geqslant k_0. \tag{34}$$

We are now in the position to complete the proof of Theorem 7. Let $N > 0$ be fixed and $i \geqslant k_0$. Let h_i be the function defined to be $k(|x|)$ for $|x|$ between the ith and $(i+1)$th the zeros of k and to be zero otherwise. Then $h_i \in H^1(\mathbf{R}^2)$, h_i are orthogonal (in $L^2(\mathbf{R}^2)$ or $H^1(\mathbf{R}^2)$) and by multiplying (33) by h_i and integrating between these zeros we see that

$$Q(h_i) = \int_{\mathbf{R}^2} \left[|\nabla h_i|^2 + \frac{\tau}{u^{1-\tau}} h_i^2 \right] = \int_{\mathbf{R}^2} \left[\frac{\mu}{r^2} + \frac{\tau}{u^{1-\tau}} \right] h_i^2$$

is strictly negative at each h_i. Hence the span of h_i is an $(N-1)$-dimensional subspace of $C_0^\infty(\mathbf{R}^2)$ such that $Q(h) < 0$. Since h_i has compact support it follows easily that there is an $(N-1)$-dimensional subspace of $H^1(\mathbf{R}^2)$ such that

$$\int_{\mathbf{R}^2} \left[\left|\nabla h(y)\right|^2 + \frac{\tau}{u^{1-\tau}} h^2 \right] < 0$$

and hence the Morse index of u must be at least N. Since N is arbitrary, the Morse index of u is infinity, a contradiction to our assumption. □

9. Existence theory; proof of Theorem 8

We now consider the existence problem of the problem (1). One can easily find the radial solutions to (1) in the whole space (see Theorem 3.2 in [31] for the case when $\tau = -1$ and Theorem 1.1 in [20] for $\tau < -1$).

Given a positive data ϕ on the bounded smooth domain Ω. Consider the boundary problem of positive solutions to (1) on Ω with the boundary condition $u = \phi$ on $\partial\Omega$.

If $-1 < \tau < 0$, we let

$$J(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 + \frac{1}{1+\tau} \int_{\Omega} u^{1+\tau}$$

on the space

$$\mathbf{A}_0 = \left\{ u \in H^1(\Omega);\ u = \phi \text{ on } \partial\Omega \right\}.$$

Since

$$\int_{\Omega} u^{1+\tau} \leqslant |\Omega|^{-\tau} \left(\int_{\Omega} u \right)^{1+\tau},$$

we can get a non-negative minimizer of $J(\cdot)$ on $\mathbf{A}_0$. For such a minimizer, one need to handle with how large for its zero set. We shall not discuss this issue. One may see [23] and [19] for more (see also Appendix A for a brief discussion).

When $\tau = -1$, an existence result has been discussed in [31] by degree argument.

Theorem 8 gives another type of existence criteria. Since the proof of Theorem 8 is simple, we give it here.

Proof. Choose a large constant M such that $\bar{u} = M$ is a super-solution. Then one can use the standard super–sub solution method to get a positive solution.

We give here the variational method. Let

$$\mathbf{A} = \{u \in H^1(\Omega);\ \underline{u} \leqslant u \leqslant M \text{ in } \Omega,\ u = \phi \text{ in } \partial\Omega\}.$$

Define

$$I(u) = \frac{1}{2}\int |Du|^2 - \frac{1}{1+\tau}\int |u|^{1+\tau}. \tag{35}$$

It is easy to see that $I(\cdot)$ is bounded from below on the closed convex set $\mathbf{A}$. Since for any $u, w \in \mathbf{A}$,

$$\left|\int u^{1+\tau} - w^{1+\tau}\right| \leqslant C\int |u - w|.$$

(We get this by mean value theorem in Calculus.) Then we can use the Sobolev compactness embedding theorem to get a minimizer u of the functional $I(\cdot)$ in the set $\mathbf{A}$, which is the solution to (1) with the boundary data ϕ (see [37]). □

The advantage of variational methods is that one may prove that the minimizer on the class $\mathbf{A}$ is a stable solution in the usual sense. Since we shall not use this fact, we shall not discuss it. A natural question is how to find a sub-solution on a bounded domain. The usual way is to use one-dimensional (or any lower-dimensional) solution or radial solution on the whole space. One can also choose a large ball containing the bounded domain Ω, and solve the equation on the ball to get radial solutions on the ball. Such radial solutions are the sub-solutions to the equation on the original domain if the boundary value of the radial solutions on the domain Ω are less than the given boundary data ϕ.

In particular, as an application of Theorem 8, we have

Corollary 18. *Assume $\tau \leqslant 0$, $\tau \neq -1$, and let $\Omega \subset \mathbf{R}^n$ be a bounded smooth domain. Given any positive smooth boundary data ϕ in $\partial\Omega$. Assume that there is a radial solution $u(r)$ in lower-dimensional space $\mathbf{R}^k$ $(k < n)$ or the whole space $\mathbf{R}^n$ such that $\phi(x) > u(|x|)$ on $\partial\Omega$. Then there is smooth positive solution to* (1) *with boundary data ϕ.*

Proof. Here we need only to use $u(r)$ as a sub-solution to (1) on the domain Ω in Proposition 8. □

Although the argument in the proof of Corollary 18 is simple, it can be used to study the existence result of positive solutions for a large class of singular elliptic partial differential equations such as

$$\Delta u + au \log u = 0$$

with Dirichlet boundary data. One may see [18] and [30] for related results.

10. Symmetry properties: proof of Theorem 9

In the last two sections, we consider the integral equation (8) which is closely related to the elliptic differential equation the elliptic equation:

$$(-\Delta)^{\mu/2}(u-h) = -u^{\tau}, \quad \text{in } \mathbf{R}^n.$$

It is clear that the positive solution to (8) is bounded from above by h.

We shall use the following notation. Given any hyperplane π in $\mathbf{R}^n$ and any function $u : \mathbf{R}^n \to \mathbf{R}$. For any point $x \in \mathbf{R}^n$, let x^{π} be the reflection of x about the plane π and let $\pi(x) \in \pi$ be projection of x into π. Define

$$u^{\pi}(x) = u\bigl(x^{\pi}\bigr).$$

In the famous paper of Gidas and Spruck [13], they proved that for $n > 2$ and $\mu = 2$, and $1 \leqslant \tau < \frac{n+2}{n-2}$, the only non-negative solution to the equation

$$-\Delta u = u^{\tau}, \quad \text{in } \mathbf{R}^n,$$

is zero. However, the negative index τ case has not been treated before Xu's recent work [39]. So, Theorem 9 can be considered as a generalization of their result to Eq. (8).

In recent years, there are important progress in the study of symmetry properties of non-negative solutions to Yamabe type equations. In particular, X. Xu [39] has obtained some related results to ours. His equation is the following:

$$u(x) = \int\limits_{\mathbf{R}^n} |x-y|^{\mu-n} u(y)^{\tau}\, dy,$$

which corresponds to the elliptic equation:

$$(-\Delta)^{\mu/2} u = u^{\tau}, \quad \text{in } \mathbf{R}^n.$$

One should be caution about the negative sign before the Laplacian operator, which forces the equation to have no positive solution on the whole space. We will use a symmetry method (see also [7] and [6]) to prove our results—Theorems 9 and 10. This symmetry method is powerful in our case since we can use the behavior at infinity of the solution.

We now give a proof of Theorem 9. After using a rotation, we may assume that the hyperplane π is orthonormal to x_1 axis at the origin. So we may assume that $h(x) = h$ is a constant in the following argument.

For a given real number λ, we define

$$\Sigma_{\lambda} = \bigl\{x = (x_1, \dots, x_n) \bigm| x_1 \geqslant \lambda\bigr\},$$

and let $x^{\lambda} = (2\lambda - x_1, x_2, \dots, x_n)$ and $u_{\lambda}(x) = u(x^{\lambda})$.

We can easily get the following

Lemma 19. *For any positive solution $u(x)$ of* (8), *we have*

$$u_\lambda(x) - u(x) = -\int\limits_{\Sigma_\lambda} \left(\frac{1}{|x-y|^{n-\mu}} - \frac{1}{|x^\lambda - y|^{n-\mu}} \right) \left(u_\lambda(y)^\tau - u(y)^\tau \right) dy. \tag{36}$$

Proof. Let

$$\Sigma_\lambda^c = \left\{ x = (x_1, \ldots, x_n) \;\middle|\; x_1 < \lambda \right\}.$$

Then it is easy to see that

$$\begin{aligned} h - u(x) &= \int\limits_{\Sigma_\lambda} \frac{1}{|x-y|^{n-\mu}} u(y)^\tau \, dy + \int\limits_{\Sigma_\lambda^c} \frac{1}{|x-y|^{n-\mu}} u(y)^\tau \, dy \\ &= \int\limits_{\Sigma_\lambda} \frac{1}{|x-y|^{n-\mu}} u(y)^\tau \, dy + \int\limits_{\Sigma_\lambda} \frac{1}{|x-y^\lambda|^{n-\mu}} u\left(y^\lambda\right)^\tau \, dy \\ &= \int\limits_{\Sigma_\lambda} \frac{1}{|x-y|^{n-\mu}} u(y)^\tau \, dy + \int\limits_{\Sigma_\lambda} \frac{1}{|x^\lambda-y|^{n-\mu}} u_\lambda(y)^\tau \, dy. \end{aligned}$$

Here we have used the fact that $|x - y^\lambda| = |x^\lambda - y|$. Substituting x by x^λ, we get

$$h - u\left(x^\lambda\right) = \int\limits_{\Sigma_\lambda} \frac{1}{|x^\lambda-y|^{n-\mu}} u(y)^\tau \, dy + \int\limits_{\Sigma_\lambda} \frac{1}{|x-y|^{n-\mu}} u_\lambda(y)^\tau \, dy.$$

Thus

$$u\left(x^\lambda\right) - u(x) = -\int\limits_{\Sigma_\lambda} \left(\frac{1}{|x-y|^{n-\mu}} - \frac{1}{|x^\lambda - y|^{n-\mu}} \right) \left(u_\lambda(y)^\tau - u(y)^\tau \right) dy.$$

This implies (36). □

We shall need the following Hardy–Littlewood–Sobolev inequality (see, for example, [29])

$$\left\| \int V(x,y) f(y) \, dy \right\|_q \leqslant P_{p,n} \|f\|_p \tag{37}$$

with $V(x,y) = |x-y|^{-\nu}$ and

$$1/p + \nu/n = 1 + 1/q.$$

Proof of Theorem 9. Define

$$\Sigma_\lambda^- = \left\{ x \;\middle|\; x \in \Sigma_\lambda,\ u(x) \geqslant u_\lambda(x) \right\},$$

and

$$\Sigma_\lambda^+ = \{x \mid x \in \Sigma_\lambda,\ u(x) < u_\lambda(x)\}.$$

Then

$$\Sigma_\lambda = \Sigma_\lambda^+ \cup \Sigma_\lambda^-.$$

We want to show that for sufficiently positive values of λ, Σ_λ^- must be empty.

Note that for $y \in \Sigma_\lambda^-$, we have $u(y)^\tau \leqslant u_\lambda(y)^\tau$. Whenever $x, y \in \Sigma_\lambda$, we have that $|x - y| \leqslant |x^\lambda - y|$ and

$$|x - y|^{\mu-n} \geqslant |x^\lambda - y|^{\mu-n}.$$

Then by Lemma 19, for any $x \in \Sigma_\lambda^-$,

$$\begin{aligned} u(x) - u_\lambda(x) &= -\int_{\Sigma_\lambda} \left(|x-y|^{\mu-n} - |x^\lambda - y|^{\mu-n}\right)\left(u(y)^\tau - u_\lambda(y)^\tau\right) dy \\ &\leqslant \int_{\Sigma_\lambda^-} |x-y|^{\mu-n}\left(u_\lambda(y)^\tau - u(y)^\tau\right) dy \\ &\leqslant -\tau \int_{\Sigma_\lambda^-} |x-y|^{\mu-n}\left[u_\lambda^{\tau-1}(u - u_\lambda)\right](y)\, dy. \end{aligned} \tag{38}$$

It follows first from inequality (37) and the Hölder inequality that, for any $q > n/(n-\mu)$,

$$\begin{aligned} \|u_\lambda - u\|_{L^q(\Sigma_\lambda^-)} &\leqslant C \left\| \int_{\Sigma_\lambda^-} |x-y|^{\mu-n}\left[u_\lambda^{\tau-1}(u_\lambda - u)\right](y)\, dy \right\|_{L^q(\Sigma_\lambda^-)} \\ &\leqslant C \left(\int_{\Sigma_\lambda^-} u_\lambda(y)^{(\tau-1)\beta}\, dy \right)^{1/\beta} \|u_\lambda - u\|_{L^q(\Sigma_\lambda^-)} \\ &\leqslant C \left(\int_{\Sigma_\lambda^-} u_\lambda(y)^{(\tau-1)\beta}\, dy \right)^{1/\beta} \|u_\lambda - u\|_{L^q(\Sigma_\lambda^-)}. \end{aligned} \tag{39}$$

By condition (9), we can choose N sufficiently large, such that for $\lambda > N$, we have

$$C\left(\int_{\Sigma_\lambda} u_\lambda(y)^{(\tau-1)\beta}\, dy \right)^{1/\beta} \leqslant \frac{1}{2}.$$

Now (39) implies that

$$\|u_\lambda - u\|_{L^q(\Sigma_\lambda^-)} = 0,$$

and therefore Σ_λ^- must be measure zero, and hence empty. Then using the standard moving plane trick (see [6,7]), we know that the solution u is symmetric in the variable x_1.

So we complete the proof of Theorem 9. $\square$

Finally, we remark that in some cases, using the analysis of ODE, we see that there is no radially symmetric positive solution to (8) with the condition (9). However, we shall not discuss this here.

11. Proof of Theorem 10

Let us now introduce the Kelvin type transform of u as follows:

$$v(x) = |x|^{\mu-n} u\left(\frac{x}{|x|^2}\right)$$

for any $x \neq 0$. Then by elementary calculations, one can see that (8) is transformed into the following:

$$h|x|^{\mu-n} - v(x) = \int_{\mathbf{R}^n} |x-y|^{\mu-n} |y|^{-\alpha} v(y)^\tau \, dy, \tag{40}$$

where $\alpha = (n+\mu) - (n-\mu)\tau > 0$. Obviously, $v(x)$ has a singularity at origin. Since u is locally bounded, it is easy to see that $v(x)$ has no singularity at infinity, i.e., for any domain Ω that is a positive distance away from the origin,

$$\int_{\Omega} v^\beta(y)\, dy < \infty. \tag{41}$$

In fact, we have

$$\begin{aligned}
\int_{\Omega} v^\beta(y), dy &= \int_{\Omega} \left(|y|^{\mu-n} u\left(\frac{y}{|y|^2}\right)\right)^\beta dy \\
&= \int_{\Omega^\star} \left(|z|^{n-\mu} u(z)\right)^\beta |z|^{-2n}\, dz \\
&= \int_{\Omega^\star} |z|^{\beta(n-\mu)-2n} u(z)^\beta \, dz \\
&\leqslant C \int_{\Omega^\star} u(z)^\beta \, dz \\
&< \infty.
\end{aligned}$$

For the second equality, we have made the transform $y = z/|z|^2$. Since Ω is a positive distance away from the origin, $\Omega^\star$, the image of Ω under this transform, is bounded. Also note that $\beta(n-\mu) - 2n > 0$. Then we get the estimate (41).

For a given real number λ, we define, as before,

$$\Sigma_\lambda = \{x = (x_1, \dots, x_n) \mid x_1 \geqslant \lambda\},$$

and let $x^\lambda = (2\lambda - x_1, x_2, \dots, x_n)$ and $v_\lambda(x) = v(x^\lambda)$.

We can easily get the following lemma.

Lemma 20. *For any solution $v(x)$ of* (40)*, we have*

$$v_\lambda(x) - v(x) = h\left(\left|x^\lambda\right|^{\mu-n} - |x|^{\mu-n}\right) - \int\limits_{\Sigma_\lambda} \left(\frac{1}{|x-y|^{n-\mu}} - \frac{1}{|x^\lambda - y|^{n-\mu}}\right)\left(\frac{1}{|y^\lambda|^\alpha} v_\lambda(y)^\tau - \frac{1}{|y|^\alpha} v(y)^\tau\right) dy. \tag{42}$$

Proof. The proof is similar to Lemma 19. Let

$$\Sigma_\lambda^c = \{x = (x_1, \dots, x_n) | x_1 < \lambda\}.$$

Then it is easy to see that

$$\begin{aligned} h|x|^{\mu-n} - v(x) &= \int\limits_{\Sigma_\lambda} \frac{1}{|x-y|^{n-\mu}} \frac{1}{|y|^\alpha} v(y)^\tau\, dy + \int\limits_{\Sigma_\lambda^c} \frac{1}{|x-y|^{n-\mu}} \frac{1}{|y|^\alpha} v(y)^\tau\, dy \\ &= \int\limits_{\Sigma_\lambda} \frac{1}{|x-y|^{n-\mu}} \frac{1}{|y|^\alpha} v(y)^\tau\, dy + \int\limits_{\Sigma_\lambda} \frac{1}{|x-y^\lambda|^{n-\mu}} \frac{1}{|y^\lambda|^\alpha} v\left(y^\lambda\right)^\tau dy \\ &= \int\limits_{\Sigma_\lambda} \frac{1}{|x-y|^{n-\mu}} \frac{1}{|y|^\alpha} v(y)^\tau\, dy + \int\limits_{\Sigma_\lambda} \frac{1}{|x^\lambda-y|^{n-\mu}} \frac{1}{|y^\lambda|^\alpha} v_\lambda(y)^\tau\, dy. \end{aligned}$$

Here we have used the fact that $|x - y^\lambda| = |x^\lambda - y|$. Substituting x by x^λ, we get

$$\begin{aligned} h\left|x^\lambda\right|^{\mu-n} - v\left(x^\lambda\right) &= \int\limits_{\Sigma_\lambda} \frac{1}{|x^\lambda-y|^{n-\mu}} \frac{1}{|y|^\alpha} v(y)^\tau\, dy \\ &\quad + \int\limits_{\Sigma_\lambda} \frac{1}{|x-y|^{n-\mu}} \frac{1}{|y^\lambda|^\alpha} v_\lambda(y)^\tau\, dy. \end{aligned}$$

Thus

$$v(x) - v\left(x^\lambda\right) = h\left(\left|x^\lambda\right|^{\mu-n} - |x|^{\mu-n}\right) - \int\limits_{\Sigma_\lambda} \left(\frac{1}{|x-y|^{n-\mu}} - \frac{1}{|x^\lambda - y|^{n-\mu}}\right)\left(\frac{1}{|y^\lambda|^\alpha} v_\lambda(y)^\tau - \frac{1}{|y|^\alpha} v(y)^\tau\right) dy.$$

This implies (42). □

We shall also need the following doubly weighted Hardy–Littlewood–Sobolev inequality of Stein and Weiss (see, for example, [29])

$$\left\| \int V(x,y) f(y)\,dy \right\|_q \leqslant P_{\alpha,\beta,p,\nu,n} \|f\|_p \tag{43}$$

with $V(x,y) = |x|^{-\beta}|x-y|^{-\nu}|y|^{-\alpha}$, $0 \leqslant \alpha < n/p'$, $0 \leqslant \beta < n/q$, $1/p + 1/p' = 1$, and

$$1/p + (\nu + \alpha + \beta)/n = 1 + 1/q.$$

Proof of Theorem 10. Define

$$\Sigma_\lambda^- = \{x \mid x \in \Sigma_\lambda,\ v(x) < v_\lambda(x)\},$$

and

$$\Sigma_\lambda^+ = \{x \mid x \in \Sigma_\lambda,\ v(x) \geqslant v_\lambda(x)\}.$$

We want to show that for sufficiently negative values of λ, Σ_λ^- and Σ_λ^+ must be empty.

Whenever $x, y \in \Sigma_\lambda$, we have that $|x-y| \leqslant |x^\lambda - y|$. Moreover, since $\lambda < 0$, $|y^\lambda| \geqslant |y|$ for any $y \in \Sigma_\lambda$. Then by Lemma 20, for any $x \in \Sigma_\lambda^-$,

$$\begin{aligned} v_\lambda(x) - v(x) &\leqslant -\int_{\Sigma_\lambda} \left(|x-y|^{\mu-n} - \left|x^\lambda - y\right|^{\mu-n}\right)|y|^{-\alpha}\left(v_\lambda(y)^\tau - v(y)^\tau\right) dy \\ &\leqslant -\int_{\Sigma_\lambda^-} \left|x^\lambda - y\right|^{\mu-n}|y|^{-\alpha}\left(v_\lambda(y)^\tau - v(y)^\tau\right) dy \\ &\leqslant -\tau \int_{\Sigma_\lambda^-} |x-y|^{\mu-n}|y|^{-\alpha}\left[v^{\tau-1}(v_\lambda - v)\right](y)\,dy. \end{aligned} \tag{44}$$

It follows first from inequality (43) and then the Hölder inequality that, for any $q > n/(n-\mu)$, which will be used below in the form that $\tau < \mu$,

$$\begin{aligned} \|v_\lambda - v\|_{L^q(\Sigma_\lambda^-)} &\leqslant C \left\| \int_{\Sigma_\lambda^-} |x-y|^{\mu-n}|y|^{-\alpha}\left[v^{\tau-1}(v_\lambda - v)\right](y)\,dy \right\|_{L^p(\Sigma_\lambda^-)} \\ &\leqslant C \left(\int_{\Sigma_\lambda^-} v(y)^{(\tau-1)\beta} dy \right)^{1/\beta} \|v_\lambda - v\|_{L^q(\Sigma_\lambda^-)} \\ &\leqslant C \left(\int_{\Sigma_\lambda} v(y)^{(\tau-1)\beta}\,dy \right)^{1/\beta} \|v_\lambda - v\|_{L^q(\Sigma_\lambda^-)}. \end{aligned} \tag{45}$$

By condition (41), we can choose N sufficiently large, such that for $\lambda > N$, we have

$$C\left(\int_{\Sigma_\lambda} v(y)^{(\tau-1)\beta}\,dy\right)^{1/\beta} \leqslant \frac{1}{2}.$$

Now (45) implies that

$$\|v_\lambda - v\|_{L^q(\Sigma_\lambda^-)} = 0,$$

and therefore Σ_λ^- must be measure zero. Then we can use the moving plane trick as before to get that v is symmetric at zero with respect to the $x-1$ direction. This completes the proof of Theorem 10. □

12. Added in proof

The results of papers [4,5,9,10,14,21,28,32,33,36,38] are related to our work here. Another related work is the paper of A. Farina: On the classification of solutions of the Lane–Emden equation on unbounded domains of R^n, J. Math. Pures Appl. 87 (5) (May 2007) 537–561.

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We would like to thank Prof. C. Gui for pointing out that in our proof of Theorem 7, we can allow the kernel of E to have infinite dimension. We thank the referee(s) for helpful suggestions.

Appendix A. Regularity result for non-negative weak solution to Eq. (1)

We now discuss some regularity result for positive weak solutions to (1). Our first goal here is to get upper bound for positive weak solutions to (1) for any $\tau < 0$ by assuming a positive lower bound. As in the proof of Theorem 2, we take $R > \rho > 0$ and a cut-off function $\xi = \xi(|x|)$ such that $|\nabla\xi| \leqslant \frac{4}{R-\rho}$, and $\xi = 1$ on B_ρ. Then using $u(x) \geqslant 1$, we have as before that

$$\int u\Delta\big(|x|^2\xi\big) = \int u^\tau |x|^2\xi \leqslant \int |x|^2 \simeq R^{n+2}.$$

Using $\Delta|x|^2 = 2n$ we have

$$\int_{B_\rho} u \leqslant \frac{AR^2}{(R-\rho)^2}\int_{T_{R,\rho}} u + BR^{n+2},$$

where A, B are uniform constants and

$$T_{R,\rho} = B_R - B_\rho.$$

We can also derive some other interesting bound without the point-wise lower bound.

Take a constant $\sigma > 0$. Then we have

$$-\int \nabla u \nabla\big(u^\sigma \xi\big) = \int u^{\sigma+\tau}\xi.$$

Using integration by part, we know that the left-hand side is

$$-\int \nabla u \nabla\big(u^\sigma \xi\big) = -\sigma \int u^{\sigma-1}|\nabla u|^2\xi - \int u^\sigma \nabla u \nabla \xi.$$

Then

$$\sigma \int u^{\sigma-1}|\nabla u|^2\xi + \int u^{\sigma+\tau}\xi = -\int u^\sigma \nabla u \nabla \xi.$$

Note that

$$-\int u^\sigma \nabla u \nabla \xi = \frac{1}{1+\sigma}\int u^{1+\sigma}\Delta\xi \leqslant \frac{C}{(1+\sigma)(R-\rho)^2}\int_{T_{R,\rho}} u^{1+\sigma},$$

where $T_{R,\rho} = B_R - B_\rho$. Hence, we have

$$\sigma \int_{B_R} u^{\sigma-1}|\nabla u|^2 + \int_{B_R} u^{\sigma+\tau} \leqslant \frac{C}{(1+\sigma)(R-\rho)^2}\int_{T_{R,\rho}} u^{1+\sigma}. \tag{A.1}$$

Let us first consider two cases.

Case 1. If we choose $\sigma = -\tau$, then we get

$$-\tau \int_{B_R} u^{-\tau-1}|\nabla u|^2 + \big|B_1(0)\big|R^n \leqslant \frac{C}{(1-\tau)(R-\rho)^2}\int_{T_{R,\rho}} u^{1-\tau}.$$

Case 2. If we send $\sigma \to 0$ in (A.1), then we get

$$\int_{B_R} u^\tau \leqslant \frac{C}{(R-\rho)^2}\int_{T_{R,\rho}} u.$$

In the following, we do iteration. Let $\sigma = -\tau + p$ in (A.1). Then we have

$$(-\tau+p)\int_{B_R} u^{-\tau+p-1}|\nabla u|^2 + \int_{B_R} u^p \leqslant \frac{C}{(1-\tau+p)(R-\rho)^2}\int_{T_{R,\rho}} u^{1-\tau+p}. \tag{A.2}$$

This gives us that

$$\frac{4(-\tau+p)}{(-\tau+p+1)^2}\int_{B_R} \big|\nabla\big(u^{\frac{-\tau+p+1}{2}}\big)\big|^2 + \int_{B_R} u^p \leqslant \frac{C}{(1-\tau+p)(R-\rho)^2}\int_{T_{R,\rho}} u^{1-\tau+p}.$$

We now in the standard Nash–Moser iteration situation. Hence, for any $0 < q$, we have a uniform constant $C(q,n)$ such that

$$\sup_{B_{\theta R}} u \leqslant \frac{C(q,n)}{((1-\theta)R)^{n/q}} \left(\int_{B_R} u^q \right)^{1/q}.$$

Once we have an upper bound, we can use the standard Calderon–Zygmund L^p theory to conclude that u is a smooth solution.

Generally, it is a difficult topic to know how large the size of the zero set of the non-negative weak solutions to (1). One may see [23] and [19] for related results in this direction.

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Terme constant de fonctions sur un espace symétrique réductif p-adique

Constant term of functions on a p-adic reductive symmetric space

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Résumé

Nous établissons une généralisation de la dualité de Casselman aux espaces symétriques réductifs p-adiques et nous étudions le comportement asymptotique de certains coefficients généralisés. Nous prouvons aussi un analogue d'un Lemme de Langlands grâce auquel nous obtenons un résultat de disjonction de certaines parties de la décomposition de Cartan des espaces symétriques réductifs p-adiques.

Abstract

We generalize Casselman's pairing to p-adic reductive symmetric spaces and study the asymptotic behaviour of certain generalized coefficients. We also prove an analogue of a lemma due to Langlands which allows us to prove a disjunction result for the Cartan decomposition of the p-adic reductive symmetric spaces.

Mots-clés : Espace symétrique ; Groupe réductif p-adique ; Module de Jacquet

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Keywords: Symmetric space; p-Adic reductive group; Jacquet module

0. Introduction

0.1. Soit F un corps local non archimédien de caractéristique 0. Soit $G = \underline{G}(F)$, le groupe des points sur F d'un groupe algébrique réductif connexe, $\underline{G}$, défini sur F et soit σ une involution rationnelle définie sur F du groupe algébrique $\underline{G}$. On notera encore σ l'involution de G induite par σ. Soit $H = \underline{H}(F)$ le groupe des points sur F d'un sous-groupe ouvert au sens de Zariski du groupe des points fixes de σ. Le quotient du groupe G par H est appelé espace symétrique réductif p-adique.

La motivation de ce travail est l'analyse harmonique sur G/H. Noter que G lui-même peut être vu comme un espace symétrique en considérant l'involution de $G \times G$ donnée par l'inversion des facteurs.

Harish-Chandra a établi la formule de Plancherel pour les groupes réductifs réels (cf. [13]) et les groupes réductifs p-adiques (cf. [23]). La formule de Plancherel pour les espaces symétriques réductifs réels a été établie par deux méthodes différentes par E.P. van den Ban et H. Schlichtkrull d'une part et P. Delorme d'autre part (cf. [1] pour une présentation des deux méthodes).

Dans plusieurs cas particuliers, les fonctions sur G/H invariantes par un sous-groupe compact maximal ont été étudiées (cf. [16,17,22]). A.G. Helminck et G.F. Helminck ont obtenu des résultats de structure de G/H (cf. [14,15]). Pour l'analyse harmonique sur G/H, on s'intéresse aux triplets (π, V, ξ), où (π, V) est une représentation lisse admissible de G et $\xi \in V^{*H}$, où V^{*H} désigne l'espace vectoriel des vecteurs H-invariants du dual de V. P. Blanc et P. Delorme ont étudié le comportement de ces objets par l'induction parabolique (cf. [3]).

Enfin on dispose d'une décomposition de type Cartan, ou décomposition polaire, des espaces symétriques réductifs p-adiques (cf. [2,11]).

On considère dans un premier temps le comportement de ces objets sous les foncteurs de Jacquet, obtenant une généralisation de résultats de Casselman (cf. [9]), la preuve utilisant les résultats de celui-ci. L'expérience du cas réel conduit à s'interresser à des sous-groupes paraboliques P de G tels que P et $\bar{P} := \sigma(P)$ soient opposés, on les appelle σ-sous-groupes paraboliques. Alors $M = P \cap \sigma(P)$ est le sous-groupe de Levi σ-stable de P.

Si (π, V) est une représentation lisse admissible de G, si $\xi \in V^{*H}$ et si $v \in V$, on note $c_{\xi,v}$ le coefficient généralisé défini par $c_{\xi,v}(g) := \langle \pi^*(g)\xi, v\rangle$, $g \in G$, où (π^*, V^*) est la représentation $g \mapsto {}^t\pi(g^{-1})$ sur le dual algébrique V^* de V. On regarde $c_{\xi,v}$ comme une fonction sur G/H. Nos résultats, joints à la décomposition de Cartan, nous permettent notamment de démontrer que si π est bornée, i.e. telle que tous ses coefficients sont bornés, les coefficients généralisés $c_{\xi,v}$ sont bornés, ce qui est l'analogue d'un résultat de M. Flensted-Jensen, T. Oshima, H. Schlichtkrull pour les espaces symétriques réductifs réels (cf. [12]).

On établit l'analogue d'un Lemme de Langlands (cf. [4, Chapitre IV, Lemme 4.4]) pour les coefficients généralisés $c_{\xi,v}$, où (π, V) est une représentation induite à partir d'un σ-sous-groupe parabolique. Cela nous permet d'obtenir un résultat de disjonction de certaines parties de la décomposition de Cartan de G/H.

N.B. : Les résultats de ce travail ont été annoncés dans [21].

Indépendamment, S. Kato et K. Takano (cf. [19, Paragraphe 5]) ont démontré le Corollaire 1 du Théorème 1 et le Théorème 3 de notre travail. Nous remercions Jacques Carmona de nous l'avoir signalé.

0.2. On considère divers groupes algébriques définis sur F, et on utilisera des abus de terminologie du type suivant : « soit A un tore déployé » signifiera « soit A le groupe des points sur F d'un tore défini et déployé sur F ». Avec ces conventions, soit G un groupe linéaire algébrique réductif et connexe défini sur F. Soit A_0 un tore déployé maximal de G ; on note M_0 son centralisateur. Si P est un sous-groupe parabolique de G contenant A_0, il possède un unique sous-groupe de Levi contenant A_0, noté M. Son radical unipotent sera noté U. On note A_G le plus grand tore déployé dans le centre de G.

On note $X(G)$ le groupe des caractères non ramifiés de G et $X_*(G)$ l'ensemble des sous-groupes à un paramètre de A_G, qui est un groupe abélien libre de type fini. On fixe une fois pour toutes une uniformisante ϖ de F. On note alors $\Lambda(G)$ l'image de $X_*(G)$ dans G par le morphisme de groupes $\underline{\lambda} \mapsto \underline{\lambda}(\varpi)$, qui est isomorphe à $X_*(G)$ par ce morphisme. On adopte des notations similaires pour les sous-groupes de Levi de G.

Notons $\Sigma(A_M)$ l'ensemble des racines de A_M dans l'algèbre de Lie de G et $\Sigma(P)$ l'ensemble des racines de A_M dans l'algèbre de Lie de P et $\Delta(P)$ le sous-ensemble des racines simples de $\Sigma(P)$.

On reprend les notations et hypothèses du paragraphe 0.1 notamment pour σ et H.

Un tore déployé de G contenu dans $\{g \in G \mid \sigma(g) = g^{-1}\}$ sera dit σ-déployé, ((σ, F)-*split torus* dans [15]).

On fixe désormais un tore σ-déployé maximal, $A_\emptyset$, et on suppose A_0 choisi de telle sorte que A_0 soit un tore déployé σ-stable maximal contenant $A_\emptyset$ (cf. [15, Lemme 4.5 (iii)] pour l'existence). On note $(A_i)_{i\in I}$, un ensemble de représentants des classes de H-conjugaison de tores σ-déployés maximaux de G, qui est fini (cf. [15, 6.10 et 6.16]). On suppose que cet ensemble contient $A_\emptyset$. Les A_i sont tous conjugués sous G (cf. [14, Proposition 1.16]). On choisit, pour tout i dans I, un élément x_i de G, avec $x_i A_\emptyset x_i^{-1} = A_i$ en prenant $x_\emptyset = e$, où e est l'élément neutre de G.

On fixe $P_\emptyset$ un σ-sous-groupe parabolique minimal de G contenant $A_\emptyset$. Soit P un σ-sous-groupe parabolique de G contenant $P_\emptyset$. On note $\overline{W(A_\emptyset)}$ un ensemble de représentants du quotient $W(A_\emptyset)$ du normalisateur dans G de $A_\emptyset$ par son centralisateur dans G, noté $M_\emptyset$. On extrait de l'ensemble $\{x_i w \mid w \in \overline{W(A_\emptyset)}\}$ un ensemble de représentants $\overline{\mathcal{W}}_{M_\emptyset}^G$ (resp. $\overline{\mathcal{W}}_M^G$) de $(H, P_\emptyset)$-doubles classes ouvertes de G (resp. (H, P)-doubles classes ouvertes de G) avec $\overline{\mathcal{W}}_M^G \subset \overline{\mathcal{W}}_{M_\emptyset}^G$. Ces ensembles sont finis (cf. [15, Proposition 6.10, Corollaire 6.16]).

On note $X(M)_\sigma$ la composante neutre de l'ensemble des caractères de $X(M)$ anti-invariants par σ. On note δ_P le module de P, qui est un élément de $X(M)_\sigma$.

0.3. On considère $P = MU$ un σ-sous-groupe parabolique de G, de sous-groupe de Levi σ-stable M et de radical unipotent U. Pour tout nombre réel $\varepsilon > 0$, on note :

$$A_M^-(\varepsilon) := \{a \in A_M;\ |\alpha(a)|_F \leqslant \varepsilon,\ \alpha \in \Delta(P)\},$$

où $|.|_F$ est la valeur absolue normalisée de F. On pose $A_M^- := A_M^-(1)$.

Soit (π, V) une représentation lisse admissible de G, notons V_P le module de Jacquet de V relativement à P et $j_P : V \to V_P$ la projection naturelle. On munit V_P de la représentation lisse admissible π_P de M définie par $\pi_P(m) j_P(v) := \delta_P(m)^{-1/2} j_P(\pi(m)v)$ pour tout $m \in M$, $v \in V$.

Lemme 2. *Si K est un sous-groupe ouvert compact de G, il existe un sous-groupe ouvert compact K' de K possédant la propriété suivante :*

Pour toute représentation lisse admissible (π, V) de G, pour tout élément ξ de V^{*H} et pour tout $v \in V^K$, on a :

$$\langle \pi^*(k)\xi, \pi(a)v \rangle = \langle \xi, \pi(a)v \rangle, \quad a \in A_M^-,\ k \in K'.$$

Le Lemme précédent permet d'utiliser des résultats de Casselman (cf. [9, Théorème 4.2.4]) que nous étendons aux coefficients généralisés dans le Théorème suivant.

Soit (π, V) une représentation lisse admissible de G.

Théorème 1. *Soit $\xi \in V^{*H}$. Alors il existe un unique $j_P^*(\xi) \in (V_P)^{*M\cap H}$ vérifiant :*

Pour tout $v \in V$, il existe $\varepsilon > 0$, dépendant de v, tel que :

$$\delta_P(a)^{-1/2}\langle \xi, \pi(a)v \rangle = \langle j_P^*(\xi), \pi_P(a) j_P(v) \rangle, \quad a \in A_M^-(\varepsilon).$$

De plus, on peut choisir ε indépendamment de $\xi \in V^{*H}$.

On note $\Sigma(P_\emptyset, A_\emptyset)$ l'ensemble des racines de $A_\emptyset$ dans l'algèbre de Lie de $P_\emptyset$. On note $\Delta(P_\emptyset, A_\emptyset)$ l'ensemble des racines simples de $\Sigma(P_\emptyset, A_\emptyset)$. Soient $P = MU$ un σ-sous-groupe parabolique contenant $P_\emptyset$ et $\Delta(U, A_\emptyset)$ les racines de $A_\emptyset$ dans l'algèbre de Lie de U qui sont éléments de $\Delta(P_\emptyset, A_\emptyset)$. Pour $\varepsilon > 0$, soit $A_\emptyset^-(P, <\varepsilon)$ l'ensemble :

$$\{a \in A_\emptyset;\ |\alpha(a)|_F < \varepsilon,\ \alpha \in \Delta(U, A_\emptyset) \text{ et } |\alpha(a)|_F \leqslant 1,\ \alpha \in \Delta(P_\emptyset, A_\emptyset) \setminus \Delta(U, A_\emptyset)\}.$$

Le Théorème suivant est une extension aux coefficients généralisés du Théorème 4.3.3 de [9] pour les coefficients.

Théorème 2. *Pour tout $v \in V$, il existe $\varepsilon > 0$ tel que, pour tout $\xi \in V^{*H}$, on ait :*

$$\delta_P(a)^{-1/2}\langle \xi, \pi(a)v \rangle = \langle j_P^*(\xi), \pi_P(a) j_P(v) \rangle, \quad a \in A_\emptyset^-(P, <\varepsilon).$$

Fixons un plongement algébrique $\tau : G \to GL_n(F)$. On peut supposer, et l'on suppose, que $\tau(K_0) \subset GL_n(\mathcal{O})$ où $\mathcal{O}$ est l'anneau des entiers de F (cf. [23, I.1]). Pour $g \in G$, écrivons : $\tau(g) = (a_{i,j})_{i,j=1,\dots,n}$, $\tau(g^{-1}) = (b_{i,j})_{i,j=1,\dots,n}$, $\|g\| = \sup_{i,j} \sup(|a_{i,j}|_F, |b_{i,j}|_F)$ et :

$$\|gH\| := \|g\sigma(g^{-1})\|. \tag{0.1}$$

Théorème 4.

(i) *Soit (π, V) une représentation lisse, admissible et de type fini de G. Soit $\xi \in V^{*H}$. Il existe $c > 0$ tel que, pour tout $v \in V$, il existe $C_v > 0$ vérifiant :*

$$|\langle \pi^*(g)\xi, v \rangle| \leqslant C_v \|gH\|^c, \quad g \in G.$$

(ii) *Si (π, V) est une représentation lisse bornée irréductible de G et $\xi \in V^{*H}$, alors pour tout $v \in V$, la fonction $c_{\xi,v}$ est bornée.*

Remarque. Le point (i) permet de voir qu'une des hypothèses du Théorème 3 de [3] est toujours satisfaite.

Noter aussi que d'après un théorème de Howe, une représentation lisse est admissible de type fini si et seulement si elle est de longueur finie.

0.4. Soit P un σ-sous-groupe parabolique de G contenant $P_\emptyset$, soient M son sous-groupe de Levi σ-stable et U son radical unipotent. Soit (δ, V_δ) une représentation lisse, admissible, bornée et de type fini de M. On introduit, pour $\chi \in X(M)_\sigma$, la représentation $\delta_\chi = \delta \otimes \chi$ de M. L'espace de δ_χ s'identifie à V_δ. On étend l'action de M à P en la prenant triviale sur U. Soit $I_\chi^P(\delta)$ l'espace des applications $\varphi : G \to V_\delta$ qui sont invariantes à gauche par un sous-groupe compact ouvert et telles que :

$$\varphi(gmu) = \delta_P^{-1/2}(m)\delta_\chi\big(m^{-1}\big)\varphi(g), \quad g \in G,\ m \in M,\ u \in U.$$

Le groupe G agit par la représentation régulière gauche $\pi_{\delta,\chi}^P$ sur $I_\chi^P(\delta)$.

Si $x \in G$ et E est une partie de G, on note $x.E := xEx^{-1}$.

A tout $w \in \overline{\mathcal{W}}_M^G$, on associe l'espace : $\mathcal{V}(\delta, w) = V_\delta^{*M\cap w^{-1}.H}$. On considère la somme $\mathcal{V}(\delta) := \bigoplus_{w\in\overline{\mathcal{W}}_M^G} \mathcal{V}(\delta, w)$. La projection de $\mathcal{V}(\delta)$ sur $\mathcal{V}(\delta, w)$ parallèlement aux autres composantes sera notée $\mathrm{pr}(\delta, w)$ ou pr_w.

Soit $\chi \in X(M)_\sigma$ tel que $|\chi\ \delta_P^{-1/2}|$ soit strictement P-dominant. On associe à $\eta \in \mathcal{V}(\delta, e)$, la fonction $\varepsilon_e(P, \delta, \chi, \eta)$ définie sur G à valeurs dans V_δ^* par les relations :

(a) $\varepsilon_e(P, \delta, \chi, \eta) = 0$ en dehors de HP.
(b) Pour tout $(h, m, u) \in H \times M \times U$, on a :

$$\varepsilon_e(P, \delta, \chi, \eta)(hmu) = \delta_P^{-1/2}(m)\chi(m)\delta^*\big(m^{-1}\big)\eta.$$

Pour $w \in \overline{\mathcal{W}}_M^G$, $\eta \in \mathcal{V}(\delta, w)$, on définit également :

$$\varepsilon_w(P, \delta, \chi, \eta) = R_{w^{-1}}\varepsilon_e(w.P, w.\delta, w.\chi, \eta),$$

où R désigne la représentation régulière droite de G et $w.\delta$ (resp. $w.\chi$) la représentation de $w.M$ déduite de δ (resp. χ) par transport de structure. On définit enfin pour $\eta \in \mathcal{V}(\delta)$:

$$j(P, \delta, \chi, \eta) = \sum_{w\in\overline{\mathcal{W}}_M^G} \varepsilon_w\big(P, \delta, \chi, \mathrm{pr}(\delta, w)\eta\big).$$

On peut appliquer le Théorème 3 de [3] avec $r = 0$ grâce à notre Théorème 4 (ii). On en déduit que pour tout $v \in V_\delta$, l'application $g \mapsto \langle j(P, \delta, \chi, \eta)(g), v\rangle$ est continue sur G. Alors $\varphi \mapsto \int_K \langle j(P, \delta, \chi, \eta)(k), \varphi(k)\rangle\, dk$ définit une forme linéaire sur $I_\chi^P(\delta)$, invariante par H sous $(\pi_{\delta,\chi}^P)^*$, on la note encore $j(P, \delta, \chi, \eta)$.

On dispose des intégrales d'entrelacements $A(\bar{P}, P, \delta, \chi)$ qui entrelacent $I_\chi^P(\delta)$ et $I_\chi^{\bar{P}}(\delta)$ et définies par des intégrales convergentes pour $\chi \in X(M)$, tel que $|\chi\delta_P^{-R_\delta}|$ soit P-dominant pour $R_\delta > 0$ bien choisi (cf. [23, Théorème IV.1.1]).

Le symbole $a \to_{\bar{P}} \infty$ signifie que $a \in A_M$ et que $|\alpha(a)|_F \to +\infty$ pour tout $\alpha \in \Sigma(\bar{P})$.

Théorème 5. *On suppose de plus que (δ, V_δ) est irréductible. Soit $\chi \in X(M)_\sigma$ tel que $|\chi\delta_P^{-1/2}|$ et $|\chi\delta_P^{-R_\delta}|$ soient strictement P-dominants. Alors, pour tout $\varphi \in I_\chi^P(\delta)$, $g \in G$, $\eta \in \mathcal{V}(\delta)$:*

$$\lim_{a \to_{\bar{P}} \infty} \chi(a)\mu_\delta(a)\delta_P^{-1/2}(a)\big\langle\big(\pi_{\delta,\chi}^P\big)^*(ga)j(P,\delta,\chi,\eta),\varphi\big\rangle = \big\langle \mathrm{pr}_e\,\eta, \big(A(\bar{P},P,\delta,\chi)(\varphi)\big)(g)\big\rangle, \tag{0.2}$$

où μ_δ est le caractère central de δ.

On note $\Lambda_T^-(A_\emptyset) := \{\lambda \in \Lambda(A_\emptyset);\ |\alpha(\lambda)|_F \leqslant e^{-T},\ \alpha \in \Delta(P_\emptyset)\}$, où $T \geqslant 0$ et $\Lambda^-(A_\emptyset) := \Lambda_0^-(A_\emptyset)$.

La décomposition de Cartan (cf. [2,11]) donne l'existence d'une partie compacte Ω de G telle que :

$$G = \bigcup_{y \in \overline{\mathcal{W}}_{M_\emptyset}^G} \Omega\Lambda^-(A_\emptyset)y^{-1}H. \tag{0.3}$$

Théorème 7. *Il existe $T > 0$ tel que la réunion $\bigcup_{y \in \mathcal{W}_{M_\emptyset}^G} \Omega\Lambda_T^-(A_\emptyset)y^{-1}H$ soit disjointe.*

0.5. Dans le Paragraphe 1, on précise les notations et on fait quelques rappels, notamment sur les involutions rationnelles de G. Dans le Paragraphe 2, on établit les liens entre module de Jacquet et vecteur distribution H-invariant (Théorème 1), on définit le terme constant de fonctions sur G/H, on établit le Théorème 2 et enfin, on obtient une propriété de transitivité du terme constant dans le Théorème 3 et on introduit la notion de vecteurs distributions H-invariants cuspidaux. Dans le Paragraphe 3, on obtient des majorations qui nous conduisent à comparer la fonction $\|.\|$ à une autre fonction définie sur G/H par P. Blanc et P. Delorme (cf. [3]). On établit ensuite des majorations de certains coefficients généralisés de représentations admissibles de type fini dans le Théorème 4. Dans le Paragraphe 4, on obtient l'analogue d'un lemme de Langlands (Théorème 5) et on en déduit une propriété de disjonction dans la décomposition de Cartan des espaces symétriques réductifs p-adiques (Théorème 7).

1. Notations et rappels

1.1. Notations

On va utiliser largement des notations et conventions de [23]. Soit F un corps local non archimédien, de caractéristique 0. On considère divers groupes algébriques définis sur F, et on utilisera des abus de terminologie ou convention du type suivant :

« Soit A un tore déployé » signifiera « soit A le groupe des points sur F d'un tore $\underline{A}$ défini et déployé sur F ». (1.1)

Avec ces conventions, soit G un groupe linéaire algébrique réductif et connexe. Soit A_0 un sous-tore de G, déployé et maximal pour cette propriété, on note M_0 son centralisateur dans G. Si P

est un sous-groupe parabolique de G contenant A_0, il possède un unique sous-groupe de Levi contenant A_0, noté M (ou M_P). Son radical unipotent sera noté U ou U_P.

Si H est un groupe algébrique, on note $\mathrm{Rat}(H)$ le groupe des caractères algébriques de H définis sur F. Si E est un espace vectoriel, on note E^* son dual. S'il est réel, on note $E_{\mathbb{C}}$ son complexifié. On note A_G le plus grand tore déployé dans le centre de G.

On note $\mathfrak{a}_G = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Rat}(G), \mathbb{R})$. La restriction des caractères rationnels de G à A_G induit un isomorphisme :

$$\mathrm{Rat}(G) \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathrm{Rat}(A_G) \otimes_{\mathbb{Z}} \mathbb{R}. \tag{1.2}$$

On dispose de l'application canonique, $H_G : G \to \mathfrak{a}_G$ définie par :

$$e^{\langle H_G(x), \chi\rangle} = \left|\chi(x)\right|_F, \quad x \in G, \chi \in \mathrm{Rat}(G) \tag{1.3}$$

où $|.|_F$ est la valeur absolue normalisée de F. Le noyau de H_G, qui est noté G^1, est l'intersection des noyaux des caractères de G de la forme $|\chi|_F$, $\chi \in \mathrm{Rat}(G)$. On notera $X(G) = \mathrm{Hom}(G/G^1, \mathbb{C}^*)$. On a des notations similaires pour les sous-groupes de Levi.

Si P est un sous-groupe parabolique de G contenant A_0, on notera $\mathfrak{a}_P = \mathfrak{a}_{M_P}$, $H_P = H_{M_P}$. On note $\mathfrak{a}_0 = \mathfrak{a}_{M_0}$, $H_0 = H_{M_0}$. On note $\mathfrak{a}_{G,F}$, resp. $\tilde{\mathfrak{a}}_{G,F}$ l'image de G, resp. A_G, par H_G. Alors G/G^1 est isomorphe au réseau $\mathfrak{a}_{G,F}$. Soit M un sous-groupe de Levi contenant A_0. Les inclusions $A_G \subset A_M \subset M \subset G$, déterminent un morphisme de groupe surjectif $\mathfrak{a}_{M,F} \to \mathfrak{a}_{G,F}$, et un morphisme injectif $\tilde{\mathfrak{a}}_{G,F} \to \tilde{\mathfrak{a}}_{M,F}$. Le premier (resp. le second) se prolonge de manière unique en une application linéaire surjective entre $\mathfrak{a}_M$ et $\mathfrak{a}_G$ (resp. injective entre $\mathfrak{a}_G$ et $\mathfrak{a}_M$). La deuxième application permet d'identifier $\mathfrak{a}_G$ à un sous-espace de $\mathfrak{a}_M$ et le noyau de la première, noté $\mathfrak{a}_M^G$, vérifie :

$$\mathfrak{a}_M = \mathfrak{a}_M^G \oplus \mathfrak{a}_G. \tag{1.4}$$

Il y a une surjection :

$$\left(\mathfrak{a}_G^*\right)_{\mathbb{C}} \to X(G) \to 1 \tag{1.5}$$

qui associe à $\chi \otimes s$, le caractère $g \mapsto |\chi(g)|^s$ (cf. [23, I.1(1)]). Le noyau est un réseau et cela définit sur $X(G)$ une structure de variété algébrique complexe pour laquelle $X(G)$ est de dimension $\dim_{\mathbb{R}} \mathfrak{a}_G$. Pour $\chi \in X(G)$, soit $\nu \in (\mathfrak{a}_G^*)_{\mathbb{C}}$ un élément se projetant sur χ par l'application (1.5). La partie réelle $\mathrm{Re}\,\nu \in \mathfrak{a}_G^*$ est indépendante du choix de ν. Nous la noterons $\mathrm{Re}\,\chi$. Si $\chi \in \mathrm{Hom}(G, \mathbb{C}^*)$, le caractère $|\chi|$ appartient à $X(G)$. On pose $\mathrm{Re}\,\chi = \mathrm{Re}\,|\chi|$. De même, si $\chi \in \mathrm{Hom}(A_G, \mathbb{C}^*)$ est continu, le caractère $|\chi|$ se prolonge de façon unique en un élément de $X(G)$ à valeurs dans $\mathbb{R}^{*+}$, que l'on note encore $|\chi|$ et on pose $\mathrm{Re}\,\chi = \mathrm{Re}\,|\chi|$.

De l'isomorphisme naturel (1.2) on déduit aisément l'égalité :

$$A_G^1 = A_G \cap G^1. \tag{1.6}$$

Alors A_G^1 est le plus grand sous-groupe compact de A_G.

On note $X_*(G)$ ou $X_*(A_G)$ l'ensemble des sous-groupes à un paramètre de A_G. C'est un groupe abélien libre de type fini. On fixe une fois pour toute une uniformisante ϖ de F. On

note alors $\Lambda(G)$, l'image de $X_*(G)$ dans G par le morphisme de groupes $\underline{\lambda} \mapsto \underline{\lambda}(\varpi)$, qui est isomorphe à $X_*(G)$ par ce morphisme.

Notons $\Sigma(A_M)$ l'ensemble des racines de A_M dans l'algèbre de Lie de G, qui s'identifie à un sous-ensemble de $\mathfrak{a}_M^*$, $\Sigma(P)$ l'ensemble des racines de A_M dans l'algèbre de Lie de P et $\Delta(P)$ le sous-ensemble des racines simples de $\Sigma(P)$.

On note W^G le groupe de Weyl de G relativement à A_0, qui agit sur $\mathfrak{a}_0$. On choisit un produit scalaire sur $\mathfrak{a}_0$ invariant par W^G. On le notera $(.|.)$, et $|.|$ la norme qu'on en déduit. (1.7)

On note :

$$
\begin{aligned}
{}^+\mathfrak{a}_P^* \ \big(\text{resp. } {}^+\bar{\mathfrak{a}}_P^*\big) &= \Big\{\nu \in \mathfrak{a}_M^* \ \Big|\ \nu = \sum_{\alpha\in\Delta(P)} x_\alpha \alpha \text{ où } x_\alpha > 0 \text{ (resp. } x_\alpha \geqslant 0)\Big\},\\
\mathfrak{a}_P^{*+} \ \big(\text{resp. } \bar{\mathfrak{a}}_P^{*+}\big) &= \big\{\nu \in \mathfrak{a}_M^* \ \big|\ (\nu|\alpha) > 0 \text{ (resp. } \geqslant 0),\ \alpha \in \Sigma(P)\big\}.
\end{aligned} \tag{1.8}
$$

Comme dans [23, I.1], on fixe K_0 un sous-groupe compact maximal de G qui est le fixateur d'un point spécial de l'appartement associé à A_0 dans l'immeuble de G.

Soit P_0 un sous-groupe parabolique minimal de G contenant A_0.

On note :

$$
\begin{aligned}
&\bar{\mathfrak{a}}_P^+ \text{ (resp. } \bar{\mathfrak{a}}_P^-) := \big\{X \in \mathfrak{a}_P \ \big|\ \langle \alpha, X\rangle \geqslant 0 \text{ (resp. } \leqslant 0),\ \alpha \in \Delta(P)\big\}.\\
&\text{On écrira } \bar{\mathfrak{a}}_0^+ \text{ (resp. } \bar{\mathfrak{a}}_0^-) \text{ au lieu de } \bar{\mathfrak{a}}_{P_0}^+ \text{ (resp. } \bar{\mathfrak{a}}_{P_0}^-),\\
&\quad {}^+\bar{\mathfrak{a}}_P := \big\{X \in \mathfrak{a}_P \ \big|\ \langle \nu, X\rangle \geqslant 0,\ \nu \in \mathfrak{a}_P^{*+}\big\},\\
&\quad \overline{M}_0^+ := H_{M_0}^{-1}\big(\bar{\mathfrak{a}}_0^+\big) \text{ et } \overline{M}_0^- := H_{M_0}^{-1}\big(\bar{\mathfrak{a}}_0^-\big).
\end{aligned} \tag{1.9}
$$

Si N est un sous-groupe fermé de G, on note dn une mesure de Haar invariante à gauche sur N. Si N et N' sont deux sous-groupes fermés de G tels que $N' \subset N$, on notera $d\dot{n}$, si elle existe, une mesure sur N/N', invariante à gauche par N, positive et non nulle. C'est le cas si N et N' sont unimodulaires.

Soit P un sous-groupe parabolique de G contenant A_0, soit M son sous-groupe de Levi contenant A_0 et soit $\bar{P} = M\bar{U}$ le sous-groupe parabolique de G opposé à P relativement à M.

Soit P un sous-groupe parabolique de G de sous-groupe de Levi M. On note ρ_P la demi-somme des racines de A_M dans l'algèbre de Lie de P, on note δ_P l'élément de $X(M)_\sigma$ tel que l'on ait :

$$\delta_P(m) = e^{2\rho_P(H_M(m))}, \quad m \in M.$$

On note $C(G, P, -2\rho_P)$ l'espace des fonctions continues $f : G \to \mathbb{C}$ telles que :

$$f(gmu) = e^{-2\rho_P(H_M(m))} f(g), \quad g \in G,\ m \in M,\ u \in U.$$

On remarque que si $f \in C(G, P, -2\rho_P)$, alors f est invariante à droite par $K_0 \cap P$. En raisonnant comme dans la preuve de la Conséquence 7 de la Proposition 5.26 de [20], et en remplaçant $K \cap M$ par $K_0 \cap P$, on montre que, pour une bonne normalisation des mesures :

Pour toute fonction f de $C(G, P, -2\rho_P)$, l'intégrale $\int_{\bar{U}} f(\bar{u}) d\bar{u}$ est absolument convergente et :

$$\int_{K_0} f(k)\,dk = \int_{\bar{U}} f(\bar{u})\,d\bar{u}. \tag{1.10}$$

Il en résulte que la forme linéaire $\mathcal{M}$ sur $C(G, P, -2\rho_P)$ définie par :

$$\mathcal{M}(f) := \int_{K_0} f(k)\,dk, \quad f \in C(G, P, -2\rho_P)$$

est invariante par les translations à gauche par les éléments de K_0, de $\bar{U}$ ainsi que ceux de M. Donc :

La forme linéaire $\mathcal{M}$ sur $C(G, P, -2\rho_P)$ est invariante par les translations à gauche par les éléments de G. (1.11)

1.2. Involutions rationnelles de G

On utilisera parfois de façon implicite les deux faits suivants. Avec nos hypothèses sur F, on a (cf. [18, Théorème 34.4 (d)]) :

Si L est le groupe des points sur F d'un groupe algébrique réductif $\underline{L}$ défini sur F, alors L est Zariski dense dans $\underline{L}$. (1.12)

Il résulte facilement de ceci et du Théorème 34.4 (c) de [18], que :

Si L est comme ci-dessus et A le groupe des points sur F d'un tore déployé $\underline{A}$ de $\underline{L}$, alors le centralisateur de $\underline{A}$ est un groupe réductif défini sur F dont le groupe des points sur F est égal au centralisateur de A dans L. (1.13)

Soit σ une involution rationnelle, définie sur F, du groupe algébrique dont G est le groupe des points sur F. Soit H le groupe des points sur F d'un sous-groupe ouvert, défini sur F, du groupe des points fixes de σ.

Un tore déployé de G contenu dans $\{g \in G \mid \sigma(g) = g^{-1}\}$ sera dit σ-déployé, ((σ, F)*-split torus* dans [14]). On dira que P est un σ-sous-groupe parabolique de G si P est un sous-groupe parabolique de G tel que P et $\sigma(P)$ soient opposés, c'est à dire tel que $P \cap \sigma(P)$ soit un sous-groupe de Levi de P. C'est alors le sous-groupe de Levi σ-stable de P : en effet, tout sous-groupe σ-stable de P est inclus dans $P \cap \sigma(P)$ qui est σ-stable. On utilisera la convention suivante :

La phrase : « Soit $P = MU$ un σ-sous-groupe parabolique de G » signifiera que P est un σ-sous-groupe parabolique de G, que M est son sous-groupe de Levi σ-stable (i.e. $M = P \cap \sigma(P)$) et que U est son radical unipotent. On notera $\bar{P} = M\bar{U}$ le sous-groupe parabolique $\sigma(P)$ qui est opposé à P relativement à M. (1.14)

D'après [14, Corollaire 6.16], si P_0 est un sous-groupe parabolique minimal de G, le nombre de (H, P_0)-doubles classes est fini, donc aussi le nombre de (H, P)-doubles classes pour tout sous-groupe parabolique P de G.

On a (cf. [15, l'équivalence de (i) et (iv) de la Proposition 4.7 et Lemme 4.5]) :

> Si $P_\emptyset$ est un σ-sous-groupe parabolique minimal, son sous-groupe de Levi σ-stable, $M_\emptyset$, contient un unique tore σ-déployé maximal de G, $A_\emptyset$, et $M_\emptyset = Z_G(A_\emptyset)$. (1.15)

On fixe désormais $A_\emptyset$ un tore σ-déployé maximal de G et on note $M_\emptyset$ son centralisateur. On fixe A_0 un tore déployé maximal de $M_\emptyset$. Alors (cf. [15, Lemme 4.5 (i)]), A_0 est σ-stable et c'est un tore déployé maximal de G. Donc :

> Le tore A_0 est un tore déployé maximal σ-stable de G contenant $A_\emptyset$. (1.16)

On fixe aussi $P_\emptyset$ un σ-sous-groupe parabolique minimal contenant $A_\emptyset$. On note $(A_i)_{i\in I}$, un ensemble de représentants des classes de H-conjugaison de tores σ-déployés maximaux de G. On suppose que cet ensemble contient $A_\emptyset$. Les A_i sont tous conjugués sous G (cf. [14, Proposition 1.16]).

On choisit, pour tout i dans I, un élément x_i de G, avec $x_i A_\emptyset x_i^{-1} = A_i$ en prenant $x_\emptyset = e$. On note $\mathcal{P}_i$ l'ensemble des σ-sous-groupes paraboliques minimaux de G contenant A_i, qui est fini, et les éléments de $\mathcal{P}_i$ sont tous conjugués entre eux par un élément du normalisateur de A_i (cf. [14, Proposition 2.7]). Comme les A_i sont conjugués entre eux, tous les éléments de $\mathcal{P}_i$ sont conjugués sous G à $P_\emptyset$ et à $P_i := x_i P_\emptyset x_i^{-1}$.

On note M_i le centralisateur dans G de A_i. Si L est un sous-groupe de G, on note $W_L(A_i)$ le quotient du normalisateur dans L de A_i par son centralisateur. On note $W(A_i)$ au lieu de $W_G(A_i)$.

On note $\overline{W}_i$ un ensemble de représentants dans $N_G(A_\emptyset)$ de $W_{H_i}(A_\emptyset) \setminus W(A_\emptyset)$ où $H_i = x_i^{-1} H x_i$. Soit $\mathcal{W}_i^G$ l'ensemble $\{x_i x \mid x \in \overline{W}_i\}$ et $\mathcal{W}_{M_\emptyset}^G = \bigcup_{i\in I} \mathcal{W}_i^G$. Alors (cf. [14, Théorème 3.1]) :

> $\mathcal{W}_{M_\emptyset}^G$ forme un ensemble de représentants des $(H, P_\emptyset)$-doubles classes ouvertes dans G. (1.17)

En particulier, comme l'ensemble des $(H, P_\emptyset)$-doubles classes est fini (cf. [15, Proposition 6.10 et Corollaire 6.16]), on voit que I est fini.

Soit $P = MU$ un σ-sous-groupe parabolique de G. On remarque que A_M, le plus grand tore déployé du centre de M, est invariant par σ, donc σ agit naturellement sur $\mathfrak{a}_M$.

Si $x \in G$ et E est une partie de G, on note $x.E := xEx^{-1}$ et si f est une application définie sur E, on note $x.f$ l'application définie sur $x.E$ par : $x.f(xyx^{-1}) = f(y)$, pour $y \in E$.

On note $X(M)_\sigma$ (resp. $X_\mathbb{R}(M)_\sigma$) l'ensemble des caractères non ramifiés de M qui sont l'image par l'application de (1.5) (pour M au lieu de G), de l'ensemble des éléments de $(\mathfrak{a}_M^*)_\mathbb{C}$ (resp. $\mathfrak{a}_M^*$) anti-invariants par σ. Alors $X(M)_\sigma$ est la composante neutre de l'ensemble des caractères de $X(M)$ anti-invariants par σ. On remarque de plus que :

$$\text{Si } \chi \in X(M)_\sigma, \quad \text{alors } \chi(h) = 1, \quad h \in M \cap H, \tag{1.18}$$

En effet :

$$\chi \circ \sigma(h) = \chi(h)^{-1}, \quad h \in M \cap H.$$

On en déduit que :

$$\chi(h)^2 = 1, \quad h \in M \cap H.$$

Alors, pour $h \in M \cap H$ fixé, l'application continue sur $X(M)_\sigma$ définie par $\chi \mapsto \chi(h)$ est à valeurs dans $\{-1, 1\}$. Par connexité, on en déduit que $\chi(h) = 1$, d'où (1.18).

On fixe P_0 un sous-groupe parabolique minimal de G contenant A_0 et contenu dans $P_\emptyset$. (1.19)

On pose :

$$\begin{aligned} \Lambda_T^-(A_\emptyset) &:= \bigl\{\lambda \in \Lambda(A_\emptyset);\ \bigl|\alpha(\lambda)\bigr|_F \leqslant e^{-T},\ \alpha \in \Delta(P_\emptyset)\bigr\}, \quad \text{où } T \geqslant 0 \quad \text{et} \\ \Lambda^-(A_\emptyset) &:= \Lambda_0^-(A_\emptyset). \end{aligned} \tag{1.20}$$

La décomposition de Cartan (cf. [2, Théorème 1.1] et [11, Théorème 0.1]) donne l'existence d'un ensemble compact Ω tel que :

$$G = \bigcup_{y \in \mathcal{W}_{M_\emptyset}^G} \Omega \Lambda^-(A_\emptyset) y^{-1} H. \tag{1.21}$$

Comme A_0 est σ-invariant, σ agit naturellement sur $\mathfrak{a}_0$. On définit $\mathfrak{a}_\emptyset$ le sous-espace des éléments anti-invariants de $\mathfrak{a}_0$.

Pour des raisons de dimension, $\mathfrak{a}_\emptyset$ est égal au sous-espace vectoriel de $\mathfrak{a}_0$ engendré par $H_{M_\emptyset}(A_\emptyset)$. (1.22)

Le groupe des automorphismes de $\mathfrak{a}_0$ engendré par σ et le groupe de Weyl de G relativement à A_0, W^G, est fini car σ préserve A_0 et donc $N_G(A_0)$. En conséquence, le produit scalaire sur $\mathfrak{a}_0$ introduit en (1.7) peut être supposé également σ-invariant.

Soit P un σ-sous-groupe parabolique de G contenant $P_\emptyset$ et M son sous-groupe de Levi σ-stable. On note $A_{G,\sigma}$ (resp. $A_{M,\sigma}$) le plus grand tore σ-déployé de A_G (resp. de A_M), $\mathfrak{a}_G^\sigma$ (resp. $\mathfrak{a}_{G,\sigma}$) l'ensemble des points fixes (resp. anti-invariants) de $\mathfrak{a}_G$ sous σ.

On note p_σ la projection de $\mathfrak{a}_G$ sur $\mathfrak{a}_{G,\sigma}$ parallèlement à $\mathfrak{a}_G^\sigma$ et $H_{G,\sigma}$ la composée $p_\sigma \circ H_G$.

1.3. Représentations

1.3.1. Représentations lisses

Soit (π, V) une représentation de G sur un espace vectoriel V complexe. Si K est un sous-groupe de G, on note V^K l'espace des vecteurs de V invariants sous $\pi(K)$. On dit qu'une représentation (π, V) est lisse si tout élément v de V appartient à V^K pour un sous-groupe compact ouvert K. On dit qu'elle est admissible si elle est lisse et si V^K est de dimension finie pour tout sous-groupe compact ouvert K. On dira qu'une représentation lisse est bornée si tous ses coefficients lisses son bornés.

Une fonction de G dans $\mathbb{C}$ bi-invariante par un sous-groupe compact ouvert sera dite lisse. On note $C_c^\infty(G)$ l'espace des fonctions lisses à support compact qui est aussi l'espace des fonctions localement constantes à support compact.

On note R (resp. L) la représentation régulière droite (resp. gauche) de G sur $C^\infty(G)$.

Si (π, V) est une représentation lisse de G, on définit sa représentation duale (π^*, V^*) par la représentation $g \mapsto {}^t\pi(g^{-1})$ sur le dual algébrique V^* de V et on définit sa contragrédiente lisse $(\check{\pi}, \check{V})$ par la restriction de π^* au sous espace $\check{V}$ des éléments de V^* fixés par un sous-groupe ouvert compact.

> Si (π, V) est une représentation admissible de G et K un sous-groupe ouvert compact de G, on définit l'opérateur $\pi(e_K)$ par la formule :
>
> $$\pi(e_K)v := \int_K \pi(k)v\,dk$$
>
> où $v \in V$ et dk est la mesure de Haar normalisée de K. (1.23)

Comme v est fixé par un sous-groupe ouvert compact, cette intégrale est une somme finie.

Avec les mêmes hypothèses, pour un élément ξ de V^{*H}, on définit l'élément $\pi^*(e_K)\xi$ de $V^{*K} \subset \check{V}$ par :

$$\langle \pi^*(e_K)\xi, v\rangle := \langle \xi, \pi(e_K)v\rangle, \quad v \in V. \tag{1.24}$$

1.3.2. Représentations rationnelles

Soit Λ un élément de $\mathrm{Rat}(M_0)$, on appelle représentation de plus haut poids Λ relativement à P_0, une représentation rationnelle de G, définie sur F, de dimension finie, (π_Λ, V_Λ), irréductible et possédant un vecteur non nul v_Λ, dit de plus haut poids Λ, invariant par le radical unipotent U_0 de P_0 et se transformant par Λ sous M_0.

Une telle représentation, si elle existe, est unique à isomorphisme près (cf. [18, Théorème 31.3 (c)]). Il résulte du Théorème 31.3 (b) de [18] que :

> Pour tout élément P_0-dominant, a de A_0, la plus grande des valuations des valeurs propres de $\pi_\Lambda(a)$ est égale à $|\Lambda(a)|_F$. (1.25)

Toujours d'après ce Théorème 31.3, on a :

> Si (π_Λ, V_Λ) est une représentation de plus haut poids Λ relativement à P_0, alors sa représentation contragrédiente $(\pi_\Lambda^*, V_\Lambda^*)$ est une représentation de plus haut poids Λ^{-1} relativement à $\bar{P}_0$. (1.26)

2. Modules de Jacquet et vecteurs-distributions H-invariants

2.1. Soit $P = MU$ un σ-sous-groupe parabolique de G fixé pour ce Chapitre 2.

On associe à $\varepsilon > 0$ l'ensemble :

$$A_M^-(\varepsilon) := \{a \in A_M;\ |\alpha(a)|_F \leqslant \varepsilon,\ \alpha \in \Delta(P)\},$$

Si l'on veut expliciter la dépendance par rapport à P, on notera $A_M(P, \leqslant\varepsilon)$ au lieu de $A_M^-(\varepsilon)$.

On notera $A_M^- := A_M^-(1)$, $A_M^+(\varepsilon) := (A_M^-(\varepsilon))^{-1}$ et $A_M^+ := (A_M^-)^{-1}$.

On dit qu'un sous-groupe ouvert compact K de G admet une factorisation d'Iwahori par rapport à P s'il vérifie les deux conditions suivantes (cf. [9, 1.4]) :

(a) l'application produit de $K_{\bar{U}} \times K_M \times K_U$ dans K est une bijection, où $K_{\bar{U}} = K \cap \bar{U}$, $K_M = K \cap M$ et $K_U = K \cap U$;
(b) pour tout $a \in A_M^-$, $aK_U a^{-1} \subseteq K_U$, $a^{-1} K_{\bar{U}} a \subseteq K_{\bar{U}}$. (2.1)

Tout sous-groupe compact ouvert contient un sous-groupe compact ouvert admettant une factorisation d'Iwahori (cf. [9, Proposition1.4.4]). (2.2)

Nous remercions Joseph Bernstein pour nous avoir fourni la démonstration du Lemme suivant :

Lemme 1. *Soit K un sous-groupe ouvert compact admettant une factorisation d'Iwahori par rapport à P : $K = K_{\bar{U}} K_M K_U$. Alors il existe un sous-groupe ouvert compact K' de G, contenu dans $K \cap K_{\bar{U}} K_M H$.*

Démonstration. Avec la convention (1.1), si G est un groupe algébrique, alors G est un groupe de Lie sur F au sens de Bourbaki (cf. [6, Chapitre III, Paragraphe 1, Définition 1]). On note $\mathfrak{g}$ son algèbre de Lie.

La différentielle en (e, e) de la fonction analytique :

$$\begin{aligned} p : \bar{P} \times H &\to G \\ (\bar{p}, h) &\to \bar{p}h \end{aligned}$$

est l'application :

$$\begin{aligned} \bar{\mathfrak{p}} \times \mathfrak{h} &\to \mathfrak{g} \\ (X, Y) &\to X + Y. \end{aligned}$$

Or $\bar{\mathfrak{p}} + \mathfrak{h} = \mathfrak{g}$ (cf. [3, début du Paragraphe 2.4]), elle est donc surjective. Les propriétés des submersions (cf. [5, 5.9.1 à 5.9.4]) nous donnent alors l'existence de deux voisinages de e, l'un, V_1, dans $\bar{P}$ que l'on peut prendre dans $K_{\bar{U}} K_M$ et l'autre, V_2, dans H tels que l'image de $V_1 \times V_2$ par p contienne un voisinage de e. Quitte à restreindre, on peut supposer ce dernier contenu dans K et que c'est un sous-groupe compact ouvert. On le note K'. $\square$

Lemme 2. *Si K est un sous-groupe ouvert compact de G, il existe un sous-groupe ouvert compact K' de K possédant la propriété suivante* :

Pour toute représentation lisse (π, V) de G, pour tout élément ξ de V^{*H} et pour tout $v \in V^K$,

$$\langle \pi^*(ak)\xi, v \rangle = \langle \pi^*(a)\xi, v \rangle, \quad a \in A_M^+,\ k \in K'.$$

En particulier, pour tout sous-groupe compact ouvert $K'' \subset K'$:

$$\langle \pi^*(a)\xi, v \rangle = \langle \pi^*(a)\pi^*(e_{K''})\xi, v \rangle, \quad a \in A_M^+ \tag{2.3}$$

(*cf.* (1.23) *et* (1.24) *pour les définitions de* $\pi(e_{K''})v$ *et de* $\pi^*(e_{K''})\xi$).

Démonstration. D'après (2.2), on se ramène au cas où K admet une factorisation d'Iwahori par rapport à P, on a alors $K = K_{\bar{U}} K_M K_U$. D'après le Lemme 1, il existe un sous-groupe ouvert compact K' de G contenu dans $K \cap K_{\bar{U}} K_M H$. Soit $k \in K'$, et soient $k_{\bar{U}} \in K_{\bar{U}}$, $k_M \in K_M$ et $h \in H$ tels que $k = k_{\bar{U}} k_M h$. Puisque $\xi \in V^{*H}$, on a :

$$\langle \pi^*(ak)\xi, v \rangle = \langle \pi^*\big(ak_{\bar{U}}a^{-1}ak_Ma^{-1}a\big)\xi, v \rangle, \quad a \in A_M^+.$$

De plus $ak_Ma^{-1} = k_M \in K_M$ et $ak_{\bar{U}}a^{-1} \in K_{\bar{U}}$ car $a \in A_M^+$ donc $ak_Ma^{-1}ak_{\bar{U}}a^{-1} \in K$. Or $v \in V^K$, donc :

$$\langle \pi^*(ak)\xi, v \rangle = \langle \pi^*(a)\xi, v \rangle, \quad a \in A_M^+. \qquad \square$$

On déduit du Lemme 12 de l'appendice que :

> Soit $P = MU$ un σ-sous-groupe parabolique de G. Si φ est une fonction de A_M dans $\mathbb{C}$, lisse et A_M-finie, elle est entièrement déterminée par sa restriction à $A_M^-(\varepsilon)$ pour un $\varepsilon > 0$.
> Si de plus $A_\emptyset \subset M$ et si φ est une fonction définie sur $A_M \cap A_\emptyset$, lisse et $A_M \cap A_\emptyset$-finie, alors elle est déterminée par sa restriction à $A_M^-(\varepsilon) \cap A_\emptyset$ pour un $\varepsilon > 0$. (2.4)

Si (π, V) est une représentation admissible de G, tout vecteur est A_G-fini. En effet, si v est élément de V, il existe un sous-groupe ouvert compact K de G tel que v soit élément de V^K. Alors, pour tout élément a de A_G, $\pi(a)v$ est élément de V^K et comme (π, V) est admissible, la dimension de V^K est finie.

Soit (π, V) une représentation admissible de G, notons (π_P, V_P) le module de Jacquet de V relativement à P, où $V_P := V/V(P)$, avec $V(P) := \langle \pi(u)v - v,\ v \in V,\ u \in U \rangle$, et $j_P : V \to V_P$ la projection naturelle. On munit V_P de la représentation π_P de M définie par $\pi_P(m)j_P(v) := \delta_P(m)^{-1/2} j_P(\pi(m)v)$ pour tout $m \in M,\ v \in V$. Elle est admissible (cf. [9, Théorème 3.3.1]), donc, d'après ce qui précède :

> Tout vecteur $v_P \in V_P$ est A_M-fini. (2.5)

Soit $(\check{\pi}, \check{V})$ la contragrédiente de (π, V), on note $((\check{\pi})_{\bar{P}}, (\check{V})_{\bar{P}})$ le module de Jacquet de $\check{V}$ relativement à $\bar{P}$ et $\check{j}_{\bar{P}} : \check{V} \to (\check{V})_{\bar{P}}$ la projection naturelle. Alors (cf. [9], voir aussi [23, Théorème I.4.1]), il existe un crochet de dualité canonique entre $\check{V}_{\bar{P}}$ et V_P noté $\langle .,. \rangle_P$ qui vérifie :

> Pour tout $(v, \check{v}) \in V \times \check{V}$, il existe $\varepsilon > 0$ tel que :
> $$\big\langle \check{j}_{\bar{P}}(\check{v}), \pi_P(a) j_P(v) \big\rangle_P = \delta_P(a)^{-1/2} \langle \check{v}, \pi(a)v \rangle, \quad a \in A_M^-(\varepsilon). \qquad (2.6)$$

D'autre part,

> Pour tout $\check{v}_{\bar{P}} \in \check{V}_{\bar{P}}$ et pour tout $v_P \in V_P$, la fonction de A_M dans $\mathbb{C}$ définie par :
> $$a \mapsto \langle \check{v}_{\bar{P}}, \pi_P(a) v_P \rangle_P$$
> est lisse et A_M-finie. (2.7)

Le Théorème suivant est une extension aux coefficients généralisés du Théorème 4.3.3 de [9] pour les coefficients.

Théorème 1. *Soit $P = MU$ un σ-sous-groupe parabolique.*

(i) *Soit (π, V) une représentation admissible de G et soit $\xi \in V^{*H}$. Alors il existe un unique $j_P^*(\xi) \in (V_P)^{*M\cap H}$ vérifiant :*

Pour tout $v \in V$, il existe $\varepsilon > 0$ tel que :
$$\delta_P(a)^{-1/2}\big\langle \xi, \pi(a)v\big\rangle = \big\langle j_P^*(\xi), \pi_P(a) j_P(v)\big\rangle, \quad a \in A_M^-(\varepsilon). \tag{2.8}$$

De plus, on peut choisir ε indépendamment de $\xi \in V^{*H}$.

(ii) *Soit K un sous-groupe ouvert compact de G. Soit $K' \subset K$ un sous-groupe ouvert compact satisfaisant aux conditions du Lemme 2. Alors pour toute représentation admissible (π, V) de G, pour tout élément ξ de V^{*H} et pour tout $v \in V^K$, on a :*

$$\big\langle j_P^*(\xi), \pi_P(a) j_P(v)\big\rangle = \big\langle \check{j}_{\bar{P}}\big(\pi^*(e_{K'})\xi\big), \pi_P(a) j_P(v)\big\rangle_P, \quad a \in A_M.$$

Démonstration. (i) Soient $j_P^*(\xi)$ et $j_P^*(\xi)'$ deux éléments de $(V_P^*)^{M\cap H}$ vérifiant (2.8). Soit $v \in V$, alors la fonction ψ_v de A_M dans $\mathbb{C}$ définie par :

$$\psi_v(a) = \big\langle j_P^*(\xi)' - j_P^*(\xi), \pi_P(a) j_P(v)\big\rangle$$

est une fonction lisse A_M-finie nulle sur $A_M^-(\varepsilon)$ pour ε assez petit. Elle est donc nulle d'après le résultat (2.4). On a donc $\psi_v(e) = 0$ et ceci pour tout $v \in V$; d'où l'unicité de $j_P^*(\xi)$ s'il existe.

Soit $v \in V$, on va définir $\langle j_P^*(\xi), j_P(v)\rangle$.

On va montrer qu'il existe une unique application, φ_v, définie sur A_M à valeurs dans $\mathbb{C}$, lisse et A_M-finie valant $\delta_P(a)^{-1/2}\langle \xi, \pi(a)v\rangle$ pour $a \in A_M^-(\varepsilon)$ pour au moins un $\varepsilon > 0$. On définira alors $\langle j_P^*(\xi), j_P(v)\rangle$ comme étant la valeur en e (élément neutre du groupe G) de φ_v. Soit K un sous-groupe ouvert compact admettant une factorisation d'Iwahori par rapport à P et tel que $v \in V^K$ (K existe d'après (2.2)), et soit K' comme dans le Lemme 2.

Vérifions l'existence de φ_v. D'après le Lemme 2,

$$\big\langle \xi, \pi(a)v\big\rangle = \big\langle \pi^*(e_{K'})\xi, \pi(a)v\big\rangle, \quad a \in A_M^-. \tag{2.9}$$

D'après [23, Théorème I.4.1] (cf. (2.6)), il existe $\varepsilon > 0$ ne dépendant que de K' et pas de ξ, tel que :

$$\delta_P(a)^{-1/2}\big\langle \pi^*(e_{K'})\xi, \pi(a)v\big\rangle = \big\langle \check{j}_{\bar{P}}\big(\pi^*(e_{K'})\xi\big), \pi_P(a) j_P(v)\big\rangle_P, \quad a \in A_M^-(\varepsilon). \tag{2.10}$$

On déduit de (2.9) et (2.10) que :

$$\delta_P(a)^{-1/2}\big\langle \xi, \pi(a)v\big\rangle = \big\langle \check{j}_{\bar{P}}\big(\pi^*(e_{K'})\xi\big), \pi_P(a) j_P(v)\big\rangle_P, \quad a \in A_M^-(\varepsilon). \tag{2.11}$$

La fonction définie par :

$$\varphi_v(a) := \big\langle \check{j}_{\bar{P}}\big(\pi^*(e_{K'})\xi\big), \pi_P(a) j_P(v)\big\rangle_P, \quad a \in A_M, \tag{2.12}$$

vérifie :

$$\varphi_v(a) = \delta_P(a)^{-1/2}\big\langle \xi, \pi(a)v\big\rangle, \quad a \in A_M^-(\varepsilon). \tag{2.13}$$

Donc φ_v convient car elle est lisse et A_M-finie d'après (2.5). Cette fonction est unique d'après le résultat (2.4).

Pour P et ξ fixés, on voit sur la formule (2.12) qu'elle ne dépend que de $j_P(v)$.

On remarque grâce au Lemme 2 équation (2.3) que l'on peut remplacer K' par un quelconque de ses sous-groupes compacts ouverts dans la définition de φ_v. (2.14)

L'unicité de φ_v permet de définir une application $j_P^*(\xi)$ de V_P dans $\mathbb{C}$ qui, à $j_P(v)$, associe $\varphi_v(e)$, pour v élément de V. L'application $j_P^*(\xi)$ est linéaire grâce à (2.12) et (2.14) qui impliquent : $\varphi_{v+v'} = \varphi_v + \varphi_{v'}$, $v, v' \in V$. Montrons que $j_P^*(\xi) \in (V_P^*)^{M\cap H}$.

Si $m \in M \cap H$, pour $\varepsilon' > 0$ assez petit,

$$\varphi_v(a) = \delta_P(a)^{-1/2}\big\langle \xi, \pi(a)v\big\rangle, \quad a \in A_M^-(\varepsilon').$$

$$\varphi_{\pi(m)v}(a) = \delta_P(a)^{-1/2}\big\langle \xi, \pi(am)v\big\rangle, \quad a \in A_M^-(\varepsilon').$$

Or a et m commutent et $m \in M \cap H$, donc :

$$\varphi_{\pi(m)v}(a) = \delta_P(a)^{-1/2}\big\langle \pi^*\big(m^{-1}\big)\xi, \pi(a)v\big\rangle, \quad a \in A_M^-(\varepsilon').$$

Comme m est élément de $M \cap H$, les applications φ_v et $\varphi_{\pi(m)v}$ coincident sur $A_M^-(\varepsilon')$. Comme φ_v et $\varphi_{\pi(m)v}$ sont toutes deux lisses et A_M-finies, on déduit du résultat (2.4) que $\varphi_v(e) = \varphi_{\pi(m)v}(e)$. Comme $\delta_P \in X(M)_\sigma$, d'après (1.18), on a $\delta_P(m) = 1$, $m \in M \cap H$. On en déduit que $j_P^*(\xi) \in (V_P^*)^{M\cap H}$.

Montrons que, pour ε comme dans (2.10), on a (2.8). On a par définition de $j_P^*(\xi)$:

$$\big\langle j_P^*(\xi), \pi_P(a) j_P(v)\big\rangle = \delta_P(a)^{-1/2}\varphi_{\pi(a)v}(e), \quad a \in A_M. \tag{2.15}$$

Montrons que :

$$\varphi_{\pi(a')v}(a) = \delta_P(a')^{1/2}\varphi_v(aa'), \quad a, a' \in A_M,\ v \in V. \tag{2.16}$$

D'après (2.12) et (2.14), pour tout sous-groupe compact ouvert assez petit K'', on a :

$$\begin{aligned}
\varphi_v(a) &= \big\langle \check{j}_{\bar{P}}\big(\pi^*(e_{K''})\xi\big), \pi_P(a) j_P(v)\big\rangle_P, \quad a \in A_M,\\
\varphi_{\pi(a')v}(a) &= \big\langle \check{j}_{\bar{P}}\big(\pi^*(e_{K''})\xi\big), \pi_P(a) j_P\big(\pi(a')v\big)\big\rangle_P,\\
&= \delta_P(a')^{1/2}\big\langle \check{j}_{\bar{P}}\big(\pi^*(e_{K''})\xi\big), \pi_P(aa') j_P(v)\big\rangle_P, \quad a, a' \in A_M.
\end{aligned}$$

D'où (2.16). Finalement :

$$\varphi_{\pi(a)v}(e) = \delta_P(a)^{1/2}\varphi_v(a).$$

Donc, d'après (2.15) :

$$\begin{aligned}\bigl\langle j_P^*(\xi), \pi_P(a) j_P(v)\bigr\rangle &= \varphi_v(a)\\ &= \delta_P(a)^{-1/2}\bigl\langle \xi, \pi(a)v\bigr\rangle, \quad a \in A_M^-(\varepsilon),\end{aligned}$$

la dernière égalité provenant de (2.13). Donc $j_P^*(\xi)$ vérifie (2.8). La preuve montre que ε ne dépend pas de ξ.

(ii) D'après (2.11), il existe $\varepsilon_1 > 0$ tel que :

$$\delta_P(a)^{-1/2}\bigl\langle \xi, \pi(a)v\bigr\rangle = \bigl\langle \check{j}_{\bar P}\bigl(\pi^*(e_{K'})\xi\bigr), \pi_P(a) j_P(v)\bigr\rangle, \quad a \in A_M^-(\varepsilon_1).$$

Et d'après le Théorème 1 (i), il existe $\varepsilon_2 > 0$ tel que :

$$\delta_P(a)^{-1/2}\bigl\langle \xi, \pi(a)v\bigr\rangle = \bigl\langle j_P^*(\xi), \pi_P(a) j_P(v)\bigr\rangle, \quad a \in A_M^-(\varepsilon_2).$$

En posant $\varepsilon := \min(\varepsilon_1, \varepsilon_2)$, on obtient :

$$\bigl\langle j_P^*(\xi), \pi_P(a) j_P(v)\bigr\rangle = \bigl\langle \check{j}_{\bar P}\bigl(\pi^*(e_{K'})\xi\bigr), \pi_P(a) j_P(v)\bigr\rangle_P, \quad a \in A_M^-(\varepsilon).$$

D'après le résultat (2.4), on a donc l'égalité sur A_M. $\square$

On suppose que $A_\emptyset \subset M$ ce qui équivaut à $A_\emptyset \subset P$ car $A_\emptyset$ est σ-stable.

Corollaire 1. *Soit $P = MU$ un σ-sous-groupe parabolique.*

(i) *On suppose que (π, V) est une représentation admissible de G et soit $\xi \in V^{*H}$. Pour tout $v \in V$, $a \mapsto \langle j_P^*(\xi), \pi_P(a) j_P(v)\rangle$ est l'unique fonction sur $A_M \cap A_\emptyset$ à valeurs complexes, lisse, $A_M \cap A_\emptyset$-finie qui soit égale à $\delta_P(a)^{-1/2}\langle \xi, \pi(a)v\rangle$ sur $A_M^-(\varepsilon) \cap A_\emptyset$ pour au moins un $\varepsilon > 0$.*

(ii) *Plus généralement, on a le même résultat en remplaçant, dans l'énoncé de* (i)*, $A_\emptyset$ par un tore σ-déployé maximal contenu dans M.*

Démonstration. La fonction considérée vérifie l'égalité voulue d'après le Théorème 1. L'unicité provient du résultat (2.4), (i) en résulte. Le choix de $A_\emptyset$ (précedant l'équation (1.16)) étant indifférent, (ii) est immédiat. $\square$

Soit K un sous-groupe compact ouvert de G totalement décomposé relativement à M_0 au sens de [7, Section 1.1]. On note $\Lambda^-(P, K)$ l'ensemble des éléments strictement P-anti-dominants de A_M (i.e. dans A_M^-), qui sont (P, K) positifs au sens de [7, Section 3.1]. Les propriétés importantes de ces notions sont :

– L'ensemble $\Lambda^-(P, K)$ est non vide (cf. [8, Lemma 6.14 et l'observation de la fin de la preuve]).

– Les sous-groupes compacts ouverts totalement décomposés relativement à M_0 forment une base de voisinage de l'élément neutre de G (cf. [7, Lemma 1]).

Le résultat suivant, du à J. Bernstein, est une forme renforcée d'un résultat de Casselman [9] (cf. (2.6)). C'est une conséquence de son Théorème de stabilisation (voir [7, Théorème 1 pour une preuve publiée]) et de la description du crochet de dualité $\langle .,. \rangle_P$ entre V_P et $\check{V}_{\bar{P}}$ (cf. [7, Section 5] et [9], voir (2.6)) :

> Soit K un sous-groupe compact ouvert totalement décomposé relativement à M_0 et soit $\lambda \in \Lambda^-(P, K)$. Il existe $n_0 \in \mathbb{N}$ tel que, pour toute représentation lisse de G, (π, V), pour tout $v \in V$, $\check{v} \in \check{V}$ invariants par K :

$$\delta_P(\lambda^n)^{-1/2}\langle \check{v}, \pi(\lambda^n)v \rangle = \langle \check{j}_{\bar{P}}(\check{v}), \pi_P(\lambda^n) j_P(v) \rangle_P, \quad n \geqslant n_0. \tag{2.17}$$

On établit le résultat suivant en vue d'une application analogue à celle du Lemme 9 de [5].

Proposition 1. *Soit K un sous-groupe compact ouvert totalement décomposé relativement à M_0 et $\lambda \in \Lambda^-(P, K)$. Il existe $n_0 \in \mathbb{N}$ tel que,*

> *Pour toute représentation admissible (π, V) de G, pour tout $\xi \in V^{*H}$ et pour tout $v \in V^K$, on ait* :

$$\delta_P(\lambda^n)^{-1/2}\langle \xi, \pi(\lambda^n)v \rangle = \langle j_P^*(\xi), \pi_P(\lambda^n) j_P(v) \rangle, \quad n \geqslant n_0.$$

Démonstration. D'après le Lemme 2, il existe un sous-groupe ouvert compact K' de K tel que :

> Pour tout sous-groupe compact ouvert $K'' \subset K'$, pour toute représentation admissible (π, V) de G, pour tout élément ξ de V^{*H} et pour tout $v \in V^K$,

$$\langle \xi, \pi(\lambda^n)v \rangle = \langle \pi^*(e_{K''})\xi, \pi(\lambda^n)v \rangle, \quad \lambda \in A_M^-. \tag{2.18}$$

D'après [7, Lemme 1], on peut prendre K'' totalement décomposé, ce que l'on fait. Alors, pour tout $v \in V^K$ et tout $\xi \in V^{*H}$, v et $\pi^*(e_{K''})\xi$ sont invariants par K'', on peut donc appliquer le résultat (2.17). Donc il existe un entier $n_0 \in \mathbb{N}$ ne dépendant que de K'', donc de K, et de λ tel que :

$$\delta_P(\lambda^n)^{-1/2}\langle \pi^*(e_{K''})\xi, \pi(\lambda^n)v \rangle = \langle \check{j}_{\bar{P}}(\pi^*(e_{K''})\xi), \pi_P(\lambda^n) j_P(v) \rangle_P, \quad n \geqslant n_0. \tag{2.19}$$

D'après (2.18) et (2.19), et comme $\Lambda^-(P, K) \subset A_M^-$, on en déduit que pour toute représentation admissible (π, V) de G, pour tout $\xi \in V^{*H}$ et pour tout $v \in V^K$, on a :

$$\delta_P(\lambda^n)^{-1/2}\langle \xi, \pi(\lambda^n)v \rangle = \langle \check{j}_{\bar{P}}(\pi^*(e_{K''})\xi), \pi_P(\lambda^n) j_P(v) \rangle_P, \quad n \geqslant n_0.$$

Or d'après le Théorème 1 (ii), et en remarquant que K'' satisfait aux hypothèses du Lemme 2, on a :

$$\langle \check{j}_{\bar{P}}\big(\pi^*(e_{K''})\xi\big), \pi_P\big(\lambda^n\big) j_P(v)\rangle_P = \langle j_P^*(\xi), \pi_P\big(\lambda^n\big) j_P(v)\rangle, \quad n \geqslant n_0. \tag{2.20}$$

La propriété résulte des équations (2.19) et (2.20). $\square$

2.2. *Terme constant de fonctions sur G/H*

Considérons l'espace :

$$C^\infty(G/H) = \bigcup_K C(K \setminus G/H),$$

où K parcourt l'ensemble des sous-groupes ouverts compacts de G et où $C(K \setminus G/H)$ est l'ensemble des fonctions sur G/H invariantes à gauche par K.

Le groupe G agit sur $C^\infty(G/H)$ par la représentation régulière gauche L.

Pour toute représentation admissible (π, V_π) de G et tout $\xi \in (V_\pi^*)^H$, notons $\mathcal{A}(\pi, \xi)$ le sous-espace de $C^\infty(G/H)$ engendré par les fonctions $c_{\xi,v} : gH \mapsto \langle \pi^*(g)\xi, v\rangle,\ v \in V_\pi$.

Posons :

$$\mathcal{A}(G/H) := \bigcup_{(\pi,\xi)} \mathcal{A}(\pi, \xi), \tag{2.21}$$

où π parcourt les représentations admissibles de G et $\xi \in (V_\pi^*)^H$.

En utilisant les sommes directes de représentations, on voit que :

$$\mathcal{A}(G/H) = \sum_{(\pi,\xi)} \mathcal{A}(\pi, \xi), \tag{2.22}$$

où π parcourre les représentations admissibles de G et $\xi \in (V_\pi^*)^H$.

Le sous-espace $\mathcal{A}(G/H)$ de $C^\infty(G/H)$ est invariant par la représentation régulière gauche L.

On note que si $f \in \mathcal{A}(G/H)$, f est A_G-finie (cf. (2.5)).

Remarque 1. On peut se limiter à ce que π parcourt les représentations admissibles de type fini de G et $\xi \in (V_\pi^*)^H$ car tout vecteur d'une représentation admissible engendre une représentation admissible de type fini.

Proposition 2.

(i) *Si f est un élément de $\mathcal{A}(G/H)$, il existe un unique élément f_P de $\mathcal{A}(M/M \cap H)$ vérifiant la propriété suivante* :

Pour tout $m \in M$, il existe $\varepsilon > 0$ tel que pour tout $a \in A_M^+(\varepsilon)$, on ait l'égalité :

$$\delta_P(ma)^{1/2} f(maH) = f_P\big(ma(M \cap H)\big).$$

On appele f_P terme constant de f le long de P.

(ii) *L'application $f \mapsto f_P$ de $\mathcal{A}(G/H)$ dans $\mathcal{A}(M/M \cap H)$ est linéaire.*

(iii) *Si* (π, V_π) *est une représentation admissible de* G, *si* $\xi \in (V_\pi^*)^H$, *si* $v \in V_\pi$, *et si* $f = c_{\xi,v}$, *alors* :

$$f_P\big(m(M \cap H)\big) = \big\langle \pi_P^*(m) j_P^*(\xi), j_P(v)\big\rangle, \quad m \in M.$$

Démonstration. Prouvons (i) et (iii). Soit $f \in \mathcal{A}(G/H)$, supposons que deux éléments f_P et f'_P de $\mathcal{A}(M/M \cap H)$ vérifient les conditions de la Proposition.

Pour tout $m \in M$, les fonctions sur A_M :

$$a \mapsto f_P\big(ma(M \cap H)\big) \quad \text{et} \quad a \mapsto f'_P\big(ma(M \cap H)\big)$$

coincident sur $A_M^+(\varepsilon)$ pour un certain $\varepsilon > 0$. Le fait qu'elles soient toutes deux lisses et A_M-finies assure alors leur égalité d'après le résultat (2.4). D'où l'unicité de f_P s'il existe.

Si $f = c_{\xi,v}$, la fonction définie par $f_P(m(M \cap H)) = \langle \pi_P^*(m) j_P^*(\xi), j_P(v)\rangle$ convient d'après le Théorème 1.

Dans le cas où f est un élément quelconque de $\mathcal{A}(G/H)$, l'existence de f_P est claire par linéarité. Ceci achève de prouver (i) et (iii). La linéarité de (ii) résulte de l'unicité dans (i). □

Soit (δ, V_δ) une représentation lisse de M. On étend l'action de M à P en la prenant triviale sur U. On considère l'ensemble $\operatorname{ind}_P^G V_\delta$ des $\varphi : G \to V_\delta$ qui sont invariantes à gauche par un sous-groupe compact ouvert et telles que :

$$\varphi(gmu) = \delta_P^{-1/2}(m)\delta\big(m^{-1}\big)\varphi(g), \quad g \in G,\ m \in M,\ u \in U.$$

Le groupe G agit par la représentation régulière gauche L sur $\operatorname{ind}_P^G V_\delta$.

Lemme 3. *Pour* $f \in \mathcal{A}(G/H)$, *on définit l'application* :

$$\begin{aligned} f_P^{\mathrm{ind}} : G &\to \mathcal{A}(M/M \cap H) \\ g &\mapsto (L_{g^{-1}} f)_P. \end{aligned}$$

Alors $f_P^{\mathrm{ind}} \in \operatorname{ind}_P^G \mathcal{A}(M/M \cap H)$. (2.23)

Démonstration. Par linéarité, on se ramène à $f = c_{\xi,v}$, où $v \in V$, $\xi \in V^{*H}$ et (π, V) est une représentation admissible de type fini de G. Pour $g_1 \in G$:

$$\big(f_P^{\mathrm{ind}}(g_1)\big)\big(m(M \cap H)\big) = \big\langle \pi_P^*(m) j_P^*(\xi), j_P\big(\pi\big(g_1^{-1}\big)v\big)\big\rangle.$$

Montrons que $f_P^{\mathrm{ind}}(g_1 mu) = \delta_P^{-1/2}(m) L_{m^{-1}} f_P^{\mathrm{ind}}(g_1)$, $g_1 \in G$, $m \in M$, $u \in U$.

Cela revient à montrer que pour $m' \in M$:

$$\big(f_P^{\mathrm{ind}}(g_1 mu)\big)\big(m'(M \cap H)\big) = \delta_P^{1/2}\big(m^{-1}\big)\big(L_{m^{-1}} f_P^{\mathrm{ind}}(g_1)\big)\big(m'(M \cap H)\big). \tag{2.24}$$

On a :

$$\begin{aligned}\big(f_P^{\mathrm{ind}}(g_1mu)\big)\big(m'(M\cap H)\big) &= \big\langle \pi_P^*(m')j_P^*(\xi), j_P\big(\pi\big(u^{-1}m^{-1}g_1^{-1}\big)v\big)\big\rangle \\ &= \big\langle \pi_P^*(m')j_P^*(\xi), j_P\big(\pi\big(m^{-1}g_1^{-1}\big)v\big)\big\rangle \\ &= \delta_P^{1/2}\big(m^{-1}\big)\big\langle \pi_P^*(m')j_P^*(\xi), \pi_P\big(m^{-1}\big)j_P\big(\pi\big(g_1^{-1}\big)v\big)\big\rangle \\ &= \delta_P^{1/2}\big(m^{-1}\big)\big\langle \pi_P^*(mm')j_P^*(\xi), j_P\big(\pi\big(g_1^{-1}\big)v\big)\big\rangle \\ &= \delta_P^{1/2}\big(m^{-1}\big)\big(f_P^{\mathrm{ind}}(g_1)\big)\big(mm'(M\cap H)\big)\end{aligned}$$

d'où (2.24). Le Lemme en résulte. □

2.3. Comportements asymptotiques de certains coefficients généralisés de représentations admissibles

On rappelle qu'on a fixé $A_\emptyset$, un tore σ-déployé maximal de G, et $P_\emptyset$ un σ-sous-groupe parabolique minimal contenant $A_\emptyset$.

Soit $\Sigma(G, A_\emptyset)$ l'ensemble des racines de $A_\emptyset$ dans l'algèbre de Lie de G. Alors $\Sigma(G, A_\emptyset)$ est un système de racines dont le groupe de Weyl s'identifie au quotient du normalisateur dans G de $A_\emptyset$, $N_G(A_\emptyset)$, par son centralisateur $Z_G(A_\emptyset)$ (cf. [15, Proposition 5.9]).

On note $\Sigma(P_\emptyset, A_\emptyset)$ l'ensemble des racines de $A_\emptyset$ dans l'algèbre de Lie de $P_\emptyset$. On note $\Delta(P_\emptyset, A_\emptyset)$ l'ensemble des racines simples de $\Sigma(P_\emptyset, A_\emptyset)$. Si Θ est une partie de $\Delta(P_\emptyset, A_\emptyset)$, on note $\langle\Theta\rangle_\emptyset$ le sous-système de $\Sigma(G, A_\emptyset)$ engendré par Θ, et P_Θ le sous-groupe parabolique de G pour lequel $\Sigma(P_\emptyset, A_\emptyset) \cup \langle\Theta\rangle_\emptyset$ est l'ensemble des racines de $A_\emptyset$ dans l'algèbre de Lie de P_Θ. Alors :

$$P_\Theta \text{ contient } P_\emptyset \text{ et } P_\Theta \text{ est un } \sigma\text{-sous-groupe parabolique de } G, \tag{2.25}$$

en effet, comme les éléments de $A_\emptyset$ sont anti-invariants par σ, l'ensemble des racines de $A_\emptyset$ dans l'algèbre de Lie de $\sigma(P_\Theta)$ est égal à l'ensemble des opposés des racines de $A_\emptyset$ dans l'algèbre de Lie de P_Θ.

Soit $0 < \varepsilon \leqslant 1$, on note pour $P = P_\Theta$:

$$A_\emptyset^-(P, <\varepsilon) := \big\{a \in A_\emptyset^-;\ \big|\alpha(a)\big|_F < \varepsilon,\ \alpha \text{ racine de } A_\emptyset \text{ dans } U_P\big\}. \tag{2.26}$$

Comme $\varepsilon \leqslant 1$, on a l'égalité :

$$A_\emptyset^-(P, <\varepsilon) = \big\{a \in A_\emptyset^-;\ \big|\alpha(a)\big|_F < \varepsilon,\ \alpha \in \Delta(P_\emptyset, A_\emptyset) \setminus \Theta\big\}. \tag{2.27}$$

Elle résulte immédiatement du fait que si α est une racine de $A_\emptyset$ dans U_P, $\alpha = \sum_{\beta\in\Delta(P_\emptyset, A_\emptyset)} n_\beta\beta$, $n_\beta \in \mathbb{N}$, alors il existe $\beta_0 \in \Delta(P_\emptyset, A_\emptyset) \setminus \Theta$ tel que $n_{\beta_0} \neq 0$.

Ceci est cohérent avec les notations de l'introduction au sous-chapitre 0.3.

On définit $A_0^-(P, <\varepsilon) := \{a \in A_0^-;\ |\alpha(a)|_F < \varepsilon,\ \alpha \text{ racine de } A_0 \text{ dans } U_P\}$.

On suppose, pour la suite de ce sous-chapitre 2.3, que P contient $A_\emptyset$.

Théorème 2. *Soit (π, V) une représentation admissible de G. Pour tout $v \in V$, il existe $\varepsilon > 0$ tel que pour tout $\xi \in V^{*H}$,*

$$\delta_P(a)^{-1/2}\langle \xi, \pi(a)v\rangle = \langle j_P^*(\xi), \pi_P(a) j_P(v)\rangle, \quad a \in A_\emptyset^-(P, < \varepsilon).$$

En d'autres termes, si f est l'élément de $\mathcal{A}(G/H)$ défini par $f = c_{\xi,v}$, alors :

$$\delta_P(a)^{-1/2} f\bigl(a^{-1}H\bigr) = f_P\bigl(a^{-1}(M \cap H)\bigr), \quad a \in A_\emptyset^-(P, < \varepsilon).$$

On peut remplacer P par un σ-sous-groupe parabolique quelconque et $A_\emptyset$ par un tore σ-déployé maximal contenu dans P.

Démonstration. Soient $v \in V$ et K un sous-groupe compact ouvert tel que $v \in V^K$. On applique le Lemme 2 à $P = P_\emptyset$. Alors il existe un sous-groupe compact ouvert K' de K, tel que :

$$\langle \xi, \pi(a)v\rangle = \langle \pi^*(e_{K'})\xi, \pi(a)v\rangle, \quad a \in A_{M_\emptyset}^-.$$

Or $A_\emptyset^- \subset A_{M_\emptyset}^-$. Donc on a :

$$\langle \xi, \pi(a)v\rangle = \langle \pi^*(e_{K'})\xi, \pi(a)v\rangle, \quad a \in A_\emptyset^-. \tag{2.28}$$

D'après [9, Théorème 4.3.3], il existe $0 < \varepsilon \leqslant 1$ tel que :

$$\delta_P(a)^{-1/2}\langle \pi^*(e_{K'})\xi, \pi(a)v\rangle = \langle \check{j}_{\bar{P}}\bigl(\pi^*(e_{K'})\xi\bigr), \pi_P(a) j_P(v)\rangle_P, \quad a \in A_0^-(P, < \varepsilon). \tag{2.29}$$

Donc d'après (2.28) et (2.29) :

$$\delta_P(a)^{-1/2}\langle \xi, \pi(a)v\rangle = \langle \check{j}_{\bar{P}}\bigl(\pi^*(e_{K'})\xi\bigr), \pi_P(a) j_P(v)\rangle_P, \quad a \in A_0^-(P, < \varepsilon) \cap A_\emptyset^-. \tag{2.30}$$

On a de plus l'égalité : $A_0^-(P, < \varepsilon) \cap A_\emptyset^- = A_\emptyset^-(P, < \varepsilon)$, d'où :

$$\delta_P(a)^{-1/2}\langle \xi, \pi(a)v\rangle = \langle \check{j}_{\bar{P}}\bigl(\pi^*(e_{K'})\xi\bigr), \pi_P(a) j_P(v)\rangle_P, \quad a \in A_\emptyset^-(P, < \varepsilon). \tag{2.31}$$

Pour $a \in A_\emptyset^-(P, < \varepsilon)$ fixé, la fonction définie sur $A_M \cap A_\emptyset$ par :

$$b \mapsto \langle \check{j}_{\bar{P}}\bigl(\pi^*(e_{K'})\xi\bigr), \pi_P(ba) j_P(v)\rangle_P$$

est une fonction lisse et $A_M \cap A_\emptyset$-finie. Or, pour $b \in A_M^- \cap A_\emptyset$ et $a \in A_\emptyset^-(P, < \varepsilon)$, on a $ba \in A_\emptyset^-(P, < \varepsilon)$. Donc, d'après (2.31) appliqué à $ba \in A_\emptyset^-(P, < \varepsilon)$ au lieu de a, on a l'égalité :

$$\delta_P(ba)^{-1/2}\langle \xi, \pi(ba)v\rangle = \langle \check{j}_{\bar{P}}\bigl(\pi^*(e_{K'})\xi\bigr), \pi_P(ba) j_P(v)\rangle_P, \quad b \in A_M^- \cap A_\emptyset.$$

D'après le Corollaire 1 du Théorème 1 appliqué à $v' := \pi(a)v$, on a alors :

$$\langle \check{j}_{\bar{P}}\bigl(\pi^*(e_{K'})\xi\bigr), \pi_P(ba) j_P(v)\rangle_P = \langle j_P^*(\xi), \pi_P(ba) j_P(v)\rangle, \quad b \in A_M^- \cap A_\emptyset,\ a \in A_\emptyset^-(P, < \varepsilon).$$

En joignant (2.31) à cette dernière égalité appliquée à $b = e$, on obtient :

$$\delta_P(a)^{-1/2}\langle \xi, \pi(a)v\rangle = \langle j_P^*(\xi), \pi_P(a) j_P(v)\rangle, \quad a \in A_\emptyset^-(P, < \varepsilon). \qquad \square$$

2.4. σ-Sous-groupes paraboliques

Donnons d'autres caractérisations des σ-sous-groupes paraboliques.

Soient A un tore déployé de G, et $\lambda \in \Lambda(A)$. On note P_λ le sous-groupe parabolique de G contenant A pour lequel les poids α de A dans l'algèbre de Lie de P_λ vérifient $|\alpha(\lambda)|_F \geqslant 1$.

On a une dualité :

$$\begin{aligned} \mathrm{Rat}(A) \times X_*(A) &\to \mathbb{Z} \\ (\alpha, \underline{\lambda}) &\mapsto \langle \alpha, \underline{\lambda} \rangle \end{aligned}$$

caractérisée par : $\alpha \circ \underline{\lambda}(\varpi) = \varpi^{\langle \alpha, \underline{\lambda} \rangle}$.

On note $P(\underline{\lambda})$ le sous-groupe parabolique de G contenant A dont les racines de A dans son algèbre de Lie sont les $\alpha \in \Sigma(G, A)$ tels que $\langle \alpha, \underline{\lambda} \rangle \geqslant 0$.

Remarque 2. $P(\underline{\lambda}) = P_\lambda$, où $\lambda := \underline{\lambda}(\varpi)$.

En effet, $P(\underline{\lambda})$ et P_λ sont deux sous-groupes paraboliques de G, donc il suffit de montrer qu'ils ont la même algèbre de Lie. Mais :

$$\left|\alpha(\lambda)\right|_F = \left|\alpha\big(\underline{\lambda}(\varpi)\big)\right|_F = \left|\varpi^{\langle \alpha, \underline{\lambda} \rangle}\right|_F = |\varpi|_F^{\langle \alpha, \underline{\lambda} \rangle}.$$

Donc :

$$\left|\alpha(\lambda)\right|_F \geqslant 1 \quad \Leftrightarrow \quad \langle \alpha, \underline{\lambda} \rangle \geqslant 0.$$

D'où la remarque.

Lemme 4. *Soient P un σ-sous-groupe parabolique de G, $M = P \cap \sigma(P)$ son sous-groupe de Levi σ-stable et $A_{M,\sigma}$ le plus grand tore σ-déployé du centre de M. Il existe $\lambda \in \Lambda(A_{M,\sigma})$ tel que $P = P_\lambda$. Alors M est égal au centralisateur dans G de λ.*

Démonstration. D'après [15, Lemme 4.6] et en tenant compte de (1.13), il existe $\underline{\lambda} \in X_*(A)$ tel que $\sigma(\underline{\lambda}) = \underline{\lambda}^{-1}$, $P = P(\underline{\lambda})$ et $M = Z_G(\underline{\lambda})$.

Comme $M = Z_G(\underline{\lambda})$, $\lambda := \underline{\lambda}(\varpi)$ est un élément de $\Lambda(A_{M,\sigma})$. D'après la Remarque 2, on a $P = P_\lambda$. Clairement $Z_G(\underline{\lambda}) \subset Z_G(\lambda)$. Par ailleurs, comme $\sigma(\underline{\lambda}) = \underline{\lambda}^{-1}$, on a $\sigma(\lambda) = \lambda^{-1}$. De plus, $Z_G(\lambda) = Z_G(\lambda^{-1})$ et $\sigma(P_\lambda) = P_{\lambda^{-1}}$. Donc $Z_G(\lambda) \subset P_\lambda \cap \sigma(P_\lambda) = M = Z_G(\underline{\lambda})$. Finalement $Z_G(\lambda) = Z_G(\underline{\lambda})$. $\square$

Lemme 5.

(i) *Si $\lambda \in \Lambda(A_\emptyset)$ est $P_\emptyset$-dominant, i.e. $|\alpha(\lambda)|_F \geqslant 1, \alpha \in \Delta(P_\emptyset, A_\emptyset)$, $P_\lambda = P_\Theta$, où $\Theta = \{\alpha \in \Delta(P_\emptyset, A_\emptyset);\ |\alpha(\lambda)|_F = 1\}$.*

(ii) *Tout σ-sous-groupe parabolique de G contenant $P_\emptyset$ est de la forme P_Θ pour $\Theta \subset \Delta(P_\emptyset, A_\emptyset)$.*

(iii) *Tout σ-sous-groupe parabolique de G est conjugué par un élément de H à un σ-sous-groupe parabolique de G de la forme $x_i w.P_\Theta$, $\Theta \subset \Delta(P_\emptyset, A_\emptyset)$, $w \in W(A_\emptyset)$, $i \in I$ (cf. 1.2 pour définition de I).*

On peut remplacer $A_\emptyset$ par n'importe quel tore σ-déployé maximal et $P_\emptyset$ par n'importe quel σ-sous-groupe parabolique de G le contenant dans les trois premières assertions.

Démonstration. (i) est clair.

Montrons (ii) : soit $P = MU$ un σ-sous-groupe parabolique de G contenant $P_\emptyset$. D'après le Lemme 4, il est de la forme P_μ pour μ élément de $\Lambda(A_{M,\sigma})$. Or $A_{M,\sigma}$ est contenu dans un tore σ-déployé maximal $A'_\emptyset$ de G, donc $\mu \in \Lambda(A'_\emptyset)$. Comme $A_\emptyset$ et $A'_\emptyset$ sont σ-stable, ils sont contenus dans $M = P \cap \sigma(P)$. Alors $A_\emptyset$ et $A'_\emptyset$ sont deux tores σ-déployés maximaux de M, donc conjugués par un élément m de M (cf. [14, Proposition 1.16]). Posant $\lambda = m\mu m^{-1}$, on a $\lambda \in \Lambda(A_\emptyset)$ et $P_\lambda = m.P_\mu = P$ car $m \in M$. Enfin, comme P contient $P_\emptyset$, l'élément λ de $\Lambda(A_\emptyset)$ doit être $P_\emptyset$-dominant. Alors (ii) résulte de (i).

On montre (iii) en reprenant les notations de 1.2. Soit P un σ-sous-groupe parabolique de G, alors P est conjugué par un élément de H à un sous-groupe parabolique de G de la forme P_λ, pour un élément λ de $\Lambda(A_i)$. Soit $\mu \in \Lambda(A_\emptyset)$ tel que $\lambda = x_i \mu x_i^{-1}$; alors $P = P_\lambda = x_i.P_\mu$. Alors il existe $\nu \in \Lambda(A_\emptyset)$ qui est $P_\emptyset$-dominant, et $w \in W(A_\emptyset)$ tel que $w\nu = \mu$ et $x_i.P_\mu = x_i w.P_\nu$.

D'après (i), il existe $\Theta \subset \Delta(P_\emptyset, A_\emptyset)$ tel que $P_\nu = P_\Theta$; alors $P = x_i w.P_\Theta$ d'où (iii). □

2.5. *Transitivité du terme constant*

Théorème 3. *Soient $P = MU$ un σ-sous-groupe parabolique de G contenant $A_\emptyset$ et Q un σ-sous-groupe parabolique de M contenant $A_\emptyset$. On pose $P_Q := QU$ de sorte que P_Q est un σ-sous-groupe parabolique de G contenu dans P, on note M_{P_Q} son sous-groupe de Levi σ-stable. Soit (π, V) une représentation admissible de G et soit $\xi \in V^{*H}$, alors $j^*_{P_Q}(\xi) = j^*_Q(j^*_P(\xi))$.*

Démonstration. Soit $P_{\emptyset,M}$ un σ-sous-groupe parabolique minimal de M contenant $A_\emptyset$ et contenu dans Q. Alors $P_{\emptyset,M}U$ est un σ-sous-groupe parabolique minimal de G. Notre choix de $P_\emptyset$ contenant $A_\emptyset$ étant indifférent, on peut supposer $P_{\emptyset,M}U$ égal à $P_\emptyset$. Alors $P_{\emptyset,M} = P_\emptyset \cap M$. On pose $\Delta_G := \Delta(P_\emptyset, A_\emptyset)$, $\Delta_M := \Delta(P_{\emptyset,M}, A_\emptyset)$, $P = P_{\Theta_P}$ avec $\Theta_P \subset \Delta_G$ et $Q = P_{\Theta_Q}$ avec $\Theta_Q \subset \Delta_M$. Alors :

$$\Theta_P = \Delta_M \quad \text{et donc} \quad \Theta_Q \subset \Theta_P. \tag{2.32}$$

Soit (π, V) une représentation admissible de G. Soient $\xi \in V^{*H}$ et $v \in V$, montrons qu'il existe $\varepsilon > 0$ tel que :

$$\big\langle j^*_Q\big(j^*_P(\xi)\big), \pi_{P_Q}(a) j_{P_Q}(v)\big\rangle = \delta_{P_Q}(a)^{-1/2}\langle \xi, \pi(a)v\rangle, \quad a \in A^-_\emptyset(P_Q, < \varepsilon). \tag{2.33}$$

En remarquant que $j_{P_Q}(v) = j_Q(j_P(v))$, on a :

$$\big\langle j^*_Q\big(j^*_P(\xi)\big), \pi_{P_Q}(a) j_{P_Q}(v)\big\rangle = \big\langle j^*_Q\big(j^*_P(\xi)\big), \pi_Q(a) j_Q\big(j_P(v)\big)\big\rangle.$$

On remarque que dans le Théorème 2, on peut prendre $\varepsilon \leqslant 1$. Alors, d'après (2.27) et en appliquant le Théorème 2 au sous-groupe parabolique Q de M, à la représentation admissible de M, (π_P, V_P), et à $j^*_P(\xi) \in (V_P)^{*M\cap H}$, on trouve qu'il existe $0 < \varepsilon' \leqslant 1$ tel que :

$$\big\langle j_Q^*\big(j_P^*(\xi)\big), \pi_Q(a) j_Q\big(j_P(v)\big)\big\rangle = \delta_Q(a)^{-1/2}\big\langle j_P^*(\xi), \pi_P(a) j_P(v)\big\rangle,$$
$$a \in A_\emptyset,\ \big|\alpha(a)\big|_F < \varepsilon',\ \alpha \in \Delta_M \setminus \Theta_Q \text{ et } \big|\alpha(a)\big|_F \leqslant 1,\ \alpha \in \Delta_M. \tag{2.34}$$

En appliquant d'autre part le Théorème 2 au sous-groupe parabolique P de G, à (π, V), représentation admissible de G, et à $\xi \in V^{*H}$, on trouve qu'il existe $0 < \varepsilon'' \leqslant 1$ tel que :

$$\big\langle j_P^*(\xi), \pi_P(a) j_P(v)\big\rangle = \delta_P(a)^{-1/2}\big\langle \xi, \pi(a)v\big\rangle,$$
$$a \in A_\emptyset,\ \big|\alpha(a)\big|_F < \varepsilon'',\ \alpha \in \Delta_G \setminus \Theta_P \text{ et } \big|\alpha(a)\big|_F \leqslant 1,\ \alpha \in \Delta_G. \tag{2.35}$$

D'après (2.32), on déduit de (2.34) et de (2.35) que :

$$\big\langle j_Q^*\big(j_P^*(\xi)\big), \pi_Q(a) j_Q\big(j_P(v)\big)\big\rangle = \delta_Q(a)^{-1/2}\delta_P(a)^{-1/2}\big\langle \xi, \pi(a)v\big\rangle,$$
$$a \in A_\emptyset,\ \big|\alpha(a)\big|_F < \min(\varepsilon', \varepsilon''),\ \alpha \in \Delta_G \setminus \Theta_Q,\ \big|\alpha(a)\big|_F \leqslant 1,\ \alpha \in \Delta_G. \tag{2.36}$$

L'ensemble des racines de $A_\emptyset$ dans l'algèbre de Lie du radical unipotent de P_Q est, par définition de P_Q, la réunion disjointe de l'ensemble des racines de $A_\emptyset$ dans l'algèbre de Lie de U_P et dans celle de U_Q. Donc :

$$\delta_{P_Q}(a) = \delta_Q(a)\delta_P(a), \quad a \in A_\emptyset.$$

D'où (2.33) en posant $\varepsilon := \min(\varepsilon', \varepsilon'')$.

Or $A^-_{M_{P_Q}}(\varepsilon) \cap A_\emptyset \subset A^-_\emptyset(P_Q, < \varepsilon)$. Donc, d'après le Corollaire 1 du Théorème 1, on a :

$$j^*_{P_Q}(\xi) = j_Q^*\big(j_P^*(\xi)\big). \qquad \square$$

On reprend les notations du Théorème précédent.

Corollaire 1. *Soit f un élément de $\mathcal{A}(G/H)$, alors f_P est élément de $\mathcal{A}(M/M \cap H)$, et on peut considérer $(f_P)_Q$ élément de $\mathcal{A}(M_Q/M_Q \cap H)$. De plus, $f_{P_Q} = (f_P)_Q$.*

2.6. Vecteurs distributions H-invariants cuspidaux

Définition 1. Soit (π, V) une représentation admissible de G et soit $\xi \in V^{*H}$, on dira que la paire (π, ξ) est H-cuspidale si pour tout σ-sous-groupe parabolique P de G, distinct de G, $j_P^*(\xi) = 0$.

Proposition 3. *La paire (π, ξ) est H-cuspidale si et seulement si pour tout σ-sous-groupe parabolique P de G, distinct de G, égal à l'un des $x_i w.P_\Theta$, pour un $\Theta \subset \Delta(P_\emptyset, A_\emptyset)$, un $w \in W(A_\emptyset)$ et un $i \in I$ (voir notations en* 1.1), *$j_P^*(\xi) = 0$.*

Démonstration. Supposons que pour tout σ-sous-groupe parabolique P de G distinct de G égal à l'un des $x_i w.P_\Theta$, pour un $\Theta \subset \Delta(P_\emptyset, A_\emptyset)$, un $w \in W(A_\emptyset)$ et un $i \in I$, on ait : $j_P^*(\xi) = 0$. Soit $P \neq G$ un σ-sous-groupe parabolique de G, alors d'après le Lemme 5 (iii), il existe $h \in H$, $\Theta \subset \Delta(P_\emptyset, A_\emptyset)$, $\Theta \neq \Delta(P_\emptyset, A_\emptyset)$, $w \in W(A_\emptyset)$ et $i \in I$ tels que $h.P = x_i w.P_\Theta$. Et donc $j_{h.P}^*(\xi) = 0$.

On note $\bar{v}$ la classe d'un élément $v \in V$ dans $V_P := V/V(P)$ et $[v]$ sa classe dans $V_{h.P} := V/V(h.P)$.

On considère l'application :

$$\begin{aligned} T_h : V_P &\to V_{h.P} \\ j_P(v) &\mapsto j_{h.P}\big(\pi(h)v\big). \end{aligned}$$

T_h est bien définie, de plus, c'est un isomorphisme car $V(h.P) = \pi(h)V(P)$. En utilisant le Théorème 1, on montre par transport de structure que :

$$T_h^*\big(j_{h.P}^*(\xi)\big) = j_P^*(\xi).$$

Mais $j_{h.P}^*(\xi) = 0$ donc $j_P^*(\xi) = 0$. □

Proposition 4. *Soit (π, V) une représentation admissible de G, soit $\xi \in V^{*H}$, et soit P un σ-sous-groupe parabolique minimal de G parmi ceux tels que $j_P^*(\xi) \neq 0$, alors $(\pi_P^*, j_P^*(\xi))$ est $H \cap M$-cuspidale.*

Démonstration. Soit $Q \subset M$ un σ-sous-groupe parabolique de M, montrons que $j_Q^*(j_P^*(\xi)) = 0$. En reprenant les notations du Théorème 3, on a $j_Q^*(j_P^*(\xi)) = j_{P_Q}^*(\xi)$. Or $P_Q \subset P$ et $P_Q \neq P$, par définition de P, on a donc $j_{P_Q}^*(\xi) = 0$, d'où la Proposition. □

On rappelle qu'une représentation admissible de type fini de G, (π, V), est cuspidale si pour tout sous-groupe parabolique P de G, distinct de G, $j_P(V) = \{0\}$.

Proposition 5. *Si (π, V) est une représentation admissible de type fini de G cuspidale et si $\xi \in V^{*H}$, alors (π, ξ) est H-cuspidale.*

Démonstration. Soit P un σ-sous-groupe parabolique de G, comme $j_P^*(\xi)$ est une forme linéaire sur $j_P(V) = \{0\}$, elle est nulle. D'où la Proposition. □

3. Majorations

3.1. Fonctions Θ_G et N_d, $d \in \mathbb{N}$ sur G/H

Fixons un plongement algébrique

$$\tau : G \to GL_n(F). \tag{3.1}$$

On peut supposer, et l'on suppose, que $\tau(K_0) \subset GL_n(\mathcal{O})$ où $\mathcal{O}$ est l'anneau des entiers de F (cf. [23, I.1]). Pour $g \in G$, écrivons :

$$\tau(g) = (a_{i,j})_{i,j=1,\dots,n}, \qquad \tau\big(g^{-1}\big) = (b_{i,j})_{i,j=1,\dots,n}.$$

Posons

$$\|g\| = \sup_{i,j} \sup\big(|a_{i,j}|_F, |b_{i,j}|_F\big). \tag{3.2}$$

On a (cf. [23, I.1]) :

$$\|g\| \geqslant 1 \quad \text{pour tout } g \in G, \qquad \|g_1 g_2\| \leqslant \|g_1\|\|g_2\| \quad \text{pour tout } g_1, g_2 \in G \quad \text{et}$$
$$\|k_1 g k_2\| = \|g\| \quad \text{pour tout } k_1, k_2 \in K_0,\ g \in G. \tag{3.3}$$

Avec les notations du Paragraphe 1.1 et (1.19), on rappelle que P_0 est un sous-groupe parabolique minimal de G que l'on a fixé. On note $(\varepsilon_{M_0}, \mathbb{C})$ la représentation triviale de M_0, $(\pi_0, V_0) = (\mathrm{ind}_{P_0}^G \varepsilon_{M_0}, \mathrm{ind}_{P_0}^G \mathbb{C})$, e_0 l'unique élément de V_0 invariant par K_0 et tel que $e_0(e) = 1$, où e est l'élément neutre du groupe G.

Remarquons que $(\check{\pi}_0, \check{V}_0)$ est isomorphe à (π_0, V_0). Pour $g \in G$, on pose :

$$\Xi(g) = \langle \pi_0(g)e_0, e_0 \rangle.$$

Alors Ξ est bi-invariante par K_0.

> On dira que deux fonctions f_1 et f_2 définies sur un ensemble E et à valeurs dans $\mathbb{R}_+$ sont équivalentes sur un sous ensemble E' de E (on notera $f_1(x) \asymp f_2(x)$, $x \in E'$), s'il existe $C, C' > O$ tels que :

$$C' f_2(x) \leqslant f_1(x) \leqslant C f_2(x), \quad x \in E'. \tag{3.4}$$

Lemme 6. *Avec les notations de* (1.9)*, il existe $C_1, C_2 > 0$ et $d_1, d_2 \in \mathbb{N}$ tels que pour tout $y \in \mathcal{W}_{M_\emptyset}^G$, on ait* :

$$C_1 \delta_{y.P_0}(m')^{1/2}\bigl(1 + \log\|m'\|\bigr)^{-d_1} \leqslant \Xi(m') \leqslant C_2 \delta_{y.P_0}(m')^{1/2}\bigl(1 + \log\|m'\|\bigr)^{d_2}, \quad m' \in y.\overline{M}_0^+.$$

On a la même inégalité en remplaçant P_0 par $\bar{P}_0$ et $\overline{M}_0^+$ par $\overline{M}_0^-$.

Démonstration. On rappelle le Lemme II.1.2 de [23] :

> Il existe $d \in \mathbb{N}$ et, pour tous $g_1, g_2 \in G$, il existe $c > 0$ tels que, pour tout $g \in G$, on ait :

$$\Xi(g_1 g g_2) \leqslant c\,\Xi(g)\bigl(1 + \log\|g\|\bigr)^d. \tag{3.5}$$

On l'applique à ymy^{-1} (resp. $y^{-1}ymy^{-1}y$) pour obtenir l'inégalité de droite (resp. de gauche) suivante. Et du fait de la finitude de $\mathcal{W}_{M_\emptyset}^G$, il existe $c_1, c_2 > 0$ et $d_1 \in \mathbb{N}$ tels que :

$$c_1\bigl(1 + \log\|ymy^{-1}\|\bigr)^{-d_1}\,\Xi(m) \leqslant \Xi\bigl(ymy^{-1}\bigr) \leqslant c_2\bigl(1 + \log\|m\|\bigr)^{d_1}\,\Xi(m), \quad y \in \mathcal{W}_{M_\emptyset}^G,\ m \in M_0. \tag{3.6}$$

De plus, d'après [23, Lemme II.1.1], il existe $c_1', c_2' > 0$ et $d_1' \in \mathbb{N}$ tels que :

$$c_1' \delta_{P_0}(m)^{1/2} \leqslant \Xi(m) \leqslant c_2' \delta_{P_0}(m)^{1/2}\bigl(1 + \log\|m\|\bigr)^{d_1'}, \quad m \in \overline{M}_0^+. \tag{3.7}$$

En appliquant (3.7) à (3.6), on obtient l'existence de $C_1, C_2 > 0$ et $d_1, d_1' \in \mathbb{N}$ tels que pour tout élément y de $\mathcal{W}_{M_\emptyset}^G$ et pour tout élément m de $\overline{M}_0^+$, on ait :

$$C_1 \delta_{P_0}(m)^{1/2}\big(1+\log\|ymy^{-1}\|\big)^{-d_1} \leqslant \Xi\big(ymy^{-1}\big) \leqslant C_2 \delta_{P_0}(m)^{1/2}\big(1+\log\|m\|\big)^{d_1+d_1'}.$$

Or

$$\delta_{P_0}(m) = \delta_{y.P_0}\big(ymy^{-1}\big), \quad y \in \mathcal{W}_{M_\emptyset}^G,\ m \in \overline{M}_0^+,$$

et il existe $C_2' > 0$ tel que :

$$1+\log\|m\| \leqslant C_2'\big(1+\log\|ymy^{-1}\|\big), \quad y \in \mathcal{W}_{M_\emptyset}^G,\ m \in M_0,$$

puisque $\|m\| \leqslant \|y^{-1}\|\|ymy^{-1}\|\|y\|$. Donc quitte à changer C_2', l'assertion est démontrée.

L'assertion sur $\bar{P}_0$ est immédiate. □

On pose :

$$\|gH\| := \big\|g\sigma\big(g^{-1}\big)\big\|, \quad g \in G. \tag{3.8}$$

Grâce à (3.3), on voit que si Ω' est une partie compacte de G,

$$\|\omega gH\| \asymp \|gH\|, \quad \omega \in \Omega',\ g \in G. \tag{3.9}$$

On définit les fonctions Θ_G et $N_d,\ d \in \mathbb{N}$, par :

$$\Theta_G(gH) = \big(\Xi\big(g\sigma\big(g^{-1}\big)\big)\big)^{1/2}, \quad g \in G, \tag{3.10}$$

et

$$N_d(gH) = \big(1+\log\|gH\|\big)^d, \quad g \in G. \tag{3.11}$$

On notera N au lieu de N_1.

En utilisant le fait que tous les tores déployés maximaux du centralisateur $M_\emptyset$ de $A_\emptyset$ sont conjugués entre eux par des éléments de $M_\emptyset$, on voit que si n est un élément du normalisateur de $A_\emptyset$ dans G, $N_G(A_\emptyset)$, il existe un élément z de $M_\emptyset$ tel que zn normalise A_0. On en déduit que tout automorphisme de $\mathfrak{a}_\emptyset$ induit par un élément de $N_G(A_\emptyset)$ préserve la restriction du produit scalaire de $\mathfrak{a}_0$ à $\mathfrak{a}_\emptyset$ d'après (1.7). Si deux éléments $y,\ y'$ de $\mathcal{W}_{M_\emptyset}^G$ vérifient $yA_\emptyset y^{-1} = y'A_\emptyset y'^{-1}$, on a $y'^{-1}y \in N_G(A_\emptyset)$. Il résulte de ce qui précède :

> Si $y \in \mathcal{W}_{M_\emptyset}^G$, le produit scalaire sur $y.\mathfrak{a}_\emptyset$ déduit de celui de $\mathfrak{a}_\emptyset$ ne dépend que de $y.\mathfrak{a}_\emptyset$. On le note encore $(.|.)$ et $|.|$ la norme qu'on en déduit. (3.12)

Si x est un élément de G tel que $x.A_\emptyset = A_i$, l'action par conjugaison induit une application linéaire de $\mathfrak{a}_\emptyset$ dans $\mathfrak{a}_i$ notée $X \mapsto x.X$ et caractérisée par (cf. (1.22)) :

$$H_{M_i}(xax^{-1}) = x.H_{M_\emptyset}(a), \quad a \in A_\emptyset,$$

où $M_i := x_i M_\emptyset x_i^{-1}$.

Pour $a \in A_\emptyset$ et $y \in \mathcal{W}_{M_\emptyset}^G$, on notera $a_y := yay^{-1}$ et on remarque qu'alors $a_y\sigma(a_y)^{-1} = a_y^2$ car $y.A_\emptyset$ est égal à A_i pour un $i \in I$ (cf. Paragraphe 1.2), donc $y.A_\emptyset$ est σ-déployé.

Lemme 7. *Soit Ω' une partie compacte de G.*

(i) *Pour tout $y \in \mathcal{W}_{M_\emptyset}^G$:*

$$1 + \log\|m'\| \asymp 1 + |H_{y.M_0}(m')|, \quad m' \in y.M_0.$$

(ii) *On a l'égalité :*

$$H_{y.M_0}(a_y^2) = 2y.H_{M_{\emptyset,\sigma}}(a), \quad y \in \mathcal{W}_{M_\emptyset}^G,\ a \in A_\emptyset.$$

(iii) *On a :*

$$N(\omega ay^{-1}H) \asymp (1 + |H_{M_{\emptyset,\sigma}}(a)|), \quad \omega \in \Omega',\ a \in A_\emptyset^-,\ y \in \mathcal{W}_{M_\emptyset}^G.$$

Il existe des constantes $c, C, c', C' > 0$ telles que :

$$Ce^{c|H_{M_0}(a)|} \leqslant \|\omega ay^{-1}H\| \leqslant C'e^{c'|H_{M_0}(a)|}, \quad \omega \in \Omega',\ a \in A_\emptyset^-,\ y \in \mathcal{W}_{M_\emptyset}^G.$$

(iv) *Pour tout $d \in \mathbb{N}$, on a :*

$$N_d(a_yH) \asymp N_d(aH), \quad y \in \mathcal{W}_{M_\emptyset}^G,\ a \in A_\emptyset^-.$$

Démonstration. (i) cf. [23, I.1(6)], en remarquant que la formule reste vraie pour $y.M_0$.

(ii) On a :

$$H_{y.M_0}(a_y^2) = y.H_{M_0}(a^2) = 2y.H_{M_0}(a) = 2y.H_{M_{\emptyset,\sigma}}(a),$$

la dernière égalité provenant du fait que $a \in A_\emptyset$.

(iii) Montrons qu'il existe des constantes $C, C' > 0$ telles que :

$$C(1 + |H_{M_{\emptyset,\sigma}}(a)|) \leqslant N(\omega ay^{-1}H) \leqslant C'(1 + |H_{M_{\emptyset,\sigma}}(a)|), \quad \omega \in \Omega',\ a \in A_\emptyset^-,\ y \in \mathcal{W}_{M_\emptyset}^G. \tag{3.13}$$

On a :

$$N(gH) = 1 + \log\|\omega y^{-1}a_y\sigma(a_y)^{-1}\sigma(y)\sigma(\omega)^{-1}\|, \quad g = \omega ay^{-1},\ \omega \in \Omega',\ a \in A_\emptyset^-,\ y \in \mathcal{W}_{M_\emptyset}^G,$$

d'où :

$$N(gH) \leqslant 1 + \log\|\omega\| + \log\|y^{-1}\| + \log\|a_y^2\| + \log\|\sigma(y)\| + \log\|\sigma(\omega)^{-1}\|,$$

$g=\omega a y^{-1}$, $\omega\in\Omega'$, $a\in A_\emptyset^-$, $y\in\mathcal{W}_{M_\emptyset}^G$.

Par compacité de Ω' et finitude de $\mathcal{W}_{M_\emptyset}^G$, il existe $c>0$ tel que :

$$N(gH)\leqslant c+\log\|a_y^2\|,\quad g=\omega a y^{-1},\ \omega\in\Omega',\ a\in A_\emptyset^-,\ y\in\mathcal{W}_{M_\emptyset}^G.$$

On en déduit l'inégalité de droite de l'équation (3.13) en combinant (i) et (ii). L'inégalité de gauche se démontre de la même façon en partant de l'égalité

$$a_y\sigma(a_y)^{-1}=y\omega^{-1}g\sigma(g)^{-1}\sigma(\omega)\sigma(y)^{-1}.$$

L'autre inégalité de (iii) est obtenue en exponentiant les inégalités de (3.13).

(iv) Il suffit de montrer l'assertion pour $d=1$. Soit $y\in\mathcal{W}_{M_\emptyset}^G$, en prenant $\Omega'=\{y\}$ dans (iii), on a :

$$N\big(yay^{-1}H\big)\asymp 1+\big|H_{M_{\emptyset,\sigma}}(a)\big|,\quad a\in A_\emptyset^-,$$

et, en appliquant à nouveau (iii) à $y=e$ et $\Omega'=\{e\}$, on obtient :

$$1+\big|H_{M_{\emptyset,\sigma}}(a)\big|\asymp N(aH),\quad a\in A_\emptyset^-,$$

d'où (iv) d'après la finitude de $\mathcal{W}_{M_\emptyset}^G$. □

Proposition 6.

(i) *Θ_G est $K_0\cap\sigma(K_0)$ invariante à gauche.*
(ii) *Soit Ω' une partie compacte de G. Il existe $C,C'>0$ et $d,d'\in\mathbb{N}$ tels que* :

$$C\delta_{\bar P_\emptyset}^{1/2}(a)N_{-d}(aH)\leqslant\Theta_G(gH)\leqslant C'\delta_{\bar P_\emptyset}^{1/2}(a)N_{d'}(aH),$$
$$g=\omega a y^{-1},\ \omega\in\Omega',\ a\in A_\emptyset^-,\ y\in\mathcal{W}_{M_\emptyset}^G.$$

Démonstration. (i) est clair.

(ii) Par compacité de Ω', il existe un ensemble fini F tel que $\Omega'\subset(K_0\cap\sigma(K_0))F$. Par invariance de Θ_G à gauche par $K_0\cap\sigma(K_0)$ et par finitude de F, on se réduit à étudier $\Theta_G(\omega a y^{-1}H)$, $a\in A_\emptyset^-$, $y\in\mathcal{W}_{M_\emptyset}^G$ pour $\omega\in F$ fixé. On le fixe. On a :

$$\Theta_G\big(\omega a y^{-1}H\big)=\big(\Xi\big(\omega y^{-1}a_y^2\sigma(y)\sigma(\omega)^{-1}\big)\big)^{1/2},\quad a\in A_\emptyset^-,\ y\in\mathcal{W}_{M_\emptyset}^G.$$

Montrons l'inégalité de droite de (ii). D'après [23, Lemme II.1.2] (cf. (3.5)) (appliqué à $g_1=\omega y^{-1}$ et $g_2=\sigma(y)\sigma(\omega)^{-1}$) et du fait de la finitude de $\mathcal{W}_{M_\emptyset}^G$, il existe $d\in\mathbb{N}$ et $c'>0$ tels que :

$$\Theta_G\big(\omega a y^{-1}H\big)^2\leqslant c'\,\Xi\big(a_y^2\big)N_d(a_yH),\quad a\in A_\emptyset^-,\ y\in\mathcal{W}_{M_\emptyset}^G.$$

En appliquant l'inégalité de droite du Lemme 6 à $m'=a_y^2\in y.A_\emptyset^-\subset y.\overline{M}_0^-$, il existe $d_2\in\mathbb{N}$ et $c''>0$ tels que :

$$\Theta_G\big(\omega a y^{-1}H\big)^2 \leqslant c''\delta_{y.\bar{P}_0}^{1/2}\big(a_y^2\big)N_{d+d_2}(a_y H), \quad a \in A_\emptyset^-,\ y \in \mathcal{W}_{M_\emptyset}^G.$$

Or :

$$\delta_{y.\bar{P}_0}\big(a_y^2\big) = \delta_{\bar{P}_0}\big(a^2\big) = \delta_{\bar{P}_\emptyset}(a)^2, \quad a \in A_\emptyset^-,\ y \in \mathcal{W}_{M_\emptyset}^G.$$

La dernière égalité provenant du fait que $a \in A_\emptyset$. On obtient alors l'inégalité voulue en appliquant le Lemme 7 (iv).

L'inégalité de gauche se démontre de la même façon en partant de l'égalité :

$$a_y\sigma(a_y)^{-1} = y\omega^{-1}g\sigma(g)^{-1}\sigma(\omega)\sigma(y)^{-1}, \quad g = \omega a y^{-1},\ \omega \in \Omega',\ a \in A_\emptyset^-,\ y \in \mathcal{W}_{M_\emptyset}^G,$$

et en utilisant l'inégalité de gauche du Lemme 6. □

Lemme 8. *Soient f_1 et f_2 deux fonctions définies sur G/H à valeurs dans $\mathbb{R}_+$ telles que* :

(a) *Pour tout $x \in G$ fixé, on ait* :

$$f_i(xgH) \asymp f_i(gH), \quad g \in G,\ i = 1, 2.$$

(b) *Pour un sous-groupe compact ouvert K, on ait $f_i(kgH) \asymp f_i(gH)$, $g \in G$, $k \in K$, $i = 1, 2$.*

(c) *Pour tout $y \in \mathcal{W}_{M_\emptyset}^G$, on ait* :

$$f_i\big(yay^{-1}H\big) \asymp f_i(aH), \quad a \in A_\emptyset^-,\ i = 1, 2.$$

(d) *$f_1(aH) \asymp f_2(aH)$, $a \in A_\emptyset^-$, $i = 1, 2$.*

Alors f_1 et f_2 sont équivalentes sur G/H.

Démonstration. On fixe Ω comme dans la décomposition de Cartan (1.21) et on utilise cette décomposition de l'espace symétrique G/H. Par compacité de Ω, il existe un ensemble fini F' tel que $\Omega \subset KF'$. Par finitude de $\mathcal{W}_{M_\emptyset}^G$ et F', il suffit de montrer que f_1 et f_2 sont équivalentes sur $\bigcup_{y \in \mathcal{W}_{M_\emptyset}^G} yA_\emptyset^- y^{-1}H$ d'après (a) et (b). Ce qui équivaut à montrer que f_1 et f_2 sont équivalentes sur $A_\emptyset^-$ d'après (c), ce qui est donné par (d). □

3.2. *Comparaison des normes $\|.\|$ et $\|.\|_{BD}$*

On note $\Sigma(G, A_0)$ (resp. $\Sigma(P_0, A_0)$ ou $\Sigma(P_0)$) l'ensemble des racines de A_0 dans l'algèbre de Lie de G (resp. P_0). On rappelle que l'on note $\Delta(P_0)$ l'ensemble des racines simples de $\Sigma(P_0)$. Si Θ est une partie de $\Delta(P_0)$, on note $\langle\Theta\rangle$ le sous-système de $\Sigma(G, A_0)$ engendré par Θ et $P_{\langle\Theta\rangle}$ le sous-groupe parabolique de G contenant P_0 pour lequel $\Sigma(P_0) \cup \langle\Theta\rangle$ est l'ensemble des racines de A_0 dans l'algèbre de Lie de $P_{\langle\Theta\rangle}$. On définit $\Theta_\emptyset$ par l'égalité :

$$P_\emptyset = P_{\langle\Theta_\emptyset\rangle},$$

On écrit

$$\Delta(P_0) = \{\alpha_1, \dots, \alpha_k\}, \qquad \Delta(P_0) \setminus \Theta_\emptyset = \{\alpha_1, \dots, \alpha_l\},$$

avec $k \geqslant l$.

On note $\delta_1, \dots, \delta_k \in \mathfrak{a}_0^*$ les poids fondamentaux de $\Sigma(P_0, A_0)$. Ils sont nuls sur $\mathfrak{a}_G$. Alors pour $i = 1, \dots, l$, $\delta_i \in \mathfrak{a}_{M_\emptyset}^*$. Reprenons les notations de [3, Paragraphe 2.7]. Il existe des entiers $n_1, \dots, n_l \in \mathbb{N}^*$ tels que $n_i\delta_i$ corresponde à un plus haut poids $\Lambda_i \in \operatorname{Rat} M_0$ d'une représentation rationnelle irréductible de dimension finie (π_i, V_i) de G de vecteur de plus haut poids v_i relativement à P_0 (cf. 1.3.2). La droite Fv_i est $P_\emptyset$-invariante (cf. [3, (2.23) et ce qui suit]). On note v_i^* l'unique élément de V_i^* de poids Λ_i^{-1} sous M_0 et vérifiant $\langle v_i^*, v_i\rangle = 1$. Pour $i = 1, \dots, l$, on notera $\tilde{\Lambda}_i := \Lambda_i(\Lambda_i^{-1} \circ \sigma)$ et $(\tilde{\pi}_i, \tilde{V}_i)$ la représentation rationnelle de G $(\pi_i \otimes (\pi_i^* \circ \sigma), V_i \otimes V_i^*)$. On note $\tilde{v}_i := v_i \otimes v_i^*$ qui est de poids $\tilde{\Lambda}_i$ sous $\tilde{\pi}_i$ relativement à M_0. Alors, il existe un vecteur H-invariant non nul sur $\tilde{\pi}_i^*$ dans $\tilde{V}_i^* = (V_i \otimes V_i^*)^* \simeq V_i^* \otimes V_i \simeq \operatorname{End} V_i$, égal à l'identité, $\tilde{e}_{i,H}^*$, vérifiant $\langle \tilde{e}_{i,H}^*, \tilde{v}_i\rangle = 1$. On notera $\tilde{\delta}_i := \delta_i - \delta_i \circ \sigma$, alors $\tilde{\delta}_i \in \mathfrak{a}_\emptyset^*$. Pour $i = 1, \dots, l$, on fixe une base de V_i formée de vecteurs poids sous A_0, ce qui permet de définir une norme sur V_i notée $\| \ \|_i$ en prenant le maximum des valuations des coordonnées dans cette base, puis une norme sur $\operatorname{End} V_i = \tilde{V}_i^*$ encore notée $\| \ \|_i$.

On pose, pour $g \in G$:

$$\|gH\|_i = \left\|\tilde{\pi}_i^*(g)\tilde{e}_{i,H}^*\right\|_i = \left\|\pi_i\left(g^\sigma g^{-1}\right)\right\|_i, \quad i = 1, \dots, l, \tag{3.14}$$

$$\|gH\|_0 = e^{|H_{G,\sigma}(g)|}, \tag{3.15}$$

où l'on a muni $\mathfrak{a}_{G,\sigma}$ de la norme provenant du produit scalaire sur $\mathfrak{a}_0$.

On définit :

$$\|gH\|_{BD} = \prod_{i=o}^{l} \|gH\|_i, \quad g \in G. \tag{3.16}$$

Proposition 7. *Il existe $c, C, c', C' > 0$ tels que :*

$$C\|gH\|_{BD}^c \leqslant \|gH\| \leqslant C'\|gH\|_{BD}^{c'}, \quad g \in G. \tag{3.17}$$

Démonstration. Soit V un espace vectoriel sur F de dimension finie et soit $\| \ \|$ une norme sur $\operatorname{End} V$ déduite d'une norme sur V. Alors si Γ est une partie compacte de $GL(V)$, on a :

$$\|\gamma T\gamma'\| \asymp \|T\|, \quad T \in \operatorname{End} V. \tag{3.18}$$

Par ailleurs,

$$-\left|H_{G,\sigma}(g)\right| + \left|H_{G,\sigma}(g')\right| \leqslant \left|H_{G,\sigma}(gg')\right| \leqslant \left|H_{G,\sigma}(g)\right| + \left|H_{G,\sigma}(g')\right|, \quad g, g' \in G. \tag{3.19}$$

Tenant compte de la deuxième égalité de (3.14), de (3.18) et de (3.19), et en écrivant $\omega a y^{-1} = \omega y^{-1}(yay^{-1})$, on déduit de ce qui précède :

$$\left\|\omega a y^{-1}H\right\|_{BD} \asymp \left\|yay^{-1}H\right\|_{BD}, \quad \omega \in \Omega,\ a \in A_\emptyset^-,\ y \in \mathcal{W}_{M_\emptyset}^G.$$

Et d'après (3.3) :

$$\left\|\omega a y^{-1} H\right\| \asymp \left\|y a y^{-1} H\right\|, \quad \omega \in \Omega,\ a \in A_{\emptyset}^{-},\ y \in \mathcal{W}_{M_{\emptyset}}^{G}.$$

Montrons que :

$$\left\|y a y^{-1} H\right\| \asymp \|aH\|, \quad a \in A_{\emptyset}^{-},\ y \in \mathcal{W}_{M_{\emptyset}}^{G}. \tag{3.20}$$

En effet, l'égalité $\sigma(a_y) = a_y^{-1}$ implique $\|yay^{-1}H\| = \|a_y^2\|$, donc $\|yay^{-1}H\| = \|ya^2y^{-1}\|$. Or, d'après (3.3) :

$$\left\|y a^2 y^{-1}\right\| \asymp \left\|a^2\right\| = \|aH\|, \quad a \in A_{\emptyset}^{-},\ y \in \mathcal{W}_{M_{\emptyset}}^{G}.$$

D'où (3.20), ce qui implique :

$$\left\|\omega a y^{-1} H\right\| \asymp \|aH\|, \quad \omega \in \Omega,\ a \in A_{\emptyset}^{-},\ y \in \mathcal{W}_{M_{\emptyset}}^{G}. \tag{3.21}$$

On a le même résultat pour $\|\ \|_{BD}$.

Tenant compte de la décomposition de Cartan (1.21), on est ramené à prouver les inégalités de la Proposition pour $g = a \in A_{\emptyset}^{-}$.

Pour $i = 1, \dots, l$ on fixe une base, $f_{i,1}, \dots, f_{i,r}$, de $\tilde{V}_i^*$ formée de vecteurs poids sous A_0 telle que $f_{i,1} = v_i^* \otimes v_i$ (donc de poids $\tilde{\Lambda}_i^{-1}$ sous $\tilde{\pi}_i^*$) et telle que $\tilde{e}_{i,H}^* = \sum_{j=1}^r c_{i,j} f_{i,j}$, où $c_{i,1} = 1$ et pour $j = 2, \dots, r$, $c_{i,j} = 0$ ou 1. On note $\|\ \|_i'$ la norme sur $\tilde{V}_i^*$ qu'on en déduit. D'après l'équivalence des normes en dimension finie, $\|\ \|_i \asymp \|\ \|_i'$. Alors, en utilisant la première égalité de (3.14), on a :

$$\|aH\|_i \asymp \left\|\sum_{j=1}^{r} c_{i,j} \tilde{\pi}_i^*(a) f_{i,j}\right\|_i', \quad a \in A_{\emptyset}^{-}.$$

Soit encore :

$$\|aH\|_i \asymp \left\|\tilde{\Lambda}_i^{-1}(a) f_{i,1} + \sum_{j=2}^{r} c_{i,j} \tilde{\pi}_i^*(a) f_{i,j}\right\|_i', \quad a \in A_{\emptyset}^{-}.$$

Comme (π_i, V_i) est une représentation de plus haut poids Λ_i, l'inspection des poids sous $A_{\emptyset}$ de $\tilde{\pi}_i = \pi_i \otimes (\pi_i^* \circ \sigma)$ montre que $|\tilde{\Lambda}_i^{-1}(a)|_F = e^{-n_i \tilde{\delta}_i(H_{M_0}(a))}$ est la plus grande valuation des valeurs propres de $\tilde{\pi}_i(a)$ pour $a \in A_{\emptyset}^{-}$. Puisque pour $i = 2, \dots, r$, $c_{i,j} = 0$ ou 1, on a donc :

$$\|aH\|_i \asymp e^{-n_i \tilde{\delta}_i(H_{M_0}(a))}, \quad a \in A_{\emptyset}^{-},\ i = 1, \dots, l. \tag{3.22}$$

D'où :

$$\|aH\|_{BD} \asymp e^{-\sum_{i=1}^{l} n_i \tilde{\delta}_i(H_{M_0}(a)) + |H_{G,\sigma}(a)|}, \quad a \in A_{\emptyset}^{-}. \tag{3.23}$$

On note $\mathfrak{a}_{\emptyset,G} := \mathfrak{a}_\emptyset \cap \mathfrak{a}_G$, $\mathfrak{a}_\emptyset^G := \mathfrak{a}_{M_\emptyset}^G \cap \mathfrak{a}_\emptyset$ (cf. (1.4)), $\bar{\mathfrak{a}}_\emptyset^+ := \bar{\mathfrak{a}}_{P_\emptyset}^+ \cap \mathfrak{a}_\emptyset$ et $\mathfrak{a}_\emptyset^{*+} := \mathfrak{a}_\emptyset^* \cap \mathfrak{a}_{P_\emptyset}^{*+}$. On remarque que $\mathfrak{a}_\emptyset^{*+}$ est non vide car $\rho_{P_\emptyset} \in \mathfrak{a}_\emptyset^{*+}$: en effet, $\rho_{P_\emptyset}$ est élément de $\mathfrak{a}_{P_\emptyset}^{*+}$ et $\rho_{P_\emptyset} \in \mathfrak{a}_\emptyset^*$ car $\sigma(\rho_{P_\emptyset}) = -\rho_{P_\emptyset}$ puisque $P_\emptyset$ est un σ-sous-groupe parabolique.

Montrons que :

$$\begin{aligned}&\text{Pour } \mu \in \mathfrak{a}_\emptyset^{*+} \text{ et } X \in \bar{\mathfrak{a}}_\emptyset^+, \text{ si l'on note } X = X^G + X_G \text{ où } X^G \in \mathfrak{a}_\emptyset^G \text{ et } X_G \in \mathfrak{a}_{\emptyset,G},\\ &\text{on a} : |X| \asymp \mu(X^G) + |X_G|,\ X \in \bar{\mathfrak{a}}_\emptyset^+.\end{aligned} \tag{3.24}$$

En remarquant que la norme $|.|$ sur $\mathfrak{a}_\emptyset$ est équivalente à la norme $X \mapsto |X^G| + |X_G|$, où $X = X^G + X_G$, avec $X^G \in \mathfrak{a}_\emptyset^G$ et $X_G \in \mathfrak{a}_{\emptyset,G}$ on se ramène à prouver qu'il existe des constantes $C_1, C_2 > 0$ telles que :

$$C_1|X| > \mu(X) > C_2|X|, \quad X \in \bar{\mathfrak{a}}_\emptyset^+ \cap \mathfrak{a}_\emptyset^G,$$

ce qui résulte du fait que la fonction μ est continue et ne s'annule pas sur le compact $\{X \in \bar{\mathfrak{a}}_\emptyset^+ \cap \mathfrak{a}_\emptyset^G;\ |X| = 1\}$. C'est une conséquence du fait que les poids fondamentaux sont des produits scalaires positifs ou nuls.

On applique (3.24) à $\mu := \sum_{i=1}^l n_i \tilde{\delta}_i$ et $X := -H_{M_0}(a), a \in A_\emptyset^-$. Alors $X_G = -H_{G,\sigma}(a)$ et $\tilde{\delta}_i(X) = \tilde{\delta}_i(X^G)$, on obtient :

$$-\sum_{i=1}^l n_i \tilde{\delta}_i\bigl(H_{M_0}(a)\bigr) + \bigl|H_{G,\sigma}(a)\bigr| \asymp \bigl|-H_{M_0}(a)\bigr|, \quad a \in A_\emptyset^-.$$

En exponentiant cette dernière équivalence jointe à (3.23), on trouve qu'il existe des constantes $c_1, C_1, c_1', C_1' > 0$ telles que :

$$C_1 e^{c_1|H_{M_0}(a)|} \leqslant \|aH\|_{BD} \leqslant C_1' e^{c_1'|H_{M_0}(a)|}, \quad a \in A_\emptyset^-. \tag{3.25}$$

Or, d'après le Lemme 7 (iii), on obtient des inégalités similaires pour $\|aH\|$ et ces inégalités conduisent au résultat voulu. □

3.3. Majorations de certains coefficients généralisés de représentations admissibles de type fini

Théorème 4.

(i) *Soit (π, V) une représentation admissible de type fini de G et soit $\xi \in V^{*H}$. Il existe $c > 0$ tel que, pour tout $v \in V$, il existe $C > 0$ vérifiant* :

$$\bigl|\langle \pi^*(g)\xi, v\rangle\bigr| \leqslant C\|gH\|^c, \quad g \in G.$$

(ii) *Si (π, V) une représentation bornée irréductible de G et $\xi \in V^{*H}$, alors pour tout $v \in V$, il existe $C > 0$ vérifiant* :

$$\bigl|\langle \pi^*(g)\xi, v\rangle\bigr| \leqslant C, \quad g \in G.$$

Démonstration. (i) Montrons le Théorème par récurrence sur la dimension de G. Si $\dim G = 0$, le Théorème est trivial. Supposons maintenant qu'il soit vrai pour tout autre groupe linéaire algébrique réductif et connexe de dimension strictement inférieure à celle de G et montrons le pour G. En utilisant le fait que :

$$\langle \pi^*(g)\xi, \pi(x)v\rangle = \langle \pi^*(x^{-1}g)\xi, v\rangle, \quad x, g \in G,\ v \in V,$$

et grâce à (3.9), on se ramène à montrer l'inégalité de (i) pour un ensemble fini de générateurs de V. Donc il suffit de montrer que :

Pour tout $v \in V$, il existe $C > 0$ et $c > 0$ tels que :

$$\left|\langle \pi^*(g)\xi, v\rangle\right| \leqslant C\|gH\|^c, \quad g \in G. \tag{3.26}$$

Soit $v \in V$.

(a) Montrons d'abord que :

Il existe $C > 0$ et $c > 0$ vérifiant :

$$\left|\langle \pi^*(a)\xi, v\rangle\right| \leqslant C\|aH\|^c, \quad a \in A_\emptyset^-. \tag{3.27}$$

Pour $\Theta \subseteq \Delta(P_\emptyset, A_\emptyset)$ et pour $0 < \varepsilon \leqslant 1$, on définit ${}_\Theta A_\emptyset^-(\varepsilon)$ par :

$$\left\{a \in A_\emptyset;\ \left|\alpha(a)\right|_F < \varepsilon \text{ pour } \alpha \in \Delta(P_\emptyset, A_\emptyset) \setminus \Theta \text{ et } \varepsilon \leqslant \left|\alpha(a)\right|_F \leqslant 1 \text{ pour } \alpha \in \Theta\right\}.$$

Alors on a l'analogue de ce qui suit le Théorème 4.3.3 de [9] :

Pour tout $0 < \varepsilon \leqslant 1$, $A_\emptyset^-$ est l'union disjointe des ${}_\Theta A_\emptyset^-(\varepsilon)$ lorsque Θ parcourt les sous ensembles de $\Delta(P_\emptyset, A_\emptyset)$. (3.28)

D'après le Théorème 2, pour chaque $\Theta \subseteq \Delta(P_\emptyset, A_\emptyset)$, il existe $\varepsilon_\Theta > 0$ tel que :

$$\delta_{P_\Theta}(a)^{-1/2}\langle \xi, \pi(a)v\rangle = \langle j_{P_\Theta}^*(\xi), \pi_{P_\Theta}(a) j_{P_\Theta}(v)\rangle, \quad a \in A_\emptyset^-(P_\Theta, < \varepsilon_\Theta). \tag{3.29}$$

On pose $\varepsilon := \min_{\Theta \subseteq \Delta(P_\emptyset, A_\emptyset)} \varepsilon_\Theta$. D'après (3.28), $A_\emptyset^-$ est l'union disjointe des ${}_\Theta A_\emptyset^-(\varepsilon)$ lorsque Θ parcourt les sous ensembles de $\Delta(P_\emptyset, A_\emptyset)$. Or, on remarque que ${}_\Theta A_\emptyset^-(\varepsilon) \subset A_\emptyset^-(P_\Theta, < \varepsilon)$ pour $\Theta \subseteq \Delta(P_\emptyset, A_\emptyset)$. D'après (3.29), et la définition de ε, on a donc :

$$\delta_{P_\Theta}(a)^{-1/2}\langle \xi, \pi(a)v\rangle = \langle j_{P_\Theta}^*(\xi), \pi_{P_\Theta}(a) j_{P_\Theta}(v)\rangle, \quad a \in_\Theta A_\emptyset^-(\varepsilon),\ \Theta \subseteq \Delta(P_\emptyset, A_\emptyset).$$

Pour $\Theta \subseteq \Delta(P_\emptyset, A_\emptyset)$ en utilisant la restriction de τ à M_Θ qui définit un plongement de M_Θ dans $GL_n(F)$ (cf. (3.1)), on définit la norme sur M_Θ et sur $M_\Theta \cap H$ (cf. (3.2) et (3.8)). Donc :

$$\|mM_\Theta \cap H\| = \|mH\|, \quad m \in M_\Theta. \tag{3.30}$$

Pour $\Theta \subset \Delta(P_\emptyset, A_\emptyset)$, $\Theta \neq \Delta(P_\emptyset, A_\emptyset)$, on peut appliquer l'hypothèse de récurrence à la représentation admissible de type fini $(\pi_{P_\Theta}, V_{P_\Theta})$ de M_Θ et à $j^*_{P_\Theta}(\xi) \in (V_{P_\Theta})^{*M_\Theta \cap H}$. De plus, comme $\varepsilon \in]0,1]$, on a :

$$\delta_{P_\Theta}(a)^{1/2} \leqslant 1, \quad a \in_\Theta A^-_\emptyset(\varepsilon).$$

Alors :

Il existe $c_\Theta > 0$ et $C_\Theta > 0$ tels que :

$$\left|\left\langle j^*_{P_\Theta}(\xi), \pi_{P_\Theta}(a) j_{P_\Theta}(v)\right\rangle\right| \leqslant C_\Theta \|aH\|^{c_\Theta}, \quad a \in_\Theta A^-_\emptyset(\varepsilon), \tag{3.31}$$

puisque $\|a^{-1}M_\Theta \cap H\| = \|aM_\Theta \cap H\|$ et $\|aM_\Theta \cap H\| = \|aH\|$.

Soit $A_{\emptyset,G}$ le plus grand tore déployé de $A_\emptyset \cap A_G$. Alors $_{\Delta(P_\emptyset,A_\emptyset)}A^-_\emptyset(\varepsilon)$ est compact modulo $A_{\emptyset,G}$ donc de la forme $\Omega' A_{\emptyset,G}$ où Ω' est une partie compacte de G. Montrons que :

Il existe $c_\Delta > 0$ et $C_\Delta > 0$ tels que :

$$\left|\left\langle \pi^*(a)\xi, v\right\rangle\right| \leqslant C_\Delta \|aH\|^{c_\Delta}, \quad a \in_{\Delta(P_\emptyset,A_\emptyset)} A^-_\emptyset(\varepsilon). \tag{3.32}$$

L'application $a \to \langle \pi^*(a)\xi, v\rangle$ est invariante par un sous-groupe compact ouvert K' de $A_\emptyset$. Comme Ω' est contenu dans une réunion finie de classes à gauche de K' dans $A_\emptyset$, il suffit de prouver une inégalité du même type pour un nombre fini $v_1, \dots, v_l$ de translatés de v, mais seulement pour $A_{\emptyset,G}$. Montrons-le.

Soit $i = 1, \dots, l$. Comme (π, V) est admissible, la restriction de la fonction $g \to \langle \pi^*(g)\xi, v_i\rangle$ à A_G est A_G-finie. D'après [23, I.2], il existe $\chi_1, \dots, \chi_m$, des caractères non ramifiés de A_G et $r > 0$ tels que :

$$\left|\left\langle \pi^*(a)\xi, v_i\right\rangle\right| \leqslant \operatorname{Sup}\left(\left|\chi_1(a)\right|, \dots, \left|\chi_M(a)\right|\right)\left(1 + \log\|a\|\right)^r, \quad a \in A_G. \tag{3.33}$$

Mais si χ est un caractère non ramifié de A_G, il existe $\lambda \in (\mathfrak{a}^*_G)_{\mathbb{C}}$ tel que :

$$\chi(a) = e^{\lambda(H_G(a))}, \quad a \in A_G.$$

Donc :

$$\left|\chi(a)\right| \leqslant e^{|\lambda||H_G(a)|}, \quad a \in A_G, \tag{3.34}$$

où $|\lambda|$ est la norme de la forme linéaire λ. Mais (3.25) jointe à la Proposition 7 montre qu'il existe $c_0 > 0$ et $C_0 > 0$ tels que :

$$C_0 e^{c_0|H_{M_0}(a)|} \leqslant \|aH\|, \quad a \in A^-_\emptyset. \tag{3.35}$$

Comme $H_{M_0}(a) = H_G(a)$, $a \in A_G$, on déduit alors de (3.34) et (3.35) qu'il existe $c_\chi > 0$ et $C_\chi > 0$ tels que :

$$\left|\chi(a)\right| \leqslant C_\chi \|aH\|^{c_\chi}, \quad a \in A_{\emptyset,G} \subset A_G \cap A^-_\emptyset.$$

L'inégalité voulue pour v_i résulte alors de (3.33), et de l'inégalité :

$$1+\log x \leqslant x, \quad x \geqslant 1,$$

ce qui achève de prouver (3.32). Donc (3.31) est vraie aussi pour $\Theta = \Delta(P_\emptyset, A_\emptyset)$.

Or $\|aH\| \geqslant 1$ (cf. (3.3)). On pose alors

$$c := \max_{\Theta\subseteq\Delta(P_\emptyset,A_\emptyset)} c_\Theta > 0 \quad \text{et} \quad C := \max_{\Theta\subseteq\Delta(P_\emptyset,A_\emptyset)} C_\Theta > 0.$$

En utilisant (3.28), on obtient (3.27).

(b) On reprend les notations de 1.2. Par un raisonnement similaire, on peut remplacer $A_\emptyset$ par l'un des $A_i := x_i A_\emptyset x_i^{-1}$ pour $i \in I$ dans (a). Toutefois, pour définir la norme sur $M_{i,\Theta} := x_i M_\Theta x_i^{-1}$, on doit utiliser le plongement $\tau \circ \operatorname{Ad} x_i^{-1}$. Alors on a seulement :

$$\|mM_{i,\Theta} \cap H\| \asymp \|mH\|, \quad m \in M_{i,\Theta},$$

mais cela suffit à achever la démonstration de (3.27) pour A_i au lieu de $A_\emptyset$. Donc pour tout $v \in V$, il existe $c_i > 0$ et $C_i > 0$ vérifiant :

$$\left|\left\langle \pi^*(a)\xi, v\right\rangle\right| \leqslant C_i \|aH\|^{c_i}, \quad a \in A_i^- := x_i A_\emptyset^- x_i^{-1}.$$

On souhaite obtenir une telle inégalité sur A_i et non sur A_i^-. Or, $A_\emptyset$ est la réunion d'ensembles $A_\emptyset^-$ lorsque $P_\emptyset$ décrit l'ensemble des σ-sous-groupes paraboliques contenant $A_\emptyset$. Le choix de $P_\emptyset$ étant indifférent dans ce qui précède et I étant fini, on en déduit que :

Pour tout $v \in V$, il existe $c > 0$ et $C > 0$ vérifiant :

$$\left|\left\langle \pi^*(a)\xi, v\right\rangle\right| \leqslant C\|aH\|^c, \quad a \in A_i,\ i \in I. \tag{3.36}$$

On déduit de (1.21) que :

$$G = \bigcup_{y\in\mathcal{W}_{M_\emptyset}^G} \Omega y^{-1} A_y H, \quad A_y = yA_\emptyset y^{-1}.$$

Soit $v \in V$ et K un sous-groupe compact ouvert tel que $v \in V^K$. Soit $g \in \Omega y^{-1} a_y H$, $y \in \mathcal{W}_{M_\emptyset}^G$, $a_y \in A_y$. Alors :

$$\left\langle \pi^*(g)\xi, v\right\rangle \in \left\{\left\langle \pi^*(a_y)\xi, \pi\left(y\omega^{-1}\right)v\right\rangle,\ \omega \in \Omega\right\}.$$

L'ensemble $y\Omega$ étant compact et v étant invariant par K, l'ensemble $\{\pi(y\omega^{-1})v, \omega \in \Omega\}$ est fini. Par conséquent, il existe un nombre fini d'éléments de V, indépendants de a_y, notés $v_1', \dots, v_r'$ tels que :

$$\left\langle \pi^*(g)\xi, v\right\rangle \in \left\{\left\langle \pi^*(a_y)\xi, v_j'\right\rangle,\ j = 1, \dots, r\right\}. \tag{3.37}$$

Soit $j \in \{1, \dots, r\}$. Comme A_y est égal à l'un des A_i, d'après (3.36) appliqué à v'_j au lieu de v, il existe $c_{y,j} > 0$ et $C_{y,j} > 0$ tels que :

$$\left|\left\langle \pi^*(a_y)\xi, v'_j\right\rangle\right| \leqslant C_{y,j} \|a_y H\|^{c_{y,j}}, \quad a_y \in A_y.$$

En prenant $C' := \mathrm{Max}_{j\in\{1,\dots,r\},\ y\in\mathcal{W}^G_{M_\emptyset}} C_{y,j}$ et $c' := \mathrm{Max}_{j\in\{1,\dots,r\},\ y\in\mathcal{W}^G_{M_\emptyset}} c_{y,j}$, on obtient :

$$\left|\left\langle \pi^*(g)\xi, v\right\rangle\right| \leqslant C' \|a_y H\|^{c'}, \quad gH = \omega y^{-1} a_y H,\ \omega \in \Omega,\ a_y = y a y^{-1},\ a \in A^-_\emptyset,\ y \in \mathcal{W}^G_{M_\emptyset}.$$

Or, d'après (3.20), pour $a \in A^-_\emptyset$, on a : $\|a_y H\| \asymp \|aH\|$ et d'après (3.21), pour $a \in A^-_\emptyset$, on a : $\|aH\| \asymp \|gH\|$. D'où (i).

(ii) Soit $v \in V$ fixé, d'après (3.37), il suffit de montrer que :

Pour tout $i \in I$, il existe $C_i > 0$ tel que :

$$\left|\left\langle \pi^*(a_i)\xi, v\right\rangle\right| \leqslant C_i, \quad a_i \in A_i. \tag{3.38}$$

Soit K un sous-groupe ouvert compact tel que $v \in V^K$. D'après le Lemme 2, il existe un sous-groupe ouvert compact K' de K tel que :

$$\left\langle \pi^*\left(a_i^{-1}\right)\xi, v\right\rangle = \left\langle \pi^*\left(a_i^{-1}\right)\pi^*(e_{K'})\xi, v\right\rangle, \quad a_i \in A_i^- \subset A^-_{M_{i,\emptyset}},$$

où $M_{i,\emptyset} := x_i M_\emptyset x_i^{-1}$. Or (π, V), étant bornée, il existe $C'_i > 0$ tel que :

$$\left|\left\langle \pi^*\left(a_i^{-1}\right)\pi^*(e_{K'})\xi, v\right\rangle\right| \leqslant C'_i, \quad a_i \in A_i^-.$$

On en déduit (3.38) en procédant comme dans la preuve de (3.36), et (ii) en résulte alors. □

Remarque 3. Ce théorème permet de voir qu'une des hypothèses du Théorème 3 de [3] est toujours satisfaite.

4. Un analogue d'un Lemme de Langlands

4.1. Résultats préliminaires

Soit $P = MU$ un σ-sous-groupe parabolique contenant $P_\emptyset$. Par abus de notation, pour $x \in G$, on note $x = k(x)m(x)u(x)$, $k(x) \in K_0$, $m(x) \in M$, $u(x) \in U$. On note p_M la projection de P sur M de noyau U. On note $K_{0,M} := p_M(P \cap K_0)$ qui est un sous-groupe compact de M. Alors $m(x)$ est défini modulo l'action à gauche de $K_{0,M}$. On remarque que $H_M(m(x))$ est bien défini et pour $\chi \in X(M)$, $\chi(m(x))$ l'est également. De plus,

Si Ω est un compact de G, l'ensemble $\{m(x) \mid x \in \Omega\}$ est inclus dans un compact de M. (4.1)

En effet, Ω est inclus dans un nombre fini de classes à gauche modulo K_0, et on utilise ce qui précède.

Lemme 9.

(i) *Soit* (π_Λ, V) *une représentation rationnelle de* G*, de dimension finie, de plus haut poids* $\Lambda \in \mathrm{Rat}(M_0)$*, ayant un vecteur,* v_Λ*, de plus haut poids* Λ*, et un vecteur,* $e^*_{\Lambda,H}$*,* H*-invariant dans* V^* *pour* π^*_Λ*, vérifiant* $\langle e^*_{\Lambda,H}, v_\Lambda\rangle = 1$*. On note* λ *(ou* λ_Λ*) l'élément de* $\mathfrak{a}^*_0$ *tel que* $e^{\lambda(H_{M_0}(m))} = |\Lambda(m)|_F$ *pour* $m \in M_0$*. On suppose de plus que* λ *est élément de* $\mathfrak{a}^*_M$*. Alors il existe* $C \in \mathbb{R}$ *tel que :*

$$\lambda\big(H_M\big(m(h)\big)\big) \geqslant C, \quad h \in H.$$

(ii) *Si* $\chi \in X(M)_\sigma$*, avec* $\mathrm{Re}\,\chi$ *strictement* P*-dominant ; alors il existe* $C' > 0$ *tel que* $|\chi(m(h))| \geqslant C'$*,* $h \in H$*.*

Démonstration. (i) Soit $h \in H$. On déduit de l'invariance de $e^*_{\Lambda,H}$ par h^{-1} l'équation :

$$\big\langle \pi^*_\Lambda\big(\big(u(h)\big)^{-1}\big(m(h)\big)^{-1}\big(k(h)\big)^{-1}\big)e^*_{\Lambda,H}, v_\Lambda\big\rangle = 1,$$

où l'on a écrit :

$$h = k(h)m(h)u(h). \tag{4.2}$$

Comme λ est élément de $\mathfrak{a}^*_M$, il existe (cf. e.g. [3, (2.23)]) un caractère rationnel, Λ_1, de M, tel que :

$$\pi_\Lambda(m)v_\Lambda = \Lambda_1(m)v_\Lambda, \quad m \in M. \tag{4.3}$$

Il existe un élément λ_1 de $\mathfrak{a}^*_M$ tel que $e^{\lambda_1(H_M(m))} = |\Lambda_1(m)|_F$, $m \in M$. Donc :

$$e^{\lambda(H_{M_0}(m))} = e^{\lambda_1(H_M(m))}, \quad m \in M_0.$$

En utilisant (1.4) pour $M = M_0$ et $G = M$, on trouve que $\lambda_1 = \lambda$ et :

$$\big|\Lambda_1(m)\big|_F = e^{\lambda(H_M(m))}, \quad m \in M. \tag{4.4}$$

D'après (4.2), on a :

$$\big\langle \pi^*_\Lambda\big(\big(k(h)\big)^{-1}\big)e^*_{\Lambda,H}, v_\Lambda\big\rangle = \big\langle \pi^*_\Lambda\big(m(h)\big)\pi^*_\Lambda\big(u(h)\big)\pi^*_\Lambda\big(h^{-1}\big)e^*_{\Lambda,H}, v_\Lambda\big\rangle.$$

Mais $e^*_{\Lambda,H}$ est invariant par H et v_Λ vérifie (4.3) et est invariant par U, donc

$$\big\langle \pi^*_\Lambda\big(\big(k(h)\big)^{-1}\big)e^*_{\Lambda,H}, v_\Lambda\big\rangle = \Lambda_1\big(m(h)\big)^{-1}. \tag{4.5}$$

Pour des raisons de compacité, il existe une constante $C' > 0$ telle que :

$$\big|\big\langle \pi^*_\Lambda(k)e^*_{\Lambda,H}, v_\Lambda\big\rangle\big|_F \leqslant C', \quad k \in K_0.$$

De (4.4) et (4.5), on déduit :

$$e^{-\lambda(H_M(m(h)))} \leqslant C', \quad h \in H.$$

D'où l'assertion en prenant successivement l'inverse puis le logarithme de l'inégalité.

(ii) On a : $\operatorname{Re}\chi \in (\mathfrak{a}_{M,\sigma})^*$. Soient χ_1 dans $X(M)_\sigma$ et χ_2 dans $X(G)_\sigma$ tels que $\operatorname{Re}\chi_1 \in (\mathfrak{a}_{M,\sigma}/\mathfrak{a}_{G,\sigma})^*$, $\operatorname{Re}\chi_2 \in (\mathfrak{a}_{G,\sigma})^*$ et $\operatorname{Re}\chi = \operatorname{Re}\chi_1 + \operatorname{Re}\chi_2$. Alors $|\chi| = |\chi_1||\chi_2|$. D'autre part,

$$\bigl|\chi_2\bigl(m(h)\bigr)\bigr| = 1, \quad h \in H.$$

En effet, puisque $|\chi_2| \in X(G)_\sigma$ et est à valeurs dans $\mathbb{R}^+$,

$$\bigl|\chi_2(h)\bigr| = 1, \quad h \in H,$$

car $|\chi_2| \circ \sigma = |\chi_2|^{-1}$, donc $|\chi_2|(h) = |\chi_2|^{-1}(h),\ h \in H$. Or :

$$|\chi_2|(h) = |\chi_2|\bigl(k(h)m(h)u(h)\bigr) = |\chi_2|\bigl(m(h)\bigr), \quad h \in H,$$

la dernière égalité provenant du fait que les caractères non ramifiés sont triviaux sur les sous-groupes compacts et les sous-groupes unipotents. On a donc :

$$\bigl|\chi\bigl(m(h)\bigr)\bigr| = \bigl|\chi_1\bigl(m(h)\bigr)\bigr|, \quad h \in H.$$

Or $\operatorname{Re}\chi_1 \in (\mathfrak{a}_{M,\sigma}/\mathfrak{a}_{G,\sigma})^*$ donc, d'après [3, Remarque 1], il existe $n_1, \ldots, n_k \in \mathbb{R}$ et $\lambda_1, \ldots, \lambda_k \in \mathfrak{a}_0^*$ tels que $\operatorname{Re}\chi_1 = \sum_{i=1}^k n_i\lambda_i$ et tels que pour $i = 1, \ldots, k$, il existe une représentation rationnelle de G de plus haut poids Λ_i vérifiant les propriétés de (i) avec $\lambda_i := \lambda_{\Lambda_i}$. Puisque $\operatorname{Re}\chi$ est strictement P-dominant, les n_i sont positifs. On obtient alors :

$$\bigl|\chi\bigl(m(h)\bigr)\bigr| = e^{\operatorname{Re}\chi_1(H_M(m(h)))} = \prod_{i=1}^{k} e^{n_i\lambda_i(H_M(m(h)))}, \quad h \in H.$$

D'où (ii) en appliquant (i) aux λ_i. □

Lemme 10. *Soit $\chi \in X(M)_\sigma$ tel que* $\operatorname{Re}\chi$ *soit strictement P-dominant. Alors il existe $C > 0$ tel que pour tout $a \in A_M$ vérifiant $H_M(a) \in -\bar{\mathfrak{a}}_P^+$ on ait :*

$$\bigl|\chi\bigl(m\bigl(a^{-1}\bar{u}a\bigr)\bigr)\bigr| \leqslant C\bigl|\chi\bigl(m(\bar{u})\bigr)\bigr|, \quad \bar{u} \in \bar{U} := \sigma(U).$$

Démonstration. Soit $\nu \in \mathfrak{a}_P^{*+}$ (cf.(1.8)) tel que l'on ait : $e^{\nu(H_M(m))} = |\chi(m)|$, $m \in M$. Il résulte de [10, (3.14)] qu'il existe $Y_P \in \mathfrak{a}_M$ tel que l'on ait :

$$H_M\bigl(m(\bar{u})\bigr) - H_M\bigl(m\bigl(a^{-1}\bar{u}a\bigr)\bigr) \in Y_P + {}^+\bar{\mathfrak{a}}_P, \quad \bar{u} \in \bar{U},\ a \in A_M \cap H_M^{-1}\bigl(-\bar{\mathfrak{a}}_P^+\bigr).$$

Comme ν est P-dominant, on a :

$$\nu\bigl(H_M\bigl(m(\bar{u})\bigr) - H_M\bigl(m\bigl(a^{-1}\bar{u}a\bigr)\bigr)\bigr) \geqslant \nu(Y_P), \quad \bar{u} \in \bar{U},\ a \in A_M \cap H_M^{-1}\bigl(-\bar{\mathfrak{a}}_P^+\bigr).$$

Soit encore :

$$e^{\nu(H_M(m(a^{-1}\bar{u}a)))} \leqslant e^{-\nu(Y_P)} e^{\nu(H_M(m(\bar{u})))}, \quad \bar{u} \in \bar{U},\ a \in A_M \cap H_M^{-1}\left(-\bar{\mathfrak{a}}_P^+\right).$$

D'où le Lemme. □

4.2. Une propriété asymptotique des intégrales d'Eisenstein

Soit $P = MU$ un σ-sous-groupe-parabolique de G. Soit (δ, V_δ) une représentation admissible de type fini de M. On introduit pour $\chi \in X(M)_\sigma$, la représentation $\delta_\chi = \delta \otimes \chi$ de M. L'espace de δ_χ s'identifie à V_δ. On étend l'action de M à P en la prenant triviale sur U. On rappelle que $\operatorname{ind}_P^G V_\delta$, noté aussi $I_\chi^P(\delta)$, est l'espace des applications $\varphi : G \to V_\delta$ qui sont invariantes à gauche par un sous-groupe compact ouvert et telles que :

$$\varphi(gmu) = \delta_P^{-1/2}(m)\delta_\chi\left(m^{-1}\right)\varphi(g), \quad g \in G, m \in M, u \in U.$$

Le groupe G agit sur $I_\chi^P(\delta)$ par la représentation régulière gauche $\pi_{\delta,\chi}^P$. On note $\bar{I}^P(\delta)$ l'espace de $\operatorname{ind}_{K_0\cap P}^{K_0} \delta|_{K_0\cap P}$. Alors la restriction des fonctions à K_0 détermine un isomorphisme de K_0-modules entre $I_\chi^P(\delta)$ et $\bar{I}^P(\delta)$. On note $\bar{\pi}_{\delta,\chi}^P$ la représentation de G sur $\bar{I}^P(\delta)$ déduite de $\pi_{\delta,\chi}^P$ par transport de structure via cet isomorphisme.

On note $C(G, P, \delta^*, \chi)$ l'espace des applications ψ sur G, à valeurs dans le dual V_δ^* de V_δ qui sont faiblement continues, i.e. telles que :

Pour tout $v \in V_\delta$, $g \mapsto \langle \psi(g), v\rangle$ soit continue et :

$$\psi(gmu) = \delta_P^{-1/2}(m)\chi(m)\delta^*\left(m^{-1}\right)\psi(g), \quad g \in G,\ m \in M,\ u \in U. \tag{4.6}$$

Le groupe G agit par représentation régulière gauche sur cet espace.

Si $\psi \in C(G, P, \delta^*, \chi)$ et $\varphi \in I_\chi^P(\delta)$, on note $\langle \psi, \varphi\rangle = \int_{K_0} \langle \psi(k), \varphi(k)\rangle\, dk$ qui définit un crochet de dualité G-invariant sur ces espaces (cf. (1.11)). (On note que cette intégrale existe car il s'agit de fonctions localement constantes sur le compact K_0.) Ceci permet d'identifier les éléments de $C(G, P, \delta^*, \chi)$ à des éléments de $I_\chi^P(\delta)^*$.

On suppose maintenant que (δ, V_δ) est une représentation bornée et irréductible de M.

Soit $\chi \in X(M)_\sigma$ vérifiant $\operatorname{Re} \chi\delta_P^{-1/2}$ strictement P-dominant. Soit $\eta \in V_\delta^{*M\cap H}$. On lui associe l'appplication $\varepsilon_e(P, \delta, \chi, \eta)$ définie sur G à valeurs dans V_δ^* par les relations :

(a) $\varepsilon_e(P, \delta, \chi, \eta) = 0$ en dehors de HP.
(b) Pour tout $(h, m, u) \in H \times M \times U$,

$$\varepsilon_e(P, \delta, \chi, \eta)(hmu) = \delta_P^{-1/2}(m)\chi(m)\delta^*\left(m^{-1}\right)\eta. \tag{4.7}$$

Remarquons que pour $g \in HP$, la décomposition $g = hmu$, $h \in H$, $m \in M$, $u \in U$ n'est pas unique mais h varie dans une classe à droite modulo $M \cap H$ et m dans une classe à gauche modulo $M \cap H$. La fonction ε_e est donc bien définie puisque $\chi\delta_P^{-1/2} \in X(M)_\sigma$ et $\eta \in V_\delta^{*M\cap H}$. De plus δ étant bornée et $\operatorname{Re} \chi\delta_P^{-1/2}$ étant strictement P-dominant, il résulte du Théorème 4 (ii) et du Théorème 3 de [3] que :

La fonction ε_e est élément de $C(G, P, \delta^*, \chi)$. (4.8)

On suppose maintenant que P contient $P_\emptyset$. Alors M contient $A_\emptyset$. On reprend les notations de 1.2 et on note $\overline{W}_{M,i}$ un ensemble de représentants dans $N_G(A_\emptyset)$ des doubles classes $W_{H_i}(A_\emptyset) \setminus W(A_\emptyset)/W_M(A_\emptyset)$ contenant l'élément neutre e et $\mathcal{W}_{M,i}^G := \{x_i x \mid x \in \overline{W}_{M,i}\}$. Alors (cf. [3, Lemme 9]) :

Toute (H, P)-double classe ouverte de G est de la forme HyP où y est un élément de :

$$\mathcal{W}_M^G = \bigcup_{i \in I} \mathcal{W}_{M,i}^G.$$

En particulier toute (H, P)-double classe ouverte est de la forme HyP où $y.P := yPy^{-1}$ est un σ-sous-groupe parabolique de G. On notera $\overline{\mathcal{W}}_M^G$ un ensemble de représentants des (H, P)-doubles classes ouvertes, contenant e, et contenu dans $\mathcal{W}_M^G$. (4.9)

A tout $w \in \overline{\mathcal{W}}_M^G$, on associe l'espace :

$$\mathcal{V}(\delta, w) = \left(V_\delta^*\right)^{M \cap w^{-1}.H} \tag{4.10}$$

et on considère la somme :

$$\mathcal{V}(\delta) := \bigoplus_{w \in \overline{\mathcal{W}}_M^G} \mathcal{V}(\delta, w). \tag{4.11}$$

La projection de $\mathcal{V}(\delta)$ sur $\mathcal{V}(\delta, w)$ parallèlement aux autres composantes sera notée $\mathrm{pr}(\delta, w)$ ou pr_w.

Montrons que :

Si $w \in \mathcal{W}_M^G$, alors $w.P$ est un σ-sous-groupe parabolique de sous-groupe de Levi σ-stable $w.M$. (4.12)

Comme $P_\emptyset \subset P$, on a :

$$A_0 \subset M_0 \subset M_\emptyset \subset P \cap \sigma(P) = M.$$

Donc A_0 est un tore déployé maximal de M. Il en résulte que $A_M \subset A_0$, donc $A_{M,\sigma} \subset A_\emptyset$. Ceci joint au Lemme 4 montre que $P = P_\lambda,\ \lambda \in \Lambda(A_\emptyset)$. Alors $w.P = P_\mu$ où $\mu = w\lambda w^{-1} \in w.A_\emptyset$. Or $w.A_\emptyset$ est égal à l'un des A_i, c'est donc un tore σ-déployé maximal. Il en résulte que $\sigma(P_\mu) = P_{\sigma(\mu)} = P_{\mu^{-1}}$, et $P_{\mu^{-1}}$ est bien opposé à $P_\mu = w.P$. Le sous-groupe de Levi σ-stable de P (resp. $w.P$) est alors égal au centralisateur dans G de λ (resp. $\mu = w\lambda w^{-1}$). Le sous-groupe de Levi σ-stable de $w.P$ est donc égal à $w.M$. D'où (4.12).

Pour $w \in \overline{\mathcal{W}}_M^G$, $\eta \in \mathcal{V}(\delta, w)$ et $\chi \in X(M)_\sigma$ vérifiant $\mathrm{Re}\, \chi\delta_P^{-1/2}$ strictement P-dominant on définit :

$$\varepsilon_w(P,\delta,\chi,\eta) = R_{w^{-1}}\varepsilon_e(w.P, w.\delta, w.\chi, \eta) \tag{4.13}$$

où R désigne la représentation régulière droite de G et $w.\delta$ (resp. $w.\chi$) la représentation de $w.M$ déduite de δ (resp. χ) par transport de structure. Cette expression est bien définie car $\eta \in V_\delta^{*M\cap w^{-1}.H}$ équivaut à $\eta \in V_{w.\delta}^{*w.M\cap H}$ et $w.P$ est bien un σ-sous-groupe parabolique d'après (4.12).

Soit σ_w l'involution rationnelle de G définie sur F, donnée par :

$$\sigma_w(g) := w^{-1}\sigma\big(wgw^{-1}\big)w, \quad g \in G.$$

Lemme 11.

(i) *Soit $P = MU$ un σ-sous-groupe parabolique de G contenant $P_\emptyset$ et $w \in \overline{\mathcal{W}}_M^G$. Alors P est un σ_w-sous-groupe parabolique, $\sigma_w(P) = \sigma(P)$ et $P \cap \sigma_w(P) = M$.*
(ii) $X(M)_\sigma \subset X(M)_{\sigma_w}$.

Démonstration. (i) Montrons que :

$$A_\emptyset \text{ est un tore } \sigma_w\text{-déployé.} \tag{4.14}$$

Soit $i \in I$ et $x \in \overline{W}_{M,i}$ tels que $w = x_i x$, alors

$$w.A_\emptyset = x_i x.A_\emptyset = x_i.A_\emptyset = A_i,$$

en particulier,

$$w.A_\emptyset \text{ est } \sigma\text{-déployé,} \tag{4.15}$$

donc :

$$\sigma_w(a) = w^{-1}\sigma\big(waw^{-1}\big)w = w^{-1}\big(wa^{-1}w^{-1}\big)w = a^{-1}, \quad a \in A_\emptyset.$$

On déduit (4.14) de l'égalité précédente.

D'autre part, on a l'inclusion $A_{M,\sigma} \subset A_M \subset A_{M_\emptyset}$, donc $A_{M,\sigma}$ est un tore σ-déployé de $A_{M_\emptyset}$. Donc $A_{M,\sigma} \subset A_\emptyset$ puisque $A_\emptyset$ est le plus grand tore σ-déployé de $A_{M_\emptyset}$. En particulier $A_{M,\sigma}$ est un tore σ_w-déployé.

D'après le Lemme 4, il existe $\lambda \in A_{M,\sigma}$ tel que $P = P_\lambda$. Alors

$$\sigma_w(P) = P_{\sigma_w(\lambda)} = P_{\lambda^{-1}} = \sigma(P),$$

donc $M = P \cap \sigma_w(P)$, d'où (i).

(ii) Montrons que $w^{-1}\sigma(w) \in M_\emptyset$:

D'après (4.15), on a :

$$\sigma\big(waw^{-1}\big) = wa^{-1}w^{-1}, \quad a \in A_\emptyset.$$

D'autre part,

$$\sigma\big(waw^{-1}\big) = \sigma(w)\sigma(a)\sigma\big(w^{-1}\big).$$

On déduit des deux égalités précédentes et du fait que $\sigma(a) = a^{-1}$ que :

$$wa^{-1}w^{-1} = \sigma(w)a^{-1}\sigma\big(w^{-1}\big).$$

Donc $w^{-1}\sigma(w)$ est élément de $Z_G(A_\emptyset)$ qui est égal à $M_\emptyset$ d'après (1.15). Ceci montre que $w^{-1}\sigma(w)$ est élément de $M_\emptyset$ comme désiré.

Soit $\chi \in X(M)_\sigma$. Comme $M_\emptyset \subset M$, on a :

$$\chi\big(\sigma_w(m)\big) = \chi\big(w^{-1}\sigma(w)\big)\chi\big(\sigma(m)\big)\chi\big(w^{-1}\sigma(w)\big)^{-1}, \quad m \in M.$$

Soit encore :

$$\chi\big(\sigma_w(m)\big) = \chi\big(\sigma(m)\big), \quad m \in M.$$

Comme $\chi \in X(M)_\sigma$, on a donc :

$$\chi\big(\sigma_w(m)\big) = \chi^{-1}(m), \quad m \in M.$$

Donc $\chi \in X(M)^{-\sigma_w}$. Par suite, $X(M)_\sigma$ est inclus dans $X(M)^{-\sigma_w}$. Or $X(M)_\sigma$ est connexe et contient l'élément neutre et $X(M)_{-\sigma_w}$ est la composante connexe de l'élément neutre dans $X(M)^{-\sigma_w}$, d'où (ii). □

On associe enfin à tout élément η de $\mathcal{V}(\delta)$ et à tout $\chi \in X(M)_\sigma$ vérifiant $\operatorname{Re} \chi\delta_P^{-1/2}$ strictement P-dominant :

$$j(P, \delta, \chi, \eta) = \sum_{w \in \overline{\mathcal{W}}_M^G} \varepsilon_w\big(P, \delta, \chi, \operatorname{pr}(\delta, w)\eta\big). \tag{4.16}$$

On déduit de (4.8), (4.13) et de (4.16) que :

> L'application $j(P, \delta, \chi, \eta)$ est un élément de $C(G, P, \delta^*, \chi)$ et on l'identifie à un élément H-invariant de $I_\chi^P(\delta)^*$. (4.17)

Remarque 4. La condition $\operatorname{Re} \chi - 2\rho_P$ strictement P-dominant dans le Théorème 3 de [3] où l'induction est non normalisée se traduit ici (où l'induction est normalisée) par la condition $\operatorname{Re} \chi\delta_P^{-1/2}$ strictement P-dominant.

On définit « les intégrales d'Eisenstein » :

> Si φ est un élément de l'espace $\bar{I}^P(\delta)$, $E(P, \delta, \chi, \eta, \varphi)$ est la fonction sur G/H définie par :
>
> $$\big(E(P, \delta, \chi, \eta, \varphi)\big)(gH) = \big\langle \bar{j}(P, \delta, \chi, \eta), \bar{\pi}_{\delta,\chi}^P\big(g^{-1}\big)\varphi\big\rangle, \quad g \in G,$$
>
> où $\bar{j}(P, \delta, \chi, \eta)$ est l'élément de $\bar{I}^P(\delta)^*$ déduit de $j(P, \delta, \chi, \eta)$ par transport de structure à l'aide de la restriction des fonctions à K_0. (4.18)

Remarquons que si $\varphi \in I_\chi^P(\delta)$, et si $\bar\varphi$ est sa restriction à K_0,

$$\big(E(P,\delta,\chi,\eta,\bar\varphi)\big)(gH) = \big\langle j(P,\delta,\chi,\eta), \pi_{\delta,\chi}^P\big(g^{-1}\big)\varphi\big\rangle, \quad g \in G.$$

Soient $P = MU$ et $Q = MV$ deux σ-sous-groupes paraboliques de même sous-groupe de Levi, on rappelle que l'on note $\bar Q := \sigma(Q)$.

Soit ψ une fonction sur $V/V \cap U$ à valeurs dans V_δ. Supposons qu'il existe $v_0 \in V_\delta$ tel que pour tout $\check v_0 \in \check V_\delta$, l'intégrale

$$\int\limits_{V/V\cap U} \big\langle \psi(\dot v), \check v_0\big\rangle\, d\dot v$$

soit absolument convergente, égale à $\langle v_0, \check v_0\rangle$. Alors v_0 est unique car la dualité entre $\check V$ et V est non dégénérée. Dans ce cas nous dirons que l'intégrale $\int_{V/V\cap U} \psi(\dot v)\, d\dot v$ converge et nous poserons :

$$\int\limits_{V/V\cap U} \psi(\dot v)\, d\dot v := v_0. \tag{4.19}$$

Il résulte de cette définition que :

$$\int\limits_{V/V\cap U} \delta(m)\psi(\dot v)\, d\dot v = \delta(m) \int\limits_{V/V\cap U} \psi(\dot v)\, d\dot v, \quad m \in M. \tag{4.20}$$

On rappelle (cf. [23, Théorème IV.1.1 et équation IV.1 (10)]) que :

> Il existe une constante $R_\delta > 0$ telle que, pour tout $\chi \in X(M)$, vérifiant :
>
> $$\langle \operatorname{Re}\chi, \alpha\rangle > R_\delta, \quad \alpha \in \Sigma(P) \cap \Sigma(\bar Q)$$
>
> telle que, pour tout $\varphi \in I_\chi^P(\delta)$ et $g \in G$, l'intégrale $\int_{V/V\cap U} \varphi(g\dot v)\, d\dot v$ converge et l'application $g \mapsto \int_{V/V\cap U} \varphi(g\dot v)\, d\dot v$ définit un élément de $I_\chi^Q(\delta)$ noté $A(Q,P,\delta,\chi)(\varphi)$. De plus, $A(Q,P,\delta,\chi)$ est un opérateur d'entrelacement non nul entre $I_\chi^P(\delta)$ et $I_\chi^Q(\delta)$. (4.21)
>
> L'application $\chi \mapsto A(\bar P,P,\delta,\chi)$ admet un prolongement rationnel au sens de [23, IV.1] que l'on note de même. (4.22)
>
> Il existe un polynôme q non nul sur $X(M)$ tel que si $\chi \in X(M)$ et si $q(\chi) \neq 0$, l'opérateur $A(\bar P,P,\delta,\chi)$ est défini et non nul (cf. [23, IV(10)]). (4.23)

Le symbole $a \to_P \infty$ (resp. $a \to_{\bar P} \infty$) signifie que $a \in A_M$ et que $|\alpha(a)|_F \to +\infty$ pour tout $\alpha \in \Sigma(P)$ (resp. $\Sigma(\bar P)$).

On rappelle que (δ, V_δ) est une représentation bornée et irréductible de M.

Théorème 5. *Avec les notations ci-dessus, soit* $\chi \in X(M)_\sigma$ *vérifiant* (4.21) *pour* $Q = \bar{P}$ *et tel que* $\operatorname{Re} \chi \delta_P^{-1/2}$ *soit strictement* P*-dominant. Alors, pour tout* $\varphi \in I_\chi^P(\delta)$*, pour tout* $g \in G$*, et pour tout* $\eta \in \mathcal{V}(\delta)$*, on a :*

$$\lim_{a \to_{\bar{P}} \infty} \chi(a)\mu_\delta(a)\delta_P^{-1/2}(a)\bigl(E(P, \delta, \chi, \eta, \bar{\varphi})\bigr)(gaH) = \bigl\langle \mathrm{pr}_e\, \eta, \bigl(A(\bar{P}, P, \delta, \chi)(\varphi)\bigr)(g)\bigr\rangle$$

où μ_δ *est le caractère central de* δ *et* $\bar{\varphi}$ *est la restriction de* φ *à* K_0.

On remarque que l'ensemble des χ vérifiant l'hypothèse du Théorème contient l'ensemble $\{\chi \in X(M)_\sigma;\ \langle \operatorname{Re} \chi, \alpha\rangle > R, \alpha \in \Sigma(P)\}$ pour R assez grand.

Démonstration. Par linéarité, on peut supposer $\eta \in \mathcal{V}(\delta, w)$ pour un $w \in \overline{\mathcal{W}}_M^G$ ce que l'on fait dans la suite. En remplaçant φ par $L_{g^{-1}}\varphi$ pour $g \in G$, on se ramène à démontrer le Théorème pour $g = e$. Avec nos hypothèses, $j(P, \delta, \chi, \eta)$ est élément de $C(G, P, \delta^*, \chi)$ (cf. (4.17)). Donc pour $a \in A_M$, on a :

$$\bigl(E(P, \delta, \chi, \eta, \bar{\varphi})\bigr)(aH) = \int_{K_0} \bigl\langle \bigl(j(P, \delta, \chi, \eta)\bigr)(k), \varphi(ak)\bigr\rangle\, dk.$$

On pose $E := E(P, \delta, \chi, \eta, \bar{\varphi})$ et $j := j(P, \delta, \chi, \eta)$. On déduit de (1.10) que l'intégrale $\int_{\bar{U}} \langle j(\bar{u}), \varphi(a\bar{u})\rangle\, d\bar{u}$ est absolument convergente et que :

$$E(aH) = \int_{\bar{U}} \bigl\langle j(\bar{u}), \varphi(a\bar{u})\bigr\rangle\, d\bar{u}.$$

En changeant $\bar{u}$ en $a\bar{u}a^{-1}$ et en utilisant les propriétés de covariance à droite de φ, on a :

$$\chi(a)\mu_\delta(a)\delta_P^{-1/2}(a)E(aH) = \int_{\bar{U}} \bigl\langle j\bigl(a^{-1}\bar{u}a\bigr), \varphi(\bar{u})\bigr\rangle\, d\bar{u}.$$

Si $a \to_{\bar{P}} \infty$, il est clair que $a^{-1}\bar{u}a$ converge vers e et l'expression sous le signe somme converge vers $\langle \mathrm{pr}_e\, \eta, \varphi(\bar{u})\rangle$ pour tout $\bar{u} \in \bar{U}$.

Il suffit donc de vérifier que l'on peut appliquer le Théorème de convergence dominée, et de montrer que :

$$\int_{\bar{U}} \bigl\langle \mathrm{pr}_e\, \eta, \varphi(\bar{u})\bigr\rangle\, d\bar{u} = \bigl\langle \mathrm{pr}_e\, \eta, \bigl(A(\bar{P}, P, \delta, \chi)(\varphi)\bigr)(e)\bigr\rangle, \quad \varphi \in I_\chi^P(\delta). \tag{4.24}$$

Passons à la majoration de $|\langle j(a^{-1}\bar{u}a), \varphi(\bar{u})\rangle|$. Posons :

$$\chi' := \chi \delta_P^{-1/2},$$

de sorte que $\chi' \in X(M)_\sigma$ et que $\operatorname{Re} \chi'$ est strictement P-dominant.

Soit $\bar{u} \in \bar{U} \cap HwP$. Ecrivons $\bar{u} = hwmu$ où $h \in H$, $m \in M$, $u \in U$. Soit $h_w := w^{-1}hw$. On a :

$$\bar{u} = wk(h_w)m(h_w)u(h_w)mu,$$

où, avec les notations de 4.1, $h_w = k(h_w)m(h_w)u(h_w)$. Donc :

$$m(\bar{u}) = m\big(wk(h_w)\big)m(h_w)m.$$

Ce qui implique :

$$\chi'\big(m(\bar{u})\big) = \chi'\big(m\big(wk(h_w)\big)\big)\chi'\big(m(h_w)\big)\chi'(m). \tag{4.25}$$

On vérifie aisément que $w^{-1}Hw$ est le groupe des points fixes de l'involution $\sigma_w = \operatorname{Ad} w^{-1} \circ \sigma \circ \operatorname{Ad} w$. En utilisant le Lemme 11, on peut appliquer le Lemme 9 (ii) à χ' avec $w^{-1}Hw$ au lieu de H. Donc :

Il existe une constante $c_1 > 0$ telle que :

$$\big|\chi'\big(m(h_w)\big)\big| \geqslant c_1, \quad h \in H. \tag{4.26}$$

De plus, $(m(wk(h_w)))^{-1}$ reste dans une partie compacte lorsque h varie dans H. Donc :

Il existe $c_2 > 0$ telle que

$$\big|\chi'\big(\big(m\big(wk(h_w)\big)\big)^{-1}\big)\big| \leqslant c_2. \tag{4.27}$$

Donc, on déduit de (4.25) que :

$$\big|\chi'(m)\big| \leqslant c_1^{-1}c_2\big|\chi'\big(m(\bar{u})\big)\big|, \quad \bar{u} \in \bar{U} \cap HwP. \tag{4.28}$$

Soit $\bar{u} \in \bar{U}$. Si $a^{-1}\bar{u}a \notin HwP$, on a $j(a^{-1}\bar{u}a) = 0$. Si $a^{-1}\bar{u}a \in HwP$, on écrit $a^{-1}\bar{u}a = h_0wm_0u_0$ avec $h_0 \in H$, $m_0 \in M$, $u_0 \in U$, et l'on a :

$$j\big(a^{-1}\bar{u}a\big) = \delta_P^{-1/2}(m_0)\chi(m_0)\delta^*\big(m_0^{-1}\big)\eta. \tag{4.29}$$

D'autre part, on a :

$$\varphi(\bar{u}) = \delta_P^{-1/2}\big(m(\bar{u})\big)\delta_\chi\big(\big(m(\bar{u})\big)^{-1}\big)\varphi\big(k(\bar{u})\big). \tag{4.30}$$

Montrons qu'il existe une constante c', indépendante de $\varphi \in I_\chi^P(\delta)$, telle que :

$$\begin{aligned}\big|\big\langle j\big(a^{-1}\bar{u}a\big), \varphi(\bar{u})\big\rangle\big| \leqslant c'\big|\chi'\big(m(\bar{u})\big)\delta_P^{-1/2}\big(m(\bar{u})\big)\chi\big(\big(m(\bar{u})\big)^{-1}\big)\big| \\ \times \big|\big\langle \delta^*\big(m_0^{-1}\big)\eta, \delta\big(\big(m(\bar{u})\big)^{-1}\big)\varphi\big(k(\bar{u})\big)\big\rangle\big|, \quad a \in A_M^-.\end{aligned} \tag{4.31}$$

En effet, d'après (4.29) et (4.30), on a

$$\begin{aligned}\big|\big\langle j\big(a^{-1}\bar{u}a\big), \varphi(\bar{u})\big\rangle\big| &= \big|\delta_P^{-1/2}(m_0)\chi(m_0)\delta_P^{-1/2}\big(m(\bar{u})\big)\chi\big(\big(m(\bar{u})\big)^{-1}\big)\big| \\ &\quad \times \big|\big\langle \delta^*\big(m_0^{-1}\big)\eta, \delta\big(m^{-1}(\bar{u})\big)\varphi\big(k(\bar{u})\big)\big\rangle\big|.\end{aligned}$$

Or :

$$\delta_P^{-1/2}(m_0)\chi(m_0) = \chi'(m_0)$$

et d'après (4.28) appliquée à $a^{-1}\bar{u}a$:

$$\big|\chi'(m_0)\big| \leqslant c_1^{-1}c_2\big|\chi'\big(m\big(a^{-1}\bar{u}a\big)\big)\big|.$$

De plus, d'après le Lemme 10, il existe $c_3 > 0$ telle que :

$$\big|\chi'\big(m\big(a^{-1}\bar{u}a\big)\big)\big| \leqslant c_3\big|\chi'\big(m(\bar{u})\big)\big|.$$

En posant $c' := c_1^{-1}c_2c_3$, on obtient (4.31). Comme $\chi'\chi^{-1} = \delta_P^{-1/2}$, on déduit de (4.31) que :

$$\big|\big\langle j\big(a^{-1}\bar{u}a\big), \varphi(\bar{u})\big\rangle\big| \leqslant c'\big|\delta_P^{-1}\big(m(\bar{u})\big)\big|\,\big|\big\langle \delta^*\big(m(\bar{u})m_0^{-1}\big)\eta, \varphi\big(k(\bar{u})\big)\big\rangle\big|. \tag{4.32}$$

Montrer que la partie droite de l'inégalité est bornée revient à montrer que pour $\varphi \in I_\chi^P(\delta)$, $|\langle \delta^*(m(\bar{u})m_0^{-1})\eta, \varphi(k(\bar{u}))\rangle|$ est bornée indépendamment de $\bar{u}$ et de a.

La fonction φ étant invariante à gauche par un sous-groupe compact ouvert K_φ, $\varphi(k)$ ne prend qu'un nombre fini de valeurs dans V_δ lorsque k décrit K_0, que l'on note $v_1, \ldots, v_l$.

Alors, puisque δ est bornée et irréductible, d'après le Théorème 4 (ii), il existe $C > 0$ tel que :

$$\big|\big\langle \delta^*(m)\eta, v_i\big\rangle\big| \leqslant C, \quad i = 1, \ldots, l,\ m \in M.$$

On déduit alors de (4.32) que :

$$\big|\big\langle j\big(a^{-1}\bar{u}a\big), \varphi(\bar{u})\big\rangle\big| \leqslant C\big|\delta_P^{-1}\big(m(\bar{u})\big)\big|, \quad \bar{u} \in \bar{U},\ a \in A_M^-. \tag{4.33}$$

Mais (cf. [23, I.1(2)]),

$$\int_{\bar{U}} \big|\delta_P^{-1}\big(m(\bar{u})\big)\big|\,d\bar{u} < +\infty. \tag{4.34}$$

Les hypothèses du Théorème de convergence dominée sont donc réunies. Il reste à montrer (4.24).

Le Théorème de convergence dominée montre que :

$$\int_{\bar{U}} \big|\big\langle \mathrm{pr}_e\,\eta, \varphi(\bar{u})\big\rangle\big|\,d\bar{u} < +\infty.$$

L'élément φ de $I_\chi^P(\delta)$ est fixé par un sous-groupe compact ouvert de G qui contient un sous-groupe compact ouvert K_M de M. Donc :

$$\int_{\bar U} \langle \mathrm{pr}_e\, \eta, \varphi(\bar u)\rangle \, d\bar u = \int_{\bar U} \langle \mathrm{pr}_e\, \eta, \varphi(k_M \bar u)\rangle \, d\bar u, \quad k_M \in K_M.$$

En utilisant les propriétés de φ, on a :

$$\int_{\bar U} \langle \mathrm{pr}_e\, \eta, \varphi(\bar u)\rangle \, d\bar u = \delta_P^{-1/2}(k_M) \int_{\bar U} \langle \delta^*(k_M)\, \mathrm{pr}_e\, \eta, \chi\big(k_M^{-1}\big)\varphi\big(k_M \bar u k_M^{-1}\big)\rangle \, d\bar u, \quad k_M \in K_M.$$

Or, χ et δ_P étant des caractères non ramifiés, $\delta_P^{-1/2}(k_M) = \chi(k_M^{-1}) = 1$ pour $k_M \in K_M$. En changeant de variable, on a :

$$\int_{\bar U} \langle \mathrm{pr}_e\, \eta, \varphi(\bar u)\rangle \, d\bar u = \int_{\bar U} \langle \delta^*(k_M)\, \mathrm{pr}_e\, \eta, \varphi(\bar u)\rangle \, d\bar u, \quad k_M \in K_M,$$

et l'intégrale est toujours absolument convergente.

On a donc, en intégrant par rapport à k_M :

$$\int_{\bar U} \langle \mathrm{pr}_e\, \eta, \varphi(\bar u)\rangle \, d\bar u = \int_{K_M} \int_{\bar U} \langle \delta^*(k_M)\, \mathrm{pr}_e\, \eta, \varphi(\bar u)\rangle \, d\bar u \, dk_M,$$

où dk_M est la mesure de Haar normalisée sur K_M.

En procédant de la même manière, on obtient :

$$\int_{\bar U} \big|\langle \mathrm{pr}_e\, \eta, \varphi(\bar u)\rangle\big| \, d\bar u = \int_{K_M} \int_{\bar U} \big|\langle \delta^*(k_M)\, \mathrm{pr}_e\, \eta, \varphi(\bar u)\rangle\big| \, d\bar u \, dk_M < +\infty. \tag{4.35}$$

On peut donc appliquer le Théorème de Fubini à (4.35) et on obtient :

$$\int_{\bar U} \langle \mathrm{pr}_e\, \eta, \varphi(\bar u)\rangle \, d\bar u = \int_{\bar U} \int_{K_M} \langle \delta^*(k_M)\, \mathrm{pr}_e\, \eta, \varphi(\bar u)\rangle \, dk_M \, d\bar u.$$

On note $e_{K_M} \mathrm{pr}_e\, \eta$ l'élément de $\check V_\delta$ défini par :

$$\langle e_{K_M} \mathrm{pr}_e\, \eta, v\rangle = \int_{K_M} \langle \mathrm{pr}_e\, \eta, \delta(k_M) v\rangle \, dk_M, \quad v \in V_\delta,$$

l'invariance de v par rapport à un sous-groupe ouvert compact de M impliquant que l'intégrale se réduit à une somme finie. On a donc :

$$\int_{\bar U} \langle \mathrm{pr}_e\, \eta, \varphi(\bar u)\rangle \, d\bar u = \int_{\bar U} \langle e_{K_M}\, \mathrm{pr}_e\, \eta, \varphi(\bar u)\rangle \, d\bar u.$$

D'après la définition de $\int_{\bar U} \varphi(\bar u)\, d\bar u$ (cf. (4.19)), il en résulte :

$$\int_{\bar U} \langle \mathrm{pr}_e\, \eta, \varphi(\bar u)\rangle \, d\bar u = \Big\langle e_{K_M}\, \mathrm{pr}_e\, \eta, \int_{\bar U} \varphi(\bar u)\, d\bar u\Big\rangle.$$

Donc :

$$\int_{\bar U} \langle \mathrm{pr}_e\, \eta, \varphi(\bar u)\rangle \, d\bar u = \int_{K_M} \Big\langle \mathrm{pr}_e\, \eta, \delta(k_M) \int_{\bar U} \varphi(\bar u)\, d\bar u\Big\rangle dk_M.$$

Soit encore, grâce à (4.20) :

$$\int_{\bar U} \langle \mathrm{pr}_e\, \eta, \varphi(\bar u)\rangle \, d\bar u = \int_{K_M} \Big\langle \mathrm{pr}_e\, \eta, \int_{\bar U} \delta(k_M)\varphi(\bar u)\, d\bar u\Big\rangle dk_M.$$

Utilisant le fait que φ est K_M-invariante à gauche et $\varphi \in I_\chi^P(\delta)$, on a :

$$\int_{\bar U} \langle \mathrm{pr}_e\, \eta, \varphi(\bar u)\rangle \, d\bar u = \int_{K_M} \Big\langle \mathrm{pr}_e\, \eta, \int_{\bar U} \varphi\big(k_M \bar u k_M^{-1}\big)\, d\bar u\Big\rangle dk_M.$$

En changeant $\bar u$ en $k_M \bar u k_M^{-1}$, on déduit :

$$\int_{\bar U} \langle \mathrm{pr}_e\, \eta, \varphi(\bar u)\rangle \, d\bar u = \int_{K_M} \Big\langle \mathrm{pr}_e\, \eta, \int_{\bar U} \varphi(\bar u)\, d\bar u\Big\rangle dk_M.$$

Soit encore :

$$\int_{\bar U} \langle \mathrm{pr}_e\, \eta, \varphi(\bar u)\rangle \, d\bar u = \Big\langle \mathrm{pr}_e\, \eta, \int_{\bar U} \varphi(\bar u)\, d\bar u\Big\rangle.$$

Tenant compte de (4.21), on en déduit (4.24), ce qui achève la preuve du Théorème. □

Soient $P = MU$ et $Q = LV$ deux σ-sous-groupes paraboliques de G tels que $P_\emptyset \subset P \subset Q$. Alors $M \subset L$ et $V \subset U$. Soient $\bar U$ et $\bar V$ définis par $\bar P = M\bar U$ et $\bar Q = L\bar V$. On pose $\bar U' := \bar U \cap L$. On reprend les notations du Théorème 5. Soit Q' le σ-sous-groupe parabolique de G égal à $(P \cap L)\bar V$, qui admet M pour sous-groupe de Levi σ-stable.

Théorème 6. *Soit* $\chi \in X(M)_\sigma$ *vérifiant* (4.21) *pour* P *et* Q' *et tel que* $\operatorname{Re}\chi\delta_P^{-1/2}$ *soit strictement* P*-dominant. Alors* :

$$\forall\varphi \in I_\chi^P(\delta), \quad \forall g \in G, \quad \forall\eta \in \mathcal{V}(\delta),$$

$$\lim_{a\to_{\bar{Q}}\infty} \chi(a)\mu_\delta(a)\delta_P^{-1/2}(a)\bigl(E(P,\delta,\chi,\eta,\bar{\varphi})\bigr)(gaH) = \int\limits_{\bar{U}'} \bigl\langle j(\bar{u}'), \bigl(A(Q',P,\delta,\chi)(\varphi)\bigr)(\bar{u}')\bigr\rangle\, d\bar{u}'$$

où μ_δ *est le caractère central de* δ, $\bar{\varphi}$ *est la restriction de* φ *à* K_0.

Démonstration. Par linéarité, on peut supposer $\eta \in \mathcal{V}(\delta, w)$ pour un $w \in \overline{\mathcal{W}}_M^G$ ce que l'on fait dans la suite. En remplaçant φ par $L_{g^{-1}}\varphi$ pour $g \in G$, on se ramène à démontrer le Théorème pour $g = e$. D'après le début de la démonstration du Théorème 5, on a :

$$\chi(a)\mu_\delta(a)\delta_P^{-1/2}(a)E(aH) = \int\limits_{\bar{U}} \bigl\langle j\bigl(a^{-1}\bar{u}a\bigr), \varphi(\bar{u})\bigr\rangle\, d\bar{u}, \quad a \in A_L. \tag{4.36}$$

On a l'homéomorphisme $\bar{U}' \times \bar{V} \to \bar{U}$, $(\bar{u}', \bar{v}) \mapsto \bar{u}'\bar{v}$ et, pour un bon choix de mesures,

$$\int\limits_{\bar{U}} f(\bar{u})\, d\bar{u} = \int\limits_{\bar{U}'\times\bar{V}} f(\bar{u}'\bar{v})\, d\bar{u}'\, d\bar{v}, \quad f \in C_c^\infty(G). \tag{4.37}$$

On a donc, d'après (4.36) :

$$\chi(a)\mu_\delta(a)\delta_P^{-1/2}(a)E(aH) = \int\limits_{\bar{U}'\times\bar{V}} \bigl\langle j\bigl(a^{-1}\bar{u}'aa^{-1}\bar{v}a\bigr), \varphi(\bar{u}'\bar{v})\bigr\rangle\, d\bar{u}'\, d\bar{v}.$$

Or $a \in A_L$ et $\bar{u}' \in \bar{U} \cap L$. Donc $a\bar{u}' = \bar{u}'a$ et :

$$\chi(a)\mu_\delta(a)\delta_P^{-1/2}(a)E(aH) = \int\limits_{\bar{U}'\times\bar{V}} \bigl\langle j\bigl(\bar{u}'a^{-1}\bar{v}a\bigr), \varphi(\bar{u}'\bar{v})\bigr\rangle\, d\bar{u}'\, d\bar{v}.$$

Si $a \to_{\bar{Q}} \infty$, il est clair que $a^{-1}\bar{v}a$ converge vers e et l'expression sous le signe somme converge simplement vers $\langle j(\bar{u}'), \varphi(\bar{u}'\bar{v})\rangle$.

On peut appliquer le théorème de convergence dominée grâce aux équations (4.33), (4.34) et (4.37). On peut alors passer à la limite et appliquer le Théorème de Fubini. On a alors :

$$\lim_{a\to_{\bar{Q}}+\infty} \chi(a)\mu_\delta(a)\delta_P^{-1/2}(a)E(aH) = \int\limits_{\bar{U}'}\int\limits_{\bar{V}} \bigl\langle j(\bar{u}'), \varphi(\bar{u}'\bar{v})\bigr\rangle\, d\bar{v}\, d\bar{u}'.$$

On procède comme dans la preuve de (4.24), pour montrer que :

$$\int_{\bar V} \langle j(\bar u'), \varphi(\bar u'\bar v)\rangle \, d\bar v = \left\langle j(\bar u'), \int_{\bar V} \varphi(\bar u'\bar v)\, d\bar v\right\rangle.$$

On a alors :

$$\lim_{a \to_{\bar Q} \infty} \chi(a)\mu_\delta(a)\delta_P^{-1/2}(a)E(aH) = \int_{\bar U'} \left\langle j(\bar u'), \int_{\bar V} \varphi(\bar u'\bar v)\, d\bar v\right\rangle d\bar u'.$$

On remarque que :

$$\bigl(A(Q', P, \delta, \chi)(\varphi)\bigr)(\bar u') = \int_{\bar V} \varphi(\bar u'\bar v)\, d\bar v.$$

On a donc :

$$\lim_{a \to_{\bar Q} \infty} \chi(a)\mu_\delta(a)\delta_P^{-1/2}(a)E(aH) = \int_{\bar U'} \bigl\langle j(\bar u'), \bigl(A(Q', P, \delta, \chi)(\varphi)\bigr)(\bar u')\bigr\rangle \, d\bar u',$$

d'où le Théorème. □

5. Une propriété de la décomposition de Cartan

5.1. Transport de structure

Soit $P = MU$ un σ-sous-groupe parabolique de G contenant $A_\emptyset$. Soit $y = x_i x \in \mathcal{W}_M^G$ (cf. 4.2 pour les notations). Alors (cf. (4.12)), $y.P$ est un σ-sous-groupe parabolique de sous-groupe de Levi σ-stable $w.M = wMw^{-1}$. Pour ce σ-sous-groupe parabolique qui ne contient pas $A_\emptyset$ mais A_i, on peut définir $\mathcal{W}_{y.M}^G$ de même qu'en (4.9), ainsi que $\mathcal{V}(y.\delta, w)$ et $\mathcal{V}(y.\delta)$. En particulier, on peut imposer à $\mathcal{W}_{y.M}^G$ de contenir y^{-1}, qui est un représentant de la $(H, y.P)$ double classe ouverte $Hy^{-1}yPy^{-1} = HPy^{-1}$. Alors $V_\delta^{*M\cap H} = V_{y.\delta}^{*y.M\cap y.H}$ donc :

$$\mathcal{V}\bigl(y.\delta, y^{-1}\bigr) = \mathcal{V}(\delta, e).$$

Si $\eta_e \in \mathcal{V}(\delta, e)$, on notera $\eta_{y^{-1}}$ l'élément correspondant dans $\mathcal{V}(y.\delta, y^{-1})$. On note R la représentation régulière droite. Avec ces notations, soit (δ_0, V_{δ_0}) la représentation triviale de M, soit η_e un élément non nul de $V_{\delta_0}^{*M\cap H}$ et soit $\chi \in X(M)_\sigma$ tel que :

$$\chi \text{ vérifie (4.21) pour } Q = \bar P \text{ et } \operatorname{Re} \chi\delta_P^{-1/2} \text{ soit strictement } P\text{-dominant.} \tag{5.1}$$

Montrons que :

$$\bigl\langle j(P, \delta_0, \chi, \eta_e), \pi_{\delta_0,\chi}^P\bigl(g^{-1}\bigr)\varphi\bigr\rangle = \bigl\langle j(y.P, y.\delta_0, y.\chi, \eta_{y^{-1}}), \pi_{y.\delta_0,y.\chi}^{y.P}\bigl(g^{-1}\bigr)R_y\varphi\bigr\rangle, \tag{5.2}$$

pour $\varphi \in I_\chi^P(\delta_0)$, $y \in \mathcal{W}_M^G$.

En effet, comme on le vérifie aisément, R_y est un opérateur d'entrelacement entre

$$\big(\pi^P_{\delta_0,\chi}, I^P_\chi(\delta_0)\big) \quad \text{et} \quad \big(\pi^{y.P}_{y.\delta_0,y.\chi}, I^{y.P}_{y.\chi}(y.\delta_0)\big),$$

i.e. :

$$\pi^P_{\delta_0,\chi}\big(g^{-1}\big)\varphi = R_{y^{-1}}\pi^{y.P}_{y.\delta_0,y.\chi}\big(g^{-1}\big)R_y\varphi, \quad \varphi \in I^P_\chi(\delta_0),\ g \in G,\ y \in \mathcal{W}^G_{M_\emptyset}. \tag{5.3}$$

D'où :

$$\big\langle j(P,\delta_0,\chi,\eta_e), \pi^P_{\delta_0,\chi}\big(g^{-1}\big)\varphi\big\rangle = \big\langle R_y j(P,\delta_0,\chi,\eta_e), \pi^{y.P}_{y.\delta_0,y.\chi}\big(g^{-1}\big)R_y\varphi\big\rangle. \tag{5.4}$$

Or les fonctions de G dans $V^*_{y.\delta_0}$: $j(y.P, y.\delta_0, y.\chi, \eta_{y^{-1}})$ et $R_y j(P,\delta_0,\chi,\eta_e)$ sont toutes deux nulles en dehors de $Hy^{-1}(y.P)$, coincident en y^{-1}, sont H-invariantes à gauche et vérifient la même relation de covariance à droite sous P donc elles sont nécessairement égales. On a alors :

$$\big\langle j(P,\delta_0,\chi,\eta_e), \pi^P_{\delta_0,\chi}\big(g^{-1}\big)\varphi\big\rangle = \big\langle j(y.P, y.\delta_0, y.\chi, \eta_{y^{-1}}), \pi^{y.P}_{y.\delta_0,y.\chi}\big(g^{-1}\big)R_y\varphi\big\rangle. \tag{5.5}$$

Soit encore, avec les notations de (4.18) :

$$\big(E(P,\delta_0,\chi,\eta_e,\bar\varphi)\big)(gH) = \big(E(y.P, y.\delta_0, y.\chi, \eta_{y^{-1}}, \overline{R_y\varphi})\big)(gH). \tag{5.6}$$

5.2.

Théorème 7. *Il existe $T > 0$ tel que la réunion $\bigcup_{y\in\mathcal{W}^G_{M_\emptyset}} \Omega A^-_T(A_\emptyset)y^{-1}H$ soit disjointe.*

Démonstration. Supposons que ce ne soit pas le cas. Alors, comme $\mathcal{W}^G_{M_\emptyset}$ est fini :

Il existe $y,\ y' \in \mathcal{W}^G_{M_\emptyset}$ avec $y \neq y'$ et une suite (g_n) telle que :

$$g_n = \omega_n\lambda_n y^{-1} = \omega_n y^{-1}\lambda_{y,n} \quad \text{et}$$
$$g_n = \omega'_n\lambda'_n y'^{-1} = \omega'_n y'^{-1}\lambda'_{y',n}$$

où $\omega_n, \omega'_n \in \Omega$, $n \in \mathbb{N}$, $\lambda_n, \lambda'_n \in A^-_n(A_\emptyset)$, et $\lambda_{y,n} := y\lambda_n y^{-1}$, $\lambda'_{y',n} := y'\lambda'_n y'^{-1}$. (5.7)

Montrons que c'est impossible.

On note P (resp. P') le σ-sous-groupe parabolique de G égal à $y.P_\emptyset$, (resp. $y'.P_\emptyset$), et $\bar P := y.\bar P_\emptyset$ (resp. $\bar P' := y'.\bar P_\emptyset$). Alors, comme λ_n et λ'_n sont des éléments de $A^-_n(A_\emptyset)$, on a :

$$\lambda_{y,n} \to_{\bar P} \infty \quad \text{et} \quad \lambda'_{y',n} \to_{\bar P'} \infty. \tag{5.8}$$

Par extraction, on peut supposer que ω_n et ω'_n convergent dans Ω vers ω et ω' puisque Ω est compact.

Soit δ_0 la représentation triviale de $M_\emptyset$. Alors $\delta := y.\delta_0$ est la représentation triviale de M. Pour $\chi \in X(M)_\sigma$ et $\varphi \in I^P_\chi(\delta)$, on pose :

$$A_\chi(\varphi) := A(\bar{P}, P, \delta, \chi)(\varphi) \in \operatorname{Ind}_{\bar{P}}^G V_{\delta_\chi}.$$

On reprend les notations du Théorème 5, et soit $\chi_0 \in X(M_\emptyset)_\sigma$ vérifiant (5.1) et tel que $A_{y.\chi_0}$ soit non nul (cf. (4.23)). Par symétrie des rôles de y et y' et par extraction, on peut supposer que :

$$\big(\chi_0\delta_{P_\emptyset}^{-1/2}\big)\big(\lambda_n^{-1}\big) \geqslant \big(\chi_0\delta_{P_\emptyset}^{-1/2}\big)\big(\lambda_n'^{-1}\big), \quad n \in \mathbb{N}, \tag{5.9}$$

ce que l'on fait. On pose $\chi := y.\chi_0$. Alors :

$$\big(E(P,\delta,\chi,\eta_e,\bar{\varphi})\big)(g_nH) = \big\langle \bar{j}(P,\delta,\chi,\eta_e), \bar{\pi}_{\delta,\chi}^P\big(\lambda_{y,n}^{-1}\big)\bar{\pi}_{\delta,\chi}^P\big(y\omega_n^{-1}\big)\bar{\varphi}\big\rangle, \quad \varphi \in I_\chi^P(\delta).$$

On pose $v_n := y\omega_n^{-1}$. Alors v_n converge vers la limite $v = y\omega$. Donc il existe un rang $n_\varphi \in \mathbb{N}$ à partir duquel le fixateur de φ contient $v^{-1}v_n$. En remarquant que $v_n = v(v^{-1}v_n)$, on obtient :

$$\big(E(P,\delta,\chi,\eta_e,\bar{\varphi})\big)(g_nH) = \big(E\big(P,\delta,\chi,\eta_e,\bar{\pi}_{\delta,\chi}^P(v)\bar{\varphi}\big)\big)(\lambda_{y,n}H), \quad \varphi \in I_\chi^P(\delta),\ n \geqslant n_\varphi.$$

Comme $P = y.P_\emptyset$, $\chi = y.\chi_0$ et δ est la représentation triviale de M, on a $\delta_P(\lambda_{y,n}) = \delta_{P_\emptyset}(\lambda_n)$, $\chi(\lambda_{y,n}) = \chi_0(\lambda_n)$. D'après (5.8), le Théorème 5 implique :

$$\lim_{n\to+\infty} \chi_0(\lambda_n)\delta_{P_\emptyset}^{-1/2}(\lambda_n)\big(E(P,\delta,\chi,\eta_e,\bar{\varphi})\big)(g_nH) = \big\langle \mathrm{pr}_e\,\eta_e, \big(A(\bar{P},P,\delta,\chi)(\varphi)\big)(v)\big\rangle,$$

pour tout $\varphi \in I_\chi^P(\delta)$.

D'après les hypothèses, l'application $A_\chi := A_{y.\chi_0}$ est non nulle. Par G-invariance, il existe $\varphi_0 \in \bar{I}^P(\delta)$ telle que $(A_\chi(\varphi_0))(v)$ soit non nulle. On pose $C := (A_\chi(\varphi_0))(v)$. On a alors :

$$\big(E(P,\delta,\chi,\eta_e,\varphi_0)\big)(g_nH) \sim \chi_0\big(\lambda_n^{-1}\big)\delta_{P_\emptyset}^{-1/2}\big(\lambda_n^{-1}\big)C,$$

où :

$$\lim_{n\to+\infty} \chi_0\big(\lambda_n^{-1}\big)\delta_{P_\emptyset}^{-1/2}\big(\lambda_n^{-1}\big) = +\infty, \tag{5.10}$$

car $\operatorname{Re}\chi_0\delta_{P_\emptyset}^{-1/2}$ est strictement $P_\emptyset$-dominant et $\lambda_n \to_{\bar{P}_\emptyset} \infty$, puisque $\lambda_n \in \Lambda_n^-(A_\emptyset)$. En refaisant le calcul pour $g_n = \omega_n' y'^{-1}\lambda_{y',n}'$, on trouve l'existence d'un rang n_1 à partir duquel :

$$\big(E(P,\delta,\chi,\eta_e,\bar{\varphi}_0)\big)(g_nH) = \big\langle \bar{j}(P,\delta,\chi,\eta_e), \bar{\pi}_{\delta,\chi}^P\big(\lambda_{y',n}'^{-1}\big)\bar{\varphi}'_0\big\rangle,$$

pour $n \geqslant n_1$, et $\varphi_0' = \bar{\pi}_{\delta,\chi}^P(v')\varphi_0$ où $v' = \lim_{n\to+\infty} y'\omega_n'^{-1}$.

Soit $x' := y'y^{-1}$, alors $x'Px'^{-1}$ est le σ-sous-groupe parabolique P' de G. Notons que $Hx'P = Hy'(y^{-1}Py)y^{-1} = Hy'P_\emptyset y^{-1}$ donc la (H,P) double classe $Hx'P$ est ouverte. On peut alors appliquer ce qu'on a vu en 5.1 au sous-groupe parabolique P de G, en prenant pour y' le représentant de la (H,P) double classe x'. On obtient alors :

$$\big(E(P,\delta,\chi,\eta_e,\varphi_0)\big)(g_nH) = \big(E\big(x'.P,x'\delta,x'\chi,\eta_{x'^{-1}},\psi_0'\big)\big)\big(\lambda_{y',n}'H\big)$$

où l'on a posé $\psi_0' = R_{x'}(\varphi_0')$.

Comme $y \neq y' \in \mathcal{W}^G_{M_\emptyset}$, on a $HyP_\emptyset \neq Hy'P_\emptyset$. Tenant compte de la définition de P et x', on en déduit que $H(x'.P) \neq Hx'^{-1}(x'.P)$. Donc $\mathrm{pr}_e\, \eta_{x'^{-1}} = 0$. Comme $\lambda'_{y',n} \to_{\overline{x'.P}} \infty$, il résulte du Théorème 5 et des relations de conjugaison qu'il existe une suite ε_n de limite nulle telle que :

$$\big(E(P,\delta,\chi,\eta_e,\varphi_0)\big)(g_nH) = \varepsilon_n \chi_0\big(\lambda_n'^{-1}\big)\delta_{P_\emptyset}^{-1/2}\big(\lambda_n'^{-1}\big). \tag{5.11}$$

On pose :

$$x_n := \big(E(P,\delta,\chi,\eta_e,\varphi_0)\big)(g_nH), \quad n \in \mathbb{N},$$
$$z_n := \chi_0\big(\lambda_n^{-1}\big)\delta_{P_\emptyset}^{-1/2}\big(\lambda_n^{-1}\big), \quad n \in \mathbb{N},$$

et :

$$z'_n := \chi_0\big(\lambda_n'^{-1}\big)\delta_{P_\emptyset}^{-1/2}\big(\lambda_n'^{-1}\big), \quad n \in \mathbb{N}.$$

Alors, d'après (5.10) :

$$x_n \sim Cz_n. \tag{5.12}$$

Pour tout $n \in \mathbb{N}$, Cz_n est non nul, et d'après (5.12) :

$$x_n(Cz_n)^{-1} \to 1.$$

D'autre part, d'après (5.11), il existe une suite ε_n de limite nulle telle que :

$$x_n = \varepsilon_n z'_n, \quad n \in \mathbb{N}.$$

Donc :

$$\big|\varepsilon_n z'_n\big||Cz_n|^{-1} \to 1. \tag{5.13}$$

Mais, d'après (5.9), on a :

$$0 < z'_n \leqslant z_n, \quad n \in \mathbb{N}.$$

Donc :

$$\big|\varepsilon_n z'_n\big||Cz_n|^{-1} \leqslant \big|\varepsilon_n C^{-1}\big|, \quad n \in \mathbb{N}. \tag{5.14}$$

Comme ε_n est de limite nulle, on trouve alors une contradiction entre (5.13) et (5.14). On en déduit que (5.7) est impossible. D'où le Théorème. □

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Appendice A

Définition 2. Soit A un tore déployé sur F et soit $\mathcal{X} := (\chi_1, \ldots, \chi_l)$ une suite finie de caractères continus de A dans $\mathbb{C}^*$. On appelle fonction de type $\mathcal{X}$ toute fonction φ de A dans $\mathbb{C}$ telle que :

$$\bigl(\bigl(L_a - \chi_1(a)\bigr) \ldots \bigl(L_a - \chi_l(a)\bigr)\varphi\bigr)(x) = 0, \quad a, x \in A.$$

Soit A un tore déployé sur F et soit A^1 le plus grand sous-groupe ouvert compact de A, de sorte que $A = \Lambda(A)A^1$.

Lemme 12.

(i) *Soit $f : A \to \mathbb{C}$ une fonction lisse. Alors f est A-finie si et seulement s'il existe une suite finie $\mathcal{X}$ de caractères lisses de A dans $\mathbb{C}^*$ telle que f soit de type $\mathcal{X}$ sur A.*

(ii) *Soient $E := \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ et $\alpha_1, \ldots, \alpha_l \in E^*$ linéairement indépendants, on pose*

$$\Gamma := \bigl\{\lambda \in \Lambda(A) \bigm| \alpha_i(\lambda) \leqslant 0,\ i = 1, \ldots, l\bigr\}$$

et $B := A^1 \times \Gamma$.
Si f est une fonction de A dans $\mathbb{C}$, lisse, A-finie et nulle sur B, alors f est nulle sur A.

Démonstration. (i) Si f est A-finie, en triangulant l'action du groupe commutatif A dans l'espace vectoriel de dimension finie engendré par les translatés de f par les éléments de A, on trouve l'existence de caractères $(\chi_1, \ldots, \chi_l)$ de A dans $\mathbb{C}$ tels que l'action de A dans une base bien choisie s'écrive :

$$\begin{pmatrix} \chi_1 & & * \\ & \ddots & \\ 0 & & \chi_l \end{pmatrix}.$$

Ces caractères sont lisses puisque f l'est et f est donc de type $(\chi_1, \ldots, \chi_l)$.

Supposons maintenant que f soit une fonction lisse de type $\mathcal{X}$ sur A, où $\mathcal{X} = (\chi_1, \ldots, \chi_l)$ est une suite finie de caractères lisses de A dans $\mathbb{C}$. On note $A^{1'}$ le sous-groupe ouvert compact de A par lequel f est invariante. Montrons que f est A-finie. Pour cela, montrons que l'espace des fonctions lisses de A dans $\mathbb{C}$, invariantes par $A^{1'}$ et de type $\mathcal{X}$ est un espace de dimension finie. On notera $C^\infty(A/A^{1'})_{\mathcal{X}}$ cet espace. Soit $\mathbb{C}[\Lambda(A)]_{\mathcal{X}}$ l'ensemble des éléments f de l'espace $\mathbb{C}[\Lambda(A)]$ tels que :

$$\bigl(L_\lambda - \chi_1(\lambda)\bigr) \ldots \bigl(L_\lambda - \chi_l(\lambda)\bigr) f = 0, \quad \lambda \in \Lambda(A).$$

On considère l'application qui à $f \in C^\infty(A/A^{1'})_\chi$ associe la famille $(\varphi_\omega)_{\omega \in A^1/A^{1'}}$ d'éléments de $\mathbb{C}[\Lambda(A)]_{\mathcal{X}}$ définie par $\varphi_\omega(\lambda) = f(\lambda\omega)$, $\lambda \in \Lambda(A)$, $\omega \in A^1/A^{1'}$. Elle est injective, d'où le résultat puisque $A^1/A^{1'}$ est fini et que $\mathbb{C}[\Lambda(A)]_{\mathcal{X}}$ est de dimension finie (cf. [10, Lemme 3.14]). On a bien montré que $C^\infty(A/A^{1'})_{\mathcal{X}}$ est de dimension finie. Comme pour tout élément a, a' de A, $L_{a'}$ commute à $(L_a - \chi_1(a)) \ldots (L_a - \chi_l(a))$, l'espace $C^\infty(A/A^{1'})_\chi$ est stable par les

translations à gauche par les éléments de A. Donc les translatés de $f \in C^\infty(A/A^{1'})_\chi$ par les éléments de A sont encore dans cet espace. Cela prouve (i).

(ii) D'après (i), il existe une suite finie de caractères lisses de A dans $\mathbb{C}$ notée $\mathcal{X}$ telle que f soit de type $\mathcal{X}$ sur A. En reprenant les notations de la démonstration de (i), il suffit de montrer que pour $\omega \in A^1/A^{1'}$, la fonction φ_ω de type $\mathcal{X}$ sur $\Lambda(A)$, est nulle sur $\Lambda(A)$. Par hypothèse, φ_ω est nulle sur Γ. De plus (cf. [10, Lemme 14]), il existe un ensemble fini $F \subset \Lambda(A)$ tel que :

$$\text{Toute fonction } g \text{ de type } \mathcal{X} \text{ sur } \Lambda(A) \text{ nulle sur } F \text{ est nulle sur } \Lambda(A). \tag{A.1}$$

Soit $\lambda_0 \in \Lambda(A)$ tel que $\alpha_i(\lambda_0) < 0$, $i = 1, \ldots, l$, qui existe d'après l'indépendance linéaire des α_i, et soit $a := \max_{f \in F, i=1,\ldots,l} \alpha_i(f)$. Alors il existe $n_0 \in \mathbb{N}$ tel que $n_0\lambda_0 + F \subset \Gamma$. On pose $\lambda := n_0\lambda_0 \in \Lambda(A)$.

Alors, pour tout $\omega \in A^1/A^{1'}$, la fonction $L_{\lambda^{-1}}\varphi_\omega$ est nulle sur F puisque φ_ω est nulle sur Γ donc $L_{\lambda^{-1}}\varphi_\omega$ est nulle sur $\Lambda(A)$ d'après (A.1). Donc pour tout $\omega \in A^1/A^{1'}$, la fonction φ_ω est nulle sur $\Lambda(A)$, d'où (ii). □

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Sufficiency of Favard's condition for a class of band-dominated operators on the axis

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Abstract

The purpose of this paper is to show that, for a large class of band-dominated operators on $\ell^\infty(\mathbb{Z}, U)$, with U being a complex Banach space, the injectivity of all limit operators of A already implies their invertibility and the uniform boundedness of their inverses. The latter property is known to be equivalent to the invertibility at infinity of A, which, on the other hand, is often equivalent to the Fredholmness of A. As a consequence, for operators A in the Wiener algebra, we can characterize the essential spectrum of A on $\ell^p(\mathbb{Z}, U)$, regardless of $p \in [1, \infty]$, as the union of point spectra of its limit operators considered as acting on $\ell^\infty(\mathbb{Z}, U)$.

Keywords: Limit operator; Favard condition; Fredholm operator; Wiener algebra

1. Introduction

We study linear operators on the space $Y^\infty = \ell^\infty(\mathbb{Z}, U)$ of all bounded two-sided infinite sequences with values in a complex Banach space U. If M is a two-sided infinite band matrix, with entries m_{ij} in the space $L(U)$ of all bounded linear operators on U and $\sup \|m_{ij}\| < \infty$,

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then, after identifying elements of Y^∞ with infinite column vectors, M acts on Y^∞ as what we call a *band operator*. The closure of the set of all band operators in $L(Y^\infty)$ is denoted by $BDO(Y^\infty)$; we call its elements *band-dominated operators*.

Let $K(Y^\infty, \mathcal{P})$ denote the closure in $L(Y^\infty)$ of the set of all operators $A \in L(Y^\infty)$ which are induced by a matrix $M = [m_{ij}]$ with only finitely many non-zero entries. It is not hard to see that $K(Y^\infty, \mathcal{P})$ is a closed two-sided ideal in the Banach algebra $BDO(Y^\infty)$, and we say that a band-dominated operator A is *invertible at infinity* if its coset $A + K(Y^\infty, \mathcal{P})$ is invertible in the factor algebra $BDO(Y^\infty)/K(Y^\infty, \mathcal{P})$. Clearly, the coset $A + K(Y^\infty, \mathcal{P})$ only depends on the asymptotic behaviour at infinity of the matrix entries of (the matrix that induces) A. The study of this asymptotic behaviour requires the study of the so-called limit operators of A. The idea is to associate A with a family, denoted by $\sigma^{\mathsf{op}}(A)$, of linear operators on Y^∞, where each member of the family represents part of the behaviour of A at infinity. The elements of $\sigma^{\mathsf{op}}(A)$ are called *the limit operators* of A. It is known [7] that, for a fairly large class of band-dominated operators A, invertibility at infinity of A is equivalent to what we call uniform invertibility of $\sigma^{\mathsf{op}}(A)$, which means:

(C1) All limit operators of A are injective.
(C2) All limit operators of A are surjective.
(C3) The inverses of the limit operators of A are uniformly bounded.

By looking at the structure of $\sigma^{\mathsf{op}}(A)$, in particular using its compactness properties, it is now possible to reduce the set of conditions $\{(C1), (C2), (C3)\}$ to an equivalent subset. In [7] it is shown that (C3) always follows from $\{(C1), (C2)\}$, so that $\{(C1), (C2), (C3)\} = \{(C1), (C2)\}$. In [1] we then went on and partially removed (C2) under the additional assumption that $A = I + K$ with an operator K whose matrix entries form a collectively compact set in $L(U)$. Note that all results mentioned so far are shown for operators on $\ell^\infty(\mathbb{Z}^N, U)$ with $N \in \mathbb{N}$ and U a complex Banach space. The aim of this paper is to show that, under the same assumption of $A = I + K$ as was made in [1] but now for operators on the axis, i.e. for $N = 1$, condition (C2) can be fully removed so that $\{(C1), (C2), (C3)\} = \{(C1)\}$ then. The remaining condition (C1) is commonly known as *Favard's condition* in the literature [5,18,19].

Historic remarks. The story of limit operators and Favard's condition starts in spaces of functions on a continuous rather than discrete domain. The typical setting was originally that of a (ordinary or partial) differential operator with almost periodic coefficients. First of all, Favard [3] showed that the condition that was subsequently named after him guarantees the existence of almost periodic solutions to a system of ODE's with almost periodic coefficients and an almost periodic right-hand side. Later, Muhamadiev [10] proved that Favard's condition implies the invertibility of Favard's almost periodic differential operator considered as operator from $BC^1(\mathbb{R}, \mathbb{R}^n)$ to $BC(\mathbb{R}, \mathbb{R}^n)$. Extensions of Muhamadiev's result to wider classes of almost periodic operators can be found in [5,11,12,18,19], for example. For operators A with almost periodic coefficients, the connection between A and its limit operators is a lot stronger than in more general settings. In particular, all limit operators of A are norm-limits of translates of A, including the operator A itself.

In [10], Muhamadiev went on to study matrix ordinary differential operators on the real line with merely bounded and uniformly continuous coefficients which lead him to define limit operators as limits of translates of the operator A with respect to what we call $\mathcal{P}$-convergence now (see Section 2.2). In this wider setting he states the theorem that injectivity of all limit operators,

that is Favard's condition, implies their invertibility as operator from $BC^1(\mathbb{R},\mathbb{R}^n)$ to $BC(\mathbb{R},\mathbb{R}^n)$. We remark that this result is very much in the spirit of our paper; it can, in fact, via reduction to an equivalent matrix integral operator, be shown to follow from our Proposition 4.1. (We note that Muhamadiev provided no proof of his result in [10] so that we do not know whether our methods of argument are a generalization of what he had in mind.) Later on, Muhamadiev [11] and Shubin [19] studied elliptic differential operators A with almost periodic coefficients. For infinitely smooth coefficients, Shubin provides a proof of Muhamadiev's result [11] that the Favard condition is equivalent to the invertibility of A on $BC^\infty(\mathbb{R}^N,\mathbb{R})$. In [12], Muhamadiev showed that, for Hölder continuous coefficients, Favard's condition is equivalent to A being Φ_+-semi Fredholm between an appropriate pair of spaces of bounded Hölder continuous functions. Similarly and much more recently, Volpert and Volpert show that, for a general class of scalar elliptic partial differential operators A on an unbounded domain but also for systems of such, the Favard condition is equivalent to the Φ_+-semi Fredholmness of A on appropriate Hölder [21,22] or Sobolev [20,22] spaces. Lange and Rabinovich [6] state a corresponding result about semi Fredholmness of band-dominated operators in the discrete scalar-valued $\ell^\infty(\mathbb{Z}^N,\mathbb{C})$ setting.

In the last 10 years, limit operators of band-dominated operators on discrete ℓ^p spaces with values in an arbitrary complex Banach space U and $p\in(1,\infty)$ have been extensively studied by Rabinovich, Roch and Silbermann [15,16]. The second author [7,8] then extended some of their results to $p\in\{1,\infty\}$. The reformulation of the so-called 'richness' property of a band-dominated operator A in terms of a particular compactness property of the operator spectrum $\sigma^{\mathrm{op}}(A)$ of A in [7] then sparked a symbiosis of the limit operator method with the generalised collectively compact operator theory that was introduced by the first author and Zhang in [2]. The first outcomes of this symbiosis are [1] and the current paper.

Contents of the paper. In Section 2 we introduce the classes of operators that we are interested in. We then define what a limit operator is and quote the result that connects the set of all limit operators to invertibility at infinity. Concluding surjectivity from injectivity whilst working with a family of operators (rather than just a single operator) is one of the main threads of the generalised collectively compact operator theory introduced by the first author and Zhang in [2]. Here we quote a slightly weakened version of a theorem from [2] that will do most of the work for us in Section 3. Roughly speaking, the strategy to conclude surjectivity of a given operator T from its injectivity is to embed it into a set of injective operators, $\mathcal{B}$, that enjoys a type of collective compactness condition and to approximate T by a sequence of operators, for example periodic operators, for which injectivity does imply surjectivity, this sequence being such that its 'limit operators' (in a certain sense) are in the set $\mathcal{B}$.

In Section 3 we state and prove the main theorem of this paper. In a nutshell, the plot of the proof is as follows. Let A be subject to (C1). Then we prove (C2) in these three steps:

(a) If $B\in\sigma^{\mathrm{op}}(A)$ and B has a surjective limit operator C, then B is surjective itself.
(b) Every $B\in\sigma^{\mathrm{op}}(A)$ has a self-similar limit operator C.
(c) Self-similar limit operators (of A, including those of B) are surjective.

By a self-similar operator we mean an operator $C\in L(Y^\infty)$ with $C\in\sigma^{\mathrm{op}}(C)$.

Finally, in Section 4 we study a class of operators which are band-dominated on all spaces $Y^p:=\ell^p(\mathbb{Z},U)$ with $p\in[1,\infty]$ simultaneously. For this particular class of operators, the so-called Wiener algebra $\mathcal{W}$, we demonstrate how the study of Fredholmness and the essential spectrum of $A\in\mathcal{W}$ with respect to any of the spaces Y^p profits from our new results in Y^∞.

2. Preliminaries

Let $p \in [1,\infty]$ and U be a complex Banach space. By $Y^p := \ell^p(\mathbb{Z}, U)$ we denote the usual ℓ^p-space of two-sided infinite sequences $(\ldots, x(-1), x(0), x(1), \ldots)$ with values $x(i)$ in the Banach space U. If we only write the letter Y then the corresponding statement holds with any space Y^p, $p \in [1,\infty]$, in place of Y.

2.1. Operators on Y and corresponding matrices

By $L(Y)$ we denote the set of bounded linear operators on Y. To every operator $A \in L(Y)$ we will associate a two-sided infinite matrix $[A] = [a_{ij}]$ in the canonical way; that is, by the following construction. For $k \in \mathbb{Z}$ let $E_k : U \to Y$ and $R_k : Y \to U$ be extension and restriction operators, defined by $E_k y = (\ldots, 0, y, 0, \ldots)$, for $y \in U$, with the y standing at the kth place in the sequence, and by $R_k x = x(k)$, for $x = (x(j))_{j\in\mathbb{Z}^N} \in Y$. Then the matrix entries of $[A]$ are defined as

$$a_{ij} := R_i A E_j \in L(U), \quad i, j \in \mathbb{Z}, \tag{1}$$

and $[A]$ is called the *matrix representation* of A. Conversely, given a matrix $M = [m_{ij}]_{i,j\in\mathbb{Z}}$ with entries in $L(U)$, we will say that M *induces* the operator

$$(Bx)(i) = \sum_{j=-\infty}^{\infty} m_{ij} x(j), \quad i \in \mathbb{Z}, \tag{2}$$

if the sum converges in U for every $i \in \mathbb{Z}$ and every $x = (x(j))_{j\in\mathbb{Z}} \in Y$ and if the resulting operator B is a bounded mapping $Y \to Y$.

It is not hard to see that if M is an infinite matrix and B is induced, via (2), by M then the matrix representation $[B]$ from (1) is equal to M. It does not work quite like that the other way round: For $p = \infty$, there are operators $A \in L(Y^p)$ (e.g. see Example 1.26 c in [8]) for which the matrix representation $M := [A]$ induces an operator B that is different from A. However, for every $A \in L(Y^p)$ with $p \in [1,\infty)$, the matrix $M := [A]$ with entries (1) induces the operator $B = A$.

We say that $A \in L(Y)$ is a *band operator* and write $A \in BO(Y)$ if it is induced by a matrix $[m_{ij}]$ with only finitely many non-zero diagonals, and we write $A \in BDO(Y)$ and say that A is *band-dominated* if A can be approximated in the operator norm by band operators.

2.2. Invertibility at infinity and limit operators

For an arbitrary set $S \subseteq \mathbb{Z}$, let $P_S \in L(Y)$ denote the operator of multiplication by the characteristic function of S. Some frequently used special cases are $P := P_{\{0,1,\ldots\}}$, $Q := I - P$, $P_n := P_{\{-n,\ldots,n\}}$ and $Q_n := I - P_n$ for $n \in \mathbb{N}$. We then put $\mathcal{P} := \{P_1, P_2, \ldots\}$, define

$$K(Y,\mathcal{P}) := \bigl\{T \in L(Y)\colon \ \|Q_n T\| \to 0,\ \|T Q_n\| \to 0 \text{ as } n \to \infty\bigr\}$$

and say that a sequence $A_1, A_2, \ldots \in L(Y)$ is $\mathcal{P}$*-convergent* to $A \in L(Y)$ if $\|T(A_n - A)\| \to 0$ and $\|(A_n - A)T\| \to 0$ as $n \to \infty$ for every $T \in K(Y,\mathcal{P})$. From [8, Proposition 1.65] we

know that $A_n \xrightarrow{\mathcal{P}} A$ if and only if the sequence (A_n) is bounded and $\|P_k(A_n - A)\| \to 0$ and $\|(A_n - A)P_k\| \to 0$ as $n \to \infty$ for every $k \in \mathbb{N}$.

Let $K_0(Y, \mathcal{P})$ denote the set of all operators $T \in L(Y)$ which are induced by a matrix $[m_{ij}]$ that has only finitely many non-zero entries. Clearly, $K_0(Y, \mathcal{P})$ is a dense subset of $K(Y, \mathcal{P})$ since $\|T - P_n T P_n\| \to 0$ as $n \to \infty$ for all $T \in K(Y, \mathcal{P})$. The set $K(Y, \mathcal{P})$ is a closed two-sided ideal in the Banach algebra $BDO(Y)$. We say that an operator $A \in BDO(Y)$ is *invertible at infinity* if its coset $A + K(Y, \mathcal{P})$ is invertible in the factor algebra $BDO(Y)/K(Y, \mathcal{P})$. The property of invertibility at infinity is of interest for different reasons. On the one hand, it is sufficiently close to Fredholmness to be useful for the study of Fredholmness. On the other hand it is relevant to determining stability of approximation methods in numerical analysis.

For the study of invertibility at infinity, we introduce so-called limit operators. To do this, let $V_k \in L(Y)$ denote the operator of shift by $k \in \mathbb{Z}$ acting by $(V_k x)(i) = x(i - k)$ for every $x \in Y$ and $i \in \mathbb{Z}$. Given $A \in L(Y)$, we say that $B \in L(Y)$ is a *limit operator* of A if there exists a sequence $h = (h(n))_{n \in \mathbb{N}} \subseteq \mathbb{Z}$ with $|h(n)| \to \infty$ and

$$V_{-h(n)} A V_{h(n)} \xrightarrow{\mathcal{P}} B$$

as $n \to \infty$. In this case we also write A_h for B. The set of limit operators A_h of A with respect to all sequences h going to $\pm\infty$ is denoted by $\sigma_{\pm}^{\mathrm{op}}(A)$, respectively. We also put $\sigma^{\mathrm{op}}(A) := \sigma_{+}^{\mathrm{op}}(A) \cup \sigma_{-}^{\mathrm{op}}(A)$ and call it the *operator spectrum* of A. An operator $A \in L(Y)$ is called *rich* if every sequence h of integers going to infinity has a subsequence g such that the limit operator A_g exists. Here is the statement that connects invertibility at infinity with the study of limit operators.

Proposition 2.1. *(See [8, Theorem 1].) A rich operator $A \in BDO(Y^\infty)$ with a preadjoint* (*meaning that A is the adjoint of another operator that acts on a predual space of Y^∞*) *is invertible at infinity if and only if the following conditions hold*:

(C1) *All limit operators of A are injective.*
(C2) *All limit operators of A are surjective.*
(C3) *The inverses of the limit operators of A are uniformly bounded.*

Remark 2.2. It is well known that, for $A \in L(X)$ with a Banach space X in the case that X is the dual of another space Z, the statements

(i) A is the adjoint of an operator $B \in L(Z)$;
(ii) the adjoint A^* maps Z, understood as a subspace of its second dual $Z^{**} = X^*$, into itself;
(iii) A is continuous in the weak* topology on X

are equivalent.

The statement of Proposition 2.1 also holds with Y^∞ replaced by Y^p for $p \in [1, \infty)$ in which case the condition about the existence of a preadjoint is even unnecessary. We will however focus on the case when $p = \infty$ because then it is possible to slim the set of conditions $\{(C1), (C2), C(3)\}$ down quite considerably. More precisely, in Theorem 2 of [8] it was shown that (C3) always follows from (C1) + (C2), which is why we can delete (C3) in the formulation of Proposition 2.1. The purpose of this paper is to show that, for a large class of operators $A \in BDO(Y^\infty)$, already condition (C2), and hence (C3), follows from (C1). For such operators,

even both conditions (C2) and (C3) can be removed in Proposition 2.1. The remaining condition, (C1), is often [4,5,18,19] referred to as Favard's condition after Jean Aimé Favard's work [3].

Definition 2.3. We say that an operator $A \in L(Y^\infty)$ is subject to *Favard's condition*, (FC), if every limit operator of A is injective on Y^∞.

2.3. Collective compactness

A family $\mathcal{K}$ of bounded linear operators on a Banach space Z is called *collectively compact* if, for any sequences $(K_n) \subseteq \mathcal{K}$ and $(z_n) \subseteq Z$ with $\|z_n\| \leqslant 1$, there is always a subsequence of $(K_n z_n)$ that converges in the norm of Z. It is immediate that every collectively compact family $\mathcal{K}$ is bounded and that all of its members are compact operators.

Definition 2.4. For $A \in BDO(Y)$, let $\mathcal{M}(A) \subseteq L(U)$ refer to the set of all matrix entries (1) of $[A]$. Now let $UM(Y)$ denote the set of all $K \in BDO(Y)$ for which $\mathcal{M}(K)$ is collectively compact in $L(U)$. Moreover, by $UM_\$(Y)$ denote the set of all rich operators $K \in UM(Y)$ and put

$$I + UM_\$(Y) := \{I + K\colon\ K \in UM_\$(Y)\}.$$

Remark 2.5. (a) Rabinovich and Roch study Fredholmness and the Fredholm index for operators in the class $I + \mathcal{C}_E^\$$ in [13], where $\mathcal{C}_E^\$$ denotes the set of all rich band-dominated operators (on $E = \ell^p(\mathbb{Z}, U)$ with a complex Banach space U) which are induced by infinite matrices with compact entries in $L(U)$. This is clearly a superclass, precisely: a proper superclass iff $\dim U = \infty$, of $I + UM_\$(Y)$.

(b) It should be mentioned that, if $A \in I + UM(Y)$, the invertibility at infinity of A implies its Fredholmness [8, Proposition 2.15]. Together with Proposition 2.1 and the main result of our paper, Theorem 3.1, this shows that, for $A \in I + UM_\$(Y^\infty)$, the Favard condition (FC) implies Fredholmness of A.

Lemma 2.6. *If U is a finite-dimensional space then*

$$I + UM_\$(Y) = UM_\$(Y) = UM(Y) = BDO(Y).$$

Proof. Let U be finite-dimensional. From Corollary 3.24 in [8] we know that then every band-dominated operator is rich. Since $L(U)$ is finite-dimensional and $\mathcal{M}(K) \subseteq L(U)$ is bounded for every $K \in BDO(Y)$, we get that $\mathcal{M}(K)$ is collectively compact, i.e. $K \in UM(Y)$. □

We now present our main tool from the collectively compact operator theory developed in §4 of [2]. Precisely, we give an adapted version of Proposition 5.17 in [1] that is a bit weaker but still sufficient for our purposes here.

Proposition 2.7. *Let $T \in BDO(Y^\infty)$ and take a sequence $T_n \in BDO(Y^\infty)$, $n \in \mathbb{N}$, such that*

(a) $T_n \xrightarrow{\mathcal{P}} T$;
(b) *T_n injective $\Rightarrow$ T_n surjective, for each $n \in \mathbb{N}$*;
(c) *$\bigcup_{n=1}^\infty \mathcal{M}(T_n - I)$ is collectively compact*;

(d) *there exists a set* $\mathcal{B} \subset L(Y^\infty)$*, such that, for every sequence* $(k(m)) \subset \mathbb{Z}$ *and increasing sequence* $(n(m)) \subset \mathbb{N}$*, there exist subsequences, denoted again by* $(k(m))$ *and* $(n(m))$*, and* $S \in \mathcal{B}$ *such that*

$$V_{-k(m)} T_{n(m)} V_{k(m)} \xrightarrow{\mathcal{P}} S \in \mathcal{B} \quad \text{as } m \to \infty;$$

(e) *every* $S \in \mathcal{B}$ *is injective.*

Then T *is invertible and, for some* $n_0 \in \mathbb{N}$*,* T_n *is invertible for all* $n \geqslant n_0$*, and*

$$\|T^{-1}\| \leqslant \sup_{n \geqslant n_0} \|T_n^{-1}\| < \infty.$$

3. Main result

Theorem 3.1. *If* (FC) *holds for* $A \in I + UM_\$(Y^\infty)$ *then all limit operators of* A *are invertible on* Y^∞ *and their inverses are uniformly bounded.*

The rest of this section is devoted to the proof of Theorem 3.1. Since we know from [8, Theorem 2] that condition (C3) of Proposition 2.1 follows from (C1) and (C2), it remains to show that (C2) follows from (C1) alias (FC) if $A \in I + UM_\$(Y^\infty)$. We break the proof of this fact down into the following three propositions. But first we need two lemmas.

Lemma 3.2. *(See [8, Proposition 3.104].) If* $A \in L(Y)$ *is rich then* $\sigma_\pm^{\mathrm{op}}(A)$ *is sequentially compact with respect to* $\mathcal{P}$*-convergence.*

Lemma 3.3. *Let* $A \in L(Y)$ *and* B *be an arbitrary limit operator of* A*.*

(a) *If* $B \in \sigma_\pm^{\mathrm{op}}(A)$ *then* $\sigma^{\mathrm{op}}(B) \subseteq \sigma_\pm^{\mathrm{op}}(A)$*, for* σ_+^{op} *and* σ_-^{op}*, respectively.*
(b) *If* A *is rich then* B *is rich.*
(c) *If* $A \in UM(Y)$ *then* $B \in UM(Y)$*.*

Proof. (a) This is Corollary 3.97 of [8].

(b) Let $A \in L(Y)$ be rich and $B \in \sigma^{\mathrm{op}}(A)$. From Lemma 3.3(a) and [8, Proposition 3.94] we know that $\{V_{-k} B V_k \colon k \in \mathbb{Z}\} \subseteq \sigma^{\mathrm{op}}(A)$. By Lemma 3.2, we get that $\{V_{-k} B V_k\}$ is relatively $\mathcal{P}$-sequentially compact. Together with [8, Proposition 3.102] this shows that B is rich.

(c) By the definition of a limit operator, the set $\mathcal{M}(B)$ is contained in the closure of $\mathcal{M}(A)$. Consequently, $\mathcal{M}(B)$ is collectively compact if $\mathcal{M}(A)$ is collectively compact. $\square$

Proposition 3.4. *Let* $A \in I + UM_\$(Y^\infty)$ *and* $B \in \sigma_\pm^{\mathrm{op}}(A)$*. If* (FC) *holds for* A *and if* B *has one surjective limit operator,* $C \in \sigma_\pm^{\mathrm{op}}(B)$ *(with the same choice of* $+$ *or* $-$ *as for* B*), then* B *is surjective itself.*

Proof. Suppose, without loss of generality, that $B \in \sigma_+^{\mathrm{op}}(A)$. Then $B = A_h$ for some sequence h of integers $h(1), h(2), \ldots \to +\infty$. By our assumption, there exists a surjective $C \in \sigma_+^{\mathrm{op}}(B)$. By Lemma 3.3(a), we have that $C = A_{\tilde{h}}$ with some integer sequence $\tilde{h}(1), \tilde{h}(2), \ldots \to +\infty$, and by Lemma 3.3(b) and (c) we know that $C \in I + UM_\$(Y^\infty)$.

By passing to subsequences, if necessary, we can always arrange that $\tilde{h}(n-1) < h(n) < \tilde{h}(n)$ for all $n \geqslant 2$, with $\tilde{h}(n) - h(n) \to +\infty$ and $h(n) - \tilde{h}(n-1) \to +\infty$ as $n \to \infty$. Now, for every $n \in \mathbb{N}$, define $g_+(n) := \tilde{h}(n) - h(n) > 0$ and $g_-(n) := \tilde{h}(n-1) - h(n) < 0$, and put

$$A_n := V_{g_-(n)} QCV_{-g_-(n)} + V_{g_+(n)} PCV_{-g_+(n)} + V_{-h(n)} P_{\{\tilde{h}(n-1),\dots,\tilde{h}(n)-1\}} AV_{h(n)}.$$

Our plan is now to check the conditions (a)–(e) of Proposition 2.7 with $B = A_h$ in place of T and with $\mathcal{B} = \sigma^{\mathrm{op}}(A)$, in order to conclude that B is surjective.

(a) It is easy to see that $A_n \overset{\mathcal{P}}{\to} A_h = B$ since $V_{-h(n)} AV_{h(n)} \overset{\mathcal{P}}{\to} A_h$.

(b) Since C is invertible it is Fredholm of index zero. So also $D_1 := PCP + QCQ = C - PCQ - QCP$ is Fredholm of index zero since PCQ and QCP are compact for $C \in I + UM(Y^\infty)$ (note that all entries of $C - I$ are compact operators and that C can be norm-approximated by band operators C' in which case both $PC'Q$ and $QC'P$ have only finitely many non-zero entries). We claim that the same is true for $D_2 := V_{g_-(n)} QCQV_{-g_-(n)} + V_{g_+(n)} PCPV_{-g_+(n)} + P_{\{g_-(n),\dots,g_+(n)-1\}}$ and every $n \in \mathbb{N}$. Indeed, since

$$\begin{aligned} \ker D_2 &= \big\{(\dots, x_{-2}, x_{-1}, 0, \dots, 0, x_0, x_1, \dots)\colon\ (x_i) \in \ker D_1\big\}, \\ \operatorname{im} D_2 &= \big\{(\dots, x_{-2}, x_{-1}, y_{g_-(n)}, \dots, y_{g_+(n)-1}, x_0, x_1, \dots)\colon\ (x_i) \in \operatorname{im} D_1,\ y_j \in U\big\} \end{aligned}$$

hold with the zeros and y_j's in the positions $\{g_-(n), \dots, g_+(n)-1\}$ of the sequence, respectively, we get that

$$\dim\ker D_2 = \dim\ker D_1 < \infty, \qquad \operatorname{codim}\operatorname{im} D_2 = \operatorname{codim}\operatorname{im} D_1 < \infty$$

and hence D_2 is also Fredholm with the same index (namely zero) as D_1. But this proves that

$$\begin{aligned} A_n = D_2 &+ V_{g_-(n)} QCPV_{-g_-(n)} + V_{g_+(n)} PCQV_{-g_+(n)} \\ &+ V_{-h(n)} P_{\{\tilde{h}(n-1),\dots,\tilde{h}(n)-1\}} (A - I) V_{h(n)} \end{aligned}$$

is Fredholm of index zero since all of QCP, PCQ and $P_{\{\tilde{h}(n-1),\dots,\tilde{h}(n)-1\}}(A - I)$ are compact. So each A_n is surjective if injective.

(c) Clearly,

$$\bigcup_{n=1}^{\infty} \mathcal{M}(A_n - I) \subseteq \mathcal{M}(A - I) \cup \mathcal{M}(C - I)$$

is collectively compact in $L(U)$ since $A - I \in UM(Y^\infty)$ by our premise and $C - I \in UM(Y^\infty)$ by Lemma 3.3(c).

(d) Moreover, if $(k(m)) \subseteq \mathbb{Z}$ is arbitrary and $(n(m)) \subseteq \mathbb{N}$ is increasing then, since A and C are rich, there exist subsequences, denoted again by $(k(m))$ and $(n(m))$, and an operator D such that

$$V_{-k(m)} A_{n(m)} V_{k(m)} \overset{\mathcal{P}}{\to} D.$$

It is an easy exercise to check that D is either a translate of B or a limit operator of B (in particular it may be a translate or limit operator of C). In each of these cases D is a limit operator of A, and so $D \in \mathcal{B}$.

(e) Every $D \in \mathcal{B}$ is injective by assumption (FC).

We have seen that conditions (a)–(e) of Proposition 2.7 are satisfied with $\mathcal{B} := \sigma^{\mathrm{op}}(A)$ and we therefore conclude that B is surjective. □

Definition 3.5. We call $C \in L(Y)$ a *self-similar* operator if $C \in \sigma^{\mathrm{op}}(C)$.

Roughly speaking, we think of self-similar operators as containing a copy of themselves, at infinity.

Remark 3.6. A concept that is related to self-similar operators is that of a recurrent operator. An operator $C \in L(Y)$ is called *recurrent* [11] if, for every limit operator D of C, it holds that $\sigma^{\mathrm{op}}(D) = \sigma^{\mathrm{op}}(C)$. It is easy to see that, if C is recurrent, then

(a) all limit operators of C are self-similar;
(b) all limit operators of C are recurrent;
(c) the local operator spectra $\sigma_+^{\mathrm{op}}(C)$ and $\sigma_-^{\mathrm{op}}(C)$ coincide with $\sigma^{\mathrm{op}}(C)$.

We also remark that, in the proof of the following proposition, we even show the slightly stronger result that every rich operator has a recurrent limit operator (namely the operator denoted by B' in the proof). It is not difficult to see that an element $\sigma^{\mathrm{op}}(B)$ of the partially ordered set $(\mathcal{A}, \supseteq)$ in the proof below is maximal iff B is recurrent.

Proposition 3.7. *Every rich operator $B \in L(Y)$ has a self-similar limit operator C.*

Proof. Let

$$\mathcal{A} := \{\sigma^{\mathrm{op}}(B) \colon\ B \in \sigma^{\mathrm{op}}(A)\}$$

which is a partially ordered set, equipped with the order '$\supseteq$.' To be able to apply Zorn's lemma to $\mathcal{A}$, we have to check that its conditions are satisfied. So let $\mathcal{B}$ be a totally ordered subset of $\mathcal{A}$, i.e.

$$\mathcal{B} := \{\sigma^{\mathrm{op}}(B) \colon\ B \in \sigma\}$$

for a subset $\sigma \subseteq \sigma^{\mathrm{op}}(A)$, such that for any two $B_1, B_2 \in \sigma$, we either have $\sigma^{\mathrm{op}}(B_1) \supseteq \sigma^{\mathrm{op}}(B_2)$ or $\sigma^{\mathrm{op}}(B_2) \supseteq \sigma^{\mathrm{op}}(B_1)$.

On $X := \sigma^{\mathrm{op}}(A)$ we define the following family of seminorms. Let

$$\varrho_{2n-1}(T) := \|P_n T\|, \qquad \varrho_{2n}(T) := \|T P_n\|$$

for $n = 1, 2, \ldots$ and every $T \in X$, and denote the topology that is generated on X by $\{\varrho_1, \varrho_2, \ldots\}$ by $\mathcal{T}$. By [8, Proposition 1.65] and since $\|T\| \leqslant \|A\|$ for every $T \in X$, convergence in $(X, \mathcal{T})$ is equivalent to $\mathcal{P}$-convergence on X. Also, since $\mathcal{T}$ is generated by a countable family of seminorms, the topological space $(X, \mathcal{T})$ is metrizable. Therefore, the $\mathcal{P}$-sequential compactness

mentioned in Lemma 3.2 is in fact $\mathcal{P}$-compactness, by which we mean compactness in $(X,\mathcal{T})$. In particular, X itself and all elements of $\mathcal{B}$ are compact sets in $(X,\mathcal{T})$.

Now put $\Sigma := \bigcap_{B\in\sigma} \sigma^{\mathrm{op}}(B)$. We claim that Σ is nonempty. Conversely, suppose

$$\emptyset = \Sigma = \bigcap_{B\in\sigma} \sigma^{\mathrm{op}}(B).$$

Then

$$\bigcup_{B\in\sigma} \big(X \setminus \sigma^{\mathrm{op}}(B)\big) = X \setminus \bigcap_{B\in\sigma} \sigma^{\mathrm{op}}(B) = X \setminus \Sigma = X$$

is an open cover of X. Since X is compact, there is a finite subset $\{B_1,\ldots,B_n\}$ of σ such that

$$X = \bigcup_{i=1}^{n} \big(X \setminus \sigma^{\mathrm{op}}(B_i)\big) = X \setminus \bigcap_{i=1}^{n} \sigma^{\mathrm{op}}(B_i)$$

so that $\bigcap_{i=1}^{n} \sigma^{\mathrm{op}}(B_i) = \emptyset$. But that is impossible since $\{\sigma^{\mathrm{op}}(B_1),\ldots,\sigma^{\mathrm{op}}(B_n)\}$ is a finite subchain of $\mathcal{B}$ consisting of nonempty sets that contain one another.

So $\Sigma \neq \emptyset$. Take a

$$T \in \Sigma = \bigcap_{B\in\sigma} \sigma^{\mathrm{op}}(B) \subseteq \sigma^{\mathrm{op}}(A).$$

From Lemma 3.3(a) we know that $\sigma^{\mathrm{op}}(B) \supseteq \sigma^{\mathrm{op}}(T)$ for every $B \in \sigma$. So $\sigma^{\mathrm{op}}(T) \in \mathcal{A}$ is an upper bound on the chain $\mathcal{B}$.

Now we can apply Zorn's lemma to $\mathcal{A}$ and get that our partially ordered set $(\mathcal{A},\supseteq)$ has a maximal element, say $\sigma^{\mathrm{op}}(B')$ with some $B' \in \sigma^{\mathrm{op}}(A)$. Now pick any $C \in \sigma^{\mathrm{op}}(B')$. From Lemma 3.3(a) we get $\sigma^{\mathrm{op}}(B') \supseteq \sigma^{\mathrm{op}}(C)$. But the maximality of $\sigma^{\mathrm{op}}(B')$ means that $\sigma^{\mathrm{op}}(B') = \sigma^{\mathrm{op}}(C)$. So $C \in \sigma^{\mathrm{op}}(B') = \sigma^{\mathrm{op}}(C)$ is a self-similar limit operator of A. $\square$

Proposition 3.8. *If $C \in I + UM_{\$}(Y^\infty)$ is self-similar and subject to* (FC) *then C is surjective.*

Proof. Since C is self-similar, there is a sequence $h = (h(n))_{n\in\mathbb{Z}}$ with $|h(n)| \to \infty$ and $V_{-h(n)} C V_{h(n)} \stackrel{\mathcal{P}}{\to} C$ as $n \to \infty$. Suppose, for simplicity of our notations, that $h(n) \to +\infty$ and $h(n) > 0$ for all $n \in \mathbb{N}$. (The argument is completely analogous if $h(n) \to -\infty$, where we can suppose that $h(n) < 0$ for all $n \in \mathbb{N}$.)

For every $n \in \mathbb{N}$, define $C_n \in BDO(Y^\infty)$ by

$$(C_n u)(i) := (C V_{-\alpha h(n)} u)(\beta), \quad i = \alpha h(n) + \beta,\ \alpha \in \mathbb{Z},\ \beta \in \big\{0,\ldots,h(n)-1\big\},$$

so that C_n commutes with $V_{h(n)}$.

We claim that this construction is such that Proposition 2.7 applies to C (in place of T) with $\mathcal{B} = \sigma^{\mathrm{op}}(C)$ and therefore proves that C is surjective. So it remains to check that conditions (a)–(e) of Proposition 2.7 are satisfied.

(a) It holds that $C_n \xrightarrow{\mathcal{P}} C$. This can be seen as follows. Fix an arbitrary $m \in \mathbb{N}$. For every $D \in L(Y^\infty)$, it is a simple consequence of the definition of the norm in Y^∞ that

$$\|D\| = \sup_{i\in\mathbb{Z}} \|P_{\{ih(n),\dots,(i+1)h(n)-1\}} D\| \quad \text{for all } n \in \mathbb{N}.$$

Therefore, for every $n \in \mathbb{N}$, it holds that $\|P_m(C - C_n)\| = \sup_{i\in\mathbb{Z}} \gamma(m,n,i)$ with

$$\gamma(m,n,i) := \left\| P_{\{ih(n),\dots,(i+1)h(n)-1\}} P_m \left(C - V_{ih(n)} C V_{-ih(n)}\right)\right\|, \quad i \in \mathbb{Z}.$$

But then it is clear that $\|P_m(C - C_n)\| \to 0$ as $n \to \infty$ since $\gamma(m,n,0) = 0$,

$$\gamma(m,n,-1) = \left\| P_{\{-m,\dots,-1\}}\left(C - V_{-h(n)} C V_{h(n)}\right)\right\| \to 0 \quad \text{as } n \to \infty$$

and $\gamma(m,n,i) = 0$ for all $i \in \mathbb{Z} \setminus \{0,-1\}$ as soon as $|h(n)| > m$.

Analogously, for every $n \in \mathbb{N}$, we have $\|(C - C_n)P_m\| = \sup_{i\in\mathbb{Z}} \delta(m,n,i)$ with

$$\delta(m,n,i) := \left\| P_{\{ih(n),\dots,(i+1)h(n)-1\}} \left(C - V_{ih(n)} C V_{-ih(n)}\right) P_m \right\|, \quad i \in \mathbb{Z}.$$

To see that $\sup_{i\in\mathbb{Z}} \delta(m,n,i) \to 0$ as $n \to \infty$, note that $\delta(m,n,0) = 0$,

$$\delta(m,n,-1) = \left\| P_{\{-h(n),\dots,-1\}}\left(C - V_{-h(n)} C V_{h(n)}\right) P_m \right\| \to 0 \quad \text{as } n \to \infty$$

and, for all $i \in \mathbb{Z} \setminus \{0,-1\}$,

$$\begin{aligned}\delta(m,n,i) &= \left\| P_{\{ih(n),\dots,(i+1)h(n)-1\}}\left(C - V_{ih(n)} C V_{-ih(n)}\right) P_m \right\| \\ &\leqslant 2 \sup_{S,T} \|P_T C P_S\| \to 0\end{aligned}$$

as $n \to \infty$ by [8, Theorem 1.42] and $C \in BDO(Y^\infty)$, where the supremum in the last expression is taken over all sets $S, T \subset \mathbb{Z}$ with $\operatorname{dist}(S,T) \geqslant h(n) - m$.

(b) By Lemma 6.7 in [1] and $C_n V_{h(n)} = V_{h(n)} C_n$ we get that C_n is surjective if injective.

(c) Clearly,

$$\bigcup_{n=1}^{\infty} \mathcal{M}(C_n - I) \subseteq \mathcal{M}(C - I)$$

is collectively compact in $L(U)$ since $C - I \in UM(Y^\infty)$.

(d) Let $(k(m)) \subseteq \mathbb{Z}$ be arbitrary and $(m(n)) \subseteq \mathbb{N}$ be monotonically increasing. Write each $k(m)$ as $\alpha(m)h(n(m)) + \beta(m)$ with $\alpha(m) \in \mathbb{Z}$ and $\beta(m) \in \{0,\dots,h(n(m))-1\}$. Then

$$\begin{aligned}D_m := V_{-k(m)} C_{n(m)} V_{k(m)} &= V_{-\beta(m)} V_{-h(n(m))}^{\alpha(m)} C_{n(m)} V_{h(n(m))}^{\alpha(m)} V_{\beta(m)} \\ &= V_{-\beta(m)} C_{n(m)} V_{\beta(m)}\end{aligned}$$

holds for each $m \in \mathbb{N}$. If $(\beta(m))_{m\in\mathbb{N}}$ has a bounded subsequence, then it even has a constant subsequence, of value $\gamma \in \mathbb{Z}$ say, and the corresponding subsequence of (D_m) converges to $V_{-\gamma} C V_\gamma$. Being a translate of $C \in \sigma^{\mathrm{op}}(C) = \mathcal{B}$, this operator is also in $\sigma^{\mathrm{op}}(C) = \mathcal{B}$. If

$(\beta(m))_{m\in\mathbb{N}}$ goes to infinity, then, since C is rich, it has a subsequence for which the corresponding subsequence of (D_m) is $\mathcal{P}$-convergent to a limit operator of C, clearly also being an element of $\mathcal{B}$.

(e) All operators in $\mathcal{B}=\sigma^{\mathsf{op}}(C)$ are injective by our assumption that (FC) holds for C. $\square$

4. The essential spectrum of operators in the Wiener algebra

Our main result from Section 3 is only valid in Y^∞. By this we mean that there are examples of band-dominated operators all limit operators of which are injective on Y^p without all of them being surjective. But in this section we study a class of operators, the so-called Wiener algebra, which are bounded on all spaces Y^p with $p\in[1,\infty]$ and for which it is possible to profit from our Y^∞ results in the general Y^p setting.

Let $p\in[1,\infty]$ and recall that an operator $A\in L(Y)$ is called a band operator if it is induced by a banded matrix M. From the boundedness of A we get that every diagonal d_k of M is a bounded sequence of elements in $L(U)$. We then put

$$\|A\|_{\mathcal{W}} := \sum_{k=-\infty}^{+\infty}\|d_k\|_\infty = \sum_{k=-\infty}^{+\infty}\sup_{j\in\mathbb{Z}}\|a_{j+k,j}\|_{L(U)}$$

and denote by $\mathcal{W}$ the closure of $BO(Y)$ in the norm $\|.\|_{\mathcal{W}}$. The set $\mathcal{W}$, equipped with the norm $\|.\|_{\mathcal{W}}$, turns out to be a Banach algebra and is called *the Wiener algebra*.

It is easy to see that $\|A\|_{L(Y)}\leqslant\|A\|_{\mathcal{W}}$ for all band operators A, so that the closure of $BO(Y)$ in the $\|.\|_{\mathcal{W}}$ norm, i.e. $\mathcal{W}$, is contained in the closure of $BO(Y)$ in the operator norm, i.e. $BDO(Y)$. Not only are operators $A\in\mathcal{W}$ bounded and band-dominated on all Y^p, $p\in[1,\infty]$, simultaneously, one can also show that if A is invertible on one of the spaces Y, its inverse A^{-1} is automatically in $\mathcal{W}$ again and therefore acts as the inverse of A on all spaces Y. Another important result is that all limit operators of $A\in\mathcal{W}$, with respect to any of the spaces Y, are also contained in $\mathcal{W}$ so that $\sigma^{\mathsf{op}}(A)$ is contained in $\mathcal{W}$ and does not depend on the space Y under consideration.

The following two results follow immediately from Corollaries 6.43 and 6.44 in [1] and our Theorem 3.1. For illustrations of these results in the particular case of a discrete Schrödinger operator and for a class of integral operators on the axis, see the final two chapters of [1].

Proposition 4.1. *Suppose $A\in I+UM_\$(Y)$ is in the Wiener algebra $\mathcal{W}$ and A, if considered on Y^∞, has a preadjoint* (*acting on a predual space of Y^∞*). *Then the following statements are equivalent*:

(FC) *All limit operators of A are injective on Y^∞.*

(i) *All limit operators of A are invertible on* one *of the spaces Y.*

(ii) *All limit operators of A are invertible on* all *the spaces Y and*

$$\sup_{p\in[1,\infty]}\ \sup_{B\in\sigma^{\mathsf{op}}(A)}\left\|B^{-1}\right\|_{L(Y^p)}<\infty.$$

(iii) *A is invertible at infinity on* one *of the spaces Y.*

(iv) *A is invertible at infinity on* all *the spaces Y.*

(v) *A is Fredholm on* one *of the spaces Y.*

(vi) *A is Fredholm on* all *the spaces Y.*

Further, on every space Y it holds that

$$\operatorname{spec}_{\mathrm{ess}}(A) = \bigcup_{B \in \sigma^{\mathrm{op}}(A)} \operatorname{spec}(B) = \bigcup_{B \in \sigma^{\mathrm{op}}(A)} \operatorname{spec}^{\infty}_{\mathrm{point}}(B). \tag{3}$$

In equality (3) we denote by

$$\operatorname{spec}(B) = \{\lambda \in \mathbb{C}\colon\ \lambda I - B \text{ is not invertible on } Y\}$$

the (invertibility) spectrum of B, by

$$\operatorname{spec}_{\mathrm{ess}}(A) = \{\lambda \in \mathbb{C}\colon\ \lambda I - A \text{ is not Fredholm on } Y\}$$

the essential spectrum of A, and by

$$\operatorname{spec}^{\infty}_{\mathrm{point}}(B) = \left\{\lambda \in \mathbb{C}\colon\ \lambda I - B \text{ is not injective on } Y^{\infty}\right\}$$

the point spectrum of B on Y^{∞}.

Remark 4.2. (a) In [13], the Fredholm index of A (see our Remark 2.5(a) for the class of operators studied in [13]) is shown to be subject to

$$\operatorname{ind} A = \operatorname{ind}(PB_+P + Q) + \operatorname{ind}(QB_-Q + P) \tag{4}$$

for an arbitrary choice of operators $B_\pm \in \sigma^{\mathrm{op}}_\pm(A)$, respectively. The arguments there are made for operators on $\ell^p(\mathbb{Z}, U)$ with $p \in (1, \infty)$ but inspection of the proofs shows that the result carries over to $p \in [1, \infty]$. The other condition in [13] is that the Banach space U has to have what Rabinovich and Roch call the *symmetric approximation property* (*sap*). This means that there is a sequence $\Pi_1, \Pi_2, \ldots$ of finite rank projections on U such that $\Pi_n \to I$ and $\Pi_n^* \to I^*$ pointwise on U and its dual space U^*, respectively. Note that [13] extends results, in particular formula (4), from [14,17], where band-dominated operators on $\ell^p(\mathbb{Z}, \mathbb{C})$ are studied with $p = 2$ and $p \in (1, \infty)$, respectively.

(b) In [9], Fredholmness and index of operators on $\ell^p(\mathbb{Z}^N, U)$ are studied for (almost) arbitrary Banach spaces U and arbitrary operators $A \in \mathcal{W}$. In particular, it is shown that if $A \in \mathcal{W}$ is Fredholm on one of the spaces $\ell^p(\mathbb{Z}^N, U)$ with $p \in [1, \infty]$, then it is Fredholm on all of these spaces and its index does not depend on p. The key observation here is that A has a Fredholm regularizer in the Wiener algebra that acts as its regularizer on all spaces $\ell^p(\mathbb{Z}^N, U)$.

In the particularly simple case of a finite-dimensional space U we know, by Lemma 2.6, that $I + UM_{\$}(Y) \cap \mathcal{W} = \mathcal{W}$ and that the predual of $Y^{\infty} = \ell^{\infty}(\mathbb{Z}, U)$ exists and is isomorphic to $\ell^1(\mathbb{Z}, U^*)$ and the preadjoint operator of $A \in L(Y^{\infty})$ always exists and is induced by $[a^*_{ji}]$ on $\ell^1(\mathbb{Z}, U^*)$. Consequently, the conditions of Proposition 4.1 simplify, and we can even make a statement on the Fredholm index.

Corollary 4.3. *Suppose $A \in \mathcal{W}$ and U is finite-dimensional. Then statements* (FC) *and* (i)–(vi) *of Proposition* 4.1 *are all equivalent. Moreover, if A is subject to all these equivalent statements then the Fredholm index of A is the same on each space Y and is given by* (4). *Further, on every space Y,* (3) *holds.*

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